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THE BORSUK–ULAM PROPERTY FOR HOMOTOPY CLASSES OF MAPS FROM THE TORUS TO THE KLEIN BOTTLE PART 2

Daciberg Lima Gonçalves — John Guaschi
Vinicius Casteluber Laass

Abstract. Let M be a topological space that admits a free involution τ, and let N be a topological space. A homotopy class $\beta \in [M, N]$ is said to have *the Borsuk–Ulam property with respect to* τ if for every representative map $f\colon M \to N$ of β, there exists a point $x \in M$ such that $f(\tau(x)) = f(x)$. In this paper, we determine the homotopy class of maps from the 2-torus $\mathbb{T}^2$ to the Klein bottle $\mathbb{K}^2$ that possess the Borsuk–Ulam property with respect to any free involution of $\mathbb{T}^2$ for which the orbit space is $\mathbb{K}^2$. Our results are given in terms of a certain family of homomorphisms involving the fundamental groups of $\mathbb{T}^2$ and $\mathbb{K}^2$. This completes the analysis of the Borsuk–Ulam problem for the case $M = \mathbb{T}^2$ and $N = \mathbb{K}^2$, and for any free involution τ of $\mathbb{T}^2$.

1. Introduction

The classical Borsuk–Ulam theorem states that for all $n \in \mathbb{N}$ and any continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$, there exists a point $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$

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