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FRACTIONAL STOCHASTIC EVOLUTION HEMIVARIATIONAL INEQUALITIES AND OPTIMAL CONTROLS

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Abstract. This paper investigates the existence of mild solutions for fractional stochastic evolution hemivariational inequalities and optimal controls. An existence theorem concerned with the mild solution for the presented system is proved by means of the fractional calculation, stochastic analysis theory, Bohnenblust–Karlin fixed point theorem and some properties of the Clarke subdifferential. Moreover, an existence result of optimal control pair that governed by a fractional stochastic evolution hemivariational inequality is also obtained. Finally, an example is given for demonstration.

1. Introduction

It is well known that the hemivariational inequality (HVI) was first introduced by Panagiotopoulos in 1981 as weak formulations for several classes of mechanical problems involving nonsmooth and nonconvex energy functionals (superpotentials) [34], [35]. Recently, many researchers have paid a lot of attentions on the control problems of HVIs. In 2000, Migórski and Ochal [30] discussed the optimal control problems of the parabolic HVIs via Galerkin method combined

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with a regularization procedure and the direct method of the calculus of variations. In 2007, Park and Park [37] proved the existence of optimal control pairs to a hyperbolic quasi-linear HVI by using the Faedo–Galerkin method and the direct method of the calculus of variations. Recently, Liu and Li [24] investigated the approximate controllability for HVIs by applying a fixed-point theorem of multi-valued maps. Very recently, Muthukumar et al. [33] applied fixed point theorem of multi-valued maps and Balder's theorem to deal with the optimal control problem of second order stochastic evolution HVIs with Poisson jumps. Harrat et al. [14] applied fractional calculus, fixed point technique, semigroup theory and multivalued analysis to study solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential.

On the other hand, fractional differential equations (FDEs) have been a hot research topic due to their application for control theory, chemistry, engineering, biology, mechanics, physics, viscoelasticity materials, electrical circuits, neural networks and so on ([39], [15], [20], [9]). We note that FDEs also serve as an excellent tool for the description of certain materials and processes [15], [20] and there has been a significant development in FDEs over the past three decades (see, for example, [5], [17]–[19], [21], [26], [46]). It is worth mentioning that the study of fractional stochastic evolution equations (FSEEs) and their control problems has become an active area of investigation because random environmental effects should be considered in some real models. Recently, the solvability, controllability, optimal control problems with applications have been studied extensively for various FSEEs in the literature; for instance, we refer the reader to [1], [13], [25], [41]–[43], [45] and the references therein.

As mentioned above, great accomplishments have been made in the solvability, controllability and optimal control problems of HVIs and FSEEs. It would be interesting to study what happens to an optimal control problem if it does not only include HVIs but also includes FSEEs. However, up to now, the solvability and optimal controls for the fractional stochastic evolution hemivariational inequality (FSEHVI) are still untreated topics in the literature. In this paper, we will focus on the solvability and optimal controls of the following control system governed by FSEHVI:

$$
\begin{cases}
\Big\langle -{}^cD_t^\alpha x(t) + Ax(t) + Bu(t) + g(t)\dfrac{dw(t)}{dt}, v\Big\rangle + J^0(t, x(t); v) \geq 0, \\
\hspace{18em} \text{for all } v \in H,\ t \in I = [0, b], \\
x(t) = x_0,
\end{cases}
\tag{1.1}
$$

where ${}^cD_t^\alpha$ denotes the Caputo fractional time derivative of order $0 < \alpha \leq 1$, H is a separable Hilbert space with the inner product $\langle \cdot\,, \cdot \rangle$, $A\colon D(A) \subseteq H \to H$ is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$

on H (this implies that there exists $M > 0$ such that $\sup_{t\in[0,\infty} \|T(t)\| \leq M$), B is a continuous linear operators from a separable Hilbert space Y to H, the control function u takes values in Y, $g(t) \in G(t, x(t))$ ($G(t, x(t))$ is a multivalued function to be specified later), and $J^0(t, \cdot, \cdot)$ denotes the generalized directional derivative (in the sense of Clarke) for a locally Lipschitz functional $J\colon H \to \mathbb{R}$. Let K be another separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_K$ and norm $\|\cdot\|_K$, $\{w(t)\}_{t\geq 0}$ a given K-valued Brownian motion or Wiener process with finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \Gamma, \{\Gamma_t\}_{t\geq 0}, \mathbb{P})$ equipped with a normal filtration $\{\Gamma_t\}_{t\geq 0}$, which is generated by the Wiener process w; and x_0 a Γ_0 measurable H-valued random variable independent of w. Let $\mathbb{E}$ denote the expectation of a random variable or the Lebesgue integral with respect to the probability measure $\mathbb{P}$ and $\mathcal{A}_{\text{ad}}$ be a set of all admissible state control pairs (x, u). The cost functional on the set $\mathcal{A}_{\text{ad}}$ is given by

$$\mathcal{J}(x, u) = \mathbb{E} \int_0^b \mathcal{L}(t, x^u(t), u(t))\, dt. \tag{1.2}$$

The plan of this paper is organized into six sections. We list some necessary preliminaries in Section 2. We prove that problem (1.1) has a mild solution under some suitable assumptions in Section 3. We treat the optimal control problem governed by (1.1) under mild conditions in Section 4. An example is given to illustrate our main results in Section 5. Finally, we provide some concluding remarks in Section 6.

2. Preliminaries

Following [40], let $(\Omega, \Gamma, \{\Gamma_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a normal filtration $\{\Gamma_t\}_{t\geq 0}$, in which a filtration $\{\Gamma_t\}_{t\in[0,b]}$ is said to be normal if Γ_0 contains all sets $A \in \Gamma$ with $\mathbb{P}(A) = 0$ and

$$\Gamma_t = \Gamma_{t+} = \bigcap_{s>t} \Gamma_s, \quad \Gamma = \Gamma_b.$$

Let $w = \{w(t)\}_{t\geq 0}$ denote a K-valued Q-Wiener process definedc on space $(\Omega, \Gamma, \{\Gamma_t\}_{t\geq 0}, \mathbb{P})$ with covariance operator Q, that is,

$$\mathbb{E}\langle w(t), x\rangle_K \langle w(s), y\rangle_K = (t \wedge s)\langle Qx, y\rangle_K, \quad \text{for all } x, y \in K,$$

where Q is a positive, self-adjoint, and trace class operator on K.

In order to define the stochastic integral with respect to the Q-Wiener process w, we need to introduce a subspace $K_0 = Q^{1/2}(K)$ of K which together with the inner product

$$\langle u, v\rangle_{K_0} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_K$$

becomes a Hilbert space. Assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $Qe_n = \lambda_n e_n$, and a sequence β_n of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n, \quad e \in K,\ t \in [0, b],$$

and $\Gamma_t = \Gamma_t^w$, where Γ_t^w is the σ-algebra generated by $\{w(s) : 0 \le s \le t\}$.

Let $L(K, H)$ be the space of all bounded linear operators from K to H and $L(H)$ denote the Hilbert space of all bounded linear operators mapping H into H. Let $L_0^2 = L_2(K_0, H)$ be a space of all Hilbert–Schmidt operators from K_0 to H endowed with a norm

$$\|\varphi\|_{L_0^2}^2 = \mathrm{Tr}\big(\big(\varphi Q^{1/2}\big)\big(\varphi Q^{1/2}\big)^*\big), \quad \text{for all } \varphi \in L_0^2.$$

Obviously, for any bounded operators $\varphi \in L(K, H)$, this norm reduces to $\|\varphi\|_{L_0^2}^2 = \mathrm{Tr}(\varphi Q \varphi^*)$. Let $L^p(\Gamma_b, H)$ be a Banach space of all Γ_b-measurable p-th power integrable random variables with values in the Hilbert space H and $L^p(\Gamma, H)$ a Banach space of all strongly Γ-measurable and H-valued random variables satisfying $\mathbb{E}\|x\|_H^p < \infty$. For each $t \ge 0$, since the sub-σ-algebra Γ_t is complete and $L^p(\Gamma_t, H)$ is a closed subspace of $L^p(\Gamma, H)$. This shows that $L^p(\Gamma_t, H)$ is a Banach space. Let $C(I, L^p(\Gamma, H))$ be the Banach space of all continuous maps from I into $L^p(\Gamma, H)$ with the norm

$$\|x\| = \Big(\sup_{t\in I} \mathbb{E}\|x(t)\|_H^p\Big)^{1/p}.$$

In what follows, for any operator $B \in L_\infty(I, L(Y, H))$, let $\|B\|_\infty$ denote the norm of operator B on Banach space $L_\infty(I, L(Y, H))$, where $L_\infty(I, L(Y, H))$ stands for the space of operator valued functions which are measurable in the strong operator topology and uniformly bounded on the interval I. Let $L_\Gamma^p(I, Y)$ be a closed subspace of $L_\Gamma^p(I \times \Omega, Y)$, consisting of all measurable and Γ_t-adapted, Y-valued stochastic processes satisfying the condition

$$\mathbb{E}\int_0^b \|u(t)\|_Y^p\,dt < \infty$$

and equipped with the norm

$$\|u\|_{L_\Gamma^p(I,Y)} = \bigg(\mathbb{E}\int_0^b \|u(t)\|_Y^p\,dt\bigg)^{1/p}.$$

For more details, we refer the reader to [40].

In what follows, we need some concepts of the fractional calculus theory.

Definition 2.1 ([20], [39]). The fractional integral of order α with lower limit zero for a function $x(t)$ is defined as

$$I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds, \quad \alpha > 0, \quad t > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt.$$

Definition 2.2 ([20], [39]). The Riemann–Liouville fractional derivative of order α with the lower limit zero for a function $x\colon [0,\infty) \to \mathbb{R}$ is defined as follows

$${}^{L}D_t^{\alpha} x = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} x(s)\, ds, \quad t > 0,\ 0 \le n-1 < \alpha < n.$$

Definition 2.3 ([20], [39]). The Caputo fractional derivative of order α with the lower limit zero for a function $x\colon [0,\infty) \to \mathbb{R}$ is defined by

$${}^{c}D_t^{\alpha} x = {}^{L}D_t^{\alpha} \bigg(x(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} x^{(k)}(0) \bigg), \quad t > 0,\ 0 \le n-1 < \alpha < n.$$

Remark 2.4 ([20], [39]).

(a) If $x \in C^n[0,\infty)$, then

$${}^{c}D_t^{\alpha} x = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds = I_t^{n-\alpha} x^{(n)}(t),$$

for $t > 0$, $0 \le n-1 < \alpha < n$;

(b) The Caputo derivative of a constant is equal to zero;

(c) If x is an abstract function which has values in H, then integrals which appear in Definitions 2.1–2.2 are taken in Bochner's sense (for Bochner's Theorem, one can see [8, p. 366]).

For a Banach space X, we use the following notations.

$\mathcal{P}_{\mathrm{bd,cl,cv,cp}}(X) = \{D \subseteq X : D$ is nonempty, bounded, closed, convex, compact$\}$.

Definition 2.5 ([7]). Let X be a Banach space. A multi-valued map $\mathcal{F}\colon X \to \mathcal{P}(X)$ is said to

(a) have convex valued if $\mathcal{F}(x)$ is convex for every $x \in X$;

(b) be bounded on the bounded sets if $\mathcal{F}(D) = \bigcup_{x \in D} \mathcal{F}(x)$ is bounded in X for all $D \in \mathcal{P}_{bd}(X)$, i.e. $\sup_{x\in D}\{\sup\{\|y\| : y \in \mathcal{F}(x)\}\} < \infty$;

(c) be upper semicontinuous (for short u.s.c.) at $x_0 \in X$, if for any neighbourhood $O(\mathcal{F}(x_0))$ of $\mathcal{F}(x_0)$, there exists a neighbourhood; $O(x_0)$ of x_0 such that $\mathcal{F}(x) \subset O(\mathcal{F}(x_0))$ for all $x \in O(x_0)$;

(d) be compact if $\mathcal{F}(D)$ is relatively compact for every any bounded subset D of V;
(e) have a fixed point if there is an element $x \in X$ such that $x \in \mathcal{F}(x)$.

Definition 2.6 ([7]). Let X and Y be two Banach spaces. A multi-valued map $\mathcal{F}\colon X \to \mathcal{P}(Y)$ is said to be closed if its graph $\mathrm{Gr}(\mathcal{F}) = \{(x, y) \in X \times Y$ $x \in X,\ y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.7 ([7]). A multi-valued map $\mathcal{F}\colon I \to \mathcal{P}_{(\mathrm{bd})(\mathrm{cl})(\mathrm{cv})}(H)$ is said to be measurable if, for each $x \in H$, the function $\vartheta\colon I \to \mathbb{R}$, defined by

$$\vartheta(t) = d(x, \mathcal{F}(t)) = \inf\{\|x - z\| : z \in \mathcal{F}(t)\},$$

belongs to $L^1(I, \mathbb{R})$.

Definition 2.8. A multi-valued map $\mathcal{F}\colon I \times H \to \mathcal{P}_{(\mathrm{bd})(\mathrm{cl})(\mathrm{cv})}(H)$ is said to be L^2-Carathéodory if

(a) $t \mapsto \mathcal{F}(t, v)$ is measurable for each $v \in H$;
(b) $v \mapsto \mathcal{F}(t, v)$ is upper semicontinuous for almost all $t \in I$;
(c) for each $m > 0$, there exists $h_m \in L^1(I, \mathbb{R}^+)$such that

$$\|\mathcal{F}(t, v)\|_{L_0^2} = \sup_{f \in \mathcal{F}(t,v)} \mathbb{E}\|f\|^2_{L_0^2} \leq h_m(t),$$

for all $\|v\|^2 \leq m$ and for almost every $t \in I$.

Definition 2.9 ([4], [32]). Let X be a Banach space with the dual space X^* and $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional on X. The generalized directional derivative (in the sense of Clarke) of J at the point $x \in X$ in the direction $v \in X$, denoted by $J^0(x; v)$, is defined by

$$J^0(x; v) = \lim_{\lambda \to 0^+} \sup_{y \to x} \frac{J(y + \lambda v) - J(y)}{\lambda}.$$

The Clarke subdifferential or generalized gradient in the sense of Clarke of J at $x \in X$, denoted by $\partial J(x)$, is a subset of X^* given by

$$\partial J(x) = \big\{x^* \in X^* : J^0(x; v) \geq \langle x^*, v\rangle,\ \text{for all } v \in X\big\}.$$

Lemma 2.10 ([11, Proposition 14.5]). *Let X and V be two Hausdorff topological spaces. Assume that $\mathcal{F}\colon X \to \mathcal{P}(V)$ is a multi-valued map such that $\mathcal{F}(X) \subset K$ and the graph of $\mathcal{F}$ is closed, where K is a compact set. Then $\mathcal{F}$ is u.s.c.*

Lemma 2.11 ([3]). *Let $(X, \|\cdot\|)$ be a Banach space and $D \in \mathcal{P}_{\mathrm{bd,cl,cv}}(X)$. Suppose that $\mathcal{F}\colon D \to \mathcal{P}_{\mathrm{bd,cl,cv}}(D)$ is a u.s.c. multi-valued map and $\mathcal{F}(D)$ is relatively compact in X. Then $\mathcal{F}$ admits a fixed point in D.*

3. Existence of mild solutions

In this section, we study the existence of mild solutions for problem (1.1). To this end, we convert to consider the following fractional semilinear stochastic inclusion:

$$\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + Bu(t) + g(t)\,\dfrac{dw(t)}{dt} + \partial J(t, x(t)), \quad t \in I = [0, b], \\ x(t) = x_0, \end{cases} \tag{3.1}$$

where the symbol ∂J stands for the Clarke subdifferential of a locally Lipschitz functional $J(t, \cdot\,)\colon H \to \mathbb{R}$, the control function $u(\,\cdot\,)$ is a stochastic process given in $L^p_\Gamma(I, Y)$ of admissible control functions, Y is a Hilbert space, $B\colon Y \to H$ is a continuous linear operator, $g(t) \in G(t, x(t))$ ($G(t, x(t))$ is a multivalued function to be specified later), and x_0 is a Γ_0 measurable H-valued random variable independent of w.

We show that every solution of (3.1) is also a solution of (1.1). For each $x \in H$, let us define the set of selections of G by

$$g \in \mathcal{S}_{G,x} = \big\{g \in L^2([0, b], L^2_0) : g(t) \in G(t, x(t)) \ \text{ for a.e. } t \in I\big\}.$$

We say that $x \in C(I, L^2(\Gamma, H))$ is a solution of (3.1) if there exist a selection $g \in \mathcal{S}_{G,x}$ of $G(t, x)$ and a function $\eta(t) \in \partial J(t, x(t))$ such that $\eta \in L^2_\Gamma(I, H)$ and

$$\begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + Bu(t) + g(t)\,\dfrac{dw(t)}{dt} + \eta(t), \quad t \in I = [0, b], \\ x(t) = x_0, \end{cases}$$

which implies that

$$\begin{cases} \left\langle -\,{}^cD_t^\alpha x(t) + Ax(t) + Bu(t) + g(t)\,\dfrac{dw(t)}{dt}, v \right\rangle_H + \langle \eta(t), v\rangle_H = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in I, \ \text{for all } v \in H, \\ x(t) = x_0. \end{cases}$$

Since $\eta(t) \in \partial J(t, x(t))$ and $\langle \eta(t), v\rangle_H \le J^0(t, x(t); v)$, we get

$$\begin{cases} \left\langle -\,{}^cD_t^\alpha x(t) + Ax(t) + Bu(t) + g(t)\,\dfrac{dw(t)}{dt}, v \right\rangle_H + J^0(t, x(t); v) \ge 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in I, \ \text{for all } v \in H, \\ x(t) = x_0. \end{cases}$$

Therefore, in order to consider problem (1.1), it suffices to discuss problem (3.1).

Based on the work [46], [10], [38], we introduce definition as follows.

Definition 3.1. For each $u \in L^p_\Gamma(I, Y)$, a Γ_t-adapted stochastic process $x \in C(I, L^2(\Gamma, H))$ is called a mild solution of problem (3.1) if $x(0) = x_0 \in H$

and there exist a selection $g \in \mathcal{S}_{G,x}$ of $G(t,x)$ and a $\eta(t) \in \partial J(t,x(t))$ such that $\eta \in L^2_\Gamma(I,H)$ a.e. $t \in I$ and

$$x(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\big[Bu(s) + \eta(s)\big]\,ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g(s)\,dw(s) \tag{3.2}$$

for $t \in I$, where

$$P_\alpha(t) = \int_0^\infty \psi_\alpha(\theta) T(t^\alpha \theta)\,d\theta, \qquad S_\alpha(t) = \alpha \int_0^\infty \theta \psi_\alpha(\theta) T(t^\alpha \theta)\,d\theta$$

with

$$\psi_\alpha(\theta) = \frac{1}{\alpha}\,\theta^{-1-1/\alpha}\,\varpi_\alpha(\theta^{-1/\alpha}) \geq 0,$$
$$\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1}\,\frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

and ψ_α is a probability density function defined on $(0,\infty)$, i.e.

$$\psi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \psi_\alpha(\theta)\,d\theta = 1.$$

Thanks to [29], direct calculation yields that

$$\int_0^\infty \theta \psi_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

Lemma 3.2 ([46, Lemmas 3.2–3.4]). *The operators P_α and S_α admit the following properties*:

(a) *For any fixed $t \geq 0$, $P_\alpha(t)$ and $S_\alpha(t)$ are bounded linear operators such that, for any $x \in H$,*

$$\|P_\alpha x\| \leq M\|x\|, \qquad \|S_\alpha x\| \leq \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|;$$

(b) *$\{P_\alpha(t), t \geq 0\}$ and $\{S_\alpha(t), t \geq 0\}$ are strongly continuous*;

(c) *If $T(t)$ is compact, then for any $t > 0$, $P_\alpha(t)$ and $S_\alpha(t)$ are also compact operators .*

Lemma 3.3 ([38, Theorem 2.3.2]). *If $\{T(t)\}_{t\geq 0}$ is a compact C_0-semigroup for $t > 0$, then it is uniformly continuous for $t > 0$.*

Remark 3.4. A semigroup $T(t)$ is uniformly continuous if $\lim\limits_{|s-t|\to 0} \|T(t) - T(s)\| = 0$. Lemma 3.2 together with Lemma 3.5 proves that $P_\alpha(t)$ and $S_\alpha(t)$ are uniformly continuous provided $T(t)$ is compact for $t > 0$.

We also make use of the following hypotheses.

(HT) $T(t)$ is a compact operator for every $t > 0$;

(HG) G is an L^2-Carathéodory multi-valued map satisfying the following conditions:

(G1) for each $t \in I$, the function $G(t, \cdot)\colon H \to P_{\mathrm{bd,cl,cv}}(L(K,H))$ is u.s.c. such that, for each $x \in H$, the function $G(\cdot, x)$ is measurable, and each fixed $x \in C(I; L^2(\Gamma, H))$, the set

$$\mathcal{S}_{G,x} = \{g \in L^2(I, L_0^2) : g(t) \in G(t, x(t)) \text{ for a.e. } t \in I\}$$

is nonempty;

(G2) For each positive $k > 0$, there exists a positive function $M_g(k)$ independent on k such that

$$\sup_{\mathbb{E}\|x\|^2 \le k} \|G(t,x)\|^2 \le M_g(k), \quad \text{for a.e. } t \in I,$$

where $\|G(t,x)\|^2 = \sup_{g \in G(t,x)} \mathbb{E}\|g\|^2$;

(HB) The control u takes values in Y, for any operator $B \in L_\infty(I, L(Y,H))$, $\|B\|_\infty$ is the norm of operator B on the Banach space $L_\infty(I, L(Y,H))$;

(HJ) The functional $J\colon I \times H \to \mathbb{R}$ satisfies the following conditions:

(J1) $J(\cdot, x)\colon I \to \mathbb{R}$ is measurable for each $x \in H$;

(J2) $J(t, \cdot)\colon H \to \mathbb{R}$ is locally Lipschitz continuous for a.e. $t \in I$;

(J3) there exist a function $\zeta \in L^2(I, \mathbb{R}^+)$ and a constant $c > 0$ satisfying

$$\|\partial J(t,x)\|^2 = \sup\{\|y\|^2 : y \in \partial J(t,x)\} \le \zeta(t) + c\|x\|^2,$$

for all $x \in H$ and for almost every$t \in I$;

(HU) $U(\cdot)\colon I \to \mathcal{P}(Y)$ is a multi-valued map with bounded, closed, and convex values such that $U(\cdot)$ is graph measurable and $U(\cdot) \subseteq \Omega$, where $\Omega \subset Y$ is bounded.

Define the admissible set as follows:

$$U_{\mathrm{ad}} = \{u(\cdot) \in L_\Gamma^p(I, Y); u(t) \in U(t) \text{ a.e. } t \in I\}.$$

Then, by Proposition 2.1.7 and Lemma 2.3.2 of [16], we know that $U_{\mathrm{ad}} \neq \emptyset$ and U_{ad} is bounded, convex, and closed subset of $L^p(I,Y)$ with $1 < p < \infty$. Clearly, $Bu \in L^p(I,H)$ for all $u \in U_{\mathrm{ad}}$. Define an operator $\Psi\colon L_\Gamma^2(I,H) \to 2^{L_\Gamma^2(I,H)}$ as follows:

$$\Psi(x) = \{\eta \in L_\Gamma^2(I,H) : \eta(t) \in \partial J(t,x) \text{ a.e. } t \in I\}, \quad \text{for all } x \in L_\Gamma^2(I,H).$$

LEMMA 3.5 ([25, Lemma 3.3]). *If condition* (HJ) *holds, then the set* $\Psi(x)$ *is nonempty weakly compact and convex for every* $x \in L_\Gamma^2(I,H)$.

LEMMA 3.6 ([31, Lemma 11]). *Let condition* (HJ) *hold and* $f_n \in \Psi(x_n)$ *such that* $x_n \to x \in L_\Gamma^2(I,H)$ *and* $f_n \xrightarrow{w} f \in L_\Gamma^2(I,H)$. *Then* $f \in \Psi(x)$.

Lemma 3.7 ([22, Lasota and Opial]). *Let I be a compact interval and H be a Hilbert space. Let G be a multi-valued map satisfying* (HG). *If Υ is a linear continuous operator from $L^2(I,L^2(\Gamma,H))$ to $C(I,L^2(\Gamma,H))$, then the operator defined by*

$$\Upsilon\circ\mathcal{S}_G\colon C(I,L^2(\Gamma,H))\to\mathcal{P}_{\mathrm{cp,cv}}(C(I,L^2(\Gamma,H))),\quad x\mapsto\Upsilon\circ\mathcal{S}_G(x)=\Upsilon\mathcal{S}_{G,x}$$

is a closed graph in $C(I,L^2\big(\Gamma,H)\big)\times C\big(I,L^2(\Gamma,H)\big)$.

The next lemma contains a Burkholder–Davis–Gundy-type inequality and is a special case of [40, Lemma 7.2].

Lemma 3.8 ([40, Lemma 7.2]). *For any $p\geq 1$ and arbitrary $L_0^2(K,H)$-valued predictable process $\varphi(\,\cdot\,)$, one has*

$$\sup_{s\in[0,t]}\mathbb{E}\bigg\|\int_0^s\varphi(\tau)\,dw(\tau)\bigg\|_H^{2p}\leq(p(2p-1))^p\bigg(\int_0^t\big(\mathbb{E}\|\varphi(s)\|_{L_0^2}^{2p}\big)^{1/p}\bigg)^p,\quad t\in I.$$

In the rest of this paper, we set $C_p=(p(2p-1))^p$.

Theorem 3.9. *If conditions* (HT), (HG), (HB), (HJ) *and* (HU) *hold, then for each $u(t)\in U_{\mathrm{ad}}$ and some p with $p\alpha>1$, problem* (1.1) *has a mild solution $x\in C(I,L^2(\Gamma,H))$.*

Proof. For any $x\in C\big(I,L^2(\Gamma,H)\big)\subset L^2\big(I,L^2(\Gamma,H)\big)$, from Lemma 3.5, we can define a multi-valued map $\mathcal{F}\colon C(I,L^2(\Gamma,H))\to\mathcal{P}(C(I,L^2(\Gamma,H)))$ by

$$\begin{aligned}\mathcal{F}(x)=\bigg\{y\in C(I,H):y(t)&=P_\alpha(t)x_0\\&+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)[Bu(s)+\eta(s)]\,ds\\&+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)g(s)\,dw(s),\ g\in\mathcal{S}_{G,x},\ \eta\in\Psi(x)\bigg\}.\end{aligned}$$

In order to show that problem (1.1) has a mild solution, we only need to prove that $\mathcal{F}$ admits a fixed point in $C(I,L^2(\Gamma,H))$. Now we divide our proof into four steps.

Step 1. In view of Lemma 3.5 and (HG), it is easy to verify that $\mathcal{F}(x)$ is convex.

Step 2. We show that $\mathcal{F}(B_r)$ is a bounded subset of $C\big(I,L^2(\Gamma,H)\big)$ for any $r>0$, where

$$B_r=\big\{x\in C(I,L^2(\Gamma,H)):\|x\|^2\leq r\big\}.$$

We claim that there exists a number $l_0>0$ such that, for each $y\in\mathcal{F}(x)$ with $x\in B_r$, $\|y\|^2\leq l_0$. In fact, for any $y\in\mathcal{F}(x)$, there exist a selection $g\in\mathcal{S}_{G,x}$

of $G(t,x)$ and a function $\eta(t) \in \partial J(t,x(t))$ such that

$$y(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[Bu(s)+\eta(s)]\,ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g(s)\,dw(s),$$

for $t \in I$. By conditions (HB), (HG), (J3), Lemma 3.2 (a), Lemma 3.8 and Hölder's inequality, one has

$$\begin{aligned}
(3.3)\quad \mathbb{E}\|y(t)\|^2 \le{}& 4\mathbb{E}\|P_\alpha(t)x_0\|^2] \\
&+ 4\mathbb{E}\bigg\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\eta(s)\,ds \bigg\|^2 \\
&+ 4\mathbb{E}\bigg\| \int_0^t (t-s)^{2(\alpha-1)}\,ds \int_0^t S_\alpha(t-s)Bu(s)\,ds \bigg\|^2 \\
&+ 4\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g(s) dw(s)\|^2 \\
\le{}& 4M^2\mathbb{E}\|x_0\|^2 + 4\bigg(\frac{\alpha M\|B\|_\infty}{\Gamma(\alpha+1)}\bigg)^2 \\
&\times \bigg(\int_0^t (t-s)^{p(\alpha-1)/(p-1)}\,ds\bigg)^{(2p-2)/p} \bigg(\mathbb{E}\int_0^t \|u(s)\|_Y^p\,ds\bigg)^{2/p} \\
&+ 4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \int_0^t (t-s)^{2(\alpha-1)}\,ds \bigg(\mathbb{E}\int_0^t \|\eta(s)\|^2\,ds\bigg) \\
&+ 4tr(Q)\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \\
&\times \int_0^t (t-s)^{2(\alpha-1)}\,ds \bigg(\mathbb{E}\int_0^t \|g(s)\|^2\,ds\bigg) \\
\le{}& 4M^2\mathbb{E}\|x_0\|^2 + 4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \\
&\times \bigg(\frac{p-1}{p\alpha-1}\bigg)^{(2p-2)/p} b^{2\alpha-2/p}\|B\|_\infty^2 \|u\|_{L^p_\Gamma(I,Y)}^2 \\
&+ 4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \frac{b^{2\alpha-1}}{2\alpha-1}\bigg(\int_0^t \zeta(s)\,ds + \int_0^t c\mathbb{E}\|x(s)\|^2\,ds\bigg) \\
&+ 4tr(Q)\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \frac{b^{2\alpha-1}}{2\alpha-1} M_g(k) \\
\le{}& 4M^2\mathbb{E}\|x_0\|^2 + 4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2 \\
&\times \bigg(\frac{p-1}{p\alpha-1}\bigg)^{(2p-2)/p} b^{2\alpha-2/p}\|B\|_\infty^2 \|u\|_{L^p_\Gamma(I,Y)}^2
\end{aligned}$$

$$+4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\frac{b^{2\alpha-1}}{2\alpha-1}$$
$$\times\big(\sqrt{b}\|\zeta\|_{L^2(I,\mathbb{R}^+)}+bcr+tr(Q)M_g(k)\big).$$

Step 3. $\{\mathcal{F}(x) : x\in B_r\}$ is equicontinuous. Indeed, for any $\varepsilon>0$, when $t_1=0$ and $0<t_2\le\delta_0$, similar to the proof of (3.3), we obtain

$$\begin{aligned}\mathbb{E}\|y(t_2)-yx(t_1)\|^2&=\|y(t_2)-x_0\|^2\le 4\mathbb{E}\|P_\alpha(t)x_0-x_0\|^2\\&+4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\bigg(\frac{p-1}{p\alpha-1}\bigg)^{(2p-2)/p}\delta_0^{2\alpha-2/p}\|B\|_\infty^2\|u\|^2_{L^p_\Gamma(I,Y)}\\&+4\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\frac{\delta_0^{2\alpha-1}}{2\alpha-1}\big(\sqrt{\delta_0}\|\zeta\|_{L^2(I,\mathbb{R}^+)}+bcr+tr(Q)M_g(k)\big).\end{aligned}$$

This together with uniform continuity of $P_\alpha(t_2)$ (see Remark 3.4) infers that $\mathbb{E}\|y(t_2)-y(t_1)\|^2$ tends to zero independently of $x\in B_r$ as $\delta_0\to 0$.

In what follows, for any $x\in B_r$ and $\delta_0/2\le t_1<t_2\le b$, one has

$$\begin{aligned}\mathbb{E}\|y(t_2)-y(t_1)\|^2&\le 4\mathbb{E}\|P_\alpha(t_2)x_0-P_\alpha(t_1)x_0\|^2\\&+4\bigg(\mathbb{E}\bigg\|\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})Q_\alpha(t_2-s)\eta(s)\,ds\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}(Q_\alpha(t_2-s)-Q_\alpha(t_1-s))\eta(s)\,ds\\&+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Q_\alpha(t_2-s)\eta(s)\,ds\bigg\|^2\bigg)\\&+4\bigg(\mathbb{E}\bigg\|\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})Q_\alpha(t_2-s)Bu(s)\,ds\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}(Q_\alpha(t_2-s)-Q_\alpha(t_1-s))Bu(s)\,ds\\&+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Q_\alpha(t_2-s)Bu(s)\,ds\bigg\|^2\bigg)\\&+4\bigg(\mathbb{E}\bigg\|\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})Q_\alpha(t_2-s)g(s)\,dw(s)\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}(Q_\alpha(t_2-s)-Q_\alpha(t_1-s))g(s)\,dw(s)\\&+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Q_\alpha(t_2-s)g(s)\,dw(s)\bigg\|^2\bigg)\\&:=\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4.\end{aligned}\tag{3.4}$$

It yields from condition (J3), Lemma 3.2, Hölder's inequality and same methods used in the proof of Lemma 3.1 in [44] that

$$
\begin{aligned}
(3.5)\quad \Lambda_2 \leq 12\bigg[&\int_0^{t_1} \big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2\,ds\\
&\times\int_0^{t_1} \|Q_\alpha(t_2-s)\|^2\mathbb{E}\|\eta(s)\|^2\,ds\\
&+\int_0^{t_1}(t_1-s)^{2(\alpha-1)}\,ds\int_0^{t_1}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\,\mathbb{E}\|\eta(s)\|^2\,ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{2(\alpha-1)}\,ds\int_{t_1}^{t_2}\|Q_\alpha(t_2-s)\|^2\,\mathbb{E}\|\eta(s)\|^2\,ds\bigg]\\
\leq 12\bigg[&\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\\
&\times\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2\,ds\int_0^{t_1}\mathbb{E}\|\eta(s)\|^2\,ds\\
&+\frac{t_1^{2\alpha-1}}{2\alpha-1}\sup_{s\in[0,t_1]}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\int_0^{t_1}\mathbb{E}\|\eta(s)\|^2\,ds\\
&+\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1}\int_{t_1}^{t_2}\mathbb{E}\|\eta(s)\|^2\,ds\bigg].
\end{aligned}
$$

Similar to the proof of (3.5), we get

$$
\begin{aligned}
(3.6)\qquad \Lambda_3 \leq 12\bigg[&\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\|B\|_\infty^2\,\|u\|^2_{L^p_\Gamma(I,Y)}\\
&\times\bigg(\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^{p/(p-1)}\,ds\bigg)^{(2p-2)/p}\\
&+\bigg(\frac{p-1}{p\alpha-1}\bigg)^{(2p-2)/p} t_1^{2\alpha-2/p}\|B\|_\infty^2\,\|u\|^2_{L^p_\Gamma(I,Y)}\\
&\times\sup_{s\in[0,t_1]}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\\
&+\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\bigg(\frac{p-1}{p\alpha-1}\bigg)^{(2p-2)/p}\\
&\times(t_2-t_1)^{2\alpha-2/p}\|B\|_\infty^2\,\|u\|^2_{L^p_\Gamma(I,Y)}\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7)\qquad \Lambda_4 \leq 12\bigg[&\bigg(\frac{\alpha M}{\Gamma(\alpha+1)}\bigg)^2\\
&\times\int_0^{[}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2\,ds\int_0^{t_1}\mathbb{E}\|g(s)\|^2_{L_0^2}\,ds
\end{aligned}
$$

$$+ \frac{t_1^{2\alpha-1}}{2\alpha-1} \sup_{s\in[0,t_1]} \|Q_\alpha(t_2-s) - Q_\alpha(t_1-s)\|^2 \int_0^{t_1} \mathbb{E}\|g(s)\|_{L_0^2}^2 \, ds$$
$$+ \left(\frac{\alpha M}{\Gamma(\alpha+1)}\right)^2 \frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1} \int_{t_1}^{t_2} \mathbb{E}\|g(s)\|_{L_0^2}^2 \, ds\bigg].$$

It is easy to see that Λ_2, Λ_3, Λ_4 tend to zero independently of $x \in B_r$ as $t_2 \to t_1$. Moreover, by uniform continuity of $P_\alpha(t)$ (see Remark 3.4), we deduce that Λ_1 tends to zero independently of $x \in B_r$ as $t_2 \to t_1$.

Summarizing the above, we know that $\{\mathcal{F}(x) : x \in B_r\}$ is equicontinuous.

Step 4. The multi-valued map $\mathcal{F}$ is compact. We shall show that, for any $t \in [0,b]$, $\Pi(t) = \{y(t) : y \in \mathcal{F}(B_r)\}$ is a relatively compact subset of H. Obviously, $\Pi(0) = \{x_0\}$ is compact. So it suffices to consider $t > 0$. Let $0 < t \le b$ be fixed. Then, for any $x \in B_r$, there exist a selection $g \in \mathcal{S}_{G,x}$ of $G(t,x)$ and a function $\eta(t) \in \Psi(x)$ satisfying

$$y(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[Bu(s) + \eta(s)] \, ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g(s) \, dw(s),$$

for $t \in I$. For any $\delta \in (0,t)$ with $0 < t \le b$ and $x \in B_r$, define $\Pi_\delta(t) = \{y_\delta(t) : y_\delta \in \mathcal{F}(B_r)\}$, where

$$y_\delta(t) = P_\alpha(t)x_0 + \int_0^{t-\delta} (t-s)^{\alpha-1} S_\alpha(t-s)[Bu(s) + \eta(s)] \, ds + \int_0^{t-\delta} (t-s)^{\alpha-1} S_\alpha(t-s) g(s) \, dw(s),$$

for $t \in I$. Since

$$\int_0^{t-\delta} (t-s)^{\alpha-1} S_\alpha(t-s)[Bu(s) + \eta(s)] \, ds$$

and

$$\int_0^{t-\delta} (t-s)^{\alpha-1} S_\alpha(t-s) g(s) \, dw(s)$$

are bounded, it follows from the compactness of $P_\alpha(t)$ and $S_\alpha(t)$ that $\Pi_\delta(t)$ is a relatively compact subset of H for any $\delta \in (0,t)$. Furthermore, one has

$$\mathbb{E}\|y_\delta(t) - y(t)\|^2 \le 3\mathbb{E}\left\| \int_{t-\delta}^t (t-s)^{\alpha-1} S_\alpha(t-s) Bu(s) \, ds \right\|^2 + 3\mathbb{E}\left\| \int_{t-\delta}^t (t-s)^{\alpha-1} S_\alpha(t-s) \eta(s) \, ds \right\|^2 + 3\mathbb{E}\left\| \int_{t-\delta}^t (t-s)^{\alpha-1} S_\alpha(t-s) g(s) \, dw(s) \right\|^2 \tag{3.8}$$

$$\leq 3\left(\frac{\alpha M}{\Gamma(\alpha+1)}\right)^2\left(\frac{p-1}{p\alpha-1}\right)^{(2p-2)/p}\delta^{2\alpha-2/p}\|B\|_\infty^2\|u\|^2_{L^p_\Gamma(I,Y)}$$
$$+3\left(\frac{\alpha M}{\Gamma(\alpha+1)}\right)^2\frac{\delta^{2\alpha-1}}{2\alpha-1}\big(\sqrt{\delta}\|\zeta\|_{L^2(I,\mathbb{R}^+)}+bcr+tr(Q)M_g(k)\big).$$

The last inequality tends to zero as $\delta\to 0$.

The discussion above shows that there are relatively compact sets arbitrarily close to the set $\Pi(t)$. Thus, we know that $\Pi(t)$, $t>0$ is also a relatively compact subset of H. By Steps 1–3, the relative compactness of $\Pi(t)$, $t>0$ and Ascoli–Arzelà theorem (see [6, Proposition 1.7.3]), we deduce that $\{\mathcal{F}(x) : x\in B_r\}$ is relatively compact subset of H. This shows that the multi-valued map $\mathcal{F}$ is compact.

Step 5. $\mathcal{F}$ has a closed graph. Let $x_n\to x_*$ in $C(I,L^2(\Gamma,H))$ and $y_n\in\mathcal{F}(x_n)$. We shall show that $y_*\in\mathcal{F}(x)$. The fact $y_n\in\mathcal{F}(x_n)\subset B_r$ implies that there exist $g_n\in\mathcal{S}_{G,x_n}$ and $\eta_n\in\Psi(x_n)$ such that, for every $t\in I$,

$$y_n(t)=P_\alpha(t)x_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)[Bu(s)+\eta_n(s)]\,ds+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)g_n(s)\,dw(s). \tag{3.9}$$

According to (J3) and (HG), we obtain the boundedness of $\{\eta_n\}$ and $\{g_n\}$. In view of the reflexivity of $L^2_\Gamma(I,H)$ and L^2_0, without loss of generality, we can suppose that

$$(\eta_n,g_n)\xrightarrow{\text{weakly}}(\eta_*,g_*)\in L^2_\Gamma(I,H)\times L^2_0. \tag{3.10}$$

Define an operator $\Upsilon\colon L^2_0\to C(I,L^2(\Gamma,H))$ by

$$\Upsilon(g)(\,\cdot\,)=\int_0^{\cdot}(\,\cdot\,-s)^{\alpha-1}Q_\alpha(\,\cdot\,-s)g(s)\,dw(s),\quad \text{for all } g\in L^2_0.$$

Obviously, Υ is linear and continuous. Moreover, similar to the proof of (3.3), condition (HG) infers

$$\mathbb{E}\|\Upsilon(g)(t)\|^2\leq\left(\frac{\alpha M}{\Gamma(\alpha+1)}\right)^2\frac{b^{2\alpha-1}}{2\alpha-1}tr(Q)M_g(k). \tag{3.11}$$

By the compactness of S_α, it follows from (J3), (HG), (3.9)–(3.11) that

$$y_n(t)\to P_\alpha(t)x_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)[Bu(s)+\eta_*(s)]\,ds+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)g_*(s)\,dw(s). \tag{3.12}$$

We note that $y_n\to y_*$ in $C(I,L^2(\Gamma,H))$, $g_n\in\mathcal{S}_{G,x_n}$ and $\eta_n\in\Psi(x_n)$ from Lemmas 3.6 and 3.7, (3.10), we have $\eta_*\in\Psi(x_*)$ and $g_*\in\mathcal{S}_{G,x_*}$. Hence, $y_*\in\mathcal{F}(x)$, which shows that graph $\mathcal{F}$ is closed.

According to Steps 4 and 5, it yields from Lemma 2.10 that $\mathcal{F}$ is u.s.c. Therefore, due to Steps 1–5, we prove that the multivalued map $\mathcal{F}$ is compact and u.s.c. and has convex closed values. Therefore, summarizing the above, we deduce that the multi-valued map $\mathcal{F}$ fulfills all the conditions of Lemma 2.11. Consequently, it follows from Lemma 2.11 that $\mathcal{F}$ has fixed point on B_r. □

4. Existence of optimal control

In this section, we consider the following Lagrange problem:

(P) Find a pair $(x^0, u^0) \in C(I, L^2(\Gamma, H)) \times U_{\rm ad}$ such that

$$\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x, u), \quad \text{for all } (x, u) \in C(I, L^2(\Gamma, H)) \times U_{\rm ad},$$

where

$$\mathcal{J}(x, u) = \mathbb{E} \int_0^b \mathcal{L}(t, x^u(t), u(t))\, dt.$$

Here x^u stands for the mild solution of system (1.1) corresponding to the control $u \in U_{\rm ad}$. To discuss Lagrange problem (P), we make use of the following hypothesis (HL):

(L1) The functional $\mathcal{L}\colon I \times H \times Y \to \mathbb{R} \cup \{\infty\}$ is Borel measurable;
(L2) For almost all $t \in I$, $\mathcal{L}$ is sequentially lower semicontinuous on $H \times Y$;
(L3) For each $x \in H$ and almost all $t \in I$, $\mathcal{L}(t, x, \cdot\,)$ is convex on Y;
(L4) There exist constants $d_1 \geq 0$, $d_2 > 0$, ξ is nonnegative and $\xi \in L^1(I, \mathbb{R})$ such that

$$\mathcal{L}(t, x, u) \geq \xi(t) + d_1 \mathbb{E}\|x\|_H^2 + d_2 \mathbb{E}\|u\|_Y^p.$$

Theorem 4.1. *If all the hypotheses of Theorem* 3.9 *and* (L1)–(L4) *hold, then Lagrange problem* (P) *admits at least one optimal pair.*

Proof. If $\inf\{\mathcal{J}(x, u) | u \in U_{\rm ad}\} = +\infty$, then we easily know Lagrange problem (P) has one optimal pair. Without loss of generality, we suppose that $\inf\{\mathcal{J}(x, u) : u \in U_{\rm ad}\} = \nu < +\infty$. Then condition (L4) implies that $\nu > -\infty$. By definition of infimum, there exists a minimizing sequence of feasible pair $\{(x^n, u^n)\} \subset \mathcal{A}_{\rm ad}$ such that $\mathcal{J}(x^n, u^n) \to \nu$ as $n \to +\infty$. Since $\{u^n\} \subseteq U_{\rm ad}$, $n = 1, 2, \ldots$, $\{u^n\}$ is bounded on $L^p_\Gamma(I, Y)$, by the reflexivity of $L^p_\Gamma(I, Y)$, there exist a subsequence of $\{u^n\}$, denoted again by $\{u^n\}$, and $u^* \in L^p_\Gamma(I, Y)$ satisfying

$$u^n \xrightarrow{\text{weakly}} u^* \in L^p_\Gamma(I, Y).$$

Since $U_{\rm ad}$ is closed and convex, it follows from Mazur's lemma that $u^* \in U_{\rm ad}$. Let $\{x^n\}$ denote the corresponding sequence of solutions of the following integral equation:

$$x^n(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[Bu^n(s) + \eta^n(s)]\,ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g^n(s)\,dw(s), \tag{4.1}$$

where $g^n \in \mathcal{S}_{G,x^n}$ and $\eta_n \in \Psi(x^n)$.

Next we prove that $\{x^n\}$ is a relatively compact subset of $C\big(I, L^2(\Gamma, H)\big)$. Firstly, similar to the proof of (3.3), we obtain

$$\begin{aligned} \mathbb{E}\|x^n(t)\|^2 &\le 4M^2\mathbb{E}\|x_0\|^2 \\ &+ 4\Big(\frac{\alpha M}{\Gamma(\alpha+1)}\Big)^2 \Big(\frac{p-1}{p\alpha-1}\Big)^{(2p-2)/p} b^{2\alpha-2/p}\|B\|_\infty^2 \|u^n\|^2_{L^p_\Gamma(I,Y)} \\ &+ 4\Big(\frac{\alpha M}{\Gamma(\alpha+1)}\Big)^2 \frac{b^{2\alpha-1}}{2\alpha-1}\Big(\int_0^t \zeta(s)\,ds + \int_0^t c\mathbb{E}\|x^n(s)\|^2\,ds\Big) \\ &+ 4tr(Q)\Big(\frac{\alpha M}{\Gamma(\alpha+1)}\Big)^2 \frac{b^{2\alpha-1}}{2\alpha-1} M_g(k). \end{aligned} \tag{4.2}$$

Thanks to the boundedness of $\{u_n\}$, (4.2) and Gronwall's inequality [?], we infer that there exists a constant $\lambda > 0$ such that $\|x^n\| \le \lambda$, which implies that $\{x^n\}$ is uniformly bounded. Next, similar to the argument of Steps 3, 4 in Theorem 3.9, we can prove that $\{x^n(t)\}$ is equicontinuous on I and $\{x^n(t)\}$ is relatively compact for any $t \in I$. Thus, the Ascoli–Arzelà theorem implies that $\{x^n\}$ is a relatively compact subset of $C(I, L^2(\Gamma, H))$ and so there exists a function $x^* \in C(I, L^2(\Gamma, H))$ such that

$$x^n \to x^* \quad \text{in } C\big(I, L^2(\Gamma, H)\big) \subset L^2\big(I, L^2(\Gamma, H\big). \tag{4.3}$$

The boundedness of $\{u_n\}$ and compactness of $Q_\alpha(t-s)$ together with the dominated convergence theorem imply that

$$\int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) Bu^n(s)\,ds \to \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) Bu^*(s)\,ds. \tag{4.4}$$

Similar to the proof of Step 5 in Theorem 3.9, according to the compactness of S_α, (J3), (HG), (4.3), Lemmas 3.6 and 3.7, one has

$$\begin{aligned} &P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\eta^n(s)\,ds \\ &\qquad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g^n(s)\,dw(s) \\ &\to P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\eta^*(s)\,ds \\ &\qquad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g^*(s)\,dw(s), \end{aligned} \tag{4.5}$$

where $g^* \in \mathcal{S}_{G,x^*}$ and $\eta^* \in \Psi(x^*)$. Hence, it follows from (4.4) and (4.5) that

$$x^*(t) = P_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[Bu^*(s) + \eta^*(s)]\,ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) g^*(s)\,dw(s).$$

This proves that x^* is a mild solution of problem (1.1) corresponding to the control $u^* \in U_{\mathrm{ad}}$.

We notice that (L1)–(L4) satisfy all the hypotheses of Balder's theorem (see Theorem 2.1 of [2]). Thus, Balder's theorem shows that the functional

$$(x,u) \mapsto \mathbb{E}\int_0^b \mathcal{L}(t, x^u(t), u(t))\,dt$$

is sequentially lower semicontinuous in the strong topology of $L^1_\Gamma(I,H)$ and weak topology of $L^p_\Gamma(I,H) \subset L^1_\Gamma(I,Y)$. Since $L^p_\Gamma(I,Y) \subset L^1_\Gamma(I,Y)$, we deduce that $\mathcal{J}$ is weakly lower semicontinuous on $L^p_\Gamma(I,Y)$. By (L4), we know that $\mathcal{J} > -\infty$. Thus, we conclude that $\mathcal{J}$ attains its infimum at $u^* \in U_{\mathrm{ad}}$ and so

$$\nu = \lim_{n\to\infty} \mathbb{E}\int_0^b \mathcal{L}(t, x^n(t), u^n(t))dt \geq \mathbb{E}\int_0^b \mathcal{L}(t, x^*(t), u^*(t))\,dt \geq \nu.$$

This completes the proof. □

5. An example

In this section, we give an example to illustrate our main results. Let $H = Y := L^2[0,\pi]$. Define $A\colon D(A) \subset H \to H$ by $Ax = x_{yy}$, where domain $D(A)$ is given by

$$D(A) = \{x \in H : x, x_y \text{ are absolutely continuous}, \quad x_{yy} \in H,\ x(t,0) = x(t,\pi) = 0\}.$$

From [23], A can be written as follows:

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle e_n, \quad x \in D(A),$$

where

$$e_n(y) = \sqrt{\frac{2}{\pi}}\sin ny, \quad n = 1, 2, \ldots$$

is the orthonormal basis of eigenfunctions of A. It is known that A is the infinitesimal generator of a compact semigroup $T(t)(t > 0)$ on H given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x \in H, \quad \|T(t)\| \leq e^{-t} < 1 = M.$$

The admissible controls set U_{ad} is defined by $U_{\mathrm{ad}} = \{u \in Y | \|u\|_{L^2([0,1],Y)} \le 1\}$. Choose $\alpha = 3/4$ and $p = 2$. Consider the problem of finding the controls $u(t,y)$ to minimize the cost function

$$\mathcal{J}(x,u) = \mathbb{E}\int_0^1 \bigg[\int_0^\pi |x(t,y)|^2\,dy\,dt + \int_0^1\int_0^\pi |u(t,y)|^2\,dy\bigg]\,dt$$

subject to the following system:

$$\tag{5.1} \begin{cases} \bigg\langle -{}^cD_t^\alpha x(t,y) + x_{yy}(t,y) + \displaystyle\int_0^1 q(y,\sigma)u(\sigma,t)\,d\sigma \\ \qquad\qquad + g(t)\dfrac{d\widehat{w}(t)}{dt}, v\bigg\rangle + J^0(t,y,x(t,y);v) \ge 0, \\ \qquad\qquad\qquad\qquad v \in H\ y \in (0,\pi),\ t \in (0,1) = (0,b); \\ x(t,0) = x(t,\pi) = 0, \quad t \in (0,1); \\ x(0,y) = x_0(t,y), \qquad y \in (0,\pi), \end{cases}$$

where $\{\widehat{w}(t)\}_{t\in\mathbb{R}}$ is a two sided and standard one dimensional Brownian motion defined on the filtered probability space $(\Omega,\Gamma,\{\Gamma_t\}_{t\ge 0},\mathbb{P})$; $g(t) \in G(t,x(t))$ and $G\colon I \times H \to \mathcal{P}(H)$ is a nonempty, bounded, closed and convex multi-valued map which satisfies the assumptions (HG); $q\colon [0,1]\times[0,1] \to \mathbb{R}$ is continuous.

Let a functional $J\colon (0,1)\times H \to \mathbb{R}$ be defined by

$$J(t,x) = \int_0^1 j(t,y,x(y))\,dy, \quad t \in (0,1),\ x \in H,$$

where

$$j(t,y,z) = \int_0^z \gamma(t,y,\theta)d\theta, \quad (t,y) \in (0,1)\times(0,\pi),\ z \in \mathbb{R}.$$

Suppose that $\gamma\colon (0,\pi)\times(0,1)\times\mathbb{R} \to \mathbb{R}$ is a functional such that

(i) for all $y \in (0,\pi)$, $z \in \mathbb{R}$, $\gamma(\,\cdot\,,y,z)\colon (0,1) \to \mathbb{R}$ is measurable;
(ii) for all $t \in (0,1)$, $z \in \mathbb{R}$, $\gamma(t,\,\cdot\,,z)\colon (0,\pi) \to \mathbb{R}$ is continuous;
(iii) for all $z \in \mathbb{R}$, there exists a constant $c_1 > 0$ satisfying

$$|\gamma(\,\cdot\,,\,\cdot\,,z)| \le c_1(1+|z|);$$

(iv) for each $z \in \mathbb{R}$, $\gamma(\,\cdot\,,\,\cdot\,,z\pm 0)$ exists.

If γ satisfies (iii), then one has $\partial j(z) \subset [\underline{\gamma}(z),\overline{\gamma}(z)]$ for $z \in \mathbb{R}$ (we omit (t,y) here), where $\underline{\gamma}(z)$ and $\overline{\gamma}(z)$ stand for the essential infimum and essential supremum of η at a point z (see [4, p. 34]). If γ satisfies (i)–(iv), then the function $j(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ defined above has properties (see [36] for more details) as follows:

(i) for all $y \in (0,\pi)$, $z \in \mathbb{R}$, $j(\,\cdot\,,y,z)\colon (0,1) \to \mathbb{R}$ is measurable and $j(\,\cdot\,,\,\cdot\,,0) \in L^2((0,1)\times(0,\pi))$;
(ii) for all $t \in (0,1)$, $z \in \mathbb{R}$, $j(t,\,\cdot\,,z)\colon (0,\pi) \to \mathbb{R}$ is continuous;
(iii) for all $(t,y) \in (0,1)\times(0,\pi)$, $j(t,y,\,\cdot\,)\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz;

(iv) there exists a constant $c_2 > 0$ satisfying

$$|\zeta| \leq c_2(1 + |z|), \quad \text{for all } \zeta \in \partial j(t, y, z),\ (t, y) \in (0, 1) \times (0, \pi);$$

(v) there exists a constant $c_3 > 0$ satisfying

$$j^0(t, y, z, -z) \leq c_3(1 + |z|), \quad \text{for all } (t, y) \in (0, 1) \times (0, \pi).$$

Moreover, the functional $J(\,\cdot\,,\,\cdot\,)$ defined above satisfies conditions (J1)–(J3) (see [36]) or the example of [17]).

Define

$$x(t, y) = x(y)(t), \qquad B(t)u(t)(y) = \int_0^1 q(y, \sigma)u(\sigma, t)\, d\sigma.$$

We also suppose that condition (HL) hold. Thus, problem (5.1) can be written as the abstract form of problem (1.1) with the cost function

$$\mathcal{J}(x, u) = \mathbb{E}\int_0^1 \left[\|x(t)\|^2 dt + \|u(t)\|^2\right] dt.$$

Summarizing the above, we know that (HT), (HG), (HB), (HJ), (HU) and (L1)–(L4) hold. Finally, we can check that $p\alpha = 2 \times (3/4) > 1$. Thus, all conditions of Theorems 3.9 and 4.1 are satisfied and we conclude problem (5.1) has a mild solution and at least one optimal pair.

6. Concluding remarks

To our best knowledge, very few papers are available in the literature for dealing with the solvability and optimal control of FSEHVIs. Due to its importance in both theoretical and real-life applications point of view, it is meaningful to study its existence, controllability, and other properties. In this work, we adopt a new method to study the solvability and optimal control of the system governed by FSEHVIs. We would like to mention that the results of this paper are obtained without the assumptions of Lipschitz continuity with respect to the second variable for the Clarke subdifferential $\partial J(t,\,\cdot\,)$, which however, is an important and necessary condition to establish similar results in [33].

On the other hand, It is well known that the problem of the approximate controllability have close relations to observer design quadratic optimal control (see Lemma 4 and Theorem 2 in [28]). We noted that Lu and Liu mainly applied a fixed point theorem of multivalued maps to obtain existence and controllability results for stochastic fractional evolution hemivariational inequalities, recently, Mahmudov [27] applied variational approach to study finite approximate controllability of Sobolev-type fractional systems, as Mahmudov [27] pointed out that finite approximate controllability of stochastic evolution systems leaves open some issues. However, up to now, there is no results for finite approximate controllability governed by fractional stochastic evolution HVIs. Thus, this topic would be interesting and deserve further investigation.

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