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THE RELATIVE CUP-LENGTH IN LOCAL MORSE COHOMOLOGY

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ABSTRACT. Local Morse cohomology associates cohomology groups to isolating neighbourhoods of gradient flows of Morse functions on (generally non-compact) Riemannian manifolds M . We show that local Morse cohomology is a module over the cohomology of the isolating neighbourhood, which allows us to define a cup-length relative to the cohomology of the isolating neighbourhood that gives a lower bound on the number of critical points of functions on M that are not necessarily Morse. Finally, we illustrate by an example that this lower bound can indeed be stronger than the lower bound given by the absolute cup-length.

1. Introduction

Let f be a Morse function on a closed manifold M . For a generic choice of a metric g on M , one can define Morse cohomology, which is isomorphic to the singular cohomology of M , and consequently its isomorphism class does neither depend on f nor g . The generators of the Morse complex are the critical points of f , and the ranks of the cohomology groups of M give lower bounds on the minimal number of critical points of Morse functions on M . These lower bounds are captured by the celebrated Morse inequalities.

There are also bounds for the number of critical points of possibly degenerate functions. Via the Lusternik-Schnirelman category, the cup-length gives a lower bound on the number of critical points for any function (not necessarily Morse)

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