

**ADDENDUM AND CORRIGENDUM TO
“ON THE CHAOS GAME
OF ITERATED FUNCTION SYSTEMS”**

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ABSTRACT. We provide a counter-example to Theorem 1.4 (a) in Topol. Methods Nonlinear Anal. **49** (2017), no. 1, 105–132, by showing that the closure of the Γ -orbit of a point x in the pointwise basin of Ls-attraction of a quasi-attractor A is not compact. In order to fix this gap, we modified the definition of Ls-basin of attraction. In addition, we propose a better place to play the chaos game and as a consequence we get some additional results on strongly-fibred quasi-attractors and Conley attractors.

1. On the concept of the basin of attraction

We retain the notation of [3]. Proposition 2.3 claims that for a compact subset A of X if $x \in \mathcal{B}_p^*(A)$ then $\overline{\Gamma(x)}$ is a forward invariant compact set. This proposition, and in particular, this claim, is one of the tools used to prove the main result Theorem 1.4 (a). However, as the second author of this note notified, there is an unsolvable bug as the following counterexample demonstrates:

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EXAMPLE 1.1. Consider $X = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with the trivial metric and define the IFS on X generated by the family $\mathcal{F} = \{f_1, f_2\}$ where

$$f_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ x + 1 & \text{if } x \neq 0, \end{cases} \quad \text{and} \quad f_2(x) = 0.$$

Every function is continuous with respect to this metric. Furthermore, the set $A = \{0\}$ is a quasi-attractor of this IFS. Recall that a quasi-attractor is a forward minimal self-similar compact subset of X . We also recall that in [3] we have introduced the pointwise basin of Ls-attraction of A as

$$\mathcal{B}_p^*(A) \stackrel{\text{def}}{=} \{x \in X : \text{Ls } F^n(\{x\}) = A\}.$$

If $x \in \mathbb{N}$, $F^n(\{x\}) = \{0, x + n\}$ for all $n \geq 1$ and thus

$$\begin{aligned} \text{Ls } F^n(\{x\}) &\stackrel{\text{def}}{=} \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} F^n(\{x\})} \\ &= \bigcap_{m \geq 1} \{0\} \cup \{x + n : n \geq m\} = \{0\} = A. \end{aligned}$$

This implies that $\mathbb{N} \subset \mathcal{B}_p^*(A)$. On the other hand, the closure of the Γ -orbit $\Gamma(x) = \{0\} \cup \{x + n : n \geq 1\}$ of any $x \in \mathbb{N}$ is an infinite set and so is not compact.

The mistake in the proof of Proposition 2.3 is essentially in the assertion that for every open neighbourhood U of A ,

$$(1.1) \quad \text{there is } n_0 \in \mathbb{N} \text{ such that } F^n(\{x\}) \subset U \text{ for all } n \geq n_0.$$

Observe that (1.1) is technically equivalent to

$$(1.2) \quad \overline{\bigcup_{m \geq n} F^m(\{x\})} \text{ converges to } A \text{ in the upper Vietoris topology.}$$

Although this mistake applies to some other parts of the paper we can *save all the results* in [3] by means of the following modification of the definition of $\mathcal{B}_p^*(A)$. We redefine

$$\mathcal{B}_p^*(A) \stackrel{\text{def}}{=} \{x \in X : \text{Ls } F^n(\{x\}) = A \text{ and (1.2) holds.}\}.$$

In general the the upper Kuratowski convergence (Ls-limit) and upper Vietoris convergence are not comparable. However, the following general lemma concludes that

$$(1.3) \quad \mathcal{B}_p^*(A) = \left\{ x \in X : \overline{\bigcup_{m \geq n} F^m(\{x\})} \rightarrow A \text{ in Vietoris topology} \right\}.$$

LEMMA 1.2. Consider a sequence $(A_n)_n$ converging in the upper Vietoris topology to a compact set A of X . Then $\emptyset \neq \text{Ls } A_n \subset A$. In addition, if

$$(1.4) \quad A_n = \overline{\bigcup_{m \geq n} B_m} \quad \text{where } B_n \text{ is a finite subset of } X \text{ for all } n \in \mathbb{N}$$

then

- (a) A_n is compact and $\text{Ls } A_n = \text{Ls } B_n$;
- (b) $\text{Ls } A_n = A$ if and only if $A_n \rightarrow A$ in the Vietoris topology.

PROOF. First of all, we will prove that $\text{Ls } A_n \subset A$. By contradiction, assume that there is $b \in \text{Ls } A_n$ which does not belong to A . Since X is a Hausdorff space and A is compact, we find a pair of open sets U and V such that $A \subset U$, $b \in V$ and $U \cap V = \emptyset$. Since $A_n \rightarrow A$ in the upper Vietoris topology, $A_n \subset U$ for all n large enough. But since $b \in \text{Ls } A_n$ then $A_n \cap V \neq \emptyset$ for infinitely many $n \in \mathbb{N}$ finding a contradiction with the fact that $U \cap V = \emptyset$.

Now, we will prove that $\text{Ls } A_n \neq \emptyset$. Since $\text{Ls } A_n \subset A$, if one assume by contradiction that $\text{Ls } A_n = \emptyset$, then for each $a \in A$ there are an open neighbourhood U_a and an integer $n(a) \geq 1$ such that $A_n \cap U_a = \emptyset$ for all $n \geq n(a)$. Then we obtain an open cover of A and, since A is compact, we can extract a finite subcover $U = U_{a_1} \cup \dots \cup U_{a_m}$ of A . Set $n_0 = \max\{n(a_1), \dots, n(a_m)\} \geq 1$. Hence, $A_n \cap U = \emptyset$ for all $n \geq n_0$ which contradicts that A_n converges to A in the upper Vietoris topology.

Finally, we will prove (a) and (b) assuming that A_n is given by (1.4). Observe that since A_n is attracted in the upper Vietoris topology by a compact set A , then the tails A_n are all compact sets. Additionally, since $A_{n+1} \subset A_n$ for all $n \geq 1$ then $\text{Ls } A_n = \bigcap_n A_n = \text{Ls } B_n$. This concludes the first item. To prove the second item we only need to show that if $A = \text{Ls } A_n$, then $A_n \rightarrow A$ in the Vietoris topology, since the reciprocal implication is clear. To do this, it suffices to prove that A_n converges to A in the lower Vietoris topology. Hence, let V be an open set such that $V \cap A \neq \emptyset$. Since $A = \text{Ls } A_n = \bigcap_n A_n$ we have $A_n \cap V \neq \emptyset$ for all n . This concludes that $A_n \rightarrow A$ and completes the proof. \square

As mentioned, with the redefinition (1.3) of the pointwise basin of attraction, all ⁽¹⁾ the results in [3] hold with the same proofs. In particular, we have that a compact set A is a quasi-attractor (i.e. $F(A) = A$ and $A = \overline{\Gamma(x)}$ for all $x \in A$) if and only if A is included in the Ls-basin of attraction. Moreover, compact minimal IFSSs, attractors (c.f. [1]) and compact semi-attractors introduced by Lasota and Myjak in [8] are quasi-attractors; but not vice versa.

⁽¹⁾ Theorem 1.4(b) as written is not true. However, later in this note we present Corollary 1.9 which solves all errors related to this issue.

We realized that although the above redefinition fixed the gap in [3], we have that $\mathcal{B}_p^*(A) = A$ for the quasi-attractor $A = \{0\}$ in Example 1.1. Since A is a well-fibred attractor then it is renderable by the deterministic (or disjunctive) chaos game as Theorem 1.6 implies. But, this result is not useful to play the disjunctive chaos game. Since we must to pick x in $\mathcal{B}_p^*(A) = A = \{0\}$; while it is easy to see that we can render the attractor picking any $x \in X$. Consequently, we should propose a new place to play the game. To do this, observe that in view of Theorem 1.4 (c), a quasi-attractor is renderable by the deterministic chaos game (c.f. [3, Definition 1.3]) if and only if

$$(1.5) \quad A \subset \overline{O_\omega^+(x)} \quad \text{for all } x \in \mathcal{B}_p^*(A) \text{ and disjunctive sequence } \omega \in \Omega.$$

Notice that although the notion of the disjunctive chaos game involves a priori probability in its definition, we observe that (1.5) is fully deterministic. This fact derandomizes the algorithm of the chaos game and for this reason was called the deterministic chaos game in [3]. Moreover, in view of (1.5) and the redefinition of $\mathcal{B}_p^*(A)$, if A is renderable from a point $x \in \mathcal{B}_p^*(A)$ by the deterministic chaos game, then for every open neighbourhood U of A and every disjunctive sequence $\omega \in \Omega$, there is $n_0 \in \mathbb{N}$ such that $f_\omega^n(x) \in U$ for all $n \geq n_0$. Again, this condition is equivalent to ask that

$$(1.6) \quad A_n = \overline{\{f_\omega^m(x) : m \geq n\}} \text{ converges to } A \text{ in the upper Vietoris topology.}$$

Thus, we propose the following set

$$\mathcal{B}_p^{**}(A) = \{x \in X : (1.6) \text{ holds for all disjunctive sequence } \omega \in \Omega\}.$$

The first trivial observation is that $\mathcal{B}_p^*(A) \subset \mathcal{B}_p^{**}(A)$. It easy to see that in the case of a contracting IFS we have a unique strict attractor A for which $\mathcal{B}_p^{**}(A) = \mathcal{B}_p^*(A) = \mathcal{B}(A) = X$. If we add the identity map to the IFS, then $\mathcal{B}_p^*(A) = A$ (notice that now A is only a quasi-attractor) but $\mathcal{B}_p^{**}(A) = X$. The same also holds for Example 1.1. Thus $\mathcal{B}_p^{**}(A)$ seems more appropriate than $\mathcal{B}_p^*(A)$ to play the chaos game. We note, the basins $\mathcal{B}_p^*(A)$ and $\mathcal{B}_p^{**}(A)$ need not be open or contain an open set unlike the basins involved with attractors. To see this, consider the IFS generated by the family $\mathcal{F} = \{f_1(x) = 1 - x, f_2(x) = x\}$ on \mathbb{R} with the usual topology. Every two element set of the form $\{x, 1 - x\} = A$ is a quasi-attractor and $A = \mathcal{B}_p^{**}(A) = \mathcal{B}_p^*(A)$.

In the next section, we will see that several results on the chaos game in [3] remain valid replacing in Definition 1.3 the set $\mathcal{B}_p^*(A)$ by $\mathcal{B}_p^{**}(A)$. For the readability of this note we recall this (modified) definition here:

DEFINITION 1.3. Let A be a quasi-attractor of an IFS. We say that A is renderable by

- (a) the *probabilistic chaos game* if for any $x \in \mathcal{B}_p^{**}(A)$ there is $\Omega(x) \subset \Omega$ with $\mathbb{P}(\Omega(x)) = 1$ such that

$$A \subset \overline{O_\omega^+(x)} \quad \text{for all } \omega \in \Omega(x);$$

- (b) the *deterministic chaos game* (or *disjunctive chaos game*) if there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$A \subset \overline{O_\omega^+(x)} \quad \text{for all } \omega \in \Omega_0 \text{ and } x \in \mathcal{B}_p^{**}(A).$$

If the IFS is forward minimal (consequently A is the whole space X and $A = \mathcal{B}_p^{**}(A) = X$), we simply say that the IFS is renderable by the probabilistic/disjunctive(deterministic) chaos game. Since the set Δ of disjunctive sequences has full probability measure one can ask that $\Omega(x), \Omega_0 \subset \Delta$.

Firstly, we must see that if a quasi-attractor A satisfies the above definition then it actually can be rendered by the tails of the fiberwise orbits. That is, if Theorem 1.4 (b) holds:

$$(1.7) \quad A \subset \overline{O_\omega^+(x)} \text{ if and only if } \lim_{n \rightarrow \infty} A_n(\omega, x) = A \text{ in the Vietoris topology}$$

where

$$(1.8) \quad A_n(\omega, x) \stackrel{\text{def}}{=} \overline{\{f_\omega^m(x) : m \geq n\}} \quad \text{with } x \in \mathcal{B}_p^*(A) \text{ and } \omega \in \Omega.$$

But, we also must indicate a mistake in the above statement as the following example shows.

EXAMPLE 1.4. Consider the IFS on $X = \mathbb{R}$ generated by the family of contractions

$$f_1(x) = \frac{x}{2}, \quad f_2(x) = \begin{cases} \frac{x}{2} + \frac{1}{4} & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if } x > \frac{1}{2}, \end{cases} \quad \text{and} \quad f_3(x) = 1.$$

Since it is a contracting IFS, we have a unique strict attractor given by $A = [0, 1/2] \cup \{1\}$ and $\mathcal{B}_p^*(A) = X$. Moreover, given $\omega = (\omega_n)_{n \in \mathbb{N}}$ with $\omega_1 = 3$ and so that $\sigma\omega = (\omega_{n+1})_{n \in \mathbb{N}}$ is a disjunctive word in $\{1, 2\}^{\mathbb{N}}$ we have that

$$A = \overline{O_\omega^+(1)} = A_1 \quad \text{and} \quad 1 \notin A_n \quad \text{for all } n \geq 2 \text{ where } A_n = \overline{\{f_\omega^m(1) : m \geq n\}}.$$

Thus one finds a neighbourhood V of 1 so that $A_n \cap V = \emptyset$ for all $n \geq 2$. In particular, (1.7) does not hold for *any* $\omega \in \Omega$ as claimed in (1.8). However, it is not difficult to see that A is renderable in the Vietoris topology by the tails of the fiberwise orbit of any disjunctive sequence $\omega \in \{1, 2, 3\}^{\mathbb{N}}$.

The gap in the proof of Theorem 1.4 (b) appears in Proposition 2.7 where we claim that, for every open set V such that $A \cap V \neq \emptyset$,

$$\text{if } A \subset \overline{O_\omega^+(x)} \text{ then } A_n \cap V \neq \emptyset$$

for all n large enough with $x \in \mathcal{B}_p^*(A)$ and $\omega \in \Omega$. This claim does not follow immediately and it is wrong when the compact set A has isolated points as the above example shows. However, in the following proposition we conclude (1.7) under the hypothesis that A is without isolated points:

PROPOSITION 1.5. *If A is without isolated points, then (1.7) holds in both cases:*

- (a) *for all $x \in \mathcal{B}_p^*(A)$ and $\omega \in \Omega$;*
- (b) *for any $x \in \mathcal{B}_p^{**}(A)$ and disjunctive sequence $\omega \in \Omega$.*

PROOF. Let $A_n = A_n(\omega, x) = \overline{\{f_\omega^m(x) : m \geq n\}}$. Observe that if x belongs to $\mathcal{B}_p^{**}(A)$, then A_n converges to A in the upper Vietoris for all disjunctive sequence ω in Ω while if x belongs to $\mathcal{B}_p^*(A)$ this convergence holds for any sequence $\omega \in \Omega$. Thus, in both cases, by taking $B_n = \{x_n\}$ where $x_n = f_\omega^n(x)$, we are under the assumptions of Lemma 1.2. Hence the proposition follows if we prove that

$$A \subset A_1 \quad \text{if and only if} \quad A_n \rightarrow A \text{ in the Vietoris topology.}$$

To see this, it suffices to prove that if $A \subset A_1$ then $A_n \rightarrow A$ in the Vietoris topology. To do this, observe that $A \subset A_1 = A_2 \cup \{x_1\}$. Hence $A \setminus \{x_1\} \subset A_2$. If $x_1 \notin A$ then $A \subset A_2$. But also if $x_1 \in A$, since A is without isolated points, $x_1 \in \overline{A \setminus \{x_1\}} \subset A_2$. Thus, in any case we get that $A \subset A_2$. Arguing recursively, we get that $A \subset A_n$ for all $n \geq 1$. Then $A \subset \bigcap A_n = \text{Ls } A_n$. From Lemma 1.2, we get that $A = \text{Ls } A_n$ and therefore A_n converges to A in the Vietoris topology. \square

If A has isolated points, Proposition 1.5 does not guarantee that A is renderable by means of the tails of the fiberwise orbits in the Vietoris topology. In the following theorem we solve this problem under the assumption that A is renderable by the probabilistic chaos game (Definition 1.3):

THEOREM 1.6. *If the set I of isolated points of A is nonempty and A is renderable by the probabilistic chaos game, then there is $\Sigma \subset \Omega$ with $\mathbb{P}(\Sigma) = 1$ and such that*

$$A_n(\omega, x) \rightarrow A \quad \text{for all } \omega \in \Sigma \text{ and } x \in \mathcal{B}_p^{**}(A) \text{ with } A \subset \overline{O_\omega^+(x)}.$$

PROOF. The first observation is the following:

CLAIM 1.7. *I is a numerable set.*

PROOF. Since \mathcal{F} is finite, it must be the case that Γ is numerable. Hence for all $a \in A$ we have that $\Gamma(a)$ is numerable. We will show that $I \subset \Gamma(a)$ for all $a \in A$; concluding I is numerable as it is a subset of a numerable set. We prove this by contraposition, that is if $b \notin \Gamma(a)$ for some $a \in A$ and $b \in A$, then b is a limit point of A . So suppose that $b \in A$ with $b \notin \Gamma(a)$ for some $a \in A$. Then $\Gamma(a) \subset A \setminus \{b\}$. However, by taking the closure of both sides

of the above inclusion we have $A = \overline{\Gamma(a)} \subset \overline{A \setminus \{b\}}$ noting that the inclusion $A \supset \overline{A \setminus \{b\}}$ always holds, we have $A = \overline{A \setminus \{b\}}$. Hence, b is a limit point of A . Thus $I \subset \Gamma(a)$ for all $a \in A$. \square

By assumption, since A is renderable by the probabilistic chaos game, for each $z \in I$ we have a set $\Omega(z) \subset \Omega$ with $\mathbb{P}(\Omega(z)) = 1$ such that

$$A \subset \overline{O_\omega^+(z)} \quad \text{for all } \omega \in \Omega(z).$$

Since, according to Claim 1.7, I is a numerable set, we can consider the full probability set

$$\Sigma = \bigcap_{z \in I} \bigcap_n \sigma^{-n}(\Omega(z))$$

where σ denotes the lateral shift. Now suppose that $\omega \in \Sigma$, $x \in \mathcal{B}_p^{**}(A)$ with $A \subset A_1(\omega, x)$.

CLAIM 1.8. $E \stackrel{\text{def}}{=} A \setminus \text{Ls } A_n(\omega, x) \subset I$.

PROOF. For short, we denote $A_n = A_n(\omega, x)$. Let $z \in E = A \setminus \text{Ls } A_n$. Suppose that $z \notin I$. Then $z \in \overline{A \setminus \{z\}}$. Since $A \subset A_1 = O_\omega^+(x) \cup \text{Ls } A_n$ then $z \in \overline{O_\omega^+(x) \setminus \{z\}}$ or $z \in \overline{\text{Ls } A_n \setminus \{z\}}$. In the first case, $z \in A_n$ for all $n \geq 1$ and thus $z \in \bigcap A_n = \text{Ls } A_n$. Also, if z is an accumulation point of $\text{Ls } A_n$, since $\text{Ls } A_n$ is closed, then $z \in \text{Ls } A_n$. Therefore, in both cases, since $z \in E$ we get a contradiction. \square

Suppose that $z \in E = A \setminus \text{Ls } A_n(\omega, x) \subset A_1(\omega, x) = O_\omega^+(x) \cup \text{Ls } A_n(\omega, x)$ and note that $E \subset I$ by Claim 1.8. Thus $z \in O_\omega^+(x) \cap I$ and so $z = f_\omega^n(x)$ for some $n \in \mathbb{N}$. But $\omega \in \Sigma$ so $A \subset A_1(\sigma^n \omega, z) = A_{n+1}(\omega, x)$; this means that $z \in O_{\sigma^n \omega}^+(z)$ (recalling that $z \in E$) so there is a $n_2 \in \mathbb{N}$ with $z = f_\omega^{n_2+n}(x) = f_{\sigma^{n_2} \omega}^n(x)$ and like before $A \subset A_1(\sigma^{n+n_2} \omega, z) = A_{n+n_2+1}(\omega, x)$. Thus we can continue this process $N \in \mathbb{N}$ times and define

$$K = \sum_{k=2}^N n_k + n \geq N.$$

Therefore $A \subset A_{1+K}(\omega, x) \subset A_N(\omega, x)$ for all $N \in \mathbb{N}$ so $A \subset \text{Ls } A_n(\omega, x)$ and $E = \emptyset$ which is contradiction. Hence E is empty and $A = \text{Ls } A_n(\omega, x)$. Therefore Lemma 1.2 implies that $A_n(\omega, x) \rightarrow A$ in the Vietoris topology as required concluding the proof. \square

Keeping Proposition 1.5 and Theorem 1.6 in mind allows us conclude a satisfactory alternative to (1.7):

COROLLARY 1.9. *The following hold:*

- (a) *If A is renderable by the probabilistic chaos game, then for all $x \in \mathcal{B}_p^{**}(A)$ there exists a full probability set $\Sigma(x) \subset \Omega$ such that for any $\omega \in \Sigma(x)$ we have $A_n(\omega, x) \rightarrow A$ in the Vietoris topology.*

- (b) *If A is renderable by the deterministic chaos game, then there exists a full probability set $\Sigma_0 \subset \Omega$ such that for any $\omega \in \Sigma_0$ and any $x \in \mathcal{B}_p^{**}(A)$ we have $A_n(\omega, x) \rightarrow A$ in the Vietoris topology.*

PROOF. Note that if $I = \emptyset$ by Proposition 1.5, since as the set of disjunctive sequences in Ω has full probability, we have the result for both (a) and (b). Thus we consider $I \neq \emptyset$ in which case we can take $\Sigma(x) = \Sigma \cap \Omega(x)$ for (a) and $\Sigma_0 = \Sigma \cap \Omega_0$ for (b) where Σ is given in Theorem 1.6 and $\Omega(x)$ and Ω_0 are the full probability sets given in Definition 1.3. \square

2. Chaos game played on the new basin of attraction

First of all we notice that probabilistic chaos game for quasi-attractors proved in Theorem A.1 holds line by line on $\mathcal{B}_p^{**}(A)$. Now, we show that the deterministic chaos game renders a well-fibred quasi-attractor with the starting point in $\mathcal{B}_p^{**}(A)$ – extending Theorem 1.6. To do this, it suffices to note that Lemma 3.15 holds, with essentially the same proof when $x \in \mathcal{B}_p^{**}(A)$. This lemma also was used to prove Theorem A.2 and Theorem 1.7 where we show that the quasi-attractors of non-expansive IFSs and symmetric IFSs respectively are renderable by the deterministic chaos game. The rest of the proofs follow without modifications and thus we conclude that Theorem A.2 as well as Theorem 1.7 holds on $\mathcal{B}_p^{**}(A)$. Finally, observe that in the case that the IFS is forward minimal then $A = \mathcal{B}^*(A) = \mathcal{B}^{**}(A) = X$ and thus we do not need to make any modifications. Thus, the deterministic chaos game played on $\mathcal{B}_p^{**}(A)$ holds in every situation considered in [3]:

- (1) well-fibred quasi-attractors of IFSs on Hausdorff topological spaces,
- (2) quasi-attractors of non-expansive IFSs on metric spaces,
- (3) quasi-attractor of a symmetric IFS on a Hausdorff topological space,
- (4) forward and backward minimal IFSs of homeomorphisms of the circle,
- (5) IFSs on a compact metric space having a minimal map.

We also remark that the counterexample given in Corollary 1.10 still proves that the non-equivalence between the deterministic and probabilistic chaos game as redefined in Definition 1.3. Namely any forward minimal but not backward minimal IFS of circle homeomorphisms is a counterexample. An explicit and transparent example of this kind of systems can be founded in [7].

We add to the above list a new type of attractor that satisfies the deterministic chaos game:

THEOREM 2.1. *If a quasi-attractor is a finite set, then it is renderable by the deterministic chaos game.*

PROOF. Since A is a finite set then A is first-countable. Thus, according to [3, Theorem A.1] satisfies the probabilistic chaos game. Then, for each $x \in A$ we

have a full probability set $\Omega(x)$ of disjunctive sequence such that the fiberwise orbit $O_\omega^+(x)$ is dense in A for all $\omega \in \Omega(x)$. The set $\Omega_0 = \bigcap_{x \in A} \Omega(x)$ has full probability. Since A is finite and $f_\omega^n(z)$ is attracted in the upper Vietoris topology by A for all $z \in \mathcal{B}_p^{**}(A)$ and $\omega \in \Omega_0$ we get that $A \subset \overline{O_\omega^+(z)}$ concluding the deterministic chaos game. \square

3. Consequences of the chaos game played on the new basin

Recently, some new results on the disjunctive chaos game have appeared in the literature [4]. However, we will show that the main theorem on the disjunctive chaos in [4] is a consequence of the results in Section 2. To do this we firstly need to introduce some definitions.

Consider an IFS generated by a family $\mathcal{F} = \{f_1, \dots, f_k\}$ of continuous maps from a Hausdorff topological space X to itself. Let $Y \subset X$ be a forward invariant set, i.e. $f(Y) \subset Y$ for all $f \in \mathcal{F}$. Given a sequence $\omega = (\omega_n)_{n \geq 1} \in \Omega = \{1, \dots, k\}^{\mathbb{N}}$ we denote by

$$Y_\omega = \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \dots \circ f_{\omega_n}(Y)$$

and define the *target set* over Y as

$$A_{\text{tar}}(Y) = \{x \in Y : \text{there is } \omega \in \Omega \text{ with } \{x\} = Y_\omega\}.$$

Following the notation of [4] we define

$$S_{\text{wh}}(Y) = \{\omega \in \Omega : \text{there is } x \in Y \text{ such that } \{x\} = Y_\omega\}.$$

We have that $S_{\text{wh}}(Y) \neq \emptyset$ if and only if $A_{\text{tar}}(Y) \neq \emptyset$. Note that unlike [4] we do not ask here that X is a compact metric space. The motivation behind of [4] is to study classes of IFSs with a unique quasi-attractor. For instance contractive IFSs in compact metric space. They introduced the class of IFSs where $S_{\text{wh}}(X) \neq \emptyset$ which generalize contractive systems and obtain that, indeed, these family of systems have a unique quasi-attractor. We extend this result (see [4, Theorem 1]) to the not necessarily compact topological setting:

PROPOSITION 3.1. *If $A \stackrel{\text{def}}{=} \overline{A_{\text{tar}}(X)}$ is a non-empty compact set, then A is the unique quasi-attractor in X .*

PROOF. If $x \in A_{\text{tar}}(X)$, then there is $\omega \in \Omega$ such that $\{x\} = X_\omega$. Hence for every $i = 1, \dots, k$ it holds that $\{f_i(x)\} = X_{i\omega}$ where $i\omega$ denotes the sequences obtained by means of the concatenation of i and ω . Thus $f_i(x) \in A_{\text{tar}}(X)$ for all $i = 1, \dots, k$ and consequently $F(A_{\text{tar}}(X)) \subset A_{\text{tar}}(X)$ where F is the Hutchinson operator of the IFS generated by \mathcal{F} . Since the maps f_i are continuous it follows that $F(A) \subset A$. Observe that by assumption A is a compact set. Now we will show that A is a minimal forward invariant non-empty compact set with respect

to inclusion. In other words, A is a forward invariant minimal compact set. Then we conclude that A is a quasi-attractor.

To see the minimality of A with respect to inclusion, let K be a forward invariant non-empty compact set. Take $x \in A_{\text{tar}}(X)$. Hence $\{x\} = X_\omega \supset K_\omega$ for some $\omega \in \Omega$. Since K is a forward invariant compact set then $\emptyset \neq K_\omega \subset K$ and hence $x \in K$. Thus $A_{\text{tar}}(X) \subset K$ and by taking closure we have that $A \subset K$. Now take K to be minimal; this means K can contain no proper nonempty compact forward invariant sets. Thus $A = K$. Thus A is the unique quasi-attractor in X . \square

The second observation is the following:

PROPOSITION 3.2. *Let A be a quasi-attractor of the IFS generated by \mathcal{F} . If $S_{\text{wh}}(A) \neq \emptyset$, then $A = \overline{A_{\text{tar}}(A)}$ is a strongly-fibred quasi-attractor.*

PROOF. Since A is a quasi-attractor then is a self-similar compact set and thus we can restrict the family \mathcal{F} to this set. Hence, by Proposition 3.1 we get that the closure of $A_{\text{tar}}(A)$ is a non-empty quasi-attractor contained in A provided $S_{\text{wh}}(A) \neq \emptyset$. Thus, necessarily $A = \overline{A_{\text{tar}}(A)}$. Moreover, trivially in this case it follows that A is also strongly-fibred. \square

Recall that every strongly-fibred quasi-attractor is a well-fibred quasi-attractor (see [3, Proposition 3.11]). Thus results on the disjunctive chaos game obtained in Section 2 apply. However, using Definition 1.3 we provide a very short and direct proof of this fact:

PROPOSITION 3.3. *Every strongly-fibred quasi-attractor is renderable by the disjunctive chaos game.*

PROOF. Let V be an open set such that $A \cap V \neq \emptyset$. Since A is a strongly-fibred quasi-attractor there is $\sigma = (\sigma_n)_{n \geq 1} \in \Omega$ such that $A_\sigma \subset V$. Then for n large enough $f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(A) \subset V$. By continuity of $\mathcal{F} = \{f_1, \dots, f_k\}$ and the set $U = (f_{\sigma_1} \circ \dots \circ f_{\sigma_n})^{-1}(V)$ is an open neighbourhood of A such that $f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(U) \subset V$. Hence, if $x \in \mathcal{B}_p^{**}(A)$, then for every disjunctive sequence $\omega \in \Omega$ it holds that $f_\omega^m(x) \in U$ for all m large enough. Since ω is disjunctive, we have an arbitrarily large m such that $f_\omega^{n+m}(x) = f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(f_\omega^m(x)) \in f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(U) \subset V$. This proves that $O_\omega^+(x)$ is dense on A for all disjunctive sequences and $x \in \mathcal{B}_p^{**}(A)$. \square

As an immediate consequence of the previous propositions we get the following:

COROLLARY 3.4. *If $A \stackrel{\text{def}}{=} \overline{A_{\text{tar}}(X)}$ is a non-empty compact set of X , then A is a strongly fibred quasi-attractor and consequently is renderable by the disjunctive chaos game (played on $\mathcal{B}_p^{**}(A)$).*

Recall that a compact set K of X is called F -stable (or Lyapunov stable for F) if for every open neighbourhood U of A there is a neighbourhood V of A such that $F^n(\{x\}) \subset U$ for all $n \geq 0$ and $x \in V$. Finally we find [4, Theorem 4] as a consequence of the following more general result.

PROPOSITION 3.5. *If $A \stackrel{\text{def}}{=} \overline{A_{\text{tar}}(X)}$ is a non-empty compact set of X which is F -stable, then $\mathcal{B}_p^{**}(A) = X$.*

PROOF. Take a point $x \in X$, a disjunctive sequence $\omega \in \Omega$ and an open neighbourhood U of A . Since $S_{\text{tar}}(X)$ is non-empty we find a sequence σ such that X_σ is a singleton in A . Thus, there is finite word $\sigma_1 \dots \sigma_n$ such that $f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(X) \subset V$ where V is the open neighbourhood of A obtained by the F -stability for the given set U . Then, since ω is disjunctive we find $\sigma_1 \dots \sigma_n$ as a sub-word of ω . Thus, $f_\omega^{m+n}(x) = f_{\sigma_1} \circ \dots \circ f_{\sigma_n}(f_\omega^m(x)) \in V$. By the F -stability we have that $F^\ell(\{y\}) \subset U$ for all $y \in V$ and $\ell \geq 0$. Consequently, we get that $f_\omega^{\ell+n+m}(x) \in U$ for all $\ell \geq 0$. Therefore $x \in \mathcal{B}_p^{**}(A)$ and we conclude the proof of the proposition. \square

EXAMPLE 3.6. Proposition 3.5 implies, in particular, that the attractor of a contractive IFS is renderable by the disjunctive chaos game starting in any point of X . Observe that if X is a non-compact space and the contracting maps were bijections of X , then $A_{\text{tar}}(X) = \emptyset$. This is the case for a family of contracting maps on \mathbb{R}^n . However, we could apply the result to a forward invariant compact set $Y \subset X$ obtaining that $\mathcal{B}_p^{**}(A_{\text{tar}}(Y)) = Y$. By exhaustion, if possible, we will also get that

$$\bigcup_Y \mathcal{B}_p^{**}(A_{\text{tar}}(Y)) = X.$$

The following example shows an application of Proposition 3.5 in the non-compact case which is not possible to get by exhaustion from the corresponding result in the compact case.

EXAMPLE 3.7. Consider the IFS on $X = \mathbb{R}$ with the usual topology generated by the function

$$f_1(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ e^x - 1 & \text{if } x < 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} e^{-x} - 1 & \text{if } x \geq 0, \\ x/2 & \text{if } x < 0. \end{cases}$$

Taking, into account that $f_1(X) = (-1, \infty)$ and $f_2(X) = (-\infty, 0]$ we get that $A_{\text{tar}}(X) \subset [-1, 0]$. Since f_1 and f_2 are topological contracting maps on $[-1, 0]$ having $x = 0$ as a fixed point in common then, necessarily $A_{\text{tar}}(X) = \{0\}$. However, there is no a forward invariant compact set $Y \subset X$ with $Y \cap [0, \infty) \neq \emptyset$

(expect $\{0\}$). We only find forward invariant compact sets of the form $Y = [-a, 0]$ with $a > 0$. Observe that $\mathcal{B}_p^{**}(A_{\text{tar}}(X)) = \mathbb{R}$ but $\mathcal{B}_p^{**}(A_{\text{tar}}(Y)) = [-a, 0]$. Thus we can not recover the result by exhaustion as we did in Example 3.6.

4. On strongly-fibred and Conley attractors

Combining Proposition 3.2 and the following proposition we get a characterization of strongly-fibred attractors in metric spaces.

PROPOSITION 4.1. *If A is a strongly-fibred quasi-attractor on a metric space, then $S_{wh}(A) \neq \emptyset$.*

PROOF. We claim that there are a sequence $(g_n)_n$ of maps on the semigroup Γ generated by \mathcal{F} and a sequence of points $(p_n)_n$ on A such that $g_1 \circ \dots \circ g_n(A) \subset B(p_n, 1/n)$ where $B(x, r)$ denotes the open ball of radius $r > 0$ centered at x . Observe that this claim concludes the proof of the proposition. Indeed, we get $\omega \in \Omega$ describing the infinite composition $g_1 \circ \dots \circ g_k \circ \dots$ and such that the diameter of $f_{\omega_1} \circ \dots \circ f_{\omega_n}(A)$ tends to zero. Thus A_ω is a singleton.

To prove this claim, let $p_1 \in A$ be any point in A . Since A is strongly fibred we get a map $g_1 \in \Gamma$ such that $g_1(A) \subset B(p_1, 1)$. By induction, assume that g_1, \dots, g_{n-1} and p_1, \dots, p_{n-1} are defined and satisfy $g_1 \circ \dots \circ g_i(A) \subset B(p_i, 1/i)$ for all $i = 1, \dots, n-1$. Take an arbitrary point $p_n \in g_1 \circ \dots \circ g_{n-1}(A)$ and consider the set $V = (g_1 \circ \dots \circ g_{n-1})^{-1}(B(p_n, 1/n))$. It follows from continuity and the choice of p_n that V is an open set and $A \cap V \neq \emptyset$. Since A is strongly-fibred quasi-attractor, there is $g_n \in \Gamma$ such that $g_n(A) \subset V$. In particular $g_1 \circ \dots \circ g_n(A) \subset B(p_n, 1/n)$ and we conclude the proof. \square

According to [2], we present here the notion of Conley attractor:

DEFINITION 4.2. A compact set $A \subset X$ is a *Conley attractor* of the Hutchinson operator F if there exists an open neighbourhood U of A such that $F^n(\overline{U}) \rightarrow A$ in the upper Vietoris topology (on the set of non-empty closed subsets of X). The largest set U where the above convergence holds is called *basin of attraction*.

The first observation is the following:

PROPOSITION 4.3. *If X is a locally compact Hausdorff topological space, then every strict attractor is a Conley attractor.*

PROOF. Let A be a strict attractor with basin of attraction B . Since for any compact set K in B , $F^n(K)$ converges in the Vietoris topology to A , we only need to find an open set U such that $A \subset U$ and \overline{U} a compact set in B . But the existence of this compact neighbourhood of A in B is general well-known consequence of the assumptions on X . Indeed, first observe that since A is compact we only need to cover A by compact neighbourhoods of its points

that miss the the closed set $C = X \setminus B$. Then, to see that for each $x \in A$ there is such a compact neighbourhood, since X is a locally compact space, we start by considering a compact neighbourhood K of x . If K does not miss C we have a compact Hausdorff space K and a closed set $K \cap C$ in K that does not contain x . Now apply that K (as a subspace) is regular, and proceed from there by the standard separation arguments one get the required compact neighbourhood. \square

In order to study whether a Conley attractor is a strict attractor we will need the following general lemma which is interesting in its own right (c.f. [6]).

LEMMA 4.4. *Let X and Y be Hausdorff topological spaces. Consider a decreasing nested sequence $(A_n)_{n \in \mathbb{N}}$ of closed subsets of X and assume that A_n converges to a compact set A in the upper Vietoris topology. Then, for every continuous map $f: X \rightarrow Y$ we have*

$$\bigcap_{n \in \mathbb{N}} f(A_n) = f\left(\bigcap_{n \in \mathbb{N}} A_n\right) \subset f(A).$$

PROOF. First observe that since X is a Hausdorff space and A_n is a nested decreasing sequence of closed sets that converges in the upper Vietoris topology it holds that

$$\bigcap_{n \in \mathbb{N}} A_n = \text{Ls } A_n \subset A \quad \text{and} \quad f(\text{Ls } A_n) \subset \bigcap_{n \in \mathbb{N}} f(A_n).$$

Thus, we only need to prove the reverse inclusion. To do this, consider $y \in f(A_n)$ for all $n \in \mathbb{N}$. Hence, $y = f(a_n)$ with $a_n \in A_n$ for all $n \in \mathbb{N}$ and thus, $\{a_n\} \rightarrow A$ in the upper Vietoris topology. According to Lemma 1.2, $\text{Ls}\{a_n\} \neq \emptyset$. Thus, we find $a \in \text{Ls}\{a_n\} \subset A$ and, hence, for every open neighbourhood U of a there is $a_n \in U$. This implies that $y = f(a_n) \in f(U)$ for all neighbourhoods U of a . Suppose for a moment that

$$(4.1) \quad \bigcap \{f(U) : U \text{ open neighbourhood of } a\} = \{f(a)\}.$$

Because of this, $y = f(a)$ and therefore $y \in f(\text{Ls}\{a_n\}) \subset f(\text{Ls } A_n)$ concluding the lemma. Then one only need to justify (4.1). To show this, since Y is a Hausdorff space, we only need to prove the inclusion

$$\begin{aligned} & \bigcap \{f(U) : U \text{ open neighbourhood of } a\} \\ & \subset \bigcap \{V : f(a) \in V, V \text{ open}\} = \{f(a)\}. \end{aligned}$$

Let $z \in f(U)$ for all open neighbourhood U of a . Given any open neighbourhood V of $f(a)$, by continuity of f , we find an open neighbourhood U of a such that $f(U) \subset V$. Since $z \in f(U)$, then we have that $z \in V$ and we get the required inclusion completing the proof. \square

Again, using the results in [3], the following proposition extends [4, Theorem 2 (ii)] to topological contexts, while shorting the proof.

THEOREM 4.5. *Let A be a strongly-fibred quasi-attractor. If A is a Conley attractor, then A is also a strict attractor.*

PROOF. Let K be a compact set in the basin of attraction U of the Conley attractor A . We need to see that $F^n(K) \rightarrow A$ in the Vietoris topology. To do this, we fix open sets V and W such that $A \subset V$ and $W \cap A \neq \emptyset$. Since $K \subset U$ and $F^n(\bar{U}) \rightarrow A$ in the (upper) Vietoris topology then $F^n(K) \subset V$ for all n large enough. Thus we only need to prove that $F^n(K) \cap W \neq \emptyset$ for all n large enough. Let

$$S \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} F^n(\bar{U}) \cup A.$$

Observe that $F(K) \subset S$ and $F(S) \subset S$.

CLAIM 4.6. $A_\omega = S_\omega$ for all $\omega \in \Omega$.

PROOF. Since $A \subset S$, it is clear that $A_\omega \subset S_\omega$. Now, we will see the reverse inclusion. First of all, let us denote by $\hat{f}_\omega^n = f_{\omega_1} \circ \dots \circ f_{\omega_n}$. Hence,

$$(4.2) \quad S_\omega = \bigcap_{n \in \mathbb{N}} \hat{f}_\omega^n(S) = \bigcap_{n \in \mathbb{N}} \hat{f}_\omega^{k+n}(S) \quad \text{for all } k \in \mathbb{N}.$$

Recall that since A is a quasi-attractor we have $F(A) = A$ and so

$$(4.3) \quad \begin{aligned} \hat{f}_\omega^{k+n}(S) &\subset \hat{f}_\omega^k(F^n(S)) \\ &= \hat{f}_\omega^k\left(\bigcup_{\ell \in \mathbb{N}} F^{n+\ell}(\bar{U}) \cup A\right) = \hat{f}_\omega^k\left(\bigcup_{m > n} F^m(\bar{U})\right) \cup \hat{f}_\omega^k(A). \end{aligned}$$

According to Lemma 1.2, $\text{Ls } F^n(\bar{U}) \subset A$ and from Lemma 4.4, it holds that

$$(4.4) \quad \begin{aligned} \bigcap_{n \in \mathbb{N}} \hat{f}_\omega^k\left(\bigcup_{m > n} F^m(\bar{U})\right) &\subset \bigcap_{n \in \mathbb{N}} \hat{f}_\omega^k\left(\overline{\bigcup_{m > n} F^m(\bar{U})}\right) \\ &= \hat{f}_\omega^k\left(\overline{\bigcap_{n \in \mathbb{N}} \bigcup_{m > n} F^m(\bar{U})}\right) = \hat{f}_\omega^k(\text{Ls } F^n(\bar{U})) \subset \hat{f}_\omega^k(A). \end{aligned}$$

Then, substituting (4.3) and (4.4) in (4.2), one gets that $S_\omega \subset \hat{f}_\omega^k(A)$ for all $k \in \mathbb{N}$ and from this $S_\omega \subset A_\omega$ as required. □

Now, since A is strongly-fibred quasi-attractor then there is $\omega \in \Omega$ such that $S_\omega = A_\omega \subset W$. This implies that for every n large enough $f_{\omega_1} \circ \dots \circ f_{\omega_n}(S) \subset W$. Thus $F^n(K) \cap W \neq \emptyset$ for all n large enough. This completes the proof that $F^n(K) \rightarrow A$ in the Vietoris topology and concludes the theorem. □

As a consequence we also extend [4, Theorem 3].

COROLLARY 4.7. *Assume that X is a Hausdorff compact topological space. Then*

$$X^* \stackrel{\text{def}}{=} \bigcap_{n \in \mathbb{N}} F^n(X)$$

is a Conley attractor of the Hutchinson operator F . Moreover, if A is a strongly-fibred quasi-attractor, the following assertions are equivalent:

- (a) $A = X^*$,
- (b) A is the unique fixed point of F ,
- (c) A is strict attractor (with basin containing X).

PROOF. The first part of the corollary follows by the same kind of argument for the analogous result for metric space [5, Proposition 2.4.7]. For completeness, we provide the details here.

Let U be an open set of X^* and set

$$X_n = F^n(X) \quad \text{for all } n \in \mathbb{N}.$$

Since $X_{n+1} \subset X_n$ and $X^* \subset X_n$ for all $n \in \mathbb{N}$, the family

$$\{U\} \cup \{X \setminus X_n : n \in \mathbb{N}\}$$

is an open cover of X_1 . Since X_1 is compact, there is a finite subcover. This means that, for some $N \in \mathbb{N}$, we have $X^* \subset U \cup (X \setminus X_n)$ for all $n \geq N$. Thus $X_n \subset U$ for all $n \geq N$. Therefore, $F^n(X)$ converges to X^* in the upper Vietoris topology.

Now, we will prove the equivalence. First of all, notice that according to [6, Theorem 4], the set X^* is a non-empty fixed point of F . Thus, it is clear that (b) implies (a) because X^* and A are both fixed points of F . The implication, (a) implies (c), follows from Theorem 4.5. To get (c) implies (b) is enough to note that if there is a strict attractor, then this is the unique fixed point in the basin of attraction. \square

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