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# DEFORMATIONAL SYMMETRIES OF SMOOTH FUNCTIONS ON NON-ORIENTABLE SURFACES

Iryna Kuznietsova — Sergiy Maksymenko

Abstract. Given a compact surface $M$, consider the natural right action of the group of diffeomorphisms $\mathcal{D}(M)$ of $M$ on $\mathcal{C}^\infty(M,\mathbb{R})$ defined by the rule: $(f,h)\mapsto f\circ h$ for $f\in\mathcal{C}^\infty(M,\mathbb{R})$ and $h\in\mathcal{D}(M)$. Denote by $\mathcal{F}(M)$ the subset of $\mathcal{C}^\infty(M,\mathbb{R})$ consisting of functions $f\colon M\to\mathbb{R}$ taking constant values on connected components of $\partial M$, having no critical points on $\partial M$, and such that at each of its critical points $z$ the function $f$ is $\mathcal{C}^\infty$ equivalent to some homogenenous polynomial without multiple factors. In particular, $\mathcal{F}(M)$ contains all Morse maps. Let also $\mathcal{O}(f)=\{f\circ h\mid h\in\mathcal{D}(M)\}$ be the orbit of $f$. Previously, the algebraic structure of $\pi_1\mathcal{O}(f)$ was computed for all $f\in\mathcal{F}(M)$, where $M$ is any orientable compact surface distinct from 2-sphere. In the present paper we compute the group $\pi_0\mathcal{S}(f,\partial\mathbb{M})$, where $\mathbb{M}$ is a Möbius band, and $\mathcal{S}(f,\partial\mathbb{M})=\{h\in\mathcal{D}(\mathbb{M})\mid f\circ h=f,\ h|_{\partial\mathbb{M}}=\mathrm{id}_{\mathbb{M}}\}$ is the subgroup of the corresponding stabilizer of $f$ consisting of diffeomorphisms fixed on the boundary $\partial\mathbb{M}$. As a consequence we obtain an explicit algebraic description of $\pi_1\mathcal{O}(f)$ for all non-orientable surfaces distinct from Klein bottle and projective plane.

## 1. Introduction

The present paper continues a series of works by many authors [25], [26], [29], [33]–[35], [31], [7], [16]–[19] and others devoted to the study of the natural

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