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SPECTRAL NUMBERS AND MANIFOLDS WITH BOUNDARY

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Abstract. We consider a smooth submanifold N with a smooth boundary in an ambient closed manifold M and assign a spectral invariant $c(\alpha, H)$ to every singular homological class $\alpha \in H_*(N)$ and a Hamiltonian H defined on the cotangent bundle T^*M. We also derive certain properties of spectral numbers, for example we prove that spectral invariants $c_\pm(H,N)$ associated to the whole Floer homology $HF_*(H, N : M)$ of the submanifold N, are the limits of decreasing nested family of open sets.

1. Introduction

1.1. Floer Homology for submanifold with boundary. Let N be a compact submanifold with a smooth boundary of an ambient closed manifold M. Consider a conormal bundle of ∂N, $\nu^*(\partial N)$, defined as

$$\nu^*(\partial N) = \{(\mathbf{q}, \mathbf{p}) \in T^*M \mid \mathbf{q} \in \partial N,\ \mathbf{p}|_{T_{\mathbf{q}}\partial N} = 0\},$$

which is a Lagrangian submanifold of the cotangent bundle T^*M. Let $\nu^*_-(\partial N)$ denote the negative part of the conormal bundle to the boundary:

$$\nu^*_-(\partial N) := \big\{(\mathbf{q}, \mathbf{p}) \in \nu^*(\partial N) \mid \mathbf{p}(\overrightarrow{n}) \geq 0,\ \text{for } \overrightarrow{n} \in TN \text{ inner normal to } \partial N\big\}$$

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and define a *negative conormal to* N, as

$$\overline{\nu}^* N := \nu^*_-(\partial N) \cup \nu^*\mathrm{Int}(N).$$

Note that we use the notation $\overline{\nu}^* N$ instead of $\nu^* N$ in order to emphasize that it is not a conormal bundle in a classical sense (in fact it is not a bundle at all, but just a set). The set $\overline{\nu}^* N$, called a *negative conormal to* N, is a singular Lagrangian submanifold that allows a smooth approximation by exact Lagrangian submanifolds. If Υ is such an approximation, we can define the Floer homology groups for the pair (O_M, Υ), denoted by $HF_*(O_M, \Upsilon : H, J)$ where O_M is the zero section of T^*M. After establishing canonical isomorphisms between Floer homology with two different approximations Υ_1 and Υ_2 (possibly with a change of almost complex structure J), we define the Floer homology for N as a direct limit of the Floer groups $HF_*(O_M, \Upsilon : H, J)$, by taking the canonical isomorphisms for morphisms that define the direct limit. Floer homology obtained in this way is isomorphic to the Morse homology of the set N, with appropriately chosen Morse function that satisfies certain conditions near the boundary ∂N.

This construction when N is an open subset of M with a smooth boundary is done by Kasturirangan and Oh in [10] (see also [21] for the sheaf theoretic point of view). Our construction differs from the one in [10] for open subsets by the fact that the tubular neighbourhood of ∂U with respect to U is always a trivial vector bundle, which makes the situation of a submanifold of codimension zero simpler.

1.2. Spectral numbers. Spectral numbers or spectral invariants were originally defined by Viterbo in [30], in the case of a cotangent bundle, in terms of generating functions for Lagrangian submanifolds. In [22], [23], Oh used Viterbo's ideas to define spectral invariants as homologically visible critical values of the action functional

$$a_H(x) := \int_x \theta - \int_0^1 H(x(t), t)\, dt,$$

which is well defined in a cotangent bundle. Here θ is the canonical Liouville form. We assume that the domain of a_H is the set of smooth paths with the ends on Lagrangian submanifolds L_0 and L_1. Let L_0 be the zero section O_M and $L_1 = \phi_H^1(L_0)$, where ϕ_H^1 is a time-one-map generated by a Hamiltonian H. The set of critical points $\mathrm{Crit}(a_H)$ of a_H consists of Hamiltonian paths starting at O_M and ending on $\phi_H^1(O_M)$. Let

$$HF_*^\lambda(O_M, \phi_H^1(O_M) : H, J)$$

denotes the filtrated homology defined via filtrated Floer complex:

$$CF_*^\lambda(O_M, \phi_H^1(O_M) : H) := \mathbb{Z}_2\langle \{x \in \mathrm{Crit}(a_H) \mid a_H(x) < \lambda\}\rangle.$$

These homology groups are well defined since the boundary map preserves the filtration:

$$\partial\colon CF_*^\lambda(O_M, \phi_H^1(O_M) : H) \to CF_*^\lambda(O_M, \phi_H^1(O_M) : H),$$

due to the well defined action functional that decreases along its "gradient flows". For a singular homology class α define

$$\sigma(\alpha, H) := \inf\left\{\lambda \in \mathbb{R} \mid F_H(\alpha) \in \operatorname{Im}\bigl(\imath_*^\lambda\bigr)\right\} \tag{1.1}$$

where

$$\imath_*^\lambda\colon HF_*^\lambda(O_M, \phi_H^1(O_M) : H, J) \to HF_*(O_M, \phi_H^1(O_M) : H, J)$$

is the homomorphism induced by inclusion and

$$F_H\colon H_*(M) \to HF_*(O_M, \phi_H^1(O_M) : H, J)$$

is an isomorphism between singular and Floer homology groups. The construction of spectral invariants is done in [22] in case of cotangent bundle, and in [23] for cohomology classes.

It turns out that Oh's invariants and those of Viterbo are in fact the same, see [16], [17].

Using spectral numbers, Oh derived the non-degeneracy of Hofer's metric for Lagrangian submanifolds, a result earlier proved by Chekanov [4] using different methods. Another application to Hofer geometry is given in [18], [19] in the characterization of geodesics in Hofer's metric for Lagrangian submanifolds of the cotangent bundle via quasi-autonomous Hamiltonians. Spectral invariants in cotangent bundles were also studied by Monzner, Vichery and Zapolsky in [20].

Spectral numbers also appear in contexts different (more general) from cotangent bundles. In [15], Leclercq constructed spectral invariants for Lagrangian Floer theory in the case when L is a closed submanifold of a compact (or convex in infinity) symplectic manifold P and $\omega|_{\pi_2(P,L)} = 0$, $\mu|_{\pi_2(P,L)} = 0$, where μ is the Maslov index. Schwarz constructed spectral invariants for contractible periodic orbits when (P, ω) is a symplectic manifold with $\omega|_{\pi_2(P)} = 0$ and $c_1|_{\pi_2(P)} = 0$ (see [29]).

Symplectic invariants were further investigated by Eliashberg and Polterovich [6], Polterovich and Rosen [26], Oh [25], Humilière, Leclercq and Seyfaddini [8], Lanzat [14] and also in [5], [16], [17].

1.3. Results of the paper. Let N be a compact submanifold with boundary ∂N of a closed manifold M. In Section 2 we describe how to construct a singular Lagrangian submanifold $\overline{\nu}^*N \subset T^*M$ associated to N, as well as a smooth exact Lagrangian approximation Υ of $\overline{\nu}^*N$. The construction of a Floer homology assigned to N as a direct limit of Floer homology of the pairs (O_M, Υ) is also done in Section 2.

Next, we construct a PSS-type isomorphism between Floer homology for N and Morse homology $HM_*(f_N, N)$ of N. Here we choose a Morse function to be admissible, i.e. to have gradient trajectories that behave well near the boundary ∂N (see the definition at the beginning of the Section 3). We also impose some transversality conditions to a Hamiltonian H. More precisely, we have the following theorem.

Theorem A. *Let N be a compact submanifold of M with a smooth boundary ∂N. Let f_N be an admissible Morse function on N and H as in* (2.6). *There exists a* PSS *type isomorphism*

$$\Phi\colon HM_*(f_N, N) \xrightarrow{\cong} HF_*(H, N : M). \qquad \square$$

This theorem is reformulated as Theorem 3.2 and proven in Section 3.

In Section 4 we assign spectral invariant to every non-zero homology class $[\alpha] \in HF_*(N)$. We define it via PSS isomorphism and we show that it is a limit of spectral invariants in Floer homology of approximations (Theorem 4.4). We also prove the continuity of spectral invariants with respect to Hofer's norm (Theorem 4.6).

There are numerous products in Floer and Morse homologies of different types, that define different algebraic structures on homologies. Here we introduce two pair-of-pants type products that behave naturally with respect to spectral invariants. More precisely, we prove the following theorem (formulated as Theorem 4.7 in Section 4.

Theorem B. *Let $HF_*(H_2 : M)$ denotes Floer homology for of the pair $(O_M, \phi^1_H(O_M))$* (*see Subsection* 4.2). *There exist pair-of-pants type products*:

$$\circ\colon HF_*(H_1, N : M) \otimes HF_*(H_2 : M) \to HF_*(H_3, N : M),$$
$$\star\colon HM_*(f, N) \otimes HF_*(H, N : M) \to HF_*(H, N : M),$$

that turns Floer homology $HF_*(H, N : M)$ into a $HM_*(f, N)$-module. The above product satisfy:

$$\Phi(\alpha \,\widetilde{\cap}\, \beta) = \Phi(\alpha) \circ \Phi(\beta).$$

Here, $\widetilde{\cap}$ is the exterior intersection product in Morse homology

$$\widetilde{\cap}\colon HM_*(f_1, N) \otimes HM_*(f_2, M) \to HM_*(f_3, N),$$

*and Φ is the*PSS *type isomorphism between Morse homology of M and Floer homology for the zero section* (*see* [11]).

Further, we prove the triangle inequality:

$$c^3_N(\alpha \,\widetilde{\cap}\, \beta, H_1 \sharp H_2) \leq c^1_N(\alpha, H_1) + \sigma^2(\beta, H_2),$$

for every $\alpha \in HM_*(f_1, N)$ and $\beta \in HM_*(f_2, M)$. Index j in c^j_N and σ^j emphasizes the corresponding Morse function f_j. This is done in Theorem 4.8.

Using Albers' chimney maps [2], we compare spectral invariants for submanifold with boundary N and periodic orbits in T^*M (Theorem 4.9).

Spectral numbers $c_\pm$ are defined as the infimum (respectively, supremum) of all $\lambda \in \mathbb{R}$ for which the inclusion map $\imath_*^\lambda$ is surjective (respectively, trivial), see the definitions on the pages 642 and 650. Finally, we prove the continuity of spectral numbers $c_\pm$ in the following sense.

THEOREM C. *Let submanifold $N \subset M$ with boundary ∂N be framed* (*see the definition in the page* 642). *Let U_n be a decreasing sequence of open subsets with smooth boundaries and $\bigcap_n U_n = N$, where N is a smooth submanifold with a smooth boundary. Then $\lim_{n\to\infty} c_+(H, U_n) = c_+(H, N)$.*

Theorem C is also true for c_- and therefore for $\gamma(H,N) := c_+(H,N) - c_-(H,N)$. As a consequence of Theorem C we conclude that

$$c_+(H, \partial N) \le c_+(H, N),$$

see Remark 5.11. Note that it is easy to see that this inequality is true in the special case when the Hamiltonian H is the pullback of a Morse function $f\colon M \to \mathbb{R}$, i.e. $H = f \circ \pi$ (where $\pi\colon T^*M \to M$ is the canonical projection), if we restrict to the Morse function with negative gradient pointing inwards (which is precisely one of the conditions in the definition of an admissible Morse function, see the beginning of the Section 3). Indeed, the critical points of the action functional are precisely the critical points of the Morse function, and the holomorphic strips do not depend on t, actually they coincide with the gradient trajectories of f. Therefore

$$\begin{aligned} c_+(f, \partial N) &= \max \mathcal{A}_H|_{\{\gamma\colon [0,1]\to T^*M | \gamma(0)\in O_M,\, \gamma(1)\in \nu^*(\partial N)\}} \\ &= -\min f|_{\partial N} = -f(A), \\ c_+(f, N) &= \max \mathcal{A}_H|_{\{\gamma\colon [0,1]\to T^*M | \gamma(0)\in O_M,\, \gamma(1)\in \overline{\nu}^* N\}} = -\min f|_N = -f(B), \end{aligned}$$

for some $A \in \partial N$ and $B \in N$ (see Figure 1), where $\mathcal{A}_H$ is defined in (2.8). Since f decreases along its negative gradient trajectories, and B is the point of absolute minimum of $-f$ we have $f(B) \le f(C) \le f(A)$, where C is some point in N (negative gradient trajectories cannot leave N, since f is admissible), therefore $c_+(H, \partial N) \le c_+(H, N)$. We prove this inequality in the case of general Hamiltonian, under some topological assumptions on N and ∂N, see Remark 5.11.

Theorem C is formulated and proven as Theorem 5.3 in Section 5. The main step in the proof is the construction of a map from filtered Floer homology for N to filtered Floer homology for U_n, which will change the filtration for an arbitrary small amount.

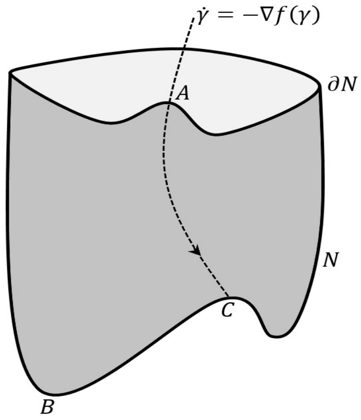

Figure 1. The case of Morse function.

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2. Approximations and Floer homology

Let M be a closed smooth manifold of dimension n and $N \subset M$ a smooth compact k-dimensional submanifold with a smooth boundary (i.e. $\partial N \subset N$). Let $\mathrm{Col}_N(\partial N)$ denote a collar neighbourhood of ∂N in N. We can assume that $\mathrm{Col}_N(\partial N)$ is a subset of a tubular neighbourhood $\mathrm{Tb}_M(\partial N)$ in M, which is isomorphic to the normal vector bundle over ∂N, therefore $\mathrm{Col}_N(\partial N)$ induces a line bundle over ∂N, the double of the collar, denote it by $\mathrm{Tb}_N(\partial N)$. Since $\mathrm{Col}_N(\partial N) \cong \partial N \times [0, +\infty)$, this bundle is trivial:

$$\mathrm{Tb}_N(\partial N) \cong \partial N \times \mathbb{R}. \tag{2.1}$$

Let us denote by E a $(n-k-1)$-rank vector sub-bundle of $\mathrm{Tb}_M(\partial N)$ such that

$$\mathrm{Tb}_M(\partial N) \cong \mathrm{Tb}_N(\partial N) \oplus E$$

and by

$$\tau\colon \mathrm{Tb}_N(\partial N) \oplus E \to \mathrm{Tb}_N(\partial N), \quad e_1 \oplus e_2 \mapsto e_1.$$

Let

$$\pi_1\colon \mathrm{Tb}_N(\partial N) \to \partial N, \qquad \pi_2\colon E \to \partial N$$

be the vector bundle projections. Let us check that $\mathrm{Tb}_M(\partial N)$ is a $(n-k-1)$-rank vector bundle over $\mathrm{Tb}_N(\partial N)$ with τ as a projection map. For every $e_1 \in \mathrm{Tb}_N(\partial N)$, we have

$$\begin{aligned}\tau^{-1}(e_1) &= \{(\pi_1(e_1), x, e_1) \in \mathrm{Tb}_N(\partial N) \oplus E \mid \pi_2(x) = \pi_1(e_1)\}\\ &\cong \{x \in E \mid \pi_2(x) = \pi_1(e_1)\} = \pi_2^{-1}(\pi_1(e_1))\end{aligned}$$

which is a fibre in the bundle E. The transition maps are defined via the transition maps of the bundle E.

We wish to define a map

$$\widetilde{\tau}\colon T^*(\mathrm{Tb}_M(\partial N)) \to T^*(\mathrm{Tb}_N(\partial N)).$$

Let us choose a connection on $\mathrm{Tb}_M(\partial N)$ which determines the horizontal and vertical space

$$T_p\mathrm{Tb}_M(\partial N) = H_p \oplus V_p$$

at any point $p \in \mathrm{Tb}_M(\partial N)$. For $X \in T\mathrm{Tb}_M(\partial N)$ denote by X_{hor} the horizontal part of X. Let $a \in T^*(\mathrm{Tb}_M(\partial N))$, $Y \in T(\mathrm{Tb}_N(\partial N))$. Define

$$\widetilde{\tau}(a)(Y) := a\big((\tau^{-1}Y)_{\mathrm{hor}}\big),$$

where τ^{-1} is any right inverse of τ. Although τ^{-1} is not uniquely defined, the term $\left(\tau^{-1}Y\right)_{\mathrm{hor}}$ does not depend on the choice of τ^{-1}, but only on the chosen connection.

It follows from (2.1)

$$T^*(\mathrm{Tb}_N(\partial N)) \cong T^*(\partial N) \times \underbrace{(\mathbb{R} \times \mathbb{R})}_{A}. \tag{2.2}$$

Let $\nu^*(\partial N)$ and $\nu^*(\mathrm{Int}\, N)$ denote the conormal bundles of ∂N and $\mathrm{Int}\, N$ in T^*M. Define

$$\nu^*_-(\partial N) := \big\{(\mathbf{q}, \mathbf{p}) \in \nu^*(\partial N) \mid \mathbf{p}(\overrightarrow{n}) \geq 0, \text{ for } \overrightarrow{n} \in TN \text{ inner normal to } \partial N\big\}$$

and

$$\overline{\nu}^* N := \nu^*_-(\partial N) \cup \nu^*(\mathrm{Int}\, N).$$

Let us describe the set $\overline{\nu}^*N$ in local coordinates. Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be local coordinates in a neighbourhood (in $T^*(\mathrm{Tb}_M(\partial N))$) of a point in ∂N such that

$$\begin{array}{ll} (q_1, \ldots, q_n) & \text{are coordinates in } M, \\ (q_1, \ldots, q_{k-1}, 0, \ldots, 0) & \text{are coordinates in } \partial N, \\ (q_1, \ldots, q_k, 0, \ldots, 0),\ q_k \leq 0 & \text{are coordinates in } N. \end{array} \tag{2.3}$$

In such coordinates:

- the set $\nu^*_-(\partial N)$ is locally described as

$$\{(q_1, \ldots, q_{k-1}, \underbrace{0, \ldots, 0}_{n-k+1}, \underbrace{0, \ldots, 0}_{k-1}, p_k, \ldots, p_n) \mid p_k \leq 0\}$$

- the set $\overline{\nu^*(\mathrm{Int}\, N)}$ is locally described as

$$\{(q_1, \ldots, q_k, \underbrace{0, \ldots, 0}_{n-k}, \underbrace{0, \ldots, 0}_{k}, p_{k+1}, \ldots, p_n) \mid q_k \leq 0\}$$

  near the boundary,

- the set $\overline{\nu}^*N$ is locally described as

$$\{(q_1,\dots,q_{k-1},\underbrace{0,\dots,0}_{n-k+1},\underbrace{0,\dots,0}_{k-1},p_k,\dots,p_n)\mid p_k\le 0\}$$
$$\cup\,\{(q_1,\dots,q_k,\underbrace{0,\dots,0}_{n-k},\underbrace{0,\dots,0}_{k},p_{k+1},\dots,p_n)\mid q_k\le 0\}.$$

In these coordinates, the set A from (2.2), is given by $\{(q_k,p_k)\}$. Note that we can choose coordinates (2.3) such that (q_k,p_k) are well defined on A. Indeed, let

$$\psi\colon \mathrm{Tb}_N(\partial N)\cong \partial N\times\mathbb{R}$$

be a diffeomorphism from (2.1) that satisfies

$$\psi^{-1}(x_1,\dots,x_{k-1},s)\in N,\quad \text{for } s\le 0.$$

For $p\in\partial N$ let $\varphi_p\colon U_p\to\mathbb{R}^n$ be some coordinates satisfying (2.3) and denote by $\pi_{n-k}\colon\mathbb{R}^n\to\mathbb{R}^{n-k}$ the projection onto the last $n-k$ coordinates. Then

$$\phi:=(\psi,\pi_{n-k}\circ\varphi_p)$$

are required coordinates (in necessary, we can decrease the neighbourhood of ∂N in T^*M in order to have all coordinates and diffeomorphisms well defined).

Denote by $\widetilde{\pi}_A\colon T^*(\partial N)\times A\to A$ the canonical projection and define

$$\pi_A\colon T^*(\mathrm{Tb}_M(\partial N))\to A,\qquad \pi_A:=\widetilde{\pi}_A\circ\widetilde{\tau}.$$

Let C be the singular curve in A defined as:

$$C:=\{(q_k,0)\mid q_k\le 0\}\cup\{(0,p_k)\mid p_k\le 0\}.$$

We have

$$\pi_A(\overline{\nu}^*N\cap T^*(\mathrm{Tb}_M(\partial N)))=C.$$

Let C_1 be any fixed smooth decreasing curve that coincides with x-axis for $x\in(-\infty,-1]$ and with y-axis for $y\in(-\infty,-1]$. Let C_ε, for $\varepsilon\in(0,\varepsilon_0)$, be the rescaling of C_1 with respect to the cone with the vertex in the corner of the curve C (see Figure 2). We say that the family $\{C_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ approximates the singular curve C and we say that any curve C_ε is an approximation of C. See also the proof of [10, Theorem 2.3] and [10, Example 2.1].

Define:

$$\Upsilon_\varepsilon:=\{(q_1,\dots,q_k,\underbrace{0,\dots,0}_{n-k},\underbrace{0,\dots,0}_{k},p_k,\dots,p_n)\mid (q_k,p_k)\in C_\varepsilon\}. \tag{2.4}$$

The approximation Υ_ε is described near the collar of ∂N in the following way. Let

$$\sigma\colon T^*(\mathrm{Tb}_N(\partial N))\to T^*(\partial N)$$

denote the projection to the first component in (2.2). Then it holds

$$\overline{\nu}^*N\cap T^*(\mathrm{Tb}_M(\partial N))=\sigma(\nu^*_-(\partial N)\cap\overline{\nu^*(\mathrm{Int}\,N)})\times C,$$

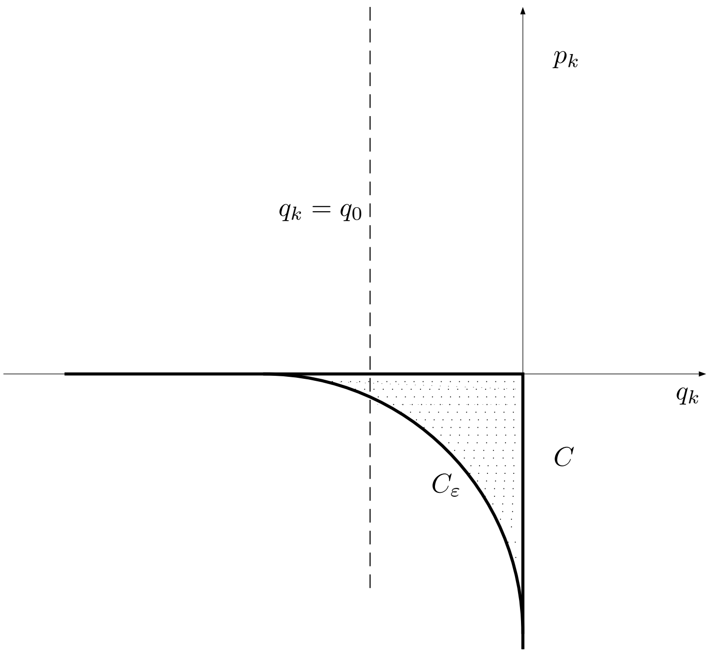

Figure 2. Function h_{Υ_ε} is the area of the shaded region.

since both objects are globally defined and coincide in the same local coordinates. We define Υ_ε globally as:

$$\Upsilon_\varepsilon := \left[\sigma(\nu^*_-(\partial N)\cap\overline{\nu^*(\operatorname{Int} N)})\times C_\varepsilon\right]\cup\left[\overline{\nu}^*N\setminus\sigma(\nu^*_-(\partial N)\cap\overline{\nu^*(\operatorname{Int} N)})\times C\right]. \tag{2.5}$$

Proposition 2.1. *Υ_ε is an exact Lagrangian submanifold of M, for every ε.*

Proof. Define the function $h_{\Upsilon_\varepsilon}\colon \Upsilon_\varepsilon \to \mathbb{R}$ as follows. Let $\pi\colon T^*M \to M$ denote the canonical projection.

(a) On $\overline{\nu}^*N\setminus\pi^{-1}(\mathrm{Tb}_M(\partial N)) = O_M|_{N\setminus\mathrm{Tb}_M(\partial N)}$: h_{Υ_ε} is equal to zero;
(b) on the intermediate region of $\overline{\nu}^*N\cap\pi^{-1}(\mathrm{Tb}_M(\partial N))$: the value of h_{Υ_ε} at the point $(q_1,\dots,q_{k-1},q_k^0,q_{k+1},\dots,q_n,p_1,\dots,p_n)$ is the negative area of the shaded region in Figure 2 (bounded by C_ε, the q_k-axis and the line $q_k = q_k^0$);
(c) on $\nu^*_-(\partial N)\cap\Upsilon_\varepsilon$: h_{Υ_ε} equals the negative area bounded by the q_k-axis, the p_k-axis and the curve C_ε.

Then $\theta|_{T\Upsilon_\varepsilon} = dh_{\Upsilon_\varepsilon}$, where θ is the canonical Liouville form on T^*M. Indeed, in the cases (a) and (c) both $\theta|_{T\Upsilon_\varepsilon}$ and $dh_{\Upsilon_\varepsilon}$ equal to zero. Regarding the case (b), if we denote by $\varphi_\varepsilon(q_k)$ the function whose graph in (q_k,p_k)-plane is the curve C_ε

(see Figure 2), we have:

$$dh_{\Upsilon_\varepsilon} = \sum_j \frac{\partial h_{\Upsilon_\varepsilon}}{\partial q_j}\, dq_j + \sum_j \frac{\partial h_{\Upsilon_\varepsilon}}{\partial p_j}\, dp_j = \varphi(q_k)dq_k,$$

while, on the other hand, we have:

$$\theta|_{T\Upsilon_\varepsilon} = \sum_j p_j\, dq_j|_{T\Upsilon_\varepsilon} = \varphi(q_k)\, dq_k$$

(the last equality follows from (2.4)). $\square$

We will occasionally drop the subscript ε and denote an approximation only by Υ, if the quantity ε in the approximation is not crucial in that moment.

Floer homology for the pair of exact Lagrangians $(O_M, \Upsilon_\varepsilon)$ is defined in the standard way. Choose a compactly supported Hamiltonian $H\colon T^*M \times [0,1] \to \mathbb{R}$ such that

$$\phi_H^1(O_M) \pitchfork O_M$$

and

$$\phi_H^1(O_M) \cap O_M|_{\partial N} = \emptyset, \quad \phi_H^1(O_M) \pitchfork \overline{\nu}^* N. \tag{2.6}$$

Both of the above conditions can be obtained by a generic choice of H. The set of the generators of $CF(O_M, \Upsilon_\varepsilon : H)$ consists of Hamiltonian paths

$$\dot{x} = X_H(x), \quad x(0) \in O_M,\, x(1) \in \Upsilon_\varepsilon, \tag{2.7}$$

which are critical points of the *effective action functional:*

$$\mathcal{A}_H^{\Upsilon_\varepsilon}(\gamma) := \int \gamma^*\theta - \int_0^1 H(\gamma(t), t)dt - h_{\Upsilon_\varepsilon}(\gamma(1)). \tag{2.8}$$

The boundary map $\partial_{J,H}$ is defined by the number of perturbed holomorphic discs with boundary on O_M and Υ_ε:

$$\begin{cases} u\colon \mathbb{R} \times [0,1] \to T^*M, \\ \dfrac{\partial u}{\partial s} + J_\varepsilon\left(\dfrac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(s,0) \in O_M, \quad u(s,1) \in \Upsilon_\varepsilon. \end{cases} \tag{2.9}$$

Floer homology of the manifold with a boundary N is defined as the direct limit of above Floer homologies for the approximations Υ_ε:

$$HF_*(H, N : M) := \varinjlim HF_*(O_M, \Upsilon_\varepsilon : H, J_\varepsilon), \tag{2.10}$$

after defining an appropriate preorder and a directed set. Take the set of all approximations and define a relation $\leq$ on it by

$$\Upsilon_{\varepsilon_1} \leq \Upsilon_{\varepsilon_2} \;\Leftrightarrow\; \varphi_{\varepsilon_1} \leq \varphi_{\varepsilon_2} \quad \text{on } U,$$

where $h_{\Upsilon_{\varepsilon_1}} = \varphi_{\varepsilon_1} \circ \pi$ and $h_{\Upsilon_{\varepsilon_2}} = \varphi_{\varepsilon_2} \circ \pi$, and $\pi\colon T^*M \to M$ is the canonical projection. It is not hard to check that the relation $\leq$ is a preorder that turns the set of all approximations into a directed set. Since h_{Υ_ε} measures the negative area between Υ_ε and $\overline{\nu}^*N$, $\Upsilon_1 \leq \Upsilon_2$ means that Υ_2 is a better approximation than Υ_1.

The connecting morphism

$$\mathbf{F}_{12}\colon HF_*(O_M, \Upsilon_1 : H, J_1) \to HF_*(O_M, \Upsilon_2 : H, J_2)$$

that define the direct limit for $\Upsilon_1 \leq \Upsilon_2$ is a canonical homomorphism constructed using the standard cobordism arguments (see [10] or (4.1) below).

3. Morse homology and PSS isomorphism

We say that a Morse function f_N defined on N is *admissible* if the following holds:

(i) there are no critical points of f_N in ∂N

(ii) for all $p \in \partial N$, $df_N(\overrightarrow{n}) < 0$ where $\overrightarrow{n} \in T_pN$ is an inner normal to ∂N.

We will first extend f_N to a Morse function f defined on $\mathrm{Tb}_N(\partial N) \cup N$ (i.e. on N extended by the outside collar of ∂N, see (2.1)), and then extend f to a Morse function f_ν defined on the tubular neighbourhood ν of the set $\mathrm{Tb}_N(\partial N) \cup N$. We consider the set $\mathrm{Tb}_N(\partial N) \cup N$ instead of just N because in the case of submanifold with a boundary, the tubular neighbourhood might not be an open subset of M (see [9]), and we need to have an extension of f_N defined on some open set in M containing N in order to apply Schwarz construction (see Lemma 3.1). Finally we will extend f_ν to a Morse function on M in a suitable way.

Let us construct the described extensions precisely. Choose an extension f of f_N such that

- $\mathrm{Crit}(f) = \mathrm{Crit}(f_N)$,
- no negative gradient trajectories starting at some point in N can leave N.

See the details in [28, Subsection 4.2.3]. Now choose a Riemannian metric g on M such that the pair (f_N, g) is Morse–Smale. For such f_N, Morse homology $HM_*(f_N, N)$ is isomorphic to the singular homology $H_*(N)$ of N.

Now we can extend f to the normal bundle of $N \cup \mathrm{Tb}_N(\partial N)$ in M, and then to the whole M to obtain Morse function $f_M : M \to \mathbb{R}$ with the following properties:

- no negative gradient trajectory $\dot{\gamma} = -\nabla f_M$ leaves N,
- there are no critical points of f_M in some neighbourhood of N in M,
- Morse complex of f_N is identified with a subcomplex of f_M.

Indeed, let $\nu := \nu(N \cup \mathrm{Tb}_N(\partial N))$ denotes a tubular neighbourhood of $N \cup \mathrm{Tb}_N(\partial N)$ in M. This neighbourhood exists since $N \cup \mathrm{Tb}_N(\partial N)$ is an embedded submanifold of M and it is possesses a $(n-k)$-dimensional vector bundle structure of normal bundle (see [9, Corollary 2.3]; if necessary, we can shrink the fibres in $\mathrm{Tb}_N(\partial N)$ to obtain that the closure of $N \cup \mathrm{Tb}_N(\partial N)$ embeds as a closed subset of M, which is assumed in [9]). We will identify the tubular neighbourhood ν with the corresponding normal bundle.

Let $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ be a Riemannian metric with associated quadratic form

$$q(\,\cdot\,) := \langle\,\cdot\,,\,\cdot\,\rangle_\nu.$$

Define

$$f_\nu(p, u_p) := f(p) + q(u_p).$$

The function f_ν is Morse function defined on ν, the critical points of f_ν are precisely the critical points of f, of the same Morse index. To define an extension f_M of f_ν, we use the following fact.

Lemma 3.1 ([28, Lemma 4.15]). *Let M be a smooth manifold, let A be closed and W open subset and $f_W \in C^\infty(W, \mathbb{R})$ be a Morse function on the submanifold W, such that*

$$\mathrm{Crit}(f_W) \subset \mathring{A} \subset A \subset W \subset M.$$

Then there exists a smooth Morse function f_M defined on M which extends f_W meaning that $f_W|_A = f_M|_A$.

For our needs, we take any closed set A satisfying: $N \subset \mathring{A} \subset A \subset \nu$ and $W := \nu$. (See also [28, Subsection 4.2.1] and [28, Subsection 4.2.3] for more details.) For two Riemannian metrics g_a and g_b there is a canonical isomorphism

$$\mathbf{G}_{ab} : HM_*(f_N, N : g_a) \to HM_*(f_N, N : g_b)$$

satisfying

$$\mathbf{G}_{ac} = \mathbf{G}_{bc} \circ \mathbf{G}_{ab}, \qquad \mathbf{G}_{aa} = \mathrm{Id}.$$

This functoriality allows us to consider the set $\{HM_*(f_N, N : g_a)\}$ as a directed system, indexed by the directed set of all generic Riemannian metrics $\{g_a\}$ with the full relation as a preorder. We define Morse homology $HM_*(f_N, N)$ as a direct limit:

$$HM_*(f_N, N) := \varinjlim HM_*(f_N, N : g_a) := \bigsqcup_s HM_*(f_N, N : g_s)/\sim$$

where $p_a \sim p_b$ if and only if $\mathbf{G}_{ac}(p_a) = \mathbf{G}_{bc}(p_b)$ for some c. The set $HM_*(f_N, N)$ obviously has a vector space structure and is isomorphic to all $HM_*(f_N, N : g_s)$.

Concerning the isomorphism between Floer homology $HF_*(H, N : M)$ and Morse homology $HM_*(f_N, N)$, we have the following theorem.

Theorem 3.2. *Let f_N be admissible and H as in* (2.6). *There exists a PSS type isomorphism*

$$\Phi\colon HM_*(f_N, N) \xrightarrow{\cong} HF_*(H, N : M).$$

Proof. We first define the PSS homomorphism for the approximations. Let us fix the approximation Υ. Let $p \in N$ be a critical point of f_N and x be a Hamiltonian path with $x(0) \in O_M$, $x(1) \in \Upsilon$. The PSS homomorphism is defined via the number of mixed objects

$$\begin{aligned}\mathcal{M}(p,x) :=& \mathcal{M}(p,x,O_M,\Upsilon : f_N, H, J, g)\\ :=& \Bigg\{(\gamma,u) \;\Bigg|\; \gamma\colon (-\infty,0] \to N,\ u\colon [0,+\infty)\times[0,1] \to T^*M,\\ &\dot\gamma(s) = -\nabla_g f_N(\gamma(s)),\\ &\frac{\partial u}{\partial s} + \left(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)\right) = 0,\\ &u(s,0), u(0,t) \in O_M,\ u(s,1) \in \Upsilon,\\ &\gamma(-\infty) = p,\ u(+\infty,t) = x(t),\\ &u(0,1) = \gamma(0)\Bigg\},\end{aligned}$$

where $\rho_R\colon [0,+\infty) \to \mathbb{R}$ is a smooth function such that

$$\rho_R(s) = \begin{cases} 1 & \text{for } s \geq R,\\ 0 & \text{for } s \leq R-1.\end{cases}$$

The loss of the compactness of the manifolds $\mathcal{M}(p,x)$ manifests in a breaking of gradient trajectories or holomorphic strips. The appearance of a bubbling is excluded due to the exact Lagrangian boundary conditions. This in particular means that the zero dimensional manifolds $\mathcal{M}(p,x)$ are compact, so the map

$$p \mapsto \sum_{x \in CF_*(O_M,\Upsilon:H)} n(p,x)x, \qquad n(p,x) := \sharp\,\mathcal{M}(p,x) \pmod 2,$$

is well defined on the chain level. On the other hand, the description of the boundary of one-dimensional manifolds of the same type provides the fact that the above map descends on the homology level:

$$\Phi^{\Upsilon}\colon HM_*(f_N, N : g) \to HF_*(O_M, \Upsilon : H, J).$$

The map Φ^{Υ} turns out to be an isomorphism; by standard cobordism arguments one can prove that its inverse is defined via the number of mixed objects of the type (u,γ). All the above holds for generic choices of almost complex structure J and Riemannian metric g. The details of the above construction are standard (see [12] for the case of an open subset of the base).

In order to prove that Φ^{Υ} defines a map between Morse and Floer homologies defined as the direct limits of approximations, one needs to check that it commutes with the morphisms defining the direct limits. This means that, if we denote:

$$\Phi^a\colon HM_* \to HF_*(O_M, \Upsilon_a : H, J_a),$$
$$\Phi^b\colon HM_*(f_N, N : g_b) \to HF_*(O_M, \Upsilon_b : H, J_b),$$

for two approximations Υ_a and Υ_b and (generically chosen) almost complex structures J_a and J_b and Riemannian metrics g_a and g_b, it holds

$$\Phi^b \circ \mathbf{G}_{ab} = \mathbf{F}_{ab} \circ \Phi^a.$$

This also can be done using suitable one-dimensional auxiliary manifolds, similarly to [12]. Let us denote by

$$\Phi\colon HM_*(f_N, N) \to HF_*(H, N : M)$$

the induced homomorphism on the direct limits.

Now, if we denote by $\Psi^a := (\Phi^a)^{-1}$, we have the same arguments as above for the maps Ψ^a. Therefore we have a well defined homomorphism

$$\Psi\colon HF_*(H, N : M) \to HM_*(f_N, N).$$

From

$$\Psi^a \circ \Phi^a = \mathrm{Id}_{HM_*(f_N, N:g_a)}, \qquad \Phi^a \circ \Psi^a = \mathrm{Id}_{HF_*(O_M, \Upsilon_a:H, J_a)}$$

we deduce:

$$\Psi \circ \Phi = \mathrm{Id}_{HM_*(f_N, N)}, \qquad \Phi \circ \Psi = \mathrm{Id}_{HF_*(H, N:M)}. \qquad \square$$

4. Spectral invariants

In order to define spectral invariants, we need to establish a filtered Floer homology, both for approximations and for direct limit homologies.

Recall that the filtered Floer homology groups for approximations are defined as homology groups of the filtered chain complex

$$CF_*^\lambda(O_M, \Upsilon : H) := \big\{x \in CF_*(O_M, \Upsilon : H) \bigm| \mathcal{A}_H^\Upsilon(x) < \lambda\big\},$$

where the effective action functional $\mathcal{A}_H^\Upsilon$ is defined in (2.8). Since $\mathcal{A}_H^\Upsilon$ decreases along the strips that define the boundary operator

$$\partial_{J,H}\colon CF_*(O_M, \Upsilon : H) \to CF_*(O_M, \Upsilon : H),$$

the boundary operator descends to $CF_*^\lambda(O_M, \Upsilon : H)$ and defines

$$\partial_{J,H}^\lambda\colon CF_*^\lambda(O_M, \Upsilon : H) \to CF_*^\lambda(O_M, \Upsilon : H).$$

We denote the corresponding homology groups by $HF_*^\lambda(O_M, \Upsilon : H, J_\Upsilon)$ and by

$$\imath_{\Upsilon*}^\lambda\colon HF_*^\lambda(O_M, \Upsilon : H, J_\Upsilon) \to HF_*(O_M, \Upsilon : H, J_\Upsilon)$$

the homomorphism induced by the inclusion map $\imath_{\Upsilon}^{\lambda}$. As usually, for $\alpha \in HM_*(f_N, N : g_{\Upsilon}) \setminus \{0\}$ define

$$c_{\Upsilon}(\alpha, H) := \inf \big\{ \lambda \mid \Phi^{\Upsilon}(\alpha) \in \mathrm{Im}\big(\imath_{\Upsilon *}^{\lambda}\big) \big\}.$$

The proof of the following proposition is the same as the proof of [20, Lemma 2.6].

Proposition 4.1. *If H and K are two Hamiltonian functions satisfying* (2.6) *such that $\phi_H^1 = \phi_K^1$, then it holds $c_{\Upsilon}(\alpha, H) = c_{\Upsilon}(\alpha, K)$.*

The spectral invariants for approximations actually do not depend on the given approximation, assuming that it is good enough. More precisely:

Proposition 4.2. *Let $\alpha \in HM_*(f_N, N : g_{\Upsilon}) \setminus \{0\}$. (Here g_{Υ} is a generic metric chosen such that the corresponding PSS for approximation Υ is well defined.) There exists an approximation $\widetilde{\Upsilon}$ such that all invariants $c_{\overline{\Upsilon}}(\mathbf{G}_{\Upsilon\overline{\Upsilon}}(\alpha), H)$ are equal for all $\overline{\Upsilon}$ with $\widetilde{\Upsilon} \leq \overline{\Upsilon}$. Here $\mathbf{G}_{\Upsilon\overline{\Upsilon}}$ is a canonical isomorphism between $HM_*(f_N, N : g_{\Upsilon})$ and $HM_*(f_N, N : g_{\overline{\Upsilon}})$.*

Proposition 4.2 will follow directly from Theorem 4.4 that we prove below.

Therefore, one natural way to define spectral invariants for Floer homology of a singular Lagrangian $\overline{\nu}^* N$ is to take a limit of spectral invariant for approximations. Let us also mention another way to define them, via PSS isomorphism.

The filtered Floer homology for a manifold with boundary N is again defined as a direct limit of corresponding filtered homologies of approximations. The direct limit homomorphisms $\mathbf{F}_{ab}$ are defined via the number of elements in

$$\mathcal{M}(x, y : \widetilde{\Upsilon}_s) := \left\{ u \;\middle|\; \begin{aligned} & u\colon \mathbb{R} \times [0,1] \to T^*M, \\ & \frac{\partial u}{\partial s} + \widetilde{J}_s\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0, \\ & u(s,0) \in O_M,\ u(s,1) \in \widetilde{\Upsilon}_s, \\ & u(-\infty, t) = x(t),\ u(+\infty, t) = y(t) \end{aligned} \right\}. \tag{4.1}$$

Here $\widetilde{\Upsilon}_s$ is a monotone homotopy for $s \in \mathbb{R}$ such that

$$\widetilde{\Upsilon}_s = \begin{cases} \Upsilon_a & \text{if } s \leq -R, \\ \Upsilon_b & \text{if } s \geq R. \end{cases}$$

(By monotone homotopy we mean $s_1 \leq s_2 \Rightarrow \Upsilon_{s_1} \leq \Upsilon_{s_2}$.) The corresponding action functional $\mathcal{A}_H^{\widetilde{\Upsilon}_s}$ decreases along perturbed holomorphic strips that define $\mathbf{F}_{ab}$, therefore

$$\mathcal{A}_H^{\Upsilon_b}(u(+\infty, t)) \leq \mathcal{A}_H^{\Upsilon_a}(u(-\infty, t))$$

whenever there exists an $u \in \mathcal{M}(x, y : \widetilde{\Upsilon}_s)$. In particular, the homomorphisms $\mathbf{F}_{ab}$ descend to the filtered chain complex. By standard arguments one shows that they are also well defined on filtered homology groups:

$$\mathbf{F}^\lambda_{ab} \colon HF^\lambda_*(O_M, \Upsilon_a : H, J_a) \to HF^\lambda_*(O_M, \Upsilon_b : H, J_b).$$

Now we define the filtered Floer homology for N as the direct limit:

$$HF^\lambda_*(H, N : M) := \varinjlim HF^\lambda_*(O_M, \Upsilon_s : H, J_s).$$

It is easy to see that $\mathbf{F}_{ab} \circ \imath^\lambda_{\Upsilon_a *} = \imath^\lambda_{\Upsilon_b *} \circ \mathbf{F}^\lambda_{ab}$, where

$$\imath^\lambda_{\Upsilon_a *} \colon HF^\lambda_*(O_M, \Upsilon_a : H, J_a) \to HF_*(O_M, \Upsilon_a : H, J_a)$$

denotes the inclusion-induced map for the approximations. Thus the induced inclusion maps

$$\imath^\lambda_* \colon HF^\lambda_*(H, N : M) \to HF_*(H, N : M)$$

are also well defined.

Definition 4.3. Let $[\alpha] \in HM_*(f_N, N) \setminus \{0\}$. By a class in $[\alpha]$ we assume a class in the direct limit, so $\alpha \in HM_*(f_N, N : g)$, for some metric g. A *spectral invariant for a manifold N with boundary* is defined as

$$c_N([\alpha], H) := \inf \big\{ \lambda \mid \Phi([\alpha]) \in \mathrm{Im}\big(\imath^\lambda_*\big) \big\}. \tag{4.2}$$

A natural question is weather $c_N([\alpha], H)$ equals to the limit of $c_\Upsilon(\alpha, H)$ in the sense of Proposition 4.2. The answer is affirmative and it is given in the following theorem.

Theorem 4.4. *Let $\alpha \in HM_*(f_N, N : g_\Upsilon) \setminus \{0\}$. There exists an approximation $\widetilde{\Upsilon}$ such that, for all $\overline{\Upsilon}$ with $\widetilde{\Upsilon} \leq \overline{\Upsilon}$ it holds*:

$$c_{\overline{\Upsilon}}(\mathbf{G}_{\Upsilon\overline{\Upsilon}}(\alpha), H) = c_N([\alpha], H).$$

Proof. Let $[\alpha] \in HM_*(f_N, N) \setminus \{0\}$, $\lambda \in \mathbb{R}$ and $[x] \in HF^\lambda_*(H, N : M)$ be such that $\Phi([\alpha]) = \imath^\lambda_*([x])$. Let $\alpha \in HM_*(f_N, N : g_\Upsilon)$, $x \in HF^\lambda_*(O_M, \Upsilon' : H, J_{\Upsilon'})$ for some Υ and Υ'. Since

$$\Phi([\alpha]) = \big[\Phi^\Upsilon(\alpha)\big] = \big[\imath^\lambda_{\Upsilon' *}(x)\big] = \imath^\lambda_*[x],$$

we have $\mathbf{F}_{\Upsilon\widehat{\Upsilon}}(\Phi^\Upsilon(\alpha)) = \mathbf{F}_{\Upsilon'\widehat{\Upsilon}}(\imath^\lambda_{\Upsilon' *}(x))$. We have the following commutative diagram:

(4.3)

$$\begin{tikzcd}
\cdots \arrow[r] & HM_*(f_N, N{:}g_\Upsilon) \arrow[r, "\mathbf{G}_{\Upsilon\widetilde{\Upsilon}}"] \arrow[d, "\Phi^\Upsilon"'] & HM_*(f_N, N{:}g_{\widetilde{\Upsilon}}) \arrow[r, "\mathbf{G}_{\widetilde{\Upsilon}\widehat{\Upsilon}}"] \arrow[d, "\Phi^{\widetilde{\Upsilon}}"] & HM_*(f_N, N{:}g_{\widehat{\Upsilon}}) \arrow[r] \arrow[d, "\Phi^{\widehat{\Upsilon}}"] & \cdots \\
\cdots \arrow[r] & HF_*(\Upsilon : J_\Upsilon) \arrow[r, "\mathbf{F}_{\Upsilon\widetilde{\Upsilon}}"] & HF_*(\widetilde{\Upsilon} : J_{\widetilde{\Upsilon}}) \arrow[r, "\mathbf{F}_{\widetilde{\Upsilon}\widehat{\Upsilon}}"] & HF_*(\widehat{\Upsilon} : J_{\widehat{\Upsilon}}) \arrow[r] & \cdots \\
\cdots \arrow[r] & HF^\lambda_*(\Upsilon : J_\Upsilon) \arrow[r, "\mathbf{F}^\lambda_{\Upsilon\widetilde{\Upsilon}}"'] \arrow[u, "\imath^\lambda_{\Upsilon *}"] & HF^\lambda_*(\widetilde{\Upsilon} : J_{\widetilde{\Upsilon}}) \arrow[r, "\mathbf{F}^\lambda_{\widetilde{\Upsilon}\widehat{\Upsilon}}"'] \arrow[u, "\imath^\lambda_{\widetilde{\Upsilon} *}"'] & HF^\lambda_*(\widehat{\Upsilon} : J_{\widehat{\Upsilon}}) \arrow[r] \arrow[u, "\imath^\lambda_{\widehat{\Upsilon} *}"'] & \cdots
\end{tikzcd}$$

In (4.3) we abbreviated $HF_*(O_M, \Upsilon : H, J_\Upsilon)$ to $HF_*(\Upsilon : J_\Upsilon)$ and so on. From this commutativity we read

$$\Phi^{\widehat{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha)\big) = \mathbf{F}_{\Upsilon\widehat{\Upsilon}}\big(\Phi^{\Upsilon}(\alpha)\big) = \mathbf{F}_{\Upsilon'\widehat{\Upsilon}}\big(\imath^\lambda_{\Upsilon'*}(x)\big) = \imath^\lambda_{\widehat{\Upsilon}*}\big(\mathbf{F}^\lambda_{\Upsilon'\widehat{\Upsilon}}(x)\big),$$

implying $\Phi^{\widehat{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha)\big) \in \operatorname{Im}\big(\imath^\lambda_{\widehat{\Upsilon}*}\big)$. We conclude

$$c_{\widehat{\Upsilon}}(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha), H) \leq c_N([\alpha], H). \tag{4.4}$$

If we take $\alpha \in HM_*(f_N, N : g_\Upsilon) \setminus \{0\}$ and $\lambda \in \mathbb{R}$ such that $\Phi^\Upsilon(\alpha) \in \operatorname{Im}\big(\imath^\lambda_{\Upsilon*}\big)$, then $\Phi^\Upsilon(\alpha) = \imath^\lambda_{\Upsilon*}(x)$ for some $x \in HF^\lambda_k(\Upsilon : H, J_\Upsilon)$. Therefore, we have

$$\Phi([\alpha]) = \big[\Phi^\Upsilon(\alpha)\big] = \big[\imath^\lambda_{\Upsilon*}(x)\big] = \imath^\lambda_*[x],$$

so we obtain the inequality

$$c_N([\alpha], H) \leq c_\Upsilon(\alpha, H). \tag{4.5}$$

The elements α and $\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha)$ represent the same element in the quotient space $HM_*(f_N, N)$. From (4.4) and (4.5) we have

$$c_{\widehat{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha), H\big) \leq c_N([\alpha], H) = c_N\big([\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha)], H\big) \leq c_{\widehat{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha), H\big), \tag{4.6}$$

so all inequalities become equalities. It also holds:

$$c_{\widetilde{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widetilde{\Upsilon}}(\alpha), H\big) \leq c_\Upsilon(\alpha, H), \tag{4.7}$$

for every $\Upsilon \leq \widetilde{\Upsilon}$. Indeed, if $\Phi^\Upsilon(\alpha) \in \operatorname{Im}\big(\imath^\lambda_{\Upsilon*}\big)$, for $\alpha \neq 0 \in HM_*(f_N, N : g_\Upsilon)$, we have

$$\Phi^\Upsilon(\alpha) = \imath^\lambda_{\Upsilon*}(x) \;\Rightarrow\; \mathbf{F}_{\Upsilon\widetilde{\Upsilon}}\big(\Phi^\Upsilon(\alpha)\big) = \mathbf{F}_{\Upsilon\widetilde{\Upsilon}}\big(\imath^\lambda_{\Upsilon*}(x)\big), \tag{4.8}$$

so, from the commutativity (4.3) it follows

$$\imath^\lambda_{\widetilde{\Upsilon}*}\big(\mathbf{F}_{\Upsilon\widetilde{\Upsilon}}(x)\big) = \mathbf{F}_{\Upsilon\widetilde{\Upsilon}}\big(\imath^\lambda_{\Upsilon*}(x)\big) \overset{(4.8)}{=} \mathbf{F}_{\Upsilon\widetilde{\Upsilon}}\big(\Phi^\Upsilon(\alpha)\big) = \Phi^{\widetilde{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widetilde{\Upsilon}}(\alpha)\big).$$

This means that

$$\Phi^{\widetilde{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widetilde{\Upsilon}}(\alpha)\big) \in \operatorname{Im}\big(\imath^\lambda_{\widetilde{\Upsilon}*}\big),$$

so (4.7) holds.

From (4.6) and (4.7) we conclude that $c_{\widehat{\Upsilon}}\big(\mathbf{G}_{\Upsilon\widehat{\Upsilon}}(\alpha), H\big)$ becomes equal to $c_N([\alpha], H)$, for every $\widehat{\Upsilon}$ with $\widetilde{\Upsilon} \leq \widehat{\Upsilon}$. □

Proposition 4.2 now follows directly from Theorem 4.4. An immediate consequence of Proposition 4.1 and Theorem 4.4 is the following.

COROLLARY 4.5. *If H and G satisfy* (2.6) *and $\phi^1_H = \phi^1_G$, then $c_N\big([\alpha], H\big) = c_N\big([\alpha], G\big)$.*

4.1. Continuity. Spectral invariants are continuous with respect to the Hofer norm.

Theorem 4.6. *Let $\|\cdot\|$ denotes the Hofer's norm:*

$$\|H\| := \int_0^1 \Big[\max_x H(x,t) - \min_x H(x,t)\Big]\,dt.$$

Relative spectral invariants for N

$$C_N([\alpha],H) := c_N([\alpha],H) - c_N(1,H)$$

are continuous with respect to $\|\cdot\|$:

$$\big|C_N([\alpha],H) - C_N([\alpha],H')\big| \leq \|H - H'\|.$$

Here 1 denotes the generator of zero homology group $HM_0(f_N,N)$.

Proof. The first step is to prove the continuity of relative invariants for approximations:

$$C_\Upsilon(\alpha,H) := c_\Upsilon(\alpha,H) - c_\Upsilon(1,H).$$

The proof of theorem then follows from the above inequality and Theorem 4.4. To prove the continuity result for the relative invariants for approximation, we consider the linear homotopy

$$H^s = (1-s)H + sH' = H + s(H' - H).$$

The canonical isomorphism $S^\Upsilon_{H,H'}$ is defined by a number of the holomorphic strips that connect a generator x of $CF_*(O_M,\Upsilon : H, J_\Upsilon)$ and a generator y of $CF_*(O_M,\Upsilon : H', J_\Upsilon)$:

$$\begin{aligned}
&\mathcal{M}(x,y,O_M,\Upsilon : H,H',J_\Upsilon)\\
&\qquad := \bigg\{ u : \mathbb{R}\times[0,1] \to T^*M \;\bigg|\; \frac{\partial u}{\partial s} + J_\Upsilon\bigg(\frac{\partial u}{\partial t} - X_{H^s}(u)\bigg) = 0\\
&\qquad\qquad u(s,0) \in O_M,\ u(s,1) \in \Upsilon,\\
&\qquad\qquad u(-\infty,t) = x(t),\ u(+\infty,t) = y(t)\bigg\}.
\end{aligned}$$

If there exists $u \in \mathcal{M}(x,y,O_M,\Upsilon : H,H',J_\Upsilon)$, for the linear homotopy H^s, then by direct computation we see that it holds

$$\begin{aligned}
\mathcal{A}^\Upsilon_{H'}(y) - \mathcal{A}^\Upsilon_H(x) &= \int_{-\infty}^{+\infty} \frac{d}{ds}\mathcal{A}^\Upsilon_{H^s}(u(s,\cdot\,))\\
&\leq E_+(H - H') := \int_0^1 \max_x (H - H')\,dt.
\end{aligned} \tag{4.9}$$

If the linear homotopy is not regular, we can approximate it by a C^1-close regular homotopy $H^s = H + \sigma(s)(H' - H)$, and obtain:

$$\mathcal{A}^\Upsilon_{H'}(y) - \mathcal{A}^\Upsilon_H(x) \leq E_+(H - H') + \varepsilon$$

for any $\varepsilon > 0$, so the estimate (4.9) holds for a regular homotopy H^s. We have

$$S^{\Upsilon}_{H,H'}(x) = \sum \pm y_j \;\Rightarrow\; \mathcal{A}^{\Upsilon}_{H'}(y_j) \leq \mathcal{A}^{\Upsilon}_{H}(x) + E_+(H - H'). \tag{4.10}$$

For $x \in HF_*(O_M, \Upsilon : H, J_\Upsilon)$ define $\sigma_\Upsilon(x, H) := \inf\{\lambda \in \mathbb{R} \mid x \in \operatorname{Im}(\imath^{\lambda}_{\Upsilon,H*})\}$. Obviously, it holds:

$$c_\Upsilon(\alpha, H) = \sigma_\Upsilon(\Phi^{\Upsilon}_H(\alpha), H).$$

It follows from (4.10):

$$\sigma_\Upsilon(S^{\Upsilon}_{H,H'}(x), H') \leq \sigma_\Upsilon(x, H) + E_+(H - H').$$

From $S^{\Upsilon}_{H,H'} \circ \Phi^{\Upsilon}_H = \Phi^{\Upsilon}_{H'}$ we conclude

$$\begin{aligned} c_\Upsilon(\alpha, H') &= \sigma_\Upsilon(\Phi^{\Upsilon}_{H'}(\alpha), H') = \sigma_\Upsilon(S^{\Upsilon}_{H,H'} \circ \Phi^{\Upsilon}_H(\alpha), H') \\ &\leq \sigma_\Upsilon(\Phi^{\Upsilon}_H(\alpha), H) + E_+(H - H') \\ &= c_\Upsilon(\alpha, H) + E_+(H - H'), \end{aligned} \tag{4.11}$$

for all $0 \neq \alpha \in HM_*(f_N, N : g_\Upsilon)$. The proof now easily follows from the inequality (4.11) and the same one applied to $\alpha = 1 \in HM_0(f_N, N : g_\Upsilon)$. □

4.2. Products in Morse and Floer homology. Products in the Morse and Floer theory were studied by various authors, e.g., Abbondandolo and Schwarz in [1], Auroux in [3], Oh in [23] and also in [13], [12].

Here we establish two products of pair of pants type. Note that, similarly to the case of Morse homology $HM_*(f, N)$ and Floer homology $HF_*(H, N : M)$, we can define Floer homology of the pair $(O_M, \phi^1_H(O_M))$ as the direct limit of Floer homologies $HF_*(O_M, \phi^1_H(O_M) : H, J)$, by choosing the canonical isomorphisms for different choices of J as the defining morphisms. We denote the obtained Floer homology by $HF_*(H : M)$.

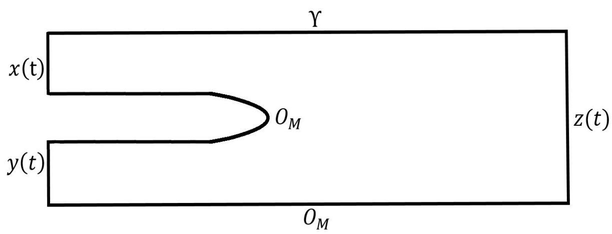

Figure 3. Riemannian surface Σ that defines the product $\circ$.

Theorem 4.7. *Let f_N and H, H_j, for $j = 1, 2, 3$ be as in Theorem* 3.2. *There exist pair-of-pants type products*:

$$\circ\colon HF_*(H_1, N : M) \otimes HF_*(H_2 : M) \to HF_*(H_3, N : M),$$
$$\star\colon HM_*(f_N, N) \otimes HF_*(H, N : M) \to HF_*(H, N : M),$$

that turns Floer homology $HF_*(H,N:M)$ into a $HM_*(f_N,N)$-module. The above products satisfy:

$$\Phi(\alpha\widetilde{\cap}\beta)=\Phi(\alpha)\circ\Phi(\beta)$$

where Φ is a PSS *type isomorphism and $\widetilde{\cap}$ denotes exterior intersection product in Morse homology.*

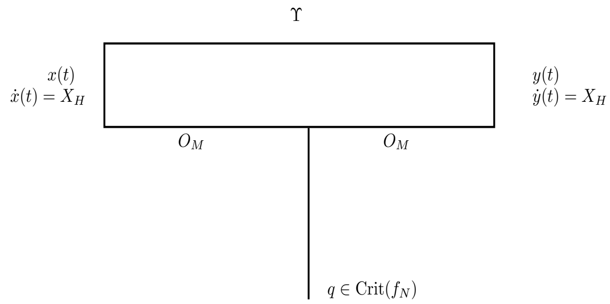

Figure 4. Mixed object that defines the product $\star$.

Proof. Both of the products are defined using the corresponding products on homology groups for approximations (or for a fixed Riemannian metric). More precisely, we first define:

$$\begin{aligned}\circ\colon HF_*(O_M,\Upsilon:H_1,J_\Upsilon)\otimes HF_*(O_M,\phi_H^1(O_M):H_2,J_\Upsilon)&\\ \to HF_*(O_M,\Upsilon:H_3,J_\Upsilon),&\\ \star\colon HM_*(f_N,N:g)\otimes HF_*(O_M,\Upsilon:H,J_\Upsilon)\to HF_*(O_M,\Upsilon:H,J_\Upsilon).&\end{aligned}$$

The above products are defined in a standard way, in the chain level, using the mappings with domains depicted in Figure 3 and Figure 4.

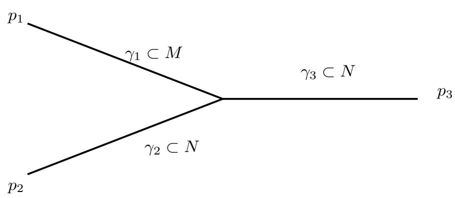

Figure 5. Tree that defines the product $\widetilde{\cap}$ (negative gradient trajectories γ_2 and γ_3 are in N and γ_1 is in M).

Next we check that it holds:

$$\begin{aligned}\mathbf{F}_{ab}^{H_3}(x_a\circ y_a)&=\mathbf{F}_{ab}^{H_1}(x_a)\circ\mathbf{F}_{ab}^{H_2}(y_a),\\ \mathbf{F}_{ab}(p_a\star x_a)&=\mathbf{G}_{ab}(p_a)\star\mathbf{F}_{ab}(x_a),\end{aligned}\tag{4.12}$$

which enables us to have the products well defined on the direct limit levels. The subscript letters H_j indicate which Hamiltonian functions we have in mind.

Finally, we check:

$$\Phi_3^{\Upsilon}(\alpha\,\widetilde{\cap}\,\beta) = \Phi_1^{\Upsilon}(\alpha)\circ\Phi_2^{\Upsilon}(\beta), \qquad (p\,\widetilde{\cap}\,q)\star x = p\star(q\star x), \tag{4.13}$$

for all $p\in HM_*(f,M:g)$, $q\in HM_*(f_N,N:g_N)$ and $x\in HF_*(O_M,\Upsilon:H,J_\Upsilon)$. The exterior intersection product in Morse homology is defined in a standard way

$$\widetilde{\cap}\colon HM_*(f_1,M)\otimes HM_*(f_2,N)\to HM_*(f_3,N),$$

see Figure 5. This gives us the claimed properties of products, on the approximation level, hence, from (4.12), the claimed properties on the direct limit level.

The details of all above proofs are the same as in [5], [12], since the constructions apply to the approximations. □

4.3. Triangle inequality. For two functions

$$H_1,H_2 : T^*M\times[0,1]\to\mathbb{R} \quad \text{with } H_1(x,1)=H_2(x,0),$$

we define their concatenation as:

$$H_1\sharp\, H_2 := \begin{cases} H_1(x,t) & \text{if } t\le 1,\\ H_2(x,t-1) & \text{if } t\ge 1.\end{cases}$$

THEOREM 4.8. *Let f_1 and f_3 be admissible Morse functions on N and let f_2 be Morse function on M. Let $[\alpha]\in HM_*(f_1,N)$, $[\beta]\in HM_*(f_2,M)$ be such that $HM_*(f_3,N)\ni[\alpha\,\widetilde{\cap}\,\beta]\ne 0$. It holds*

$$c_N^3\big([\alpha\,\widetilde{\cap}\,\beta],H_1\sharp H_2\big)\le c_N^1([\alpha],H_1)+\sigma^2([\beta],H_2),$$

where j in c_N^j and σ^j emphasizes the corresponding Morse function f_j, and σ^2 is the invariant defined similarly as the invariant (1.1)*, using the direct limit construction of $HF_*(H_2:M)$.*

PROOF. One first proves the triangle inequality for approximations:

$$c_\Upsilon^3(\alpha\,\widetilde{\cap}\,\beta,H_1\sharp H_2)\le c_\Upsilon^1(\alpha,H_1)+\sigma^2(\beta,H_2).$$

Choose a regular Hamiltonian H_3 with $\|H_3-H_1\sharp H_2\|_{C^0}<\varepsilon$. Denote by Σ the Riemannian surface $\mathbb{R}\times[-1,0]\sqcup\mathbb{R}\times[0,1]$ in Figure 3 with the identification $(s,0^-)\sim(s,0^+)$ for $s\ge 0$, and by (s,t) the coordinates defined in three ends, $\Sigma_1\approx(-\infty,0]\times[0,1]$, $\Sigma_2\approx(-\infty,0]\times[0,1]$ and $\Sigma_3\approx[0,+\infty)\times[0,1]$.

Let $K\colon T^*M\times\Sigma\to\mathbb{R}$ be a smooth family of Hamiltonians such that

$$K(\,\cdot\,,s,t)=\begin{cases} H_1(\,\cdot\,,t+1) & \text{for } s\le -1,\ 0\le t\le 1,\\ H_2(\,\cdot\,,t) & \text{for } s\le -1,\ -1\le t\le 0,\\ \dfrac{1}{2}\,H_3\Big(\,\cdot\,,\dfrac{t+1}{2}\Big) & \text{for } s\ge 1,\\ 0 & \text{for } (s,t)\notin\Sigma\setminus(\Sigma_1\cup\Sigma_2\cup\Sigma_3);\end{cases}$$

$$\Big\|\frac{\partial K}{\partial s}\Big\|\le\begin{cases}\varepsilon & \text{for } s\in[-1,1],\\ 0 & \text{elsewhere.}\end{cases}$$

Let $\widetilde{x}=\Phi_1^{\Upsilon}(\alpha)$ and $\widetilde{y}=\Phi_2(\beta)$. Since $\alpha\,\widetilde{\cap}\,\beta\neq 0$ and

$$0\neq\Phi_3^{\Upsilon}(\alpha\,\widetilde{\cap}\,\beta)=\Phi_1^{\Upsilon}(\alpha)\circ\Phi_2(\beta)=\widetilde{x}\circ\widetilde{y},$$

there exist $x\in CF_*^{\lambda}(O_M,\Upsilon:H_1,J)$ and $y\in CF_*^{\sigma}(O_M,O_M:H_2,J)$ that participate in the formal sums defining $\widetilde{x}$ and $\widetilde{y}$ respectively, as well as $z\in CF_*(O_M,\Upsilon:H_3,J)$ and $u\colon\Sigma\to T^*M$ satisfying

$$\begin{cases}\overline{\partial}_{K,J^{\Upsilon}}(u)=0,\\ u(s,-1)\in O_M,\ u(s,1)\in\Upsilon, & s\in\mathbb{R},\\ u(s,0^-),u(s,0^+)\in O_M, & s\le 0,\\ u_1(-\infty,t)=x(t),\\ u_2(-\infty,t)=y(t),\\ u_3(+\infty,t)=z(t).\end{cases}$$

Since

$$\int_\Sigma\Big\|\frac{\partial u}{\partial s}\Big\|^2 ds\,dt\ge 0,$$

$$\int_\Sigma u^*\omega=-\int x^*\theta+h_\Upsilon(x(1))-\int y^*\theta+h_\Upsilon(y(1))+\int z^*\theta-h_\Upsilon(z(1)),$$

it follows form properties of a Hamiltonian K and Stoke's formula:

$$\mathcal{A}_{H_3}^{\Upsilon}(z)\le\mathcal{A}_{H_1}^{\Upsilon}(x)+\mathcal{A}_{H_2}(y)+4\varepsilon.$$

The proof now follows from Theorem 4.6 and (4.13). □

4.4. Comparison with spectral invariants for periodic orbits in T^*M. Let us sketch the construction of spectral invariants for periodic orbit Floer homology. Since Floer homology for periodic orbits is not well defined for compactly supported Hamiltonians in T^*M, we need to consider Hamiltonians with a support in some fixed cotangent ball bundle. This is used in [7] and also in [20]. Fix $R>0$, $\varepsilon>0$ and a smooth function $h:(-\varepsilon,+\infty)\to\mathbb{R}$ such that:

- $h(t)=0$ for $t\ge 0$;
- $h'(t)\ge 0$ for $t\le 0$;

- h' is small enough so that the flow of $h(\|\mathbf{p}\| - R)$ does not have non constant periodic orbit of period less or equal to 1 for $\|\mathbf{p}\| \in (0, \varepsilon)$.

We choose $H_t(\mathbf{q}, \mathbf{p})$ to be equal to $h(\|\mathbf{p}\| - R)$ for $\|\mathbf{p}\| \geq R - \varepsilon$.

Let $HF_*(T^*M : H, J)$ and $HM_*(F, T^*M)$ denote Floer homology for periodic orbits in T^*M and Morse homology for the Morse function $F \colon T^*M \to \mathbb{R}$ respectively. The filtration in Floer homology for periodic orbits is given by the standard action functional

$$a_H(\gamma) := \int \gamma^*\theta - \int_0^1 H\, dt$$

which is well defined in the cotangent bundle setting. Denote by $HF_*^\lambda(T^*M : H, J)$ the corresponding filtered group and by $\jmath_*^\lambda$ the map induced by the inclusion map. Let PSS stands for PSS isomorphism for periodic orbits, defined in a way analogous to [27]:

$$\mathrm{PSS} : HM_*(F, T^*M : g) \xrightarrow{\cong} HF_*(T^*M : H, J)$$

and let $\alpha \in HM_*(F, T^*M : g)$. Filtered Floer homology groups $HF_*^\lambda(T^*M : H, J)$ are homology groups of a chain complex generated by

$$CF_*^\lambda(T^*M : H) := \{a \in CF_*(T^*M : H) \mid a_H(a) < \lambda\},$$

where $CF_*(T^*M : H)$ denotes the $\mathbb{Z}_2-$vector space over the set of periodic Hamiltonian $H-$orbits in T^*M.

Let $\jmath_*^\lambda \colon HF_*^\lambda(T^*M : H, J) \to HF_*(T^*M : H, J)$ denote the map induced by the inclusion map. For $\alpha \in HM_*(F, T^*M : g) \setminus \{0\}$ define

$$\rho(\alpha, H) := \inf\{\lambda \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}(\jmath_*^\lambda)\}.$$

We now choose a Morse function $F \colon T^*M \to \mathbb{R}$ in a specific way. Let f_N be an admissible Morse function on N and f_M be its extension to M as described on the page 627. We can extend f_M to $F \colon T^*M \to \mathbb{R}$ in the same way as we extended f_N to F_M to obtain the Morse function F on T^*M with no negative gradient trajectories leaving N. Now the Morse complex $CM_*(f_N)$ is a subset of the Morse complex $CM_*(F)$ with $m_{f_N}(p) = m_F(p)$, for $p \in \mathrm{Crit}(f_N)$, and the inclusion map of these complexes becomes the homomorphism $\imath_*$ on the homology level.

For two generic almost complex structures J_a and J_b, denote by $\mathbf{D}_{ab}$ a canonical isomorphism of Floer homologies for periodic orbits:

$$\mathbf{D}_{ab} : HF_*(T^*M : H, J_a) \to HF_*(T^*M : H, J_b)$$

that satisfies $\mathbf{D}_{bc} \circ \mathbf{D}_{ab} = \mathbf{D}_{ac}$.

We define a preorder on the space of compatible complex structures to be the full relation and Floer homology for periodic orbits as a direct limit

$$HF_*(T^*M : H) := \varinjlim HF_*(T^*M : H, J_s).$$

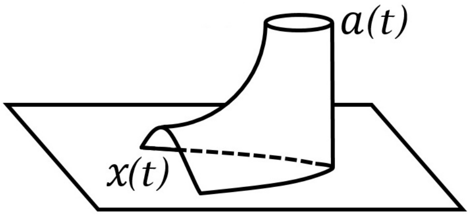

Figure 6. Chimney.

Similarly, by varying a Riemannian metrics, we define Morse homology as a direct limit

$$HM_*(F, T^*M) := \varinjlim HM_*(F, T^*M : g).$$

It is not hard to check that the inclusion map

$$\imath_*\colon HM_*(f_N, N : g) \to HM_*(F, T^*M : g)$$

induces the map, denote it the same:

$$\imath_* : HM_*(f_N, N) \to HM_*(F, T^*M).$$

Theorem 4.9. *Let* $[\alpha] \in HM_*(f_N, N) \setminus \{0\}$. *Then*

$$c_N([\alpha], H) \geq \rho(\imath_*([\alpha]), H).$$

Proof. The proof is divided to several steps, we only sketch them here and refer the reader to [12, Section 4] for the detailed construction and technicalities. The first step is to construct an Albers-type morphism:

$$\chi\colon HF_*(O_M, \Upsilon : H, J) \to HF_*(T^*M : H, J),$$

for fixed parameters and approximations. This mapping is defined via the number of perturbed holomorphic "chimneys", i.e. maps u defined on

$$\Pi := \mathbb{R} \times [0,1]/\sim, \quad \text{where } (s,0) \sim (s,1) \text{ for } s \geq 0$$

that satisfy:

$$\begin{cases} u\colon \Pi \to T^*M, \\ \partial_s u + J(\partial_t u - X_H \circ u) = 0, \\ u(s,0) \in O_M,\ u(s,1) \in \Upsilon \quad \text{for } s \leq 0, \\ u(-\infty, t) = x(t),\ u(+\infty, t) = a(t), \end{cases}$$

see Figure 6 (see also [2]).

The next step is to show that the maps χ are also well defined on the filtered groups:

$$\chi^\lambda\colon HF_*^\lambda(O_M, \Upsilon : H, J_\Upsilon) \to HF_*^\lambda(T^*M : H, J_\Upsilon).$$

This is easily done by inspection the change of the action functional along chimney. Then, one shows that the diagrams

$$\begin{array}{ccc} HF_*^\lambda(O_M,\Upsilon : H, J_\Upsilon) & \xrightarrow{\chi^\lambda} & HF_*^\lambda(T^*M : H, J_\Upsilon) \\ \downarrow{\scriptstyle \imath_*^\lambda} & & \downarrow{\scriptstyle \jmath_*^\lambda} \\ HF_*(O_M,\Upsilon : H, J_\Upsilon) & \xrightarrow{\chi} & HF_*(T^*M : H, J_\Upsilon) \\ \uparrow{\scriptstyle \Phi^\Upsilon} & & \downarrow{\scriptstyle \mathrm{PSS}^{-1}} \\ HM_*(f_N, N : g_\Upsilon) & \xrightarrow[\imath_*]{} & HM_*(F, T^*M : g_\Upsilon) \end{array} \tag{4.14}$$

and

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & HF_*^\lambda(\Upsilon_a) & \xrightarrow{\mathbf{F}_{ab}} & HF_*^\lambda(\Upsilon_b) & \xrightarrow{\mathbf{F}_{bc}} & HF_*^\lambda(\Upsilon_c) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \chi^a} & & \downarrow{\scriptstyle \chi^b} & & \downarrow{\scriptstyle \chi^c} & & \\ \cdots & \longrightarrow & HF_*^\lambda(J_a) & \xrightarrow[\mathbf{D}_{ab}]{} & HF_*^\lambda(J_b) & \xrightarrow[\mathbf{D}_{bc}]{} & HF_*^\lambda(J_c) & \longrightarrow & \cdots \end{array} \tag{4.15}$$

commute. This is technically the most demanding step and it can be done by a careful choice of auxiliary one-dimensional manifolds.

The commutativity of the diagram (4.15) enables us to define a chimney-type map

$$\chi\colon HF_*(H, N : M) \to HF_*(T^*M : H).$$

The commutativity of the diagram (4.14) implies the commutativity of

$$\begin{array}{ccc} HF_*^\lambda(H, N : M) & \xrightarrow{\chi^\lambda} & HF_*^\lambda(T^*M : H) \\ \downarrow{\scriptstyle \imath_*^\lambda} & & \downarrow{\scriptstyle \jmath_*^\lambda} \\ HF_*(H, N : M) & \xrightarrow{\chi} & HF_*(T^*M : H) \\ \uparrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{PSS}^{-1}} \\ HM_*(f, N) & \xrightarrow[\imath_*]{} & HM_*(F, T^*M) \end{array}$$

which easily implies

$$c_N([\alpha], H) \geq \rho(\imath_*([\alpha]), H). \qquad \square$$

5. The case of nested open subsets

Let U be an open subset of M, with ∂U a smooth $(n-1)$-dimensional manifold. Our construction applies in this situation ($N = U$, $\dim N = \dim M$), in fact, this case is the subject of [12].

In [24] Oh considered a spectral invariant

$$c_+(H, U) := \inf\{\lambda \in \mathbb{R} \mid \imath_*^\lambda\colon HF_*^\lambda(H, U : M) \to HF_*(H, U : M) \text{ is surjective}\}.$$

If $U \overset{\jmath_{UV}}{\hookrightarrow} V$ are two open subset of M and $\jmath_{UV*}\colon H*(U) \to H_*(V)$ is surjective, Oh proved that

$$c_+(H,V) \leq c_+(H,U). \tag{5.1}$$

In [12] we proved a slightly more precise statement, the above inequality for any homology class (with $\mathbb{Z}_2$ coefficients), using the PSS isomorphism for an open subset.

Theorem 5.1. *Let $U \hookrightarrow V$ be two open subsets of M and let*

$$\jmath_{UV*}\colon HM_*(f,U) \to HM_*(f,V)$$

(*the homomorphism induced by inclusion $\jmath_{UV}\colon U \hookrightarrow V$*) *be surjective. Let $c_U([\alpha],H)$ be as in* (4.2). *For $[\alpha] \in HM_k(f,U) \setminus \{0\}$ it holds*:

$$c_V(\jmath_{UV*}([\alpha]),H) \leq c_U([\alpha],H).$$

We generalize (5.1) in the following way. Define

$$c_+(H,N) := \inf\big\{\lambda \in \mathbb{R} \mid \imath_*^\lambda\colon HF_*^\lambda(H,N:M) \to HF_*(H,N:M) \text{ is surjective}\big\}$$

and, for a fixed approximation,

$$c_+(H,\Upsilon) := \inf\big\{\lambda \in \mathbb{R} \mid \imath_*^\lambda : HF_*^\lambda(O_M,\Upsilon:H,J) \to HF_*(O_M,\Upsilon:H,J) \text{ is surjective}\big\}.$$

Remark 5.2. It follows from Proposition 4.2 that the invariants $c_+(H,\Upsilon^N)$ all become equal to $c_+(H,N)$ for all sufficiently good approximations $\Upsilon^N \geq \Upsilon_0$.

The natural arising question is whether spectral invariants are in some way continuous with respect to a submanifold $N \subset M$? We will give an affirmative answer in the case when N is a framed submanifold in M. We say that a submanifold $N \subset M$ is *framed* if for some tubular neighbourhood $\mathrm{Tb}_N(\partial N)$, the normal bundle $\nu(N \cup \mathrm{Tb}_N(\partial N))$ of a submanifold $N \cup \mathrm{Tb}_N(\partial N)$ is trivial.

Theorem 5.3. *Suppose that a submanifold $N \subset M$ is framed. Let U_n be a decreasing sequence of open subsets with smooth boundaries and $\bigcap_n U_n = N$, where N is a smooth submanifold with a smooth boundary. Then*

$$\lim_{n\to\infty} c_+(H,U_n) = c_+(H,N).$$

Before proving Theorem 5.3, we first note that we can assume that U_n are tubular neighbourhoods of N. Indeed, otherwise, since U_n are open and N is compact, we can find a sequence V_n of tubular neighbourhoods, such that, for $n \geq n_0$ it holds $V_n \subseteq U_n \subseteq V_{n+1}$.

We will divide the proof into several steps.

The first step is to construct, for a fixed $n \in \mathbb{N}$ and $\varepsilon > 0$, an inclusion mapping from $HF_*^{\lambda}\big(O_M, \Upsilon^N : H, J\big)$ to $HF_*^{\lambda+\varepsilon}\big(O_M, \Upsilon^n : H, J\big)$, where the approximations Υ^N of $\overline{\nu}^* N$ and Υ^n of $\overline{\nu}^* U_n$ are suitably chosen.

This is done in Lemmas 5.4–5.7. The proofs of Lemmas 5.4 and 5.5 follow the ideas from [24] up to some point, with some specificities adapted to our case. Therefore we expose them in details.

Let $\|\cdot\|$ be the norm induced by a fixed Riemannian metric on T^*M. Denote by

$$D_a^* M := \{(\mathbf{q}, \mathbf{p}) \in T^*M \mid \|\mathbf{p}\| \leq a\}.$$

Let $R > 0$ be such that all the solutions of (2.9) are contained in $D_R^* M$ (this R exists by standard C^0-estimates).

Lemma 5.4. *Fix $\varepsilon > 0$, $\delta > 0$. There exists a smooth function $K\colon T^*M \to \mathbb{R}$ such that the Hamiltonian flow ϕ_K^1 has the following properties*:

(a) ϕ_K^1 *leaves O_M and $\overline{\nu}^* N$ invariant*
(b) ϕ_K^1 *maps $\overline{\nu}^* N \cap D_R^* M$ into $\overline{\nu}^* N \cap D_\delta^* M$*
(c) ϕ_K^1 *leaves $(T^*M \setminus D_{2R}^* M) \cup (T^*M \setminus T^*(N \cup \mathrm{Tb}_N(\partial N)))$ fixed.*
(d) *C^0 norm of K is smaller than ε.*

Proof. Denote $X := N \cup \mathrm{Tb}_N(\partial N)$. Since N is framed, so

$$\mathrm{Tb}_M(X) \cong X \times \mathbb{R}^{n-k} \;\Rightarrow\; T^*\big(\mathrm{Tb}_M(X)\big) \cong T^*(X) \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}, \tag{5.2}$$

we can choose coordinates $(q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$ that satisfy both (2.3) and

- $(q_{k+1}, \dots, q_n, p_{k+1}, \dots, p_n)$ are globally defined,
- (q_k, p_k) are globally defined in $T^*(\mathrm{Tb}_N(\partial N))$, say for $|q_k| < \delta_0$, where δ_0 is some constant that is smaller than δ.

Recall that, in the coordinates (2.3), the set $\nu_-^*(\partial N)$ is locally described by

$$\{(q_1, \dots, q_{k-1}, 0, \dots, 0, 0, \dots, 0, p_k, \dots, p_n) \mid p_k \leq 0\}$$

and $\overline{\nu^*(\mathrm{Int}\, N)}$ is described as

$$\{(q_1, \dots, q_k, \underbrace{0, \dots, 0}_{n-k}, \underbrace{0, \dots, 0}_{k}, p_{k+1}, \dots, p_n) \mid q_k \leq 0\}.$$

Let us first define a Hamiltonian diffeomorphism in the set $|q_k| < \delta_0/2$. Consider a Hamiltonian vectorfield

$$(q_k, \dots, q_n, p_k, \dots, p_n) \mapsto (\lambda q_k, \dots, \lambda q_n, -\lambda p_k, \dots, -\lambda p_n)$$

in the $(q_k, \dots, q_n, p_k, \dots, p_n)$-plane generated by the Hamiltonian function

$$(q_k, \dots, q_n, p_k, \dots, p_n) \mapsto \lambda q_k p_k + \ldots + \lambda q_n p_n,$$

where λ is chosen to be large enough so that the corresponding Hamiltonian diffeomorphism maps the set

$$\{(0,\dots,0,p_k,\dots,p_n) \mid \|(0,\dots,0,p_k,\dots,p_n)\| \le R\}$$

into

$$\{(0,\dots,0,p_k,\dots,p_n) \mid \|(0,\dots,0,p_k,\dots,p_n)\| \le \delta\}.$$

Outside $|q_k| < \delta$, our diffeomorphism will be constructed similarly, but without the coordinates (q_k, p_k). Therefore choose a cut-off function $\beta\colon [0,+\infty) \to [0,1]$ such that

$$\beta(t) = \begin{cases} 0 & \text{for } t \in [2,+\infty), \\ 1 & \text{for } t \in [0,1]. \end{cases}$$

For $a > 0$, denote by $\beta_a(t) := \beta(t/a)$ and define

$$K(\mathbf{q},\mathbf{p}) := \rho_R(\mathbf{q},\mathbf{p})\beta_r(\|(q_{k+1},\dots,q_n)\|)(\beta_{\delta_0/2}(|q_k|)\lambda q_k p_k + \ldots + \lambda q_n p_n). \tag{5.3}$$

Here $\rho_R\colon T^*M \to [0,1]$ is a smooth cut–off function equal to 1 in $D^*_R M$ and equal to zero in $T^*M \setminus D^*_{2R}M$ (In (5.3), as in the rest of the paper, by $\|(q_k,\dots,q_n)\|$ we assume $\|((0,\dots,0,q_k,\dots,q_n,0,\dots,0)\|$ etc.) By choosing $r > 0$ small enough we can obtain $\|K\|_{C^0} < \varepsilon$. $\square$

In the following lemma we keep the notations from Lemma 5.4.

Lemma 5.5. *For given $\varepsilon > 0$, there exist an approximation Υ^N, $\delta_1 > 0$ small enough and a Hamiltonian L such that for all $\delta < \delta_1$ the corresponding Hamiltonian isotopy ϕ^1_L satisfies*

(a) *ϕ^1_L maps $\Upsilon^N \cap D^*_\delta M$ into $O_M \cap \mathrm{Tb}_M(N \cup \mathrm{Tb}_N(\partial N)) \subset O_{U_n}$ (for some open subset $U_n \subset M$),*
(b) *ϕ^1_L leaves $T^*M \setminus D^*_{2\delta}M$ fixed,*
(c) *$\|L\|_{C^0} < \varepsilon$.*

Proof. Recall Υ^N is of the form

$$\Upsilon^N = \Big[\sigma(\nu^*_-(\partial N) \cap \overline{\nu^*(\mathrm{Int}\, N)}) \times C_\varepsilon\Big] \cup \Big[\overline{\nu}^* N \setminus \sigma(\nu^*_-(\partial N) \cap \overline{\nu^*(\mathrm{Int}\, N)}) \times C\Big]$$

(see (2.5)) where C_ε is a smooth curve in (q_k, p_k), Suppose that $\delta_1 > 0$ is small enough and Υ^N is such that

$$\Upsilon^N \cap D^*_{\delta_1} M \subset T^*(\mathrm{Tb}_M(N \cup \mathrm{Tb}_N(\partial N))),$$
$$O_{\partial N} \times C_\varepsilon \setminus \big(\{q_k = 0\} \cup \{p_k = 0\}\big) \subset T^*(\mathrm{Tb}_M(N \cup \mathrm{Tb}_N(\partial N))).$$

Since Υ^N decomposes in two components, we will construct a Hamiltonian as a sum of two Hamiltonians:

$$\begin{aligned} &L(q_1,\dots,q_n,p_1\dots,p_n) \\ &\qquad = L_1(q_k,p_k) + L_2(q_1,\dots,q_{k-1},q_{k+1},\dots,q_n,p_1,\dots,p_{k-1},p_{k+1},\dots,p_n). \end{aligned}$$

First we construct L_1. We want $\phi^1_{L_1}$ to

- rotate the negative part of p_k-axis to the positive part of q_k-axis
- leave the line $(q_k - \text{axis}) \cap \Upsilon^N$ fixed.

To get this, we can choose the Hamiltonian function of the form $-\pi(q_k^2 + p_k^2)/4$ in the region $q_k \le 0$, $p_k \le 0$ and multiply it with a suitable cut-off function, such that it has a compact support and it is equal to zero at q_k-axis $\cap\, \Upsilon^N$. Denote the corresponding Hamiltonian diffeomorphism by φ. It deforms the curve C into some smooth curve, say $\widetilde{C}$ which coincide with q_k-axis outside a compact interval. If the curve $\widetilde{C}$ is not a graph of some map $p_k = f(q_k)$, we can deform C in such way that both C and $\widetilde{C}$ become graphs. Note that the Hamiltonian diffeomorphism, say ψ with the differential of corresponding Hamiltonian function equal to f maps $\widetilde{C}$ to the q_k-axis. Again we cut-off this Hamiltonian suitably and define $\phi^1_{L_1}$ to be the composition of these two Hamiltonian isomorphisms, $\psi \circ \varphi$.

Let us now construct the Hamiltonian L_2. Similarly to the construction above, we first define a Hamiltonian $\check{L}$ in $(q_{k+1}, \dots, q_n, p_{k+1}, \dots, p_n)$-plane such that associated Hamiltonian diffeomorphism which

- rotates the negative part of p_j-axis to the positive part of q_j-axis,
- leave the line $(q_j - \text{axis}) \cap \Upsilon^N$ fixed,

for all $j = k+1, \dots, n$. This can be done by choosing

$$\check{L}(q_{k+1}, \dots, q_n, p_{k+1}, \dots, p_n) := -\frac{\pi}{4}\big(q_{k+1}^2 + p_{k+1}^2 + \ldots + q_n^2 + p_n^2\big)$$

in the region $q_j \le 0$, $p_j \le 0$, $j = k+1, \dots, n$ and by multiplying it with a suitable cut-off functions in (q_j, p_j)-planes, such that it has a compact support and it is equal to zero at $(q_j - \text{axis}) \cap \Upsilon^N$. Finally, we set

$$\begin{aligned} &L_2(q_{k+1}, \dots, q_n, p_{k+1}, \dots, p_n) \\ &\qquad := \beta_{\delta_1}\big(\|(p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_n)\|\big)\check{L}(q_{k+1}, \dots, q_n, p_{k+1}, \dots, p_n). \end{aligned}$$

By decreasing $\delta_1 > 0$ we obtain (c). □

Lemma 5.6. *Let K be as in Lemma 5.4 and L be as in Lemma 5.5. Denote*

$$\begin{aligned} \widetilde{x}(t) &:= \phi^t_L \circ \phi^t_K(x(t)), \qquad & \widetilde{u}(s,t) &:= \phi^t_L \circ \phi^t_K(u(s,t)), \\ \widetilde{H} &:= L \sharp K \sharp H, & \widetilde{J}_t &:= \big(\phi^t_L \circ \phi^t_K\big)_* J_t. \end{aligned} \tag{5.4}$$

Then, for given $\varepsilon > 0$ we can choose δ_1, Υ^N and Υ^n from Lemma 5.5 such that the mapping $x \mapsto \widetilde{x}$ maps the generating set $CF(O_M, \Upsilon^N : H, J)$ into $CF(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J})$, where Υ^n is an approximation of $\overline{\nu}^* U_n$. The mapping $u \mapsto \widetilde{u}$ maps the set $\mathcal{M}(O_M, \Upsilon^N : H, J)$ into $\mathcal{M}(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J})$. Moreover, for given $\varepsilon > 0$, we can decrease δ_1 and choose approximations Υ^n and Υ^N such *that, for every $x \in \mathcal{M}(O_M, \Upsilon^N : H, J)$, it holds*

$$\big|\mathcal{A}^{\Upsilon^n}_{\widetilde{H}}(\widetilde{x}) - \mathcal{A}^{\Upsilon^N}_{H}(x)\big| < \varepsilon. \tag{5.5}$$

Proof. For a fixed Υ^N and Υ^n, one can choose δ_1 small enough such that

$$x \in CF\big(O_M, \Upsilon^N : H, J\big) \ \Rightarrow\ \widetilde{x} \in CF\big(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J}\big).$$

This is possible since the set $CF(O_M, \Upsilon^N : H, J)$ is finite. Then, one easily checks

$$u \in \mathcal{M}\big(O_M, \Upsilon^N : H, J\big) \ \Rightarrow\ \widetilde{u} \in \mathcal{M}\big(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J}\big).$$

To prove (5.5), we estimate

$$\Big|\mathcal{A}^{\Upsilon^n}_{\widetilde{H}}(\widetilde{x}) - \mathcal{A}^{\Upsilon^N}_{H}(x)\Big| \le \big|\mathcal{A}_{\widetilde{H}}(\widetilde{x}) - \mathcal{A}_H(x)\big| + |h_{\Upsilon^n}(\widetilde{x}(1))| + |h_{\Upsilon^N}(x(1))|, \tag{5.6}$$

where

$$\mathcal{A}_H(\gamma) := \int \gamma^*\theta - \int_0^1 H(\gamma(t), t)\,dt.$$

The last two terms in (5.6) can be made smaller than $\varepsilon/4$ by choosing good enough approximations.

Next, as in [24], note that

$$\mathcal{A}_{\widetilde{H}}(\widetilde{x}) - \mathcal{A}_H(x) = \int_0^1 \frac{d}{ds}\big[\mathcal{A}_{s(L\,\sharp\,K)\,\sharp\,H}\big(\phi^t_{s(L\,\sharp\,K)}(x)\big)\big]$$

and

$$\begin{aligned}\frac{d}{ds}&\big[\mathcal{A}_{s(L\,\sharp\,K)\,\sharp\,H}\big(\phi^t_{s(L\,\sharp\,K)}(x)\big)\big]\\ &= d\mathcal{A}_{s(L\,\sharp\,K)\,\sharp\,H}\bigg(\frac{\partial\phi^t_{s(L\,\sharp\,K)}(x)}{\partial s}\bigg) - \int_0^1 L\,\sharp\,K\big(\phi^t_{s(L\,\sharp\,K)}(x)\big)\,dt.\end{aligned}$$

By decreasing δ_1 and r from (5.3), we can obtain

$$\bigg|\int_0^1 L\,\sharp\,K(\phi^t_{s(L\,\sharp\,K)}(x))\,dt\bigg| < \frac{\varepsilon}{4}.$$

To finish the proof, recall

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega(\dot\gamma - X_H(\gamma), \xi)dt + \langle\xi, \theta(\gamma(1))\rangle - \langle\xi, \theta(\gamma(0))\rangle.$$

Since $s \mapsto \phi^t_{s(L\,\sharp\,K)}(x(t))$ is a Hamiltonian orbit of $L\,\sharp\,K$ with the initial point $x(t)$ and

$$\bigg\langle\frac{\partial\phi^0_{s(L\,\sharp\,K)}(x(0))}{\partial s}, \theta\big(\phi^0_{s(L\,\sharp\,K)}(x(0))\big)\bigg\rangle = 0$$

we have

$$d\mathcal{A}_{s(L\,\sharp\,K)\,\sharp\,H}\bigg(\frac{\partial\phi^t_{s(L\,\sharp\,K)}(x)}{\partial s}\bigg) = \bigg\langle\frac{\partial\phi^1_{s(L\,\sharp\,K)}(x(1))}{\partial s}, \theta\big(\phi^1_{s(L\,\sharp\,K)}(x(1))\big)\bigg\rangle. \tag{5.7}$$

Let us check that the right side of (5.7) can be made arbitrarily small, by decreasing δ_1. Since

$$\frac{\partial \phi^1_{s(L\,\sharp\,K)}(x(1))}{\partial s} = X_{L\,\sharp\,K}$$

which is basically $X_K + X_L$, let us estimate the form θ at the latter vector field. Firstly, since the support of L is contained in $D^*_{2\delta_1}M$, by decreasing δ_1, we can obtain

$$\big|\mathbf{p}d\mathbf{q}(X_L)\big| < \frac{\varepsilon}{4}.$$

Secondly, we have

$$\begin{aligned}\frac{\partial K}{\partial p_j} &= \beta_r(\|(q_{k+1},\dots,q_n)\|) \\ &\quad\times\left(\frac{\partial \rho_R}{\partial p_j}(\mathbf{q},\mathbf{p})(\beta_{\delta_1/2}(|q_k|)\lambda q_k p_k + \ldots + \lambda q_n p_n) + \rho_R(\mathbf{q},\mathbf{p})\cdot a(\mathbf{q},\mathbf{p})\right)\end{aligned}$$

where

$$a(\mathbf{q},\mathbf{p})) := \begin{cases}\beta_{\delta_1/2}(|q_k|)\lambda q_k & \text{for } j = k,\\ \lambda q, & \text{for } j = k+1,\dots,n.\end{cases}$$

Since

$$\left\|\frac{\partial K}{\partial p_j}\right\| \le \|(q_k,\dots,q_n)\|\cdot \mathrm{const},$$

we can decrease r so we obtain

$$\big|\theta(X_K)\big| = |\mathbf{p}\,d\mathbf{q}(X_K)| = \left|\sum p_j\,\frac{\partial K}{\partial p_j}\right| < \frac{\varepsilon}{4}. \qquad \square$$

LEMMA 5.7. *It holds*

$$\widetilde{\partial x} = \partial\widetilde{x}. \tag{5.8}$$

and, for every $\varepsilon > 0$, the mapping

$$\imath^\lambda_n = \imath^\lambda_{(\Upsilon^N,\Upsilon^n)}\colon HF^\lambda(O_M,\Upsilon^N : H,J) \to HF^{\lambda+\varepsilon}(O_M,\Upsilon^n : H,J) \tag{5.9}$$

is well defined.

PROOF. The proof of (5.8) is similar to the proof in [24, Chapter 3]. We sketch the key steps for the sake of completeness.

For, $\delta_1 > 0$, denote by

$$U_{\delta_1} := \big\{\mathbf{q}\in \mathrm{Tb}_M(N\cup \mathrm{Tb}_N(\partial N)) \mid \|(q_{k+1},\dots,q_n)\| < \delta_1\big\},$$

where $(q_{k+1},\dots,q_n)$ are global coordinates from (5.2). For a exact Lagrangian manifold Υ, denote by

$$\begin{aligned}\mathcal{M}(O_M,\Upsilon : H,J) := \Bigg\{u\;\Bigg|\; & u\colon \mathbb{R}\times[0,1]\to T^*M,\\ & E(u) := \int \|\partial_s u\|^2\,dt\,ds < +\infty,\end{aligned}$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial s} + J\bigg(\frac{\partial u}{\partial t} - X_H(u)\bigg) = 0,\\ &u(s,0) \in O_M,\ u(s,1) \in \Upsilon. \end{aligned}\right\}$$

For some $\delta_1 > 0$ and λ sufficiently large (rescaling factor that we use in proof of Lemma 5.4) every $\widetilde{u} \in \mathcal{M}(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J})$ satisfies:

$$\widetilde{u}(-\infty, 1) \in U_{\delta_1} \ \Rightarrow\ \widetilde{u}(s,1) \in U_{\delta_1}, \quad \text{for all } s \in \mathbb{R}. \tag{5.10}$$

The proof of this claim follows from our construction in Lemmas 5.4–5.6 and it is the same as the proof of [24, Proposition 3.2]. Note that at some point the proof of [24, Proposition 3.2] uses the fact that $H \equiv 0$ near $t = 0$ and $t = 1$. This is not a loss of generality, since, according to [24, Lemma 3.1], we can perturb our Hamiltonian to satisfy this condition with the change in the filtration as small as desired. The condition (5.10) implies (5.8). Indeed choose λ such that $\widetilde{u}(s,1) \in U_{\delta_1}$ for all $u \in \mathcal{M}(O_M, \Upsilon^N : H, J)$. Together with (5.10), this means that for every $u \in \mathcal{M}(O_M, \Upsilon^N : H, J)$, $\widetilde{u}$ participates in the computation of $\partial\widetilde{x}$. Again from (5.10) and the fact the mappings $x \mapsto \widetilde{x}$, $u \mapsto \widetilde{u}$ are bijections from $\mathcal{M}(O_M, \Upsilon^N : H, J)$ to the set

$$\big\{u \in \mathcal{M}\big(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J}\big) \mid u(s,1) \in U_{\delta_1}\big\}$$

(since it is constructed using Hamiltonian diffeomorphism), we conclude the reverse: every $\widetilde{u}$ participating in $\widetilde{\partial x}$ comes from some $u \in \mathcal{M}(O_M, \Upsilon^N : H, J)$, for every $\widetilde{x}$ with $\widetilde{x}(1) \in U_{\delta_1}$. This implies (5.8).

Now we compose our map $x \mapsto \widetilde{x}$ with a canonical continuation homomorphism (see, for example [10]):

$$HF_*^{\lambda}\big(O_M, \Upsilon^n : \widetilde{H}, \widetilde{J}\big) \to HF_*^{\lambda+\varepsilon/2}(O_M, \Upsilon^n : H, J)$$

and finish the proof. This canonical isomorphism exists for a given ε since the norms $\|K\|_{C^0}$ and $\|L\|_{C^0}$ can be made arbitrary small. □

Lemma 5.8. *For U_n a tubular neighbourhood of N, it holds*

$$c_+(H, N) \geq c_+(H, U_n).$$

Proof. The following diagram

$$\begin{tikzcd}
HF_*^{\lambda}(O_M, \Upsilon^N : H, J) \arrow[r, "\imath_n^{\lambda}"] \arrow[d, "\jmath_{N*}^{\lambda}"'] & HF_*^{\lambda+\varepsilon}(O_M, \Upsilon^n : H, J) \arrow[d, "\jmath_{U_n*}^{\lambda+\varepsilon}"] \\
HF_*(O_M, \Upsilon^N : H, J) \arrow[r, "\imath_n"] \arrow[d, "F_{(H,N)}^{-1}"'] & HF_*(O_M, \Upsilon^n : H, J) \arrow[d, "F_{(H,U)}^{-1}"] \\
H_*(N) \arrow[r, "\jmath_{NU_n*}"'] & H_*(U)
\end{tikzcd}$$

commutes. Indeed, the upper diagram is obvious, and the commutativity of the lower one is proved in [24] in the similar situation (for $U \subset V$ open). The vertical mappings $F_{(H,N)}$ and $F_{(H,U)}$ are isomorphisms between singular and Floer homologies defined in [10] (without the use of PSS).

Since N is a deformation retract of U_n, the map

$$\jmath_{NU_n*} \colon H_*(N) \to H_*(U_n)$$

induced by the inclusion $\jmath_{NU_n} \colon N \to U_n$ is an isomorphism. Note that in this case, the mapping $\imath_n$ for $\lambda = \infty$ in (5.9) is also an isomorphism.

We claim:

$$c_+(H, \Upsilon^n) \leq c_+\big(H, \Upsilon^N\big) + \varepsilon. \tag{5.11}$$

To prove (5.11), denote by $a := c_+\big(H, \Upsilon^N\big)$ and choose $\alpha \in HF_*(O_M, \Upsilon^n : H, J)$. Since $\imath_n$ is an isomorphism, there exists $\beta \in HF_*\big(O_M, \Upsilon^N : H, J\big)$ with $\imath_n(\beta) = \alpha$, an since $\jmath^a_{N*}$ is surjective, there exists $\gamma \in HF^a_*(O_M, \Upsilon^N : H, J)$ such that $\jmath^a_{N*}(\gamma) = \beta$. We have:

$$\alpha = \imath_n(\beta) = \imath_n \circ \jmath^a_{N*}(\gamma) = \jmath^{a+\varepsilon}_{U_n} \circ \imath^a_n(\gamma) = \jmath^{a+\varepsilon}_{U_n}(\imath^a_n(\gamma)),$$

so $\alpha \in \mathrm{Im}\big(\jmath^{a+\varepsilon}_{U_n}\big)$. Therefore $\jmath^{a+\varepsilon}_{U_n}$ is surjective, so (5.11) holds.

Since the invariants c_+ all become equal, starting from some approximation (Remark 5.2), it follows from (5.11) that

$$c_+(H, U) \leq c_+(H, N) + \varepsilon$$

for every ε, so the proof is complete.

Note that in this lemma we only used the surjectivity of $\jmath_{NU_n*}$, and not the injectivity. $\square$

Proof of Theorem 5.3. In the same way as in the proof of Lemma 5.8 we can prove that $c_+(H, U_n) \leq c_+(H, U_{n+1})$. Therefore, we need to show that, given $\varepsilon > 0$ there exists n_0 such that

$$c_+(H, N) < c_+(H, U_{n_0}) + \varepsilon.$$

Choose n_0 such that N is a deformation retract of U_{n_0} and $\delta_1 > 0$, Υ^N, Υ^{n_0} such that, for $\widetilde{H}$ and $\widetilde{x}$ from Lemma 5.6 it holds

- $\left|\mathcal{A}^{\Upsilon^{n_0}}_{\widetilde{H}}(\widetilde{x}) - \mathcal{A}^{\Upsilon^N}_H(x)\right| < \varepsilon/2$ (Lemma 5.6),
- $c_+\left(H, \Upsilon^N\right) = c_+(H, N)$, $c_+\left(H, \Upsilon^{n_0}\right) = c_+(H, U_{n_0})$ (Remark 5.2),
- $c_+\big(\widetilde{H}, \Upsilon^{n_0}\big) < c_+(H, \Upsilon^{n_0}) + \varepsilon/2$.

The last item can be achieved by choosing δ and r from the proof of Lemma 5.4 small enough so that $\|H - \widetilde{H}\|_{C^0}$ norm is small enough; this fact can be proved by tracking the change of the action functional associated to the homotopy of Hamiltonians (see [22, Theorem 7.2] or [10, Proposition 4.4]).

Note that

$$c_+(H,\Upsilon^N) = \max\big\{\mathcal{A}_H^{\Upsilon^N}(x) \mid [x] \in HF_*(O_M,\Upsilon^N : H,J)\big\}$$

and similarly, for $c_+(H,\Upsilon^{n_0})$. For every $[x] \in HF_*(O_M,\Upsilon^N : H,J)$, we have

$$\mathcal{A}_H^{\Upsilon^N}(x) < \mathcal{A}_{\widetilde{H}}^{\Upsilon^n}(\widetilde{x}) + \frac{\varepsilon}{2} \leq c_+(\widetilde{H},\Upsilon^{n_0}) + \frac{\varepsilon}{2} \leq c_+(H,\Upsilon^{n_0}) + \varepsilon.$$

By taking a maximum over $[x] \in HF_*\big(O_M,\Upsilon^N : H,J\big)$ we conclude

$$c_+\big(H,\Upsilon^N\big) < c_+\left(H,\Upsilon^{n_0}\right) + \varepsilon. \qquad \square$$

Remark 5.9. In the same way we can prove that the invariants

$$c_-(H,N) := \sup\big\{\lambda \in \mathbb{R} \mid \imath_*^\lambda \colon HF_*^\lambda(H,N:M) \to HF_*(H,N:M) \text{ is trivial}\big\}$$

are continuous with respect to a decreasing sequence U_n, $\bigcap U_n = N$, therefore the same holds for the invariants:

$$\gamma(H,N) := c_+(H,N) - c_-(H,N).$$

Question 5.10. An interesting question is whether there is a similar continuity result for spectral invariants for a single homology class, defined in (4.2). If this is true, then the inequality

$$c_U\big(\imath_*^M[\alpha],H\big) \leq c_N([\alpha],H)$$

would easily follow from the commutativity of the following diagram:

$$\begin{tikzcd}
HM_*(f_N,N:g) \arrow[r, "\mathrm{PSS}^N"] \arrow[d, "\imath_*^M"'] & HF_*\big(O_M,\Upsilon^N:H,J\big) \arrow[d, "\imath_{(\Upsilon^N,\Upsilon^U)}"] & HF_*^\lambda(O_M,\Upsilon^N:H,J) \arrow[l, "\jmath_\lambda^N"'] \arrow[d, "\imath^\lambda_{(\Upsilon^N,\Upsilon^U)}"] \\
HM_*(f_U,U:g) \arrow[r, "\mathrm{PSS}^U"'] & HF_*\big(O_M,\Upsilon^U:H,J\big) & HF_*^{\lambda+\varepsilon}(O_M,\Upsilon^U:H,J) \arrow[l, "\jmath_{\lambda+\varepsilon}^U"]
\end{tikzcd}$$

Remark 5.11. The previous results apply to the case when the boundary of a submanifold is empty. Consider two closed framed submanifolds (with or without boundary) N_1 and N_2 such that $N_1 \subset N_2$. Choose two sequences of open sets U_n and V_n that satisfy:

- U_n is a tubular neighbourhood of N_2, V_n is a tubular neighbourhood of N_1,
- $U_{n+1} \subseteq U_n$, $V_{n+1} \subseteq V_n$,
- $\bigcap\limits_n U_n = N_2$, $\bigcap\limits_n V_n = N_1$,
- ∂U_n and ∂V_n are smooth submanifolds of M of codimension one,
- $V_n \subset U_n$.

Suppose that $\imath_*\colon H_*(N_1) \to H_*(N_2)$ is surjective, then $\imath_*\colon H_*(V_n) \to H_*(U_n)$ is surjective. It follows from Theorem 5.1 that

$$c_+(H,V_n) \leq c_+(H,U_n).$$

By taking the limit when $n \to \infty$ and using Theorem 4.6 we conclude

$$c_+(H, N_1) \leq c_+(H, N_2).$$

If the inclusion map $\imath_* \colon H_*(N_1) \to H_*(N_2)$ is injective (hence $\imath_* \colon H_*(V_n) \to H_*(U_n)$ is injective), Oh showed that

$$c_-(H, V_n) \geq c_-(H, U_n)$$

(see [24, Theorem 4.3 (2)]). Therefore

$$c_-(H, N_1) \geq c_-(H, N_2). \tag{5.12}$$

Finally, if $\imath_* \colon H_*(N_1) \to H_*(N_2)$ is a bijection, it holds

$$\gamma(H, N_1) \leq \gamma(H, N_2).$$

Consider the special case when $N_2 = N$, $N_1 = \partial N$. Suppose that N is framed, this implies that ∂N is framed too, since

$$\nu(\partial N) \cong \nu(N)|_{\partial N} \oplus \mathrm{Tb}_N(\partial N).$$

If $\imath_* \colon H_*(\partial N) \to H_*(N)$ is surjective, then

$$c_+(H, \partial N) \leq c_+(H, N).$$

Since we deal with $\mathbb{Z}_2$-coefficients, then $\imath_* \colon H_*(\partial N) \to H_*(N)$ is never an injection, for $* = k = \dim N$. Indeed, since $H_k(N) = \{0\}$ and $H_k(N, \partial N) = \mathbb{Z}_2$, using the long exact sequence for the pair $(N, \partial N)$ we have that

$$\mathrm{Ker}(\imath_*) = \mathbb{Z}_2, \qquad \imath_* \colon H_{k-1}(\partial N) \to H_{k-1}(N).$$

However, if N is not orientable and if we deal with $\mathbb{Z}$-coefficients, the above reasoning does not apply, and it is possible for $\imath_*$ to be injective. In order to prove the inequality (5.12) one needs to consider a coherent orientation of mixed moduli spaces and construct a PSS morphism from Theorem A with $\mathbb{Z}$-coefficients. This will be the subject of a further research.

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