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SUBHARMONIC SOLUTIONS AND MINIMAL PERIODIC SOLUTIONS OF FIRST-ORDER VARIANT SUBQUADRATIC HAMILTONIAN SYSTEMS

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Abstract. Using the homological link theorem and iteration inequalities of Maslov-type index, we prove the multiplicity of subharmonic solutions for some variant subquadratic non-autonomous Hamiltonian systems. Moreover, the minimal period problem has also been considered for the variant subquadratic autonomous Hamiltonian systems.

1. Introduction and main results

In this paper, we first consider subharmonic solutions of the following non-autonomous Hamiltonian system

$$\begin{cases} -J\dot{z} = H'_z(t,z), \\ z(k\tau) = z(0), \qquad k \in \mathbb{N}, \end{cases} \tag{1.1}$$

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where $\tau > 0$, H'_z denotes the gradient of H with respect to $z \in \mathbb{R}^{2n}$, and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with I_n being the identity matrix on $\mathbb{R}^n$.

Let $z = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n}$, for $\mu_i,\ \nu_i > 0$ with $\mu_i + \nu_i = 1$ $(i = 1, \ldots, n)$, define

$$V(z) = (\mu_1 p_1, \ldots, \mu_n p_n, \nu_1 q_1, \ldots, \nu_n q_n).$$

Assume H satisfies:

(H1) $H \in C^2\big(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}\big)$ and τ-periodic in the first variable t,

(H2) there exist $\alpha_i, \beta_i > 0$ $(i = 1, \ldots, n)$ such that

$$\lim_{|z| \to +\infty} \frac{H(t,z)}{w(z)} = 0 \quad \text{uniformly in } t,$$

where $w(z) = \sum_{i=1}^{n} \big(|p_i|^{1+\alpha_i/\beta_i} + |q_i|^{1+\beta_i/\alpha_i}\big)$,

(H3) for V defined as above, there exists $c_1, c_2 > 0$ and $\beta \in (1,2)$ such that for $(t,z) \in \mathbb{R} \times \mathbb{R}^{2n}$, there holds

$$\min\big\{H(t,z), H(t,z) - H'_z(t,z) \cdot V(z)\big\} \geq c_1 |z|^{\beta} - c_2$$

(H4) there exists $\lambda \in [1, \beta^2/(\beta+1))$ such that

$$\big|H''_{zz}(t,z)\big| \leq c_2\big(|z|^{\lambda-1} + 1\big), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n},$$

(H5) $H(t,0) = 0$, and $H(t,z) > 0$, $|H'_z(t,z)| > 0$ for $z \neq 0$.

Note that (H4) implies that $\beta \in \big(\sqrt{5} + 1/2, 2\big)$.

REMARK 1.1. Conditions (H1)–(H5) are similar to those in [17] with minor modifications, where H possesses C^1-smoothness, and $V(z)$ and $w(z)$ are of the forms $V(p,q) = (\mu p, \nu q)$ and $w(p,q) = |p|^{1+\sigma/\tau} + |q|^{1+\tau/\sigma}$ for $p, q \in \mathbb{R}^n$, where $\mu, \nu, \sigma, \tau > 0$ and $\mu + \nu = 1$. Note that for $p, q \in \mathbb{R}$, the Hamiltonian function in [17]

$$H(p,q) = \begin{cases} \dfrac{|p|^{1+\sigma/\tau}}{\ln(1+|p|^{\xi})} + \dfrac{|q|^{1+\tau/\sigma}}{\ln(1+|q|^{\xi})}, & p \neq 0,\ q \neq 0, \\[2ex] \dfrac{|p|^{1+\sigma/\tau}}{\ln(1+|p|^{\xi})}, & p \neq 0,\ q = 0, \\[2ex] \dfrac{|q|^{1+\tau/\sigma}}{\ln(1+|q|^{\xi})}, & p = 0,\ q \neq 0, \\[2ex] 0, & p = 0,\ q = 0, \end{cases}$$

satisfies (H2)–(H5) with $\mu = \tau/(\sigma+\tau)$, $\nu = \sigma/(\sigma+\tau)$, $\beta = \tau/\sigma + 1 - 2\xi$ and $\lambda = \sigma/\tau + \xi$, where $\sigma > \tau > 0$, $\tau/\sigma > 2\xi > 0$ and $(\tau/\sigma + 1 - 2\xi)^2/(\tau/\sigma + 2 - 2\xi) > \sigma/\tau + \xi$ (by continuity, the numbers σ, τ, ξ exist). Furthermore, by the truncation techniques, we can redefine a Hamiltonian function satisfying (H1)–(H5) which equals H outside some ball.

In [17], the authors only obtain the existence reslut by the general link thorem, however, using the Maslov-type index iteration theory and a homological linking theorem, we verify the multiplicity of geometrically distinct subharmonic solutions and the existence of minimal periodic solutions.

Remark 1.2. There exists a classical subquadratic growth condition in [3] stating that

$$H(z) \geq \theta H'_z(z) \cdot z > 0, \quad |z| \geq R \quad \text{and} \quad H(z) \geq c_1|z|^s - c_2,$$

where $1/2 < \theta < 1$ and $s \in (1, 1/\theta)$. While in [13], define $D(p,q) = (p/a, q/b)$, $z = (p,q) \in \mathbb{R}^{2n}$ with $1 < a,\ b \leq 2$ and $1/a + 1/b > 2$, the authors generalize the above subquadratic growth condition, that is,

$$H(z) \geq H'_z(z) \cdot D(z) > 0, \quad |z| \geq R \quad \text{and} \quad H(z) \geq c_1(|p|^{\overline{a}} + |q|\overline{b}) - c_2,$$

where $1 < \overline{a} \leq a$ and $1 < \overline{b} \leq b$.

Finally, in [17], a variant subquadratic growth condition ((H2) and (H3) in this paper) is developed, which in spirit is an isotropic growth condition having some components growing more than 2 and other components growing less than 2 and generalizing the above subquadratic growth conditions. There exist other types of subquadratic growth conditions, see [4], [5], [34].

We remind the readers that the generalization process of the subquadratic growth condition resembles that of the superquadratic growth condition, see [2], [3], [16], [28]. Moreover, the subharmonic solution problems and the minimal periodic solution problems have been considered for the superquadratic case in [28].

Given $j \in \mathbb{Z}$ and a $k\tau$-periodic solution $(z, k\tau)$ of the system (1.1), we define the phase shift $j * z$ of z by $j * z(t) = z(t + j\tau)$. Recall that two solutions $(z_1, k_1\tau)$ and $(z_2, k_2\tau)$ are geometrically distinct if $j * z_1 \neq l * z_2$ for all $j,\ l \in \mathbb{Z}$.

The goal of the present article is to obtain the multiplicity of geometrically distinct subharmonic solutions for the system (1.1) under the above subquadratic condition. Furthermore, the minimal periodic solutions of the following autonomous Hamiltonian system

$$\begin{cases} -J\dot{z} = H'_z(z), \\ z(\tau) = z(0) \end{cases} \tag{1.2}$$

is also considered. Our main results are as follows.

Theorem 1.3. *Suppose H satisfies* (H1)–(H5). *Then there exists $\tau_0 > 0$ such that, for any $\tau \geq \tau_0$ and any integer $k \geq 1$, the system* (1.1) *possesses a nontrivial $k\tau$-periodic solution z_k with its Maslov-type index satisfying*

$$i_{k\tau}(z_k) \leq n \leq i_{k\tau}(z_k) + \nu_{k\tau}(z_k).$$

Moreover, if $i_{k\tau}(z_k) + \nu_{k\tau}(z_k) > n$ *and* $p > 2n/(i_{k\tau}(z_k) + \nu_{k\tau}(z_k) - n)$*, then* z_k *and* z_{pk} *are geometrically distinct.*

Remark 1.4. Suppose $B(t)$ is a τ-periodic continuous strictly positive definite martix function, consider the following Hamiltonian system

$$\begin{cases} -J\dot{z} = B(t)z(t), \\ z(\tau) = z(0), \end{cases}$$

then there exists a periodic solution z satisfing $i_\tau(z) \geq n$ and $\nu_\tau(z) \geq 1$ (see Proposition 2.3), then $i_\tau(z)+\nu_\tau(z) > n$. Thus the assumption $i_{k\tau}(z_k)+\nu_{k\tau}(z_k) > n$ in Theorem 1.3 is reasonable.

If H contains a quadratic term, i.e. $H(t,z) = \big(\widehat{B}(t)z, z\big)/2 + \widehat{H}(t,z)$, then we have the following result.

Theorem 1.5. *Suppose* $\widehat{H}$ *satisfies* (H1)–(H4)*,* H *satisfies* (H5) *and* $\widehat{B}$ *satisfies*

(H6) $\widehat{B}(t)$ *is a* τ*-periodic, symmetric and continuous matrix function with*

$$\big(\widehat{B}(t)z, z\big) = 2(\widehat{B}(t)z, V(z)), \quad (t,z) \in \mathbb{R}\times\mathbb{R}^{2n},$$

(H7) *there exists an unbounded sequence* $\{\rho_m\} \subset (0,+\infty)$ *such that*

$$(\widehat{B}(t)B_\rho z, B_\rho z) = \rho^{\varrho-2}(\widehat{B}(t)z, z), \quad (t,z) \in \mathbb{R}\times\mathbb{R}^{2n},$$

where B_ρ *is defined in Section 3 for* $\rho \in \{\rho_m\}$ (*since it is complex, we do not give it here*).

Let $w = \max_{t\in\mathbb{R}} |\widehat{B}(t)|$*. Then there exists* $\tau_0 > 0$ *such that for any* $\tau \geq \tau_0$ *and for any integer* $1 \leq k < 2\pi/(w\tau)$*, the system* (1.1) *possesses a nontrivial* $k\tau$*-periodic solution* z_k *with its Maslov-type index satisfying*

$$i_{k\tau}(z_k) \leq n \leq i_{k\tau}(z_k) + \nu_{k\tau}(z_k).$$

Furthermore, if $i_{k\tau}(z_k)+\nu_{k\tau}(z_k) > n$*, then* z_k *and* z_{pk} *are geometrically distinct provided* $p > 2n/(i_{k\tau}(z_k) + \nu_{k\tau}(z_k) - n)$ *and* $pk < 2\pi/(w\tau)$*.*

Below is the minimal periodic solution result.

Theorem 1.6. *Suppose the autonomous Hamiltonian* $H(z)$ *satisfies* (H1)–(H5) *and*

(H8) $H''_{zz}(z)$ *is strictly positive for every* $z \in \mathbb{R}^{2n} \setminus \{0\}$*.*

Then there exists $\tau_0 > 0$ *such that for any* $\tau \geq \tau_0$*, the system* (1.2) *possesses a nontrivial periodic solution* z *with minimal period* τ*.*

The first result for the existence of subharmonic periodic solutions of the system (1.1) was obtained by Rabinowitz in [32]. Since then, many mathematicians made their contributions in this topic, see for example [7], [10], [12], [18], [21], [22], [28], [30], [34], [36]. For the brake subharmonic solutions of Hamiltonian systems we refer to [19], [20]. For the P-symmetric subharmonic solutions we refer to [27].

In the pioneer work [31], Rabinowitz proposed a conjecture on whether a superquadratic Hamiltonian system possesses a non-constant periodic solution having any prescribed minimal period. After paper [31], much work has been done in this field. We refer to [8], [9], [11], [14], [15], [36] for the minimal periodic solutions. For the minimal period problem of brake solutions of Hamiltonian systems, we refer to [23]. For the minimal P-symmetric periodic solutions of Hamiltonian systems, we refer to [24], [26], [35].

Linking theorems provide a simple but extremely powerful method to prove the existence of critical points. We follow the ideas in [28] which use the homologically link Theorem 2.9 (see [1]) to look for the critical points and estimate the corresponding Morse index. Based on those, we study the subharmonic solutions and minimal periodic solutions under subquadratic growth conditions by the method in [22] respectively. The main difficulty is to construct two sets and prove that they are homological linking which is the content of Lemma 2.8.

In Section 2, we briefly sketch some notions about the Maslov-type index and the iteration inequalities developed by C. Liu and Y. Long in [25]. We also recall the homologically link theorem in [1] from which we can find a critical point of the corresponding functional together with Morse index information. We prove that there is a homologically link structure for the functional under the conditions of Theorem 1.3. The proofs of Theorems 1.3, 1.5 and 1.6 will be given in Section 3.

2. Preliminaries

We first review the Maslov-type index and some iteration properties. Here we use the notions and results in [25], [29].

Recall that the symplectic group is defined as

$$\mathrm{Sp}(2n) = \big\{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T J M = J\big\},$$

where $\mathcal{L}(\mathbb{R}^{2n})$ is the space of $2n \times 2n$ real matrices. For $\tau > 0$, the set of symplectic paths is defined by $\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I\}$.

Let $S_\tau = \mathbb{R}/\tau\mathbb{Z}$ and $\mathfrak{L}_s(\mathbb{R}^{2n})$ denote the set of all symmetric real $2n \times 2n$ matrices. For $B(t) \in C(S_\tau, \mathfrak{L}_s(\mathbb{R}^{2n}))$, suppose γ is the fundamental solution of the linear Hamiltonian systems

$$\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}. \tag{2.1}$$

Then the Maslov-type index pair of γ is defined as a pair of integers

$$(i_\tau, \nu_\tau) \equiv (i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},$$

where i_τ is the index part and

$$\nu_\tau = \dim \ker(\gamma(\tau) - I)$$

is the nullity. We also call (i_τ, ν_τ) the Maslov-type index of $B(t)$, just as in [25] and [29]. If (z, τ) is a τ-periodic solution of (1.1), then the Maslov-type index of the solution z is defined to be the Maslov-type index of $B(t) = H''_{zz}(t, z(t))$ and denoted by $(i_\tau(z), \nu_\tau(z))$.

For $\gamma \in \mathcal{P}_\tau(2n)$, we define the m-th iteration path $\gamma_m \colon [0, m\tau] \to \mathrm{Sp}(2n)$ of γ by

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for all } j\tau \le t \le (j+1)\tau,\ 0 \le j \le m-1.$$

We denote the Maslov-type index of γ^m on $[0, m\tau]$ by $(i_{m\tau}, \nu_{m\tau})$.

Proposition 2.1 ([22]). *If z is a $k\tau$-periodic solution of the system* (1.1), *then $i_{k\tau}(j * z) = i_{k\tau}(z)$ and $\nu_{k\tau}(j * z) = \nu_{k\tau}(z)$ for all integers $0 \le j \le k$.*

Proposition 2.2 ([22], [25]). *For $m \in \mathbb{N}$, there holds*

$$m(i_\tau + \nu_\tau - n) - n \le i_{m\tau} \le m(i_\tau + n) + n - \nu_{m\tau}.$$

Proposition 2.3 ([1]). *Let $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ be τ-periodic and positive definite for all $t \in [0, \tau]$. Suppose that $B(t_0)$ is strictly positive for some $t_0 \in [0, \tau]$. Then $i_\tau(B) \ge n$.*

Proposition 2.4 ([9]). *Let $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ be τ-periodic. Suppose there exists some $m \in \mathbb{Z}$ such that $i_{m\tau}(B) \le n + 1$, $i_\tau(B) \ge n$ and $\nu_\tau(B) \ge 1$. Then $m = 1$.*

Now we introduce some concepts and conclusions which are used later. For $S_\tau = \mathbb{R}/\tau\mathbb{Z}$, let $E = W^{1/2,2}(S_\tau, \mathbb{R}^{2n})$. Recall that E consists of all the elements $z \in L^2(S_\tau, \mathbb{R}^{2n})$ satisfying

$$z(t) = \sum_{j \in \mathbb{Z}} \exp\left(\frac{2j\pi t}{\tau} J\right) a_j, \quad a_j \in \mathbb{R}^{2n},$$
$$\|z\|^2 = \tau |a_0|^2 + \tau \sum_{j \in \mathbb{Z}} |j| |a_j|^2 < +\infty.$$

The inner product in E is given by

$$\langle z_1, z_2 \rangle = \tau a_0^1 \cdot a_0^2 + \tau \sum_{j \in \mathbb{Z}} |j| a_j^1 \cdot a_j^2 \quad \text{for } z_k = \sum_{j \in \mathbb{Z}} \exp\left(\frac{2j\pi t}{\tau} J\right) a_j^k,\ k = 1, 2.$$

Lemma 2.5 ([33]). *For each $s \in [1, +\infty)$, the space E is compactly embedded in $L^s(S_\tau, \mathbb{R}^{2n})$. In particular, there is a constant $C_s > 0$ such that $\|z\|_{L^s} \le C_s \|z\|$ for all $z \in E$.*

Let $\mathfrak{L}_s(E)$ and $\mathfrak{L}_c(E)$ denote the spaces of bounded self-adjoint linear operator and compact linear operator on E, respectively. For $B(t) \in C\big(S_\tau, \mathfrak{L}_s\big(\mathbb{R}^{2n}\big)\big)$, we define two operators $A, B \in \mathfrak{L}_s(E)$ by extending the bilinear forms:

$$\langle Ax, y\rangle = \int_0^\tau -J\dot{x}(t)\cdot y(t)\,dt, \qquad \langle Bx, y\rangle = \int_0^\tau B(t)x(t)\cdot y(t)\,dt \tag{2.2}$$

on E. Then $B \in \mathfrak{L}_c(E)$. For $m \in \mathbb{N}$, set $E^0 = \mathbb{R}^{2n}$,

$$E_m = \sum_{j=-m}^{m} \exp\left(\frac{2j\pi t}{\tau}J\right)\mathbb{R}^{2n}, \qquad E^{\pm} = \sum_{\pm j>0} \exp\left(\frac{2j\pi t}{\tau}J\right)\mathbb{R}^{2n},$$

and $E_m^+ = E_m \cap E^+$, $E_m^- = E_m \cap E^-$. Obviously, $E = E^+ \oplus E^0 \oplus E^-$ and $E_m = E_m^+ \oplus E^0 \oplus E_m^-$. It is easy to check that E^+, E^0, E^- are respectively the subspaces of E on which A is positive definite, null, and negative definite, and these spaces are orthogonal with respect to A. For $z = z^+ + z^0 + z^-$ with $z^\pm \in E^\pm$ and $z^0 \in E^0$, we have

$$\begin{gathered}\langle Az, z\rangle = \langle Az^+, z^+\rangle + \langle Az^-, z^-\rangle,\\ \|z\|^2 = |z^0|^2 + \frac{1}{2}(\langle Az^+, z^+\rangle - \langle Az^-, z^-\rangle).\end{gathered}$$

Let P_0 be the orthogonal projection from E to E^0 and P_m be the orthogonal projection from E to E_m for $m \in \mathbb{N}$. Then $\{P_m\}_{m=0}^\infty$ is a Galerkin approximation sequence with respect to the operator A.

For $S \in \mathfrak{L}_s(E)$ and $d > 0$, we denote by $M_d^+(S)$, $M_d^-(S)$ and $M_d^0(S)$ the eigenspace corresponding to the eigenvalue belonging to $[d, +\infty)$, $(-\infty, -d]$ and$(-d, d)$, respectively, and denote by $M^+(S)$, $M^-(S)$, and $M^0(S)$, respectively, the positive definite, negative definite, and null subspace of S. Set $S^\sharp = (S|_{\operatorname{Im} S})^{-1}$, and $P_mSP_m \equiv (P_mSP_m)\,|_{E_m}\colon E_m \to E_m$.

In [15], Fei and Qiu studied the relation between the Maslov-type index and the Morse index by the Galerkin approximation method and got the following theorem.

THEOREM 2.6 ([15]). *For $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ with the Maslov-type index (i_τ, ν_τ) and any constant $0 < d \le \|(A-B)^\sharp\|^{-1}/4$, there exists an $m_0 > 0$ such that for $m \ge m_0$, there holds*

$$\begin{aligned}\dim M_d^+(P_m(A-B)P_m) &= \frac{1}{2}\dim E_m - i_\tau - \nu_\tau,\\ \dim M_d^-(P_m(A-B)P_m) &= \frac{1}{2}\dim E_m + i_\tau,\\ \dim M_d^0(P_m(A-B)P_m) &= \nu_\tau,\end{aligned} \tag{2.3}$$

where the operator B is the defined by (2.2) *corresponding to $B(t)$.*

Definition 2.7 ([1], [6]). Let M be a Hilbert manifold. Suppose that Q is a closed q-dimensional ball topologically embedded into M and S is a closed subset such that $\partial Q \cap S = \emptyset$. We say

(a) ∂Q and S homotopically link if $\varphi(Q) \cap S \neq \emptyset$ for each $\varphi \in C(Q, M)$ with $\varphi|_{\partial Q} = \mathrm{id}|_{\partial Q}$.

(b) ∂Q and S homologically link if ∂Q is the support of a non-vanishing homology class in $H_{q-1}(M \setminus S)$.

The following lemma is a new linking structure which is different from those in [28], [34].

Lemma 2.8. *Let* $M = M_1 \oplus M_2$ *be a Hilbert space with* $\dim M_2 = q$ *and* $P_2 \colon M \to M_2$ *be the orthogonal projection. For* $\vartheta > 0$*, let* B_ϑ *be such a bounded linear invertible operator on* M *that* $P_2 B_\vartheta \colon M_2 \to M_2$ *is invertible and* $\vartheta > \|(P_2 B_\vartheta)^{-1}\|$*. Suppose* $S = M_1 + u_0$ *with* $u_0 \in M_2$ *and* $\|u_0\| = 1$*,* $Q = \{B_\vartheta z \mid z \in M_2,\ \|z\| \leq \vartheta\}$*. Then* ∂Q *and* S *are homologically link.*

Proof. By Theorem II.1.2 in [6], we only need to prove that ∂Q and S homotopically link.

First we prove that $Q \cap S \neq \emptyset$. It suffices to prove that $\psi_0(v) = 0$ has a solution in Q, where

$$\psi_0(v) = B_0 B_\vartheta^{-1} v - u_0, \quad v \in Q.$$

Note that $v = B_\vartheta B_0^{-1} u_0$ $(\in Q \setminus \partial Q)$ is the unique solution of ψ_0 in Q, where $B_0 = P_2 B_\vartheta$ and B_0^{-1} denotes the inverse of $B_0|_{M_2}$. Therefore $\partial Q \cap S = \emptyset$, and we have $\deg(\psi_0, Q, 0) = \pm 1$ via the Brouwer's degree.

It remains to show that $\varphi(Q) \cap S \neq \emptyset$, where $\varphi \in C(Q, M)$ with $\varphi|_{\partial Q} = \mathrm{id}|_{\partial Q}$. We define $\psi \colon Q \to M_2$ as

$$\psi(v) = P_2 \varphi(v) - u_0$$

and hence it is sufficient to prove $\psi(v) = 0$ has a solution in Q. Since $\psi = \psi_0$ on ∂Q, by the Brouwer's degree theory, we have $\deg(\psi, Q, 0) = \deg(\psi_0, Q, 0) = \pm 1$. Thus $\psi(v) = 0$ has a solution in Q. □

Let f be a C^2 functional on a Hilbert manifold M. Denote by $D^2 f$ the Hessian of f. Recall that the Morse index $m(x)$ of f at a critical point x is the dimension of a maximal subspace on which $D^2 f(x)$ is strictly negative and the large Morse index $m^*(x)$ of x is $m(x) + \dim\ker D^2 f(x)$.

In order to find the critical points and get the corresponding Morse index estimates, we need the following homologically link theorem which was proved in [1].

Theorem 2.9 ([1]). *Let* M *be a Hilbert manifold. Let* $Q \subset M$ *be a topologically embedded closed* q*-dimensional ball and let* $S \subset M$ *be a closed subset such* *that* $\partial Q \cap S = \emptyset$. *Assume that* ∂Q *and* S *homologically link. Let* $f \in C^2(M)$ *be a function with Fredholm gradient such that*

(a) $\sup_{\partial Q} f < \inf_S f$;

(b) *the functional* f *satisfies* (PS) *condition on some open interval containing* $\left[\inf_S f, \sup_Q f\right]$.

Then, if Γ *denotes the set of all* q*-chains in* M *whose boundary has support* ∂Q, *the number*

$$c := \inf_{\xi \in \Gamma} \sup_{|\xi|} f$$

belongs to $\left[\inf_S f, \sup_Q f\right]$ *and is a critical value of* f, *where* $|\xi|$ *denotes the support of the chain* ξ. *Moreover,* f *has a critical point* $\overline{x}$ *such that* $f(\overline{x}) = c$ *and the following estimate on Morse index and the large Morse index of* $\overline{x}$ *holds*

$$m(\overline{x}) \leq q \leq m^*(\overline{x}).$$

3. Proof of the main results

Firstly, we give a detailed proof of Theorem 1.3, so we assume H satisfies (H1)–(H5). Define

$$G(z) = \int_0^\tau H(t,z)\,dt - \frac{1}{2}\langle Az, z\rangle \quad \text{for } z \in E.$$

By (H4), we have $G \in C^2(E,\mathbb{R})$. Hence looking for τ-periodic solutions of (1.1) is equivalent to looking for critical points of G on E. Moreover, set $G_m = G|_{E_m}$.

LEMMA 3.1. *The functional* G *satisfies* (PS)* *condition w.r.t.* $\{E_j\}_{j\in\mathbb{Z}}$, *i.e. any sequence* $\{z_j\} \subset E$ *satisfying* $z_j \in E_j$, $G_j(z_j)$ *is bounded and* $G_j'(z_j) \to 0$ *as* $j \to +\infty$ *possesses a convergent subsequence in* E.

PROOF. Let $\{z_j\}$ be such a sequence with $|G(z_j)| \leq K_1$ and $G_j'(z_j) \to 0$ as $j \to \infty$, where $K_1 > 0$. It suffices to show that $\{z_j\}$ is bounded. For large j we have

$$\begin{aligned} K_1 + \|z_j\| &\geq G(z_j) - G_j'(z_j)V(z_j) \\ &= \int_0^\tau \big(H(t,z_j) - H_z'(t,z_j)\cdot V(z_j)\big)\,dt \\ &\geq \int_0^\tau \big(c_1|z_j|^\beta - c_2\big)\,dt = c_1\|z_j\|_{L^\beta}^\beta - \tau c_2 \end{aligned} \tag{3.1}$$

via (H3). Then there exists $K_2 > 0$ such that

$$\|z_j\|_{L^\beta} \leq K_2\big(1 + \|z_j\|^{1/\beta}\big). \tag{3.2}$$

Set $z_j = z_j^+ + z_j^0 + z_j^-$, for large j we have

$$\|z_j^+\| \ge \|G_j'(z_j)\cdot z_j^+\| = \bigg|\int_0^\tau \big[H_z'(t,z_j)\cdot z_j^+ - (-J\dot z_j\cdot z_j^+)\big]\,dt\bigg|.$$

By (H4), Hölder's inequality and Lemma 2.5, we obtain

$$\begin{aligned}\|z_j^+\|^2 &= \frac{1}{2}\int_0^\tau -J\dot z_j\cdot z_j^+\,dt \\ &\le \bigg|\int_0^\tau H_z'(t,z_j)\cdot z_j^+\,dt\bigg| + \|z_j^+\| \\ &\le \int_0^\tau c_3(|z_j|^\lambda+1)|z_j^+|\,dt + \|z_j^+\| \\ &\le c_3\bigg(\int_0^\tau |z_j|^\beta\,dt\bigg)^{\lambda/\beta}\bigg(\int_0^\tau |z_j^+|^{\beta/(\beta-\lambda)}\,dt\bigg)^{(\beta-\lambda)/\beta} \\ &\quad + c_3\|z_j^+\|_{L^1} + \|z_j^+\| \\ &\le K_3(1+\|z_j\|_{L^\beta}^\lambda)\|z_j^+\|,\end{aligned}\tag{3.3}$$

where $c_3, K_3 > 0$ are suitable constants. Combining (3.2) and (3.3), we have

$$\|z_j^+\| \le K_4\big(1+\|z_j\|^{\lambda/\beta}\big),\tag{3.4}$$

where $K_4 > 0$. Similarly, we have

$$\|z_j^-\| \le K_4\big(1+\|z_j\|^{\lambda/\beta}\big).\tag{3.5}$$

Next we estimate the boundedness of $\{z_j^0\}$. Set $\widehat z_j = z_j - z_j^0 = z_j^+ + z_j^-$. Since $\lambda < \beta$, by (H4), (3.4), (3.5) and Lemma 2.5, we obtain

$$\begin{aligned}&\bigg|\int_0^\tau \big[H(t,z_j) - H\big(t,z_j^0\big)\big]\,dt\bigg| \\ &\quad = \bigg|\int_0^\tau\int_0^1 H_z'(t,z_j^0+s\widehat z_j)\cdot\widehat z_j\,ds\,dt\bigg| \\ &\quad \le \int_0^\tau 2^\lambda c_3\big(|z_j^0|^\lambda + |\widehat z_j|^\lambda + 1\big)|\widehat z_j|\,dt \le K_5\big(1+\|z_j\|^{\lambda+\lambda/\beta}\big),\end{aligned}\tag{3.6}$$

where $K_5 > 0$ is a suitable constant. From (3.4)–(3.6), we see

$$\begin{aligned}\int_0^\tau H\big(t,z_j^0\big)\,dt = G(z_j) + \frac{1}{2}\langle Az_j,z_j\rangle - \int_0^\tau \big[H(t,z_j) - H\big(t,z_j^0\big)\big]\,dt \\ \le K_6\big(1+\|z_j\|^{\lambda+\lambda/\beta}\big),\end{aligned}\tag{3.7}$$

where $K_6 > 0$. From (H3), it follows that

$$\int_0^\tau H\big(t,z_j^0\big)\,dt \ge \int_0^\tau \big(c_1|z_j^0|^\beta - c_2\big)\,dt.\tag{3.8}$$

From (3.7) and (3.8), we see that

$$\big|z_j^0\big| \le K_7\big(1+\|z_j\|^{(\lambda+\lambda\beta)/\beta^2}\big),\tag{3.9}$$

where $K_7 > 0$. From (H4), (3.4), (3.5) and (3.9), we see $\{z_j\}$ is bounded. $\square$

Choose a constant $\varrho > 0$ such that $\widetilde{\alpha}_i = \varrho\alpha_i/(\alpha_i+\beta_i) \geq 1$ and $\widetilde{\beta}_i = \varrho\beta_i/(\alpha_i+\beta_i) \geq 1$, where α_i and β_i are given by (H2). For $\rho > 0$ and $z = (p_1,\ldots,p_n,q_1,\ldots,q_n) \in E$, we define an operator $B_\rho\colon E \to E$ by

$$B_\rho z = \Big(\rho^{\widetilde{\beta}_1-1}p_1,\ldots,\rho^{\widetilde{\beta}_n-1}p_n,\rho^{\widetilde{\alpha}_1-1}q_1,\ldots,\rho^{\widetilde{\alpha}_n-1}q_n\Big).$$

It is easy to see that B_ρ is a linear bounded and invertible operator and $\|B_\rho\| \leq 1$ if $\rho \leq 1$.

For $z = z^+ + z^0 + z^- \in E$, we have

$$\langle AB_\rho z, B_\rho z\rangle = \rho^{\varrho-2}\langle Az, z\rangle = \rho^{\varrho-2}\big(\|z^+\|^2 - \|z^-\|^2\big). \tag{3.10}$$

Set $X_m = E_m^- \oplus E^0$ and $Y_m = E_m^+$. For $u_0 \in Y_1$ with $\|u_0\| = 1$, define $S = E^- \oplus E^0 + u_0$ and $Q = \{B_\vartheta z \mid z \in E^+,\ \|z\| \leq \vartheta\}$, where $\vartheta > 0$ is determined below.

Lemma 3.2. *There exists $\vartheta > 1$ large enough such that $\sup_{\partial Q} G < 0$.*

Proof. By (H2), for any $\varepsilon > 0$, there exists M_ε such that

$$H(t,z) \leq \varepsilon\sum_{i=1}^n \big(|p_i|^{1+\alpha_i/\beta_i} + |q_i|^{1+\beta_i/\alpha_i}\big) + M_\varepsilon, \quad (t,z) \in \mathbb{R}\times\mathbb{R}^{2n}. \tag{3.11}$$

For $B_\vartheta z \in \partial Q$, from (3.10) and (3.11), we have

$$\begin{aligned}G(B_\vartheta z) &= \int_0^\tau H(t,B_\vartheta z) - \frac{1}{2}\,\langle AB_\vartheta z, B_\vartheta z\rangle \\ &\leq \varepsilon\sum_{i=1}^n\int_0^\tau \Big(\vartheta^{(\widetilde{\beta}_i-1)(1+\alpha_i/\beta_i)}|p_i|^{1+\alpha_i/\beta_i} + \vartheta^{(\widetilde{\alpha_i}-1)(1+\beta_i/\alpha_i)}|q_i|^{1+\beta_i/\alpha_i}\Big)\,dt \\ &\qquad + M_\varepsilon\tau - \vartheta^\varrho \leq (2n\varepsilon K_8 - 1)\vartheta^\varrho + M_\varepsilon\tau,\end{aligned} \tag{3.12}$$

where $K_8 > 0$ is the embedding constant.

Choose $\varepsilon > 0$ such that $2n\varepsilon K_8 < 1$, then for $\vartheta > 1$ large enough, we have $\sup_{\partial Q} G < 0$. $\square$

Lemma 3.3. *There exists $\tau_0 > 0$ such that for $\tau \geq \tau_0$, there holds $\inf_S G \geq 1$.*

Proof. For $z \in S$, we have

$$\begin{aligned}G(z) &= \int_0^\tau H(t,z)\,dt + \|z^-\|^2 - \|u_0\|^2 \\ &= \lambda\int_0^{2\pi} H(\lambda t, z(\lambda t))\,dt + \|z^-\|^2 - \|u_0\|^2,\end{aligned}$$

where $\lambda = \tau/(2\pi)$. Then the proof follows the lines of [3], [17] by (H5). $\square$

Lemma 3.4. *For $\vartheta > 1$, we have ∂Q_m and S_m homologically link.*

PROOF. Since $\vartheta > 1$, $\vartheta > \|B_{\vartheta}^{-1}\| = \|B_{1/\vartheta}\|$. Let $P\colon E \to E^+$ denote the orthogonal projection. Then $PB_{\vartheta}\colon E^+ \to E^+$ is linear bounded and invertible (see [2], [16]). Note that $B_{\vartheta}(E_m) \subset E_m$ and $B_{\vartheta}|_{E_m} : E_m \to E_m$ is linear bounded and invertible. Then $\widetilde{P}_m B_{\vartheta}|_{E_m} : Y_m \to Y_m$ is also linear bounded and invertible, where $\widetilde{P}_m\colon E_m \to Y_m$ is the orthogonal projection. The assertion follows by Lemma 2.8. $\square$

THEOREM 3.5. *Suppose H satisfies* (H1)–(H5). *Then there exists $\tau_0 > 0$ such that for $\tau \geq \tau_0$, the system* (1.1) *possesses a nontrivial τ-periodic solution z satisfying*

$$i_{\tau}(z) \leq n \leq i_{\tau}(z) + \nu_{\tau}(z). \tag{3.13}$$

PROOF. For any $m \in \mathbb{N}$, Lemma 3.1 shows that G_m satisfies (PS) condition. And Lemmas 3.2–3.4 show that G_m satisfies other hypotheses of Theorem 2.9. Then G_m has a critical point z_m satisfying

$$\inf_S G \leq G(z_m) \leq \sup_Q G \quad \text{and} \quad m(z_m) \leq \dim Y_m \leq m^*(z_m). \tag{3.14}$$

By Lemmata 3.1 and 3.3, when $\tau \geq \tau_0$, we may assume $z_m \to z \in E$ as $m \to +\infty$ with $G(z) \geq 1$ and $G'(z) = 0$. By (H5), the only trivial solution of the system (1.1) is $z(t) \equiv 0$. Therefore, for $\tau \geq \tau_0$, the critical point z of G is a classical nontrivial τ-periodic solution of the system (1.1).

Now we show (3.13) holds. The proof is similar to that in [22]. Let B be the operator for $B(t) = H''_{zz}(t, z(t))$ defined by (2.2). Direct computation implies

$$\|G''(x) - (B - A)\| \to 0 \quad \text{as } \|x - z\| \to 0, \quad x \in E. \tag{3.15}$$

Let $d = \|(A - B)^{\sharp}\|^{-1}/4$. By (3.15), there exists $r_0 > 0$ such that

$$\|G''(x) - (B - A)\| < \frac{d}{2}, \quad x \in V_{r_0} = \{x \in E \mid \|x - z\| \leq r_0\}.$$

Hence, for m large enough, there holds

$$\|G''_m(x) - P_m(B - A)P_m\| < \frac{d}{2}, \quad x \in V_{r_0} \cap E_m. \tag{3.16}$$

For $x \in V_{r_0} \cap E_m$ and $w \in M_d^-(P_m(B - A)P_m) \setminus \{0\}$, (3.16) implies that

$$\langle G''_m(x)w, w\rangle \leq \langle P_m(B - A)P_m w, w\rangle + \|G''_m(x) - P_m(B - A)P_m\| \cdot \|w\|^2 \leq -\frac{d}{2}\|w\|^2 < 0.$$

Then

$$\dim M^-(G''_m(x)) \geq \dim M_d^-(P_m(B - A)P_m), \quad x \in V_{r_0} \cap E_m. \tag{3.17}$$

Similarly, we have

$$\dim M^+(G''_m(x)) \geq \dim M_d^+(P_m(B - A)P_m), \quad x \in V_{r_0} \cap E_m. \tag{3.18}$$

Note that

$$\begin{aligned}\dim M_d^-(P_m(B-A)P_m) &= \dim M_d^+(P_m(A-B)P_m),\\ \dim M_d^0(P_m(B-A)P_m) &= \dim M_d^0(P_m(A-B)P_m).\end{aligned}\tag{3.19}$$

By (3.14), (3.17)–(3.19) and Theorem 2.6, for large m, we have

$$\begin{aligned}\frac{1}{2}\dim E_m - n = \dim Y_m &\geq m(z_m)\\ &\geq \dim M_d^+(P_m(A-B)P_m) = \frac{1}{2}\dim E_m - i_\tau(z) - \nu_\tau(z)\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}\frac{1}{2}\dim E_m - n &\leq m^*(z_m)\\ &\leq \dim M_d^-(P_m(B-A)P_m) + \dim M_d^0(P_m(B-A)P_m)\\ &= \frac{1}{2}\dim E_m - i_\tau(z).\end{aligned}\tag{3.21}$$

Thus we obtain (3.13) by (3.20) and (3.21). □

Proof of Theorem 1.3. Since H is $k\tau$-periodic, Theorem 3.5 implies the system (1.1) possesses a nontrivial $k\tau$-periodic solution z_k satisfying

$$i_{k\tau}(z_k) \leq n \leq i_{k\tau}(z_k) + \nu_{k\tau}(z_k).\tag{3.22}$$

If z_k and z_{pk} are not geometrically distinct, then there exist integers l and m such that $l * z_k = m * z_{pk}$ by definition. It follows from Proposition 2.1 that

$$\begin{aligned}i_{k\tau}(l * z_k) &= i_{k\tau}(z_k), & \nu_{k\tau}(l * z_k) &= \nu_{k\tau}(z_k),\\ i_{pk\tau}(m * z_{pk}) &= i_{pk\tau}(z_{pk}), & \nu_{pk\tau}(m * z_{pk}) &= \nu_{pk\tau}(z_{pk}).\end{aligned}$$

By (3.22), we have $i_{pk\tau}(z_{pk}) \leq n$ and $i_{k\tau}(z_k) + \nu_{k\tau}(z_k) > n$. Proposition 2.2 shows that

$$p \leq \frac{2n}{i_{k\tau}(z_k) + \nu_{k\tau}(z_k) - n}$$

which contradicts with the assumption

$$p > \frac{2n}{i_{k\tau}(z_k) + \nu_{k\tau}(z_k) - n}.$$ □

Proof of Theorem 1.5. At the moment, $H(t,z)$ is defined by

$$H(t,z) = \frac{1}{2}\big(\widehat{B}(t)z, z\big) + \widehat{H}(t,z).$$

Since (H6) and (H7) hold, we have

$$G(z) - G'(z)V(z) = \int_0^\tau (\widehat{H}(t,z) - \widehat{H}_z'(t,z)\cdot V(z))\,dt$$

and

$$\big\langle \widehat{B}B_\vartheta z, B_\vartheta z\big\rangle = \vartheta^{\varrho-2}\langle \widehat{B}z, z\rangle, \quad z \in E,$$

where B_ϑ is the operator defined by $B_\vartheta(t)$, B_ϑ for $\vartheta \in \{\rho_m\}$ is defined as in this section and then we can complete the proof by applying the same arguments as above. □

Proof of Theorem 1.6. By Theorem 3.5, there exists $\tau_0 > 0$ such that for any $\tau \geq \tau_0$, the system (1.2) possesses a nontrivial τ-periodic solution z with

$$i_\tau(z) \leq n. \tag{3.23}$$

The rest proof is almost the same as that in [25]. For readers' convenience, we estimate the iteration number of the solution (z, τ).

Suppose (z, τ) has minimal period τ/k, i.e. its iteration number is $k \in \mathbb{N}$. Since the Hamiltonian system in (1.2) is autonomous and the condition (H8) holds, it follows by Proposition 2.3 that

$$\nu_{\tau/k}(z) \geq 1 \quad \text{and} \quad i_{\tau/k}(z) \geq n. \tag{3.24}$$

Thus by (3.23), (3.24) and Proposition 2.4, we obtain $k = 1$ and complete the proof (see [28]). □

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