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SOME GENERALIZATIONS OF DISTALITY

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Abstract. We incorporate the notion of a distal system into the continuum theory [14] through the notion of the *continuum-wise distal homeomorphism*. Results concerning distal homeomorphisms will be generalized to the case of cw-distal homeomorphisms. Notions of cw-distality for measures will be studied. We also analyze the variation of distality for flows obtained by making the proximal cell [1] to depend on a given subset of the full set of reparametrizations. Some properties of these reparametrized distality will be obtained.

1. Introduction

1.1. Continuum-wise distal homeomorphisms.

Let $f\colon X \to X$ be a homeomorphism of a metric space (X,d). We say that f is *expansive* [19] if there is $\varepsilon > 0$ such that if $d(f^n(x), f^n(y)) \leq \varepsilon$ for every $n \in \mathbb{Z}$ then $x = y$. Equivalently,

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if there is $\varepsilon > 0$ such that

$$C \subset X \text{ and } \sup_{n\in\mathbb{Z}} \operatorname{diam}(f^n(C)) \leq \varepsilon \;\Rightarrow\; \operatorname{diam}(C) = 0. \tag{1.1}$$

Here $\operatorname{diam}(C) = \sup\{d(x,y) : x, y \in C\}$ denotes the diameter of C. Notice that C in this equivalence runs over the set of subsets of X denoted by 2^X. By restricting C to a given class of nonempty subsets $\mathscr{C} \subset 2^X$ containing the single-point sets we obtain a corresponding notion of expansiveness. More precisely, given such a $\mathscr{C}$ we say that f is *$\mathscr{C}$-expansive* if there is $\varepsilon > 0$ such that (1.1) holds whenever $C \in \mathscr{C}$. Notice f is $\mathscr{C}$-expansive if and only if it is $\overline{\mathscr{C}}$-expansive where $\overline{\mathscr{C}} = \{\overline{C} : C \in \mathscr{C}\}$. Clearly $\mathscr{C}$-expansiveness coincides with expansiveness if we take $\mathscr{C}$ as the set of all nonempty subsets (or equivalently nonempty closed subsets) of X. Another important class $\mathscr{C}$ is the following: a *continuum* is a nonempty compact connected metric space and a *subcontinuum* is a subset which is itself a continuum under the induced topology. By taking $\mathscr{C}_c = \mathscr{C}_c(X)$ as the set of subcontinua of X we obtain that $\mathscr{C}_c$-expansiveness is precisely the continuum-wise expansiveness (or *cw-expansiveness*) introduced by Kato [9]. This definition incorporated the expansive systems into the Continuum Theory [14].

In this paper we pursue similar ideas but for the distal homeomorphisms instead. Recall that a homeomorphism of a compact metric space $f\colon X \to X$ is *distal* if $\inf_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) = 0$ implies $x = y$. This concept appears first in Hilbert [21] where attempts to give a topological characterization of the rigid group of motions were made. To be distal homeomorphism is then equivalent to say that

$$C \subset X \text{ and } \inf_{n\in\mathbb{Z}} \operatorname{diam} f^n(C) = 0 \;\Rightarrow\; \operatorname{diam}(C) = 0. \tag{1.2}$$

(The only difference between (1.1) and (1.2) is that the supremum in the former was replaced by infimum in the latter.)

Again we observe that C in the definition above runs over the set of all subsets of X. By restricting C to belong to a class $\mathscr{C} \subset 2^X$ as above we obtain the notion of $\mathscr{C}$-distality. More precisely, f is called *$\mathscr{C}$-distal* if (1.2) holds whenever $C \in \mathscr{C}$. Again f is $\mathscr{C}$-distal if and only if it is $\overline{\mathscr{C}}$-distal and $\mathscr{C}$-distality with $\mathscr{C}$ being the set of nonempty subsets of X coincides with the classical distality. By restricting C to belong to the class of subcontinua $\mathscr{C}_c$ we obtain the following definition incorporating the distal systems into the Continuum Theory.

Definition 1.1. A homeomorphism of a compact metric space $f\colon X \to X$ is *cw-distal* if it is $\mathscr{C}_c$-distal. More precisely, if every subcontinuum $C \subset X$ satisfying $\inf_{n\in\mathbb{Z}} \operatorname{diam} f^n(C) = 0$ is degenerated, i.e. reduces to a singleton.

We will obtain some properties of cw-distal homeomorphisms. For instance, a cw-expansive homeomorphism of a compact metric space of positive topological dimension cannot be cw-distal. Moreover, a homeomorphism of a compact metric space is cw-distal if and only if its proximal cells [1] are all totally disconnected. Also, a cw-distal extension of a cw-distal homeomorphism is a cw-distal homeomorphism. Moreover, the product of cw-distal homeomorphisms of compact metric spaces is cw-distal, every extension of a cw-distal homeomorphism under a light map of a compact metric space is cw-distal and every factor of a cw-distal homeomorphism under a light weakly confluent map of a compact metric space is cw-distal. We also introduce the notion of *cw-distal measure.* We prove that a Borel probability measure is cw-distal if and only if the proximal cells are totally disconnected relative to the measure. A homeomorphism of a separable metric space is cw-distal if and only if every nonatomic Borel probability measure is cw-distal. We also consider the notion of *strongly cw-distal measures.* We prove that every strongly cw-distal measure of f is cw-distal. In particular, if every nonatomic Borel probability measure is strongly cw-distal for f, then f is cw-distal. Finally we prove that the strongly cw-distal measures of a homeomorphism of a compact locally connected metric space constitute a $F_{\sigma\delta}$ subset of the space of Borel probability measures equipped with the weak* topology.

The proofs of the mains results and some examples will be given in the Section 3. The use of the continuum and hyperspace theories will be an important tool. See the references [14] for the continuum theory and [1] for some basic properties of distal homeomorphisms.

It is known that an expansive homeomorphism of an infinite compact metric space cannot be distal (e.g. [19]). Likewise, it is natural to believe that a cw-expansive homeomorphism of an infinite compact metric space cannot be cw-distal. However, this is not true: Take for instance an expansive homeomorphism of the ternary Cantor set of $[0,1]$. Nevertheless, such examples exist on the Cantor set only. More precisely, we have the following result.

Theorem 1.2. *A cw-expansive homeomorphism of a compact metric space of positive topological dimension cannot be cw-distal.*

Next we present a characterization of cw-distal homeomorphisms in terms of proximal cells. Given a homeomorphism of a metric space X and $x \in X$, we define the *proximal cell* [1]

$$P(x) = \Big\{ y \in X : \inf_{n\in\mathbb{Z}} d\big(f^n(x), f^n(y)\big) = 0 \Big\}.$$

(Notation $P_f(x)$ indicates dependence on f.) We define the *continuum-wise diameter* of $A \subseteq X$ by

$$\mathrm{diam}_{\mathrm{cw}}(A) = \sup\{\mathrm{diam}(C) : C \text{ is a subcontinuum and } C \subseteq A\}.$$

With this notation we can restate the definition of cw-expansive homeomorphism: a homeomorphism of a metric space $f\colon X\to X$ is cw-expansive if there is $\delta>0$ such that $\operatorname{diam}_{\mathrm{cw}}(\Gamma_\delta(x))=0$ for every $x\in X$, where

$$\Gamma_\delta(x)=\big\{y\in X : d\big(f^n(x),f^n(y)\big)\le\delta,\ \text{for all } n\in\mathbb{Z}\big\}.$$

Recall that a metric space is *totally disconnected* if every connected subset reduces to a single point. Equivalently, if it has zero continuum-wise diameter. A subset A is a totally disconnected if it is totally disconnected with respect to the induced topology. Equivalently, if $\operatorname{diam}_{\mathrm{cw}}(A)=0$.

With these definitions we have the following result.

Theorem 1.3. *A homeomorphism $f\colon X\to X$ of a metric space is cw-distal if and only if $P(x)$ is a totally disconnected for every $x\in X$, equivalently, if* $\operatorname{diam}_{\mathrm{cw}}(P(x))=0$ *for every $x\in X$.*

Let $g\colon Y\to Y$ be a homeomorphism of a compact metric space Y. A map $\pi\colon Y\to X$ is distal if $\inf\limits_{n\in\mathbb{Z}} d\big(g^n(y_1),g^n(y_2)\big)>0$ for distinct $y_1,y_2\in Y$ satisfying $\pi(y_1)=\pi(y_2)$. A *homomorphism* from g to a homeomorphism $f\colon X\to X$ is a continuous onto map $\pi\colon Y\to X$ satisfying $f\circ\pi=\pi\circ g$. We say that g is a *distal extension* of f if there is a distal homomorphism from g to f.

These definitions motivate the following ones: a map $\pi\colon Y\to X$ is said to be cw-distal if $\inf\limits_{n\in\mathbb{Z}}\operatorname{diam}(g^n(C))>0$ whenever C is a nondegenerated subcontinuum in $\pi^{-1}(x)$ for some $x\in X$. We say that g is a *cw-distal extension* of f if there is a cw-distal homomorphism from g to f.

As noticed in [1], g is distal if and only if it is a distal extension of the trivial (one point) homeomorphism. Likewise, g is cw-distal if and only if it is a cw-distal extension of the trivial homeomorphism. With these definitions we have the following result.

Theorem 1.4. *A cw-distal extension of a cw-distal homeomorphism is a cw-distal homeomorphism.*

Let $f\colon X\to X$ and $g\colon Y\to Y$ be homeomorphisms of compact metric spaces X and Y. The *product* of f and g is the map $f\times g\colon X\times Y\to X\times Y$ defined by $(f\times g)(x,y)=(f(x),g(y))$. The product $f\times g$ turns out to be a homeomorphism of $X\times Y$ if we equip the latter space with the metric $d^2((x,y),(x',y'))=\max\{d(x,x'),d(y,y')\}$. On the other hand, we say that f is a *factor* of g (or that g is an *extension* of f) if there is an onto continuous map $h\colon Y\to X$ such that $f\circ h=h\circ g$. In such a case we say that f is a factor of g (resp. g is an extension of f) under h.

It is known that the product of distal homeomorphisms as well as every factor of a distal homeomorphism of a compact metric space are distal. In the sequel we present a continuum-wise version of these facts. Recall that a continuous map $h\colon Y \to X$ is *light* if $\dim(h^{-1}(x)) = 0$ (i.e. $h^{-1}(x)$ is totally disconnected) for every $x \in X$; and *weakly confluent* if the induced map $\widehat{h}\colon \mathscr{C}_c(Y) \to \mathscr{C}_c(X)$ defined by $\widehat{h}(C) = h(C)$ is onto. Note that every factor of a cw-expansive homeomorphism under a light weakly confluent map of a compact metric space is cw-expansive too [9]. This motivates the result below for cw-distal homeomorphisms.

Theorem 1.5. *The following properties hold*:

(a) *The product of cw-distal homeomorphisms of compact metric spaces is cw-distal.*

(b) *Every extension of a cw-distal homeomorphism under a light map of a compact metric space is cw-distal.*

(c) *Every factor of a cw-distal homeomorphism under a light weakly confluent map of a compact metric space is cw-distal.*

Let us present some examples.

Example 1.6. Given $N \in \mathbb{N}^+$ we say that a homeomorphism $f : X \to X$ is *N-distal* if $P(x)$ has at most N elements for every $x \in X$. It follows from Theorem 1.3 that every N-distal homeomorphism is cw-distal. Since there are examples of N-distal homeomorphisms which are not distal (e.g. Example 1.1 in [12]), it follows that there are cw-distal homeomorphisms which are not distal.

Example 1.7. There are cw-distal homeomorphisms which are not N-distal for every positive integer N.

Example 1.8. Every distal homeomorphism of a compact metric space has zero topological entropy. See [16] for an elementary proof. The continuum-wise counterpart of this result is false. Indeed, every homeomorphism of a zero-dimensional compact uncountable metric space is cw-distal and there are such homeomorphisms with positive topological entropy (e.g. [2]). In particular, cw-distal homeomorphisms with positive topological entropy on the ternary Cantor set do exist.

The notion of cw-distality can be extended to Borel probability measures as follows.

Definition 1.9. A *cw-distal measure* of a homeomorphism $f\colon X \to X$ of a metric space is a Borel probability measure μ of X such that every subcontinuum C with $\mu(C) > 0$ satisfies

$$\inf_{n\in\mathbb{Z}} \operatorname{diam} f^n(C) > 0.$$

This definition is the cw-counterpart of the notion of cw-expansive measure introduced by Shin [18].

Notice that every cw-distal measure is *nonatomic* (i.e. without points of positive measures). For if μ is a cw-distal measure of $f\colon X\to X$ and $x\in X$, then the subcontinuum $C=\{x\}$ clearly satisfies $\inf_{n\in\mathbb{Z}}\operatorname{diam}(f^n(C))=0$ and so $\mu(\{x\})=\mu(C)=0$.

The *cw-part* of a Borel probability measure μ of X is the set function

$$\mu_{\mathrm{cw}}(A)=\sup\{\mu(C):C\text{ is a subcontinuum of }X\text{ and }C\subseteq A\},\quad\text{for all }A\subset X.$$

We say that A is *μ-totally disconnected* if $\mu_{\mathrm{cw}}(A)=0$. Similar to Theorem 1.3 we obtain the following result.

Theorem 1.10. *Let $f\colon X\to X$ be a homeomorphism of a metric space. A Borel probability measure μ of X is a cw-distal measure of f if and only if $P(x)$ is μ-totally disconnected for every $x\in X$.*

The next result although simple yields the link between the cw-distal homeomorphisms and the cw-distal measures. Recall that a metric space is separable if it exhibits a dense sequence.

Theorem 1.11. *A homeomorphism of a separable metric space is cw-distal if and only if every nonatomic Borel probability measure is cw-distal.*

A related example is as follows. Recall from [12] that a *distal measure* of a homeomorphism $f\colon X\to X$ of a metric space is a Borel probability measure μ satisfying $\mu(P(x))=0$ for every $x\in X$. By Theorem 1.10 every distal measure is cw-distal.

Example 1.12. There are homeomorphism of compact metric spaces exhibiting cw-distal measures which are not distal.

Given a homeomorphism of a metric space X, $x\in X$ and $\delta>0$, we define the set

$$P(x,\delta)=\Big\{y\in X:\inf_{n\in\mathbb{Z}}d(f^n(x),f^n(y))<\delta\Big\}.$$

We will also consider the following class of measures:

Definition 1.13. A *strongly cw-distal measure* of a homeomorphism $f\colon X\to X$ of a metric space is a Borel probability measure μ of X such that for every $\varepsilon>0$ there is $\delta>0$ such that $\mu_{cw}(P(x,\delta))<\varepsilon$ for every $x\in X$.

About this class of measures we have the following result.

Theorem 1.14. *Let $f\colon X\to X$ be a homeomorphism of a compact metric space. Then, every strongly cw-distal measure of f is cw-distal. In particular, if every nonatomic Borel probability measure is strongly cw-distal for f, then f is cw-distal.*

Inspired by [13] we study the set of strongly cw-distal measures of a homeomorphism. For this we need some basic concepts. Let X be a compact metric space. Denote by $\mathcal{M}(X)$ the set of Borel probability measures of X. This set is a compact metric space if equipped with the weak* topology defined by the convergence $\mu_n \to \mu$ if and only if $\int \phi \, d\mu_n \to \int \phi \, d\mu$ for every continuous map $\phi\colon X \to \mathbb{R}$.

A subset of a topological space is an F_σ *subset* if it is the union of countably many closed subsets [10]. It is an $F_{\sigma\delta}$ *subset* if it is the intersection of countably many F_σ subsets. A metric space is *locally connected* if the connected open neighbourhoods of each point form a neighbourhood base at that point.

With these definitions we can state the following result.

Theorem 1.15. *The strongly cw-distal measures of a homeomorphism of a compact locally connected metric space X constitute a $F_{\sigma\delta}$ subset of $\mathcal{M}(X)$.*

In particular, the set of strongly cw-distal measures $\mathcal{M}^s_{\mathrm{cwd}}(f)$ of a homeomorphism $f\colon X \to X$ of a compact metric space X is a Borel subset of $\mathcal{M}(X)$.

Denoting by $\mathcal{M}_{\mathrm{cwd}}(f)$ the cw-distal measures of f we have $\mathcal{M}^s_{\mathrm{cwd}}(f) \subseteq \mathcal{M}_{\mathrm{cwd}}(f)$ (by Theorem 1.14) and the question is about the relative size of $\mathcal{M}^s_{\mathrm{cwd}}(f)$ in $\mathcal{M}_{\mathrm{cwd}}(f)$. More precisely, we are interested in finding necessary and sufficient conditions for $\mathcal{M}^s_{\mathrm{cwd}}(f) = \mathcal{M}_{\mathrm{cwd}}(f)$, or, for $\mathcal{M}^s_{\mathrm{cwd}}(f)$ to be dense in $\mathcal{M}_{\mathrm{cwd}}(f)$. A partial answer is as follows.

We say that a homeomorphism of a metric space $f : X \to X$ is *equicontinuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \in \mathbb{Z}$.

Theorem 1.16. *If $f\colon X \to X$ is an equicontinuous homeomorphism of a compact metric space, then $\mathcal{M}^s_{\mathrm{cwd}}(f) = \mathcal{M}_{\mathrm{cwd}}(f)$.*

1.2. Distal flows with reparametrizations. In this subsection we will present some generalized notions of distality for $\mathbb{R}$-actions namely flows.

Recall that a *flow* of a metric space X is a continuous map $\phi\colon \mathbb{R} \times X \to X$ satisfying $\phi(0, x) = x$ and $\phi(t, \phi(r, x)) = \phi(t+r, x)$ for every $t, r \in \mathbb{R}$ and $x \in X$. We denote by $\phi_t\colon X \to X$ the homeomorphism $\phi_t(x) = \phi(t, x)$. A flow ϕ is *distal* if $x = y$ whenever $x, y \in X$ satisfy $\inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_t(y)) = 0$. The distal flows have been widely study in topological dynamics [1], [5], [6]. We can restate the definition of distal flow as follows. A flow ϕ is distal if $P(x) = \{x\}$ for every $x \in X$ where $P(x)$ is the *proximal cell* defined by

$$P(x) = \Big\{ y \in X : \inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_t(y)) = 0 \Big\}.$$

This definition suggests a similar one depending on a continuous map $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$. More precisely, given a flow ϕ of X and such a map s we define the s-dependent proximal cell,

$$P_s(x) = \Big\{ y \in X : \inf_{t\in\mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0 \Big\}.$$

Clearly, $P(x) = P_{Id}(x)$ where $\mathrm{Id}\colon \mathbb{R} \to \mathbb{R}$ is the identity. Therefore, ϕ is distal if and only if

$$P_{\mathrm{Id}}(x) = \{x\}, \quad \text{for all } x \in X.$$

The natural question is: Which flows ϕ satisfy this equation but with the identity Id replaced by another continuous map $s\colon \mathbb{R} \to \mathbb{R}$ given in advance? More precisely, given $s\colon \mathbb{R} \to \mathbb{R}$, we want to know which flows ϕ can satisfy

$$P_s(x) = \{x\}, \quad \text{for all } x \in X. \tag{1.3}$$

How "distal" these flows are? Our first result gives an answer for this question. We say that a flow ϕ is *trivial* (resp. *closed*) if $\phi_t(x) = x$ for every $x \in X$ and every $t \in \mathbb{R}$ (resp. every orbit $O(x) = \{\phi_t(x) : t \in \mathbb{R}\}$ is closed). We say that ϕ is *uniformly closed* if there is $T \neq 0$ such that $\phi_T(x) = x$ for every $x \in X$.

Let $\mathcal{C}$ denote the *set of reparametrizations* i.e. the space of continuous maps $s\colon \mathbb{R} \to \mathbb{R}$ fixing 0.

Theorem 1.17. *Let ϕ be a flow of a metric space X. If there is $s \in \mathcal{C} \setminus \{\mathrm{Id}\}$ satisfying* (1.3)*, then ϕ is uniformly closed. If additionally s is bounded, then ϕ is trivial.*

Another problem would be to consider the equation (1.3) but replacing $\{x\}$ by another set depending on x. Of course, natural candidates for such a replacement are the orbit $O(x)$ or the *orbit closure* $Cl(O(x))$ where $Cl(\,\cdot\,)$ denotes the closure operation. The first of these candidates yields the following equation:

$$P_s(x) = O(x), \quad \text{for all } x \in X. \tag{1.4}$$

Related examples are as follows.

Example 1.18. Clearly $P_0(x) = Cl(O(x))$ for every $x \in X$ and every flow ϕ of a metric space X, where 0 here is the zero map ($t \in \mathbb{R} \mapsto 0$). Then, the sole flows ϕ satisfying (1.4) with $s = 0$ are the closed ones.

Example 1.19. For every uniformly closed flow ϕ there is $s \in \mathcal{C}$ bounded satisfying (1.4) (take for instance $s = 0$).

These examples motivate the results below.

Theorem 1.20. *The following properties are equivalent for every flow ϕ of a compact metric space X:*

(a) *ϕ is closed.*
(b) *ϕ satisfies* (1.4) *for every $s \in \mathcal{C}$ bounded.*
(c) *ϕ satisfies* (1.4) *for some $s \in \mathcal{C}$ bounded.*

We say that $x \in X$ is a *singularity* of a flow ϕ if $\phi_t(x) = x$ for every $t \in \mathbb{R}$.

Theorem 1.21. *If a flow ϕ of a compact metric space X satisfies* (1.4) *with $s = \mathrm{Id}$, then ϕ has singularities.*

The next step would be to consider flows ϕ satisfying (1.3) but with the orbit closure $\mathrm{Cl}(O(x))$ instead of $\{x\}$, namely, satisfying the equation

$$P_s(x) = \mathrm{Cl}(O(x)), \quad \forall x \in X. \tag{1.5}$$

However, we have the following example.

Example 1.22. Since $P_0(x) = \mathrm{Cl}(O(x))$ for $x \in X$ and every flow ϕ of X (e.g. Example 1.18), every flow ϕ satisfies (1.5) with $s = 0$.

This example suggests a different approach to obtain concrete results from the Equation (1.5). To motivate it we recall the following definition.

Following Bowen and Walters [3], a flow ϕ is *expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $y = \phi_r(x)$ for some $-\varepsilon \le r \le \varepsilon$ whenever $x, y \in X$ satisfy $d(\phi_t(x), \phi_{s(t)}(y)) \le \delta$ for every $t \in \mathbb{R}$ and some $s \in \mathcal{C}$.

Keynes and Sears [11] generalized this definition by replacing $\mathcal{C}$ by a subset $\mathcal{F}$ of $\mathcal{C}$. More precisely, given $\mathcal{F} \subseteq \mathcal{C}$ we say that ϕ is *$\mathcal{F}$-expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $y = \phi_r(x)$ for some $-\varepsilon \le r \le \varepsilon$ whenever $x, y \in X$ satisfy $d(\phi_t(x), \phi_{s(t)}(y)) \le \delta$ for every $t \in \mathbb{R}$ and some $s \in \mathcal{F}$. (A similar approach for Borel measures was considered recently by Villavicencio [20]).

In the light of these definitions it is natural to consider proximal cells depending not on a single map $s\colon \mathbb{R} \to \mathbb{R}$ but rather on a subset $\mathcal{F} \subseteq \mathcal{C}$. More precisely, given a flow ϕ, $x \in X$ and $\mathcal{F} \subseteq \mathcal{C}$ we define the $\mathcal{F}$-depending proximal cell

$$P_{\mathcal{F}}(x) = \Big\{ y \in X : \inf_{t \in \mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0, \text{ for some } s \in \mathcal{F} \Big\}.$$

Precisely, $P_s(x) = P_{\{s\}}(x)$ when $\mathcal{F}$ consists of a single map s. Moreover,

$$P_{\mathcal{F}}(x) = \bigcup_{s \in \mathcal{F}} P_s(x), \quad \text{for all } x \in X.$$

Natural candidates for $\mathcal{F}$ are $\mathcal{C}$ itself or else $\mathcal{H}$, the set of homeomorphisms of $\mathbb{R}$ fixing 0. The next result is about the latter set.

Theorem 1.23. *A flow ϕ of a compact metric space X is trivial if and only if $P_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$.*

Finally we will consider equation (1.5) but for general subsets of reparametrizations $\mathcal{F}$ (instead of s). More precisely, given such an $\mathcal{F}$ we would like to know which flows ϕ satisfy

$$P_{\mathcal{F}}(x) = \mathrm{Cl}(O(x)), \quad \text{for all } x \in X. \tag{1.6}$$

To state a concrete result we introduce the following basic concepts. A subset $A \subseteq \mathbb{R}$ is *syndetic* if there is $K \subseteq \mathbb{R}$ compact such that $\mathbb{R} = \{a + k : (a,k) \in A \times K\}$. We say that $x \in X$ is an *almost periodic point* of ϕ if $\{t \in \mathbb{R} : \phi_t(x) \in U\}$ is syndetic for every neighbourhood U of x. A flow ϕ is *pointwise almost periodic* (also called *semisimple flow* [15]) if every $x \in X$ is an almost periodic point of ϕ. Every distal flow is pointwise almost periodic but not conversely (c.f. Remark 2.24 in [4]).

With these definitions we can state the result below.

Theorem 1.24. *The following properties are equivalent for every flow ϕ of a compact metric space X:*

(a) *ϕ is pointwise almost periodic.*
(b) *ϕ satisfies equation* (1.6) *with $\mathcal{F} = \mathcal{C}$.*
(c) *ϕ satisfies equation* (1.6) *with $\mathcal{F} = \mathcal{H}$.*

This paper is organized as follows. In Section 2 we introduce some notations and prove some preliminary lemmas. In Section 3 we prove our results.

2. Preliminary lemmas for flows

To prove theorems 1.17 and 1.20 we will use the following notations and related lemmas. Given a flow ϕ of a metric space X, $x \in X$ and $s : \mathbb{R} \to \mathbb{R}$ we define

$$\omega_s(x) = \Big\{y \in X : \lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) = 0 \text{ for some sequence } t_n \to \infty\Big\},$$
$$\alpha_s(x) = \Big\{y \in X : \lim_{n\to\infty} d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) = 0 \text{ for some sequence } t_n \to -\infty\Big\}$$

and

$$O_s(x) = \{\phi_{t-s(t)}(x) : t \in \mathbb{R}\}.$$

If $s = 0$ is the zero map, then $\omega_0(x) = \omega(x)$, $O_0(x) = O(x)$ and $\alpha_0(x) = \alpha(x)$, where $\omega(x)$ and $\alpha(x)$ are the classical *omega* and *alpha limit sets*,

$$\omega(x) = \Big\{y \in X : y = \lim_{n\to\infty} \phi_{t_n}(x) \text{ for some sequence } t_n \to \infty\Big\},$$
$$\alpha(x) = \Big\{y \in X : y = \lim_{n\to\infty} \phi_{t_n}(x) \text{ for some sequence } t_n \to -\infty\Big\}.$$

Both the alpha and the omega-limit sets are closed, *invariant* (i.e. $\phi_t(I) = I$ for all $t \in \mathbb{R}$ and $I = \omega(x), \alpha(x)$) and satisfy the formula

$$Cl(O(x)) = \alpha(x) \cup O(x) \cup \omega(x), \quad \text{for all } x \in X. \tag{2.1}$$

For general maps $s \in \mathcal{C}$ we obtain a related formula.

Lemma 2.1. *For every flow ϕ of a metric space X one has*

$$P_s(x) = \alpha_s(x) \cup O_s(x) \cup \omega_s(x), \quad \textit{for all } x \in X \textit{ and all } s \in \mathcal{C}.$$

Proof. It follows easily from the definitions that $\alpha_s(x)\cup\omega_s(x)\subseteq P_s(x)$. If $t\in\mathbb{R}$, the constant sequence $t_n=t$ satisfies

$$\lim_{n\to\infty} d(\phi_{t_n}(x),\phi_{s(t_n)}(\phi_{t-s(t)}(x)))=\lim_{n\to\infty} d(\phi_t(x),\phi_t(x))=0.$$

Hence $\phi_{t-s(t)}(x)\in P_s(x)$ and so

$$\alpha_s(x)\cup O_s(x)\cup\omega_s(x)\subseteq P_s(x).$$

Now take $y\in P_s(x)$. Then, there is a sequence $t_n\in\mathbb{R}$ such that

$$\lim_{n\to\infty} d(\phi_{t_n}(x),\phi_{s(t_n)}(y))=0. \tag{2.2}$$

If t_n is unbounded, we can assume by passing to a subsequence if necessary that $t_n\to\infty$ or $t_n\to-\infty$. Hence $y\in\alpha_s(x)\cup\omega_s(x)$. Otherwise, t_n is bounded and so we can assume that $t_n\to t$ for some $t\in\mathbb{R}$ by passing to a subsequence if necessary. Then, (2.2) implies $d(\phi_t(x),\phi_{s(t)}(y))=0$ i.e. $y=\phi_{t-s(t)}(x)\in O_s(x)$. All together imply

$$P_s(x)\subseteq\alpha_s(x)\cup O_s(x)\cup\omega_s(x)$$

proving the result. □

Lemma 2.2. *If ϕ is a flow of a metric space X and $s\colon\mathbb{R}\to\mathbb{R}$ is bounded, then*

$$\omega(x)\subseteq\bigcup_{a\in\mathbb{R}}\phi_a(\omega_s(x))\quad\textit{and}\quad\alpha(x)\subseteq\bigcup_{a\in\mathbb{R}}\phi_a(\alpha_s(x))\quad\textit{for every}\quad x\in X.$$

Proof. Fix $x\in X$. If $y\in\omega(x)$, there is a sequence $t_n\to\infty$ such that $d(\phi_{t_n}(x),y)\to0$ as $n\to\infty$. Since s is bounded, we can assume up to passing to a subsequence if necessary that $s(t_n)\to a$ for some $a\in\mathbb{R}$ as $n\to\infty$. Since

$$d(\phi_{t_n}(x),\phi_{s(t_n)}(\phi_{-a}(y)))\le d(\phi_{t_n}(x),y)+d(y,\phi_{s(t_n)}(\phi_{-a}(y))),$$

$d(\phi_{t_n}(x),y)\to0$ and

$$d(y,\phi_{s(t_n)}(\phi_{-a}(y)))\to d(y,\phi_a(\phi_{-a}(y)))=d(y,y)=0$$

as $n\to\infty$, we obtain

$$\lim_{n\to\infty} d(\phi_{t_n}(x),\phi_{s(t_n)}(\phi_{-a}(y)))=0$$

and so $\phi_{-a}(y)\in\omega_s(x)$. Hence $y\in\phi_a(\omega_s(x))$ proving the first inclusion. The proof of the second inclusion is analogous. □

We have the following corollary.

Corollary 2.3. *Let ϕ be a flow of a compact metric space X. If $x\in X$ satisfies $P_s(x)=O(x)$ for some bounded function $s\colon\mathbb{R}\to\mathbb{R}$, then $O(x)$ is closed.*

Proof. By Lemma 2.1 one has $\omega_s(x) \cup \alpha_s(x) \subseteq O(x)$. Then, Lemma 2.2 implies

$$\omega(x) \cup \alpha(x) \subseteq \bigcup_{a\in\mathbb{R}} \phi_a(O(x)) = O(x)$$

and so $O(x)$ is closed by (2.1). $\square$

Lemma 2.4. *For every flow ϕ of a compact metric space X and every continuous bounded function $s : \mathbb{R} \to \mathbb{R}$ one has*

$$O(x) \subseteq P_s(x) \subseteq \mathrm{Cl}(O(x)), \quad \textit{for all } x \in X.$$

Proof. Fix $x \in X$. Obviously $O_s(x) \subseteq O(x)$. Now take $y \in \omega_s(x)$. Then, there is a sequence $t_n \to \infty$ such that $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ as $n \to \infty$. Since s is bounded, we can assume by passing to a subsequence if necessary that $s(t_n) \to a$ for some $a \in \mathbb{R}$ as $n \to \infty$. Since X is compact, we can also assume that $\phi_{t_n}(x) \to z$ for some $z \in X$ as $n \to \infty$. Clearly $z \in \omega(x)$. Since

$$d(z, \phi_a(y)) \leq d(z, \phi_{t_n}(x)) + d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) + d(\phi_{s(t_n)}(y), \phi_a(y)),$$

$d(z, \phi_{t_n}(x)) \to 0$, $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ and

$$d(\phi_{s(t_n)}(y), \phi_a(y)) \to d(\phi_a(y), \phi_a(y)) = 0$$

as $n \to \infty$, we obtain $d(z, \phi_a(y)) = 0$, i.e. $\phi_a(y) = z \in \omega(x)$. Therefore, $y \in \omega(x)$ proving $\omega_s(x) \subseteq \omega(x)$. Similarly we prove $\alpha_s(x) \subseteq \alpha(x)$. By Lemma 2.1 and (2.1) we conclude that

$$P_s(x) \subseteq \mathrm{Cl}(O(x)).$$

On the other hand, since s is bounded continuous, the map $t \in \mathbb{R} \to t - s(t)$ is onto. Take $y \in O(x)$. Then, $y = \phi_r(x)$ for some $r \in \mathbb{R}$. Taking $t \in \mathbb{R}$ such that $r = t - s(t)$ we get $y = \phi_r(x) = \phi_{t-s(t)}(x) \in O_s(x)$. Then, Lemma 3.1 implies

$$O(x) \subseteq P_s(x)$$

and the proof follows. $\square$

Remark 2.5. Lemma 2.4 is false if $s\colon \mathbb{R} \to \mathbb{R}$ were unbounded. Take for instance a minimal distal flow ϕ of a compact metric space with more than one point and $s = \mathrm{Id}$.

To prove Theorems 1.23 and 1.24 we use the following lemma.

Lemma 2.6. *The following properties hold for every flow ϕ of a compact metric space X and every $x \in X$:*

(a) *If $y \in P_{\mathcal{H}}(x)$, then $x \in P_{\mathcal{H}}(y)$.*
(b) $\mathrm{Cl}(O(x)) \subseteq P_{\mathcal{H}}(x)$.

PROOF. To prove (a), take $x \in X$ and $y \in P_{\mathcal{H}}(x)$. Then, there are a homeomorphism $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ and a sequence $t_n \in \mathbb{R}$ such that

$$d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0 \quad \text{as } n \to \infty.$$

The homeomorphism $s^{-1}\colon \mathbb{R} \to \mathbb{R}$ and the sequence $t'_n = s(t_n)$ satisfies $s^{-1}(0) = 0$ and $d(\phi_{t'_n}(y), \phi_{s^{-1}(t'_n)}(x)) \to 0$ as $n \to \infty$ so $x \in P_{\mathcal{H}}(y)$. This proves (a).

To prove (b), we first observe that $x \in P_{\mathcal{H}}(x)$ (just take $s(t) = t$ in the definition of $P_{\mathcal{H}}(x)$). Now, take $y \in O(x)$ thus $y = \phi_r(x)$ for some $r \in \mathbb{R}$. By the previous observation we can assume that $r \neq 0$. First suppose $r > 0$ and define

$$s(t) = \begin{cases} t & \text{if } t \leq r, \\ \dfrac{1}{2}(t + r) & \text{if } r \leq t \leq 3r, \\ t - r & \text{if } 3r \leq t. \end{cases}$$

Clearly $s\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $s(0) = 0$. Since

$$d(\phi_t(x), \phi_{s(t)}(y)) = d(\phi_t(x), \phi_{t-r}(\phi_r(x))) = d(\phi_t(x), \phi_t(x)) = 0, \quad \text{for all } t \geq 3r,$$

we get $\inf\limits_{t \in \mathbb{R}} d(\phi_t(x), \phi_{s(t)}(y)) = 0$ proving $y \in P_{\mathcal{H}}(x)$. Similarly, $y \in P_{\mathcal{H}}(x)$ when $r < 0$ hence $O(x) \subseteq P_{\mathcal{H}}(x)$. Now take $y \in \omega(x)$. Since X is compact, we can choose $q \in \omega(y)$. Hence $q \in \omega(x)$ and so there is a sequence $t_n \to \infty$ such that $d(\phi_{t_n}(x), q) \to 0$ as $n \to \infty$. But $q \in \omega(y)$ so there is another sequence $s_n \to \infty$ such that $d(\phi_{s_n}(y), q) \to 0$ as $n \to \infty$. Then, $d(\phi_{t_n}(x), \phi_{s_n}(y)) \to 0$ as $n \to \infty$ by the triangle inequality. Clearly, we can assume that both s_n and t_n converge monotonically to infinity as $n \to \infty$. Hence, by defining $s(t_n) = s_n$ and extending linearly to all $\mathbb{R}$ we obtain a homeomorphism $s\colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$. This homeomorphism satisfies $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to \infty$ as $n \to \infty$ and so $y \in P_{\mathcal{H}}(x)$. Therefore, $\omega(x) \subseteq P_{\mathcal{H}}(x)$. Similarly, $\alpha(x) \subseteq P_{\mathcal{H}}(x)$ and then (b) holds by (2.1). ☐

3. Proof of the main results

PROOF OF THEOREM 1.2. We follow the arguments in [9]. Let $f\colon X \to X$ be a cw-expansive homeomorphism of a compact metric space X with topological dimension $\dim(X) > 0$. Here we consider the hyperspace of X is the set $2^X_c = \{A \subseteq X : A \text{ is a compact subset of } X\}$. Recall that $\mathscr{C}_c = \{C \in 2^X : C \text{ is a subcontinuum of } X\}$ denotes the set of subcontinua of X. Clearly $\mathcal{C}_c \subset 2^X_c$. We endow 2^X_c (and then $\mathcal{C}_c$) with the Hausdorff metric $d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq U_\varepsilon(B) \text{ and } B \subseteq U_\varepsilon(A)\}$ where $U_\varepsilon(A)$ is the ε-neighbourhood of A. It is known that both 2^X_c and $\mathscr{C}_c$ are compact [14].

If $\varepsilon > 0$, we define

$$W^s_\varepsilon = \big\{C \in \mathscr{C}_c : \operatorname{diam} f^n(C) \leq \varepsilon, \text{ for all } n \geq 0\big\},$$
$$W^u_\varepsilon = \big\{C \in \mathscr{C}_c : \operatorname{diam} f^n(C) \leq \varepsilon, \text{ for all } n \leq 0\big\},$$

$$\begin{aligned}W^s &= \big\{C \in \mathscr{C}_c : \lim_{n\to\infty} \operatorname{diam} f^n(C) = 0\big\},\\ W^u &= \big\{C \in \mathscr{C}_c : \lim_{n\to-\infty} \operatorname{diam} f^n(C) = 0\big\}.\end{aligned}$$

We claim that if c is a cw-expansivity constant of f, then $W^s_\varepsilon \subseteq W^s$ and $W^u_\varepsilon \subseteq W^u$ for every $0 < \varepsilon \leq c$. Otherwise, there is $A \in W^s_\varepsilon \setminus W^s$ (say). Since $A \notin W^s$ there are a sequence $n(1) < \ldots < n(i) < \ldots \to \infty$ and $\delta > 0$ such that

$$\operatorname{diam}(f^{n(i)}(A)) \geq \delta, \quad \text{for all } i \in \mathbb{N}^+. \tag{3.1}$$

Since $\mathscr{C}_c$ is compact, we can assume that $f^{n(i)}(A) \to B$ for some $B \in \mathscr{C}_c$. Because of (2.1) we have

$$\operatorname{diam}(B) \geq \delta.$$

In particular, B is nondegenerated. On the other hand, fix $n \in \mathbb{Z}$. As $n(i) \to \infty$, $n(i) + n \geq 0$ for all i large. As $A \in W^s_\varepsilon$, $\operatorname{diam}(f^j(A)) \leq \varepsilon$ for every $j \geq 0$. By taking $j = n(i) + n$ we get

$$\operatorname{diam}(f^{n(i)+n}(A)) \leq \varepsilon, \quad \text{for } i \text{ large}.$$

Letting $i \to \infty$ above we obtain $\operatorname{diam}(f^n(B)) \leq \varepsilon$ for every $n \in \mathbb{Z}$. Since B is nondegenerated and $\varepsilon \leq c$, we get a contradiction. Therefore, the claim is proved.

We conclude the proof of the theorem once we prove that there is a nondegenerated subcontinuum in $W^s \cup W^u$. By the claim it suffices to prove that there is a nondegfenerated subcontinuum in $W^s_\varepsilon \cup W^u_\varepsilon$ where $\varepsilon = c/2$. Suppose this is not true. Since $\dim(X) > 0$, there are sequences of subcontinua $A \supseteq A_2 \supseteq \ldots \supseteq A_i \supseteq \ dots$ and of integers $n(1) < \ldots < n(i) < \ldots \to \infty$ such that

$$\operatorname{diam}(f^j(A_i)) \leq \varepsilon \quad \text{and} \quad \operatorname{diam}(f^{n(i)}(A_i)) > \varepsilon, \tag{3.2}$$

for all $0 \leq j \leq n(i) - 1$ and all $i \in \mathbb{N}^+$. Since $\mathscr{C}_c$ is compact, we can assume that $f^{n(i)}(A_i) \to B$ for some $B \in \mathscr{C}_c$. As before the second inequality in (3.2) implies $diam(B) \geq \varepsilon$. In particular, B is nondegenerated. On the other hand, fix an integer $n \geq 1$. As $n(i) \to \infty$, $n < n(i)$ for every i large, and so $0 \leq n(i) - n \leq n(i) - 1$. Then, we can put $j = n(i) - n$ to obtain

$$\operatorname{diam}(f^{n(i)-n}(A_i)) \leq \varepsilon, \quad \text{for all } i \text{ large}.$$

Letting $i \to \infty$ we get

$$\operatorname{diam}(f^{-n}(B)) \leq \varepsilon, \quad \text{for all } n \geq 1.$$

It follows that $f^{-1}(B) \in W^u_\varepsilon$. As B is nondegenerated, $f^{-1}(B)$ also is contradicting that there are no nondegenerated subcontinua in $W^s_\varepsilon \cup W^u_\varepsilon$. $\square$

Proof of Theorem 1.3. First suppose that f is cw-distal and take $x \in X$. Fix $\varepsilon > 0$ and a subcontinuum $C \subseteq P(x)$. It follows from the definition of $P(x)$ that there is a map $N\colon C \to \mathbb{N}$ such that, for all $y \in C$,

$$N(y) = \sup \bigl\{ n \in \mathbb{N} : d(f^i(x), f^i(y)) > \varepsilon, \text{ for all } -n \le i \le n \bigr\}.$$

Clearly, N is continuous and so $N(y) = N$ is constant since C is connected. It follows that $\operatorname{diam}(f^N(C)) \le \varepsilon$ proving

$$\inf_{n \in \mathbb{Z}} \operatorname{diam}(f^n(C)) = 0. \tag{3.3}$$

Since f is cw-distal, C reduces to a singleton and so $P(x)$ is totally disconnected.

Conversely, suppose that $P(x)$ is totally disconnected for every $x \in X$. Then, if C is a subcontinuum satisfying (3.3), by taking $x \in C$ we get $C \subseteq P(x)$. As $P(x)$ is totally disconnected, C reduces to a singleton proving that f is cw-distal. □

Let Z be any compact metric space. We denote by $\mathscr{C}_c(Z)$ as the set of subcontinua of Z.

Proof of Theorem 1.4. Let X and Y be compact metric spaces. Let $g\colon Y \to Y$ be a cw-distal extension of a cw-distal homeomorphism $f\colon X \to X$. Then, there is a cw-distal homomorphism $\pi\colon Y \to X$ from g to X. We shall prove that g is cw-distal. For this we take $C \in \mathscr{C}_c(Y)$ such that $\operatorname{diam}(g^{i_n}(C)) \to 0$ as $n \to \infty$ for some sequence $i_n \in \mathbb{Z}$. Since Y is compact, $g^{i_n}(C) \to y$ with respect to the Hausdorff metric for some $y \in Y$ as $n \to \infty$. Since π is continuous, $\pi(g^{i_n}(C)) \to \pi(y)$ as $n \to \infty$. As π is a homomorphism, $\pi(g^{i_n}(C)) = f^{i_n}(\pi(C))$ and so $f^{i_n}(\pi(C)) \to \pi(y)$ with respect to the Hausdorff metric as $n \to \infty$. Hence $\operatorname{diam}(f^{i_n}(\pi(C))) \to 0$ as $n \to \infty$ and so $\pi(C)$ reduces to a point x because f is cw-distal. Therefore, $C \subseteq \pi^{-1}(x)$ is a subcontinuum of Y satisfying $\operatorname{diam}(g^{i_n}(C)) \to 0$ as $n \to \infty$ and so C reduces to a singleton because π is cw-distal. Then, g is cw-distal. □

Proof of Theorem 1.5. Let $f\colon X \to X$ and $g\colon Y \to Y$ be homeomorphisms of compact metric spaces X and Y. First suppose that f and g are cw-distal. Denote by $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ the natural projections and fix $(x, y) \in X \times Y$. It is easy to see that

$$P_{f\times g}(x, y) \subseteq P_f(x) \times P_g(y).$$

Hence $\pi_1(C) \subseteq P_f(x)$ and $\pi_2(C) \subseteq P_g(y)$ for every subcontinuum $C \subseteq P_{f\times g}(x, y)$. Since f and g are cw-distal, both $\pi_1(C)$ and $\pi_2(C)$ reduce to singleton. Hence C reduces to a singleton too proving that $P_{f\times g}(x, y)$ is totally disconnected. As (x, y) is arbitrary, $f \times g$ is cw-distal by Theorem 1.3.

Next we suppose that f is cw-distal and that g is an extension of f under a light map $h\colon Y \to X$. Take $C \in \mathscr{C}_c(Y)$ such that $\operatorname{diam}(g^{i_n}(C)) \to 0$ as

$n \to \infty$ for some sequence $i_n \in \mathbb{Z}$. Since Y is compact, $g^{i_n}(C) \to y$ with respect to the Hausdorff metric as $n \to \infty$ for some $y \in Y$. By continuity, $h(g^{i_n}(C)) \to x = h(y)$ and so $f^{i_n}(h(C)) \to x$ as $n \to \infty$ with respect to the Hausdorff metric too. In particular, $\operatorname{diam}(f^{i_n}(h(C)) \to 0$ as $n \to \infty$ and then $h(C) = \{x'\}$ for some $x' \in X$ because f is cw-distal. It follows that C is a subcontinuum of $h^{-1}(x')$. Since h is light, C reduces to a point too and then g is cw-distal.

Finally, suppose that g is cw-distal and that f is a factor of g under a light weakly confluent map $h\colon Y \to X$. Take $C \in \mathscr{C}_c(X)$ such that $\operatorname{diam}(f^{i_n}(C)) \to 0$ as $n \to \infty$ for some sequence $i_n \in \mathbb{Z}$. Since h is weakly confluent, there is $A \in \mathscr{C}_c(Y)$ such that $h(A) = C$. Since $f \circ h = h \circ g$, we have $f^{i_n}(C) = f^{i_n}(h(A)) = h(g^{i_n}(A))$ and so $\operatorname{diam}(h(g^{i_n}(A))) \to 0$ as $n \to \infty$. Then, by passing to a subsequence if necessary, we can assume that $h(g^{i_n}(A)) \to x$ with respect to the Hausdorff metric $n \to \infty$. On the other hand, since $\mathscr{C}_c(Y)$ is compact (for Y is), we can assume that there is $A_\infty \in \mathscr{C}_c(Y)$ such that $g^{i_n}(A) \to A_\infty$ with respect to the Hausdorff metric as $n \to \infty$. But $h(g^{i_n}(A)) \to x$ so $A_\infty \subseteq h^{-1}(x)$ hence A_∞ is degenerated because h is light. It follows that $\operatorname{diam}(g^{i_n}(A)) \to 0$ as $n \to \infty$ and then A reduces to a point since g is cw-distal. Therefore, $C = h(A)$ reduces to a point too and the proof follows. □

Proof of Example 1.7. A homeomorphism is *countably distal* if its proximal cells are all countable. Consider $X = [0,1] \times C$ where C is the ternary Cantor set of $[0,1]$. By [7] there is a homeomorphism $g\colon C \to C$ such that $g(0) = 0$, $g(1) = 1$ and $\{g^n(y) : n \in \mathbb{Z}\}$ is dense in C for every $y \in C \setminus \{0,1\}$. Define $f\colon X \to X$ by $f(x,y) = (x, g(y))$ for $(x,y) \in X$. Clearly, $P(x,y) \subseteq x \times C$ and so $P(x,y)$ is totally disconnected for every $(x,y) \in X$. It follows that f is cw-distal. On the other hand, since $g(0) = 0$ and $\{g^n(y) : n \in \mathbb{Z}\}$ is dense in C for every $y \in C \setminus \{0,1\}$, we get that $P(x,0) = x \times (C \setminus \{0,1\})$ is uncountable for every $x \in [0,1]$. Therefore, f is not countably distal and so it is not N-distal for every positive integer N. □

Proof of Theorem 1.11. Let $f\colon X \to X$ be a homeomorphism of a separable metric space. First suppose that f is cw-distal. Take any nonatomic Borel probability measure μ and a subcontinuum C such that $\inf_{n\in\mathbb{Z}} \operatorname{diam}(f^n(C)) = 0$. Since f is cw-distal, C is degenerated (i.e. a singleton) and so $\mu(C) = 0$ since μ is nonatomic. Therefore, μ is cw-distal.

Conversely, suppose that every nonatomic Borel probability measure is cw-distal. If f were not cw-distal, there would exist a nondegenerated subcontinuum C such that $\inf\limits_{n\in\mathbb{N}} \operatorname{diam}(f^n(C)) = 0$. As C is nondegenerated and X separable, C is uncountable and, since it is compact, there would exist a nonatomic Borel probability measure μ with $\mu(C) = 1$. But then μ is cw-distal and so $\mu(C) = 0$ a contradiction. □

Proof of Example 1.12. As in the proof of Example 1.7, take a homeomorphism $f\colon C \to C$ of the ternary Cantor set in $[0,1]$ such that $f(0)=0$, $f(1)=1$ and $\{f^n(x) : n \in \mathbb{Z}\}$ is dense in C for every $x \in C \setminus \{0,1\}$. It follows that $P(0) = C \setminus \{0,1\}$ and so $P(0)$ is not countable. Then, by Theorem 1.3 in [12], there is a nonatomic Borel probability measure μ of C which is non-distal for f. Since C is totally disconnected, f is cw-distal and so μ is cw-distal by Theorem 1.11. $\square$

Proof of Theorem 1.14. The second part of the theorem follows from the first and Theorem 1.11. For the first part, let μ be a strongly cw-distal measure of f. Take a subcontinuum $C \subset P(x)$ for some $x \in X$ and fix $\varepsilon > 0$. Let $\delta > 0$ be given by the strongly cw-distality of μ for this ε. Since $C \subseteq P(x) \subseteq P(x,\delta)$, $\mu(C) \le \mu_{\mathrm{cw}}(P(x,\delta)) < \varepsilon$. Since ε is arbitrary, $\mu(C) = 0$ proving the result. $\square$

Proof of Theorem 1.15. Let $\mathcal{M}^s_{\mathrm{cwd}}(f)$ denote the set of strongly cw-distal measures of a homeomorphism $f\colon X \to X$. Given $\varepsilon, \delta > 0$ we define

$$C(\varepsilon,\delta) = \bigl\{\mu \in \mathcal{M}(X) : \mu_{\mathrm{cw}}(P(x,\delta)) > \varepsilon, \text{ for some } x \in X\bigr\}.$$

It follows easily from the definition that

$$\mathcal{M}(X) \setminus \mathcal{M}^s_{\mathrm{cwd}}(f) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} C\bigl(m^{-1}, n^{-1}\bigr). \tag{3.4}$$

We shall prove that $C(\varepsilon,\delta)$ is open in $\mathcal{M}(X)$ for every $\varepsilon, \delta > 0$.

Take a sequence $\mu_n \in \mathcal{M}(X)$ converging to $\mu \in C(\varepsilon,\delta)$ for some $\varepsilon, \delta > 0$. Since $\mu \in C(\varepsilon,\delta)$, there are $x \in X$ and a subcontinuum $A \subseteq P(x,\delta)$ such that $\mu(A) > \varepsilon$. Since $P(x,\delta)$ is open, contains A and X is locally connected, there is a *connected* open subset $O \subseteq X$ such that $A \subseteq O \subseteq \overline{O} \subseteq P(x,\delta)$ (e.g. Corollary 9.13 in [8, p. 112]). Since O is open, well known properties of the weak* convergence (e.g. [17]) imply

$$\liminf_{n\to\infty} \mu_n(O) \ge \mu(O) \ge \mu(A) > \varepsilon$$

Therefore, $\mu_n(O) > \varepsilon$ for n large. Since $O \subset P(x,\delta)$, $\mu_n \in C(\varepsilon,\delta)$ for n large proving that $C(\varepsilon,\delta)$ is open as asserted. Then, (3.4) implies that $\mathcal{M}^s_{\mathrm{cwd}}(f)$ is $F_{\sigma\delta}$ and the result follows. $\square$

Proof of Theorem 1.16. Since every cw-distal measure is nonatomic, Theorem 1.11 implies that $\mathcal{M}_{\mathrm{cwd}}(f) = \mathcal{M}_{\mathrm{na}}(X)$ where $\mathcal{M}_{\mathrm{na}}(X)$ is the set of nonatomic Borel probability measures of X. Then, it suffices to show $\mathcal{M}^s_{\mathrm{cwd}}(f) = \mathcal{M}_{\mathrm{na}}(X)$ for every equicontinuous homeomorphism $f\colon X \to X$.

Fix $\mu \in \mathcal{M}_{\mathrm{na}}(X)$ and $\varepsilon > 0$. Since μ is nonatomic, there is $\varepsilon_0 > 0$ such that $\operatorname{diam}(A) \le \varepsilon_0$ implies $\mu(A) < \varepsilon$ for every measurable subset A. Let δ_0 be given by the equicontinuity of f for $\varepsilon_0/2$. Take $y \in P(x,\delta_0)$ for some $x \in X$. It follows that $d(f^i(x), f^i(y)) < \delta_0$ for some $i \in \mathbb{Z}$. By the choice of δ_0 we get

$d(f^n(f^i(x)), f^n(f^i(y))) < \varepsilon_0/2$ for every $n \in \mathbb{Z}$. Replacing $n = -i$ we obtain $d(x, y) < \varepsilon_0/2$. Since $y \in P(x, \delta_0)$ is arbitrary, $\operatorname{diam}(P(x, \delta_0)) \leq \varepsilon_0$ for every $x \in X$. By the choice of ε_0 we get $\mu(P(x, \delta_0)) < \varepsilon$ hence $\mu_{\mathrm{cw}}(P(x, \delta_0)) < \varepsilon$ for every $x \in X$ proving $\mathcal{M}_{\mathrm{na}}(X) \subseteq \mathcal{M}^s_{\mathrm{cwd}}(f)$. Since every strongly cw-distal measure is cw-distal (hence nonatomic), we obtain $\mathcal{M}^s_{\mathrm{cwd}}(f) \subseteq \mathcal{M}_{\mathrm{na}}(X)$ proving the result. □

Proof of Theorem 1.17. Let ϕ be a flow of a metric space X. Suppose that there is $s \in \mathcal{C} \setminus \{\mathrm{Id}\}$ satisfying (1.3). Then, Lemma 2.1 implies $O_s(x) \subseteq \{x\}$ and so $\phi_{t-s(t)}(x) = x$ for every $x \in X$ and every $t \in \mathbb{R}$. Since $s \neq \mathrm{Id}$, $T = t - s(t) \neq 0$ for some $t \in \mathbb{R}$. For this T we obtain $\phi_T(x) = x$ for every $x \in X$ and then ϕ is uniformly closed. Finally, if s is bounded, then $O(x) = \{x\}$ for every $x \in X$ by Lemma 2.4 and so ϕ is trivial. □

Proof of Theorem 1.20. Let ϕ be a flow of a compact metric space X. If ϕ is closed, $\mathrm{Cl}(O(x)) = O(x)$ and then $P_s(x) = O(x)$ for every $x \in X$ and every $s \in \mathcal{C}$ bounded by Lemma 2.4. Therefore, item (a) implies (b). Obviously item (b) implies (c) and (c) implies (a) by Corollary 2.3. □

Proof of Theorem 1.21. Suppose by contradiction that there is a flow without singularities ϕ of a compact metric space X satisfying (1.4) holds with $s = \mathrm{Id}$. Since ϕ has no singularities, and X is compact, there is $r > 0$ such that $\phi_r(x) \neq x$ for every $x \in X$. Otherwise, it would exist sequences $r_i \to 0^+$ and $x_i \in X$ with $\phi_{r_i}(x_i) = x_i$. By compactness we can assume $x_i \to x$ for some $x \in X$. Then, given $t > 0$ we write $t = k_i r_i + l_i$ for some $k_i \in \mathbb{N}$ and $0 \leq l_i < r_i$ so $d(\phi_t(x), x) \leq d(\phi_t(x), \phi_t(x_i)) + d(\phi_{l_i}(x_i), x) \to 0$ as $i \to \infty$ thus $\phi_t(x) = x$ for every $t \geq 0$ proving that x is a singularity which is absurd.

Now, take any $x \in X$. It follows from (1.4) with $s = Id$ that there is a sequence $t_n \in \mathbb{R}$ such that $d(\phi_{t_n}(x), \phi_{t_n + r}(x)) \to 0$ as $n \to \infty$. By compactness we can assume that $\phi_{t_n}(x) \to z$ for some $z \in X$ as $n \to \infty$. Then, $\phi_r(z) = z$ which is impossible since ϕ has no singularities. □

Proof of Theorem 1.23. Clearly $P_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$ whenever ϕ is trivial. Conversely, if $P_{\mathcal{H}}(x) = \{x\}$ for every $x \in X$, then $Cl(O(x)) = \{x\}$ for every $x \in X$ by Lemma 2.6 and so ϕ is trivial. □

Proof of Theorem 1.24. As is well-known [4], a point $z \in X$ is almost periodic for ϕ if and only if the orbit closure $\mathrm{Cl}(O(z))$ is *minimal* (i.e. compact, nonempty, invariant and no proper subset of it is compact nonempty invariant).

First we prove that item (a) implies (b). Take $x \in X$ and $y \in P_{\mathcal{C}}(x)$. Then, there are $s\colon \mathbb{R} \to \mathbb{R}$ continuous with $s(0) = 0$ and a sequence $t_n \in \mathbb{R}$ such that $d(\phi_{t_n}(x), \phi_{s(t_n)}(y)) \to 0$ as $n \to \infty$. Since X is compact, we can assume by taking a subsequence if necessary that $\phi_{t_n}(x) \to z$ as $n \to \infty$ for some $z \in X$.

It follows that $\phi_{s(t_n)}(y) \to z$ as $n \to \infty$ thus $z \in \mathrm{Cl}(O(x)) \cap \mathrm{Cl}(O(y))$. Then, $\mathrm{Cl}(O(x)) \cap \mathrm{Cl}(O(y)) \neq \emptyset$ and so $\mathrm{Cl}(O(x)) = \mathrm{Cl}(O(y))$ since both $\mathrm{Cl}(O(x))$ and $\mathrm{Cl}(O(y))$ are minimal sets. In particular, $y \in \mathrm{Cl}(O(x))$ and so $P_{\mathcal{C}}(x) \subseteq \mathrm{Cl}(O(x))$. Since $\mathrm{Cl}(O(x)) \subseteq P_{\mathcal{H}}(x)$ (by Lemma 2.6) and $P_{\mathcal{H}}(x) \subseteq P_{\mathcal{C}}(x)$ (by definition because $\mathcal{H} \subseteq \mathcal{C}$), we obtain $\mathrm{Cl}(O(x)) \subseteq P_{\mathcal{C}}(x)$. Then, $\mathrm{Cl}(O(x)) = P_{\mathcal{C}}(x)$ for every $x \in X$ proving (b).

By Lemma 2.6 we have that item (b) implies (c).

Finally we prove that (c) implies (a). Take $y \in \mathrm{Cl}(O(x))$ for some $x \in X$. Then, $y \in P_{\mathcal{H}}(x)$ and so $x \in P_{\mathcal{H}}(y) = \mathrm{Cl}(O(x))$ by Lemma 2.6. It follows that $\mathrm{Cl}(O(x)) \subseteq \mathrm{Cl}(O(y))$ hence $\mathrm{Cl}(O(x))$ is minimal thus x is almost periodic. Since x is arbitrary, item (a) holds. ☐

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