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SPECIFICATION PROPERTIES FOR NON-AUTONOMOUS DISCRETE SYSTEMS

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Abstract. In this paper notions of strong specification property and quasi-weak specification property for non-autonomous discrete systems are introduced and studied. It is shown that these properties are dynamical properties and are preserved under finite product. It is proved that a k-periodic non-autonomous system on intervals having weak specification is Devaney chaotic. Moreover, it is shown that if the system has strong specification then the result is true in general. Specification properties of induced systems on hyperspaces and probability measures spaces are also studied. Examples/counterexamples are provided wherever necessary to support results obtained.

1. Introduction

Dynamical system is a very well developed branch of mathematics. In its contemporary formulation, the theory grows directly from advances in understanding complex and nonlinear systems in physics and mathematics. Over the last 40 years with the discovery of chaos lots of research has been done in autonomous dynamical systems. The first paper that described chaos in a mathematically rigorous way is that of Li and Yorke [10]. Since then the research on chaos has had a great influence on modern science. Specification property

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is an interesting and rather stronger notion of chaos. This property is closely related to the study of hyperbolic systems. Roughly speaking, by specification property we mean that any k finite pieces of orbits can be approximated by some periodic orbit, provided the period is large enough. In 1971, Bowen introduced the specification property on Axiom A diffeomorphisms [3]. This property seems technical but it is satisfied by many important maps like shift maps, Anosov diffeomorphisms, etc. For more details, one can refer to [21]. The relevance of the specification property is that it plays a key role in the study of the uniqueness of equilibrium states, large deviation theory, multi-fractal analysis, etc. In [22], it is observed that the irregular set for maps with the specification property has full topological pressure. Various types of specification properties including weak specification property, almost specification property [5], [9], [11], approximate product property [14], for autonomous discrete dynamical systems have been intensively studied from an ergodic view point as well as from an algebraic view point. In [1], authors have studied specification on operators and proved that this property is equivalent to the notion of Devaney chaos for backward shift operators on Banach sequence spaces. Recently, specification property for uniform spaces has been defined and studied in [19].

Most of the real-world problems like weather and climate prediction, heartbeat patterns, spread of infectious diseases, etc., are time variant, that is, they involve time-dependent parameters, modulation and various other effects and are known as non-autonomous dynamical systems. They turn out to be more flexible tools for the description and study of real world phenomena. Dynamics of such systems are more complicated than autonomous dynamical systems and the variety of dynamical behaviour that can be represented is much richer. The concept of non-autonomous discrete dynamical systems was introduced by Kolyada and Snoha [8], in 1996. Since then non-autonomous systems are widely studied and have remarkable applications [4], [25]. Chaos for non-autonomous dynamical systems was introduced by Tian and Chen [23]. In 2014, authors have introduced the concept of weak specification property (WSP) for non-autonomous systems [12]. They have related specification property to topological mixing, shadowing and distality in non-autonomous systems. In 2017, authors have proved that any non-autonomous system having the weak specification property has positive topological entropy and that each uniformly expanding non-autonomous system satisfies the shadowing and the specification property [18]. Recently, authors have studied stronger forms of sensitivity for non-autonomous discrete dynamical systems [15], [16], [24].

The paper is organized as follows. In Section 2, we give prerequisites for development of rest of the paper. In Section 3, we introduce the concepts of strong

(SSP) and quasi-weak specification (QSP) properties for non-autonomous systems. It is shown that if a k-periodic non-autonomous system has specification properties, then the corresponding autonomous system generated by the composition of k members also has specification properties. Moreover, specification properties are dynamical properties and preserved under a finite product. In Section 4, we show that for a non-autonomous system QSP is equivalent to topological mixing on a compact metric space. It is shown that a k-periodic non-autonomous system on intervals having WSP or QSP is Devaney chaotic and if the system has SSP, then this result is true in general. In Section 5, we prove that if a non-autonomous system has SSP, then the corresponding system induced on its hyperspace also has SSP and this result holds both ways for WSP as well as QSP. It is proved that if a non-autonomous system has SSP, then the corresponding system induced on the probability measures space also has SSP and the result is true both ways for QSP.

2. Preliminaries

In the present paper, we consider the following non-autonomous discrete dynamical system:

$$x_{n+1} = f_n(x_n), \quad n \geq 1, \tag{2.1}$$

where (X, d) is a *compact metric space* and $f_n \colon X \to X$ is a continuous map, for each $n \geq 1$. When $f_n = f$, for each $n \geq 1$, then the system (2.1) becomes an autonomous system. Denote $f_{1,\infty} := \{f_n\}_{n=1}^{\infty}$, and for all positive integers i and n,

$$f_n^i := f_{n+i-1} \circ \ldots \circ f_n, \qquad f_n^0 := \mathrm{id}.$$

For the system (2.1), the *orbit* of any point $x \in X$ is a set

$$\{f_1^n(x) : n \geq 0\} = \mathcal{O}_{f_{1,\infty}}(x).$$

For $k \in \mathbb{N}$, we say that $(X, f_{1,\infty})$ is a *k-periodic* discrete system, if $f_{n+k}(x) = f_n(x)$, for each $x \in X$ and for each $n \in \mathbb{N}$. Throughout this paper, $f_{1,\infty}$ denotes a surjective family, that is, each f_i is surjective. For a non-autonomous system $(X, f_{1,\infty})$, we put

$$X^2 = X \times X \quad \text{and} \quad (f_{1,\infty})^2 = (g_1, \ldots, g_n, \ldots),$$

where $g_n = f_n \times f_n$, for each positive integer n. Therefore, $(X^2, (f_{1,\infty})^2)$ is a non-autonomous dynamical system. We have,

$$g_1^n = g_n \circ \ldots \circ g_1 = (f_n \times f_n) \circ \ldots \circ (f_1 \times f_1) = f_1^n \times f_1^n.$$

Similarly we can define $(X^m, (f_{1,\infty})^m)$ in general for any positive integer m. Let (X, d_1) and (Y, d_2) be two metric spaces, then the product metric $\widetilde{d}$ on $X \times Y$ is defined by

$$\widetilde{d}((x_1,y_1),(x_2,y_2)) = \max\{d_1(x_1,x_2), d_2(y_1,y_2)\},$$

for all $(x_1,y_1),(x_2,y_2) \in X \times Y$. Let $B_d(x,\varepsilon)$ denote the open ball of radius $\varepsilon > 0$ with center x and $\mathbb{N}$ denote the set of natural numbers.

The symbol $\mathcal{K}(X)$ denotes the hyperspace of all non-empty compact subsets of X endowed with the *Vietoris topology.* A basis for Vietoris topology is given by the sets

$$\langle U_1,\ldots,U_k\rangle = \bigg\{K \in \mathcal{K}(X) : K \subseteq \bigcup_{i=1}^{k} U_i \quad \text{and } K \cap U_i \neq \emptyset, \text{ for each } i \in \{1,\ldots,k\}\bigg\},$$

where $U_1,\ldots,U_k$ are non-empty open subsets of X. Let $x \in X$, $A \in \mathcal{K}(X)$ and $N(A,\varepsilon) = \bigcup_{a\in A} B_d(a,\varepsilon)$. The Hausdorff metric in $\mathcal{K}(X)$ induced by d, denoted by $\mathcal{H}$ is defined by $\mathcal{H}(A,B) = \inf\{\varepsilon > 0 : A \subseteq N(B,\varepsilon) \text{ and } B \subseteq N(A,\varepsilon)\}$, where $A,B \in \mathcal{K}(X)$. The topology induced by the Hausdorff metric on $\mathcal{K}(X)$ coincides with the Vietoris topology if and only if the space X is compact [7]. Under this topology, the set of all finite subsets of X, $\mathcal{F}(X)$, is dense in $\mathcal{K}(X)$. Let $(X,f_{1,\infty})$ be a non-autonomous dynamical system and $\overline{f}_n$ the continuous function on $\mathcal{K}(X)$ induced by f_n, for each $n \in \mathbb{N}$. Then the sequence $\overline{f}_{1,\infty} = (\overline{f}_1,\ldots,\overline{f}_n,\ldots)$ induces a non-autonomous discrete dynamical system $(\mathcal{K}(X),\overline{f}_{1,\infty})$, where $\overline{f}_1^n = \overline{f}_n \circ \ldots \circ \overline{f}_2 \circ \overline{f}_1$. Clearly, $\overline{f}_1^n = \overline{f_1^n}$.

Let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X and $\mathcal{M}(X)$ be the set of all *Borel probability measures* on $(X,\mathcal{B}(X))$ and $\mathcal{M}(X)$ be equipped with the *Prokhorov metric* $\mathcal{D}$ defined by

$$\mathcal{D}(\mu,\nu) = \inf\{\varepsilon : \mu(A) \leq \nu(N(A,\varepsilon)) + \varepsilon \text{ and } \nu(A) \leq \mu(N(A,\varepsilon)) + \varepsilon,$$

for each $A \in \mathcal{B}(X)\}$. It is known that topology induced by $\mathcal{D}$ is weak*-topology [6]. For $x \in X$, $\delta_x \in \mathcal{M}(X)$ denotes *Dirac point measure*, given by $\delta_x(A) = 1$, if $x \in A$ and 0 otherwise. Let

$$\mathcal{M}_n(X) = \bigg\{\frac{1}{n}\bigg(\sum_{i=1}^{n} \delta_{x_i}\bigg) : x_i \in X \text{ (not necessarily distinct)}\bigg\},$$

$$\mathcal{M}_\infty(X) = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n(X).$$

It is known that $\mathcal{M}_\infty(X)$ is dense in $\mathcal{M}(X)$ and each $\mathcal{M}_n(X)$ is closed in $\mathcal{M}(X)$ [2]. For a non-autonomous system $(X,f_{1,\infty})$, we consider the non-autonomous induced system $\big(\mathcal{M}(X),\widetilde{f}_{1,\infty}\big)$, where each $\widetilde{f}_i\colon \mathcal{M}(X) \to \mathcal{M}(X)$ is the induced continuous function and $\widetilde{f}_1^n(\mu)(A) = \mu(f_1^{-n}(A))$, $\mu \in \mathcal{M}(X)$, $A \in \mathcal{B}(X)$ and $f_1^{-n} = (f_1^n)^{-1}$.

Definition 2.1 ([23]). A non-autonomous system $(X, f_{1,\infty})$ is said to be *topologically transitive*, if for each pair of non-empty open subsets U, V of X, there exists $n \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$. For any pair of non-empty open subsets U, V of X denote, $N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$. Therefore, $(X, f_{1,\infty})$ is transitive if $N_{f_{1,\infty}}(U, V) \neq \emptyset$, for any pair of non-empty open subsets U, V of X.

Definition 2.2 ([13]). A point $x \in X$ is said to be *periodic*, for a non-autonomous system $(X, f_{1,\infty})$, if there exists $n \in \mathbb{N}$ such that $f_1^{nk}(x) = x$, for every $k \in \mathbb{N}$.

Definition 2.3 ([23]). A system $(X, f_{1,\infty})$ is said to exhibit *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for every $x \in X$ and any neighborhood U of x, there exist $y \in U$ and $n \in \mathbb{N}$ with $d(f_1^n(x), f_1^n(y)) > \delta$; δ is called a constant of sensitivity.

Definition 2.4. A non-autonomous system $(X, f_{1,\infty})$ is said to be *chaotic in the sense of Devaney* on X if

(a) It is topologically transitive on X;
(b) It has a dense set of periodic points;
(c) It has sensitive dependence on initial conditions on X.

A non-autonomous system $(X, f_{1,\infty})$ is *Wiggins chaotic*, if it satisfies conditions (a) and (c) only in the above definition.

Definition 2.5. A non-autonomous system $(X, f_{1,\infty})$ is said to be *topologically mixing*, if there exists $n \in \mathbb{N}$ such that $N_{f_{1,\infty}}(U, V) \supseteq [n, \infty)$, for any pair of non-empty open subsets U, V of X.

Definition 2.6 ([12]). A non-autonomous system $(X, f_{1,\infty})$ is said to have *weak specification property* (WSP), if for every $\varepsilon > 0$, there exists N such that for every choice of points $x_1, x_2, \ldots, x_s \in X$ and any sequence $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_s \leq b_s$ of non-negative integers with $a_j - b_{j-1} > N$, $(2 \leq j \leq s)$, there is a point $z \in X$ satisfying

$$d\big(f_1^j(z), f_1^j(x_i)\big) < \varepsilon, \quad \text{for all } a_i \leq j \leq b_i,\ 1 \leq i \leq s.$$

Definition 2.7. Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two non-autonomous discrete dynamical systems. Let $h\colon X \to Y$ be such that $g_n(h(x)) = h(f_n(x))$, for each $n \in \mathbb{N}$ and for each $x \in X$. If h is a continuous, surjective map (homeomorphism), then $f_{1,\infty}$ and $g_{1,\infty}$ are said to be *topologically semi-conjugate* (*topologically conjugate*).

3. Strong and quasi-weak specification properties for non-autonomous systems

In this section, we first introduce concepts of strong and quasi-weak specification properties for non-autonomous systems. It is shown that if a k-periodic non-autonomous system has specification properties, then the corresponding autonomous system generated by the composition of k members, also has specification properties. It is proved that specification properties are dynamical properties and are preserved under any finite product.

Definition 3.1. A non-autonomous system $(X, f_{1,\infty})$ is said to have *strong specification property* (SSP), if for every $\varepsilon > 0$, there exists $M(\varepsilon)$ such that, for every choice of points $x_1, \ldots, x_k \in X$ and any sequence $a_1 \leq b_1 < \ldots < a_k \leq b_k$ of non-negative integers with $a_j - b_{j-1} > M(\varepsilon)$, $(2 \leq j \leq k)$ and any $p > M(\varepsilon) + b_k - a_1$, there exists a periodic point $z \in X$ with period p satisfying

$$d\big(f_1^j(z), f_1^j(x_i)\big) < \varepsilon, \quad \text{for all } a_i \leq j \leq b_i,\ 1 \leq i \leq k.$$

We shall call the above defined specification property for $k = 2$ as the *periodic specification property* (PSP). Now, we define a weaker form of specification property.

Definition 3.2. A non-autonomous system $(X, f_{1,\infty})$ is said to have the *quasi-weak specification property* (QSP), if for every $\varepsilon > 0$, there exists a positive integer $M(\varepsilon)$ such that for all $x_1, x_2 \in X$ and for every $n \geq M(\varepsilon)$, there is a point $z \in X$ such that $d(z, x_1) < \varepsilon$ and $d(f_1^n(z), f_1^n(x_2)) < \varepsilon$. Clearly, we have SSP $\Rightarrow$ WSP $\Rightarrow$ QSP (in fact, to ensure QSP, it is sufficient to satisfy Definition 2.6 only for $s = 2$).

Remark 3.3. In [12], authors have proved that a non-autonomous system having WSP is topologically mixing and justified that the converse is not true in general. Since SSP implies WSP, therefore if a non-autonomous system $(X, f_{1,\infty})$ has SSP, then it is topologically mixing but converse is not true in general.

Theorem 3.4. *Let $(X, f_{1,\infty})$ be a k-periodic non-autonomous system and $g = f_k \circ \ldots \circ f_1$. If $(X, f_{1,\infty})$ has* SSP*, then the corresponding autonomous system (X, g) also has* SSP.

Proof. Let $\varepsilon > 0$ be arbitrary and $M(\varepsilon)$ be a positive integer corresponding to ε as in the definition of SSP. Let $M_1 = [M(\varepsilon)/k] + 1$, where $[\,\cdot\,]$ denotes the greatest integer function. Consider a sequence $a_1 \leq b_1 < \ldots < a_l \leq b_l$ of non-negative integers with $a_j - b_{j-1} > M_1$ $(2 \leq j \leq l)$ and take any $p > M_1 + b_l - a_1$, then $ka_1 \leq kb_1 < \ldots < ka_l \leq kb_l$ and $ka_j - kb_{j-1} > kM_1 > M(\varepsilon)$ and $pk > M(\varepsilon) + kb_l - ka_1$. Since $(X, f_{1,\infty})$ has SSP, therefore there exists a periodic point $z \in X$ with period kp such that $d\big(f_1^i(z), f_1^i(x_m)\big) < \varepsilon$, for all

$ka_m \leq i \leq kb_m$, $1 \leq m \leq l$. Taking $i = jk$, we get $a_m \leq j \leq b_m$, $1 \leq m \leq l$ and $d\big(f_1^{jk}(z), f_1^{jk}(x_m)\big) < \varepsilon$. Now, the system $(X, f_{1,\infty})$ is k-periodic, so $f_1^{jk} = (f_1^k)^j$. Therefore, $d\big((f_1^k)^j(z), (f_1^k)^j(x_m)\big) < \varepsilon$, that is, $d\big(g^j(z), g^j(x_m)\big) < \varepsilon$, for all $a_m \leq j \leq b_m$, $1 \leq m \leq l$. Also, $f_1^{kps}(z) = z$, for each $s \in \mathbb{N}$ and in particular for $s = 1$, we get that $(f_1^k)^p(z) = z$ and hence $g^p(z) = z$, for $p > M_1 + b_l - a_1$. Thus, the autonomous system (X, g) has SSP. □

Theorem 3.5. *Let (X, d_1) and (Y, d_2) be two metric spaces and $(X, f_{1,\infty})$, $(Y, g_{1,\infty})$ be two non-autonomous systems such that $f_{1,\infty}$ is topologically semi-conjugate to $g_{1,\infty}$. If $f_{1,\infty}$ has* SSP, *then $g_{1,\infty}$ also has* SSP.

Proof. Since $f_{1,\infty}$ is topologically semi-conjugate to $g_{1,\infty}$, therefore there exists a continuous surjective map $h \colon X \to Y$ such that $h \circ f_n = g_n \circ h$, for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary, then by uniform continuity of h, for given $\varepsilon > 0$, there exists $\delta > 0$, such that

$$d_1(x, y) < \delta \;\Rightarrow\; d_2(h(x), h(y)) < \varepsilon, \quad \text{for all } x, y \in X. \tag{3.1}$$

Let $M(\delta)$ be as in the definition of SSP for $f_{1,\infty}$ and for the given ε, take $M(\varepsilon) = M(\delta)$. Consider a sequence $a_1 \leq b_1 < \ldots < a_k \leq b_k$ of non-negative integers with $a_j - b_{j-1} > M(\delta)$ $(2 \leq j \leq k)$ and $y_1, y_2, \ldots, y_k \in Y$. Now as h is surjective, so corresponding to each y_i, there exists $x_i \in X$ such that $h(x_i) = y_i$, for each $i = 1, \ldots, k$. Since $f_{1,\infty}$ has SSP, therefore there exists $x \in X$ such that $d_1(f_1^j(x), f_1^j(x_i)) < \delta$, for $a_i \leq j \leq b_i$ and $f_1^{pm}(x) = x$, for any $p > M(\varepsilon) + b_k - a_1$ and for each $m \in \mathbb{N}$. Therefore, by (3.1), we have

$$d_2(h(f_1^j(x)), h(f_1^j(x_i))) < \varepsilon, \quad \text{for all } a_i \leq j \leq b_i,\ 1 \leq i \leq k. \tag{3.2}$$

Taking $y = h(x)$ and using $h \circ f_n = g_n \circ h$, we get that, for every $j \in \mathbb{N}$, $g_1^j(y) = g_j \circ \ldots \circ g_1(h(x)) = g_j \circ \ldots \circ h(f_1(x)) = \ldots = h(f_1^j(x))$. Thus, using (3.2), we get $d_2\big(g_1^j(y), g_1^j(y_i)\big) = d_2\big(h(f_1^j(x)), h(f_1^j(x_i))\big) < \varepsilon$, for $a_i \leq j \leq b_i$, $1 \leq i \leq k$. Also $g_1^{pm}(y) = h(f_1^{pm}(x)) = h(x) = y$, for each $m \in \mathbb{N}$. Therefore, $(Y, g_{1,\infty})$ has SSP. □

Corollary 3.6. *Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two non-autonomous systems such that $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$. Then $f_{1,\infty}$ has* SSP *if and only if $g_{1,\infty}$ has* SSP.

Theorem 3.7. *Let (X, d_1) and (Y, d_2) be two metric spaces. Then non-autonomous systems $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ have* SSP *if and only if the non-autonomous system $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ has* SSP.

Proof. Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ have SSP and $h_{1,\infty} = f_{1,\infty} \times g_{1,\infty}$. We show that $h_{1,\infty}$ has SSP. Let $\varepsilon > 0$ be arbitrary and $M_1(\varepsilon)$, $M_2(\varepsilon)$ be as in the definition of SSP for $f_{1,\infty}$ and $g_{1,\infty}$, respectively and $M(\varepsilon) = \max\{M_1(\varepsilon), M_2(\varepsilon)\}$. Consider a sequence $a_1 \leq b_1 < \ldots < a_k \leq b_k$ of non-negative integers with

$a_j - b_{j-1} > M(\varepsilon)$ $(2 \leq j \leq k)$ and take any $p > M(\varepsilon) + b_k - a_1$. Let $(x_i, y_i) \in X \times Y$, for $i = 1, \ldots, k$ be arbitrary. Now, as $M(\varepsilon) \geq M_1(\varepsilon)$ and $M(\varepsilon) \geq M_2(\varepsilon)$ and $p > M(\varepsilon) + b_k - a_1 \geq M_l(\varepsilon) + b_k - a_1$, for $l = 1, 2$, therefore using SSP of $f_{1,\infty}$ and $g_{1,\infty}$, there exist $x \in X$ and $y \in Y$ such that

$$d_1(f_1^j(x), f_1^j(x_i)) < \varepsilon \quad \text{and} \quad f_1^{mp}(x) = x, \tag{3.3}$$

$$d_2(g_1^j(y), g_1^j(y_i)) < \varepsilon \quad \text{and} \quad g_1^{mp}(y) = y, \tag{3.4}$$

for each $a_i \leq j \leq b_i$, $i = 1, \ldots, k$ and for each $m \in \mathbb{N}$. Hence, $h_1^{pm}(x, y) = (f_1^{pm} \times g_1^{pm})(x, y) = (f_1^{pm}(x), g_1^{pm}(y)) = (x, y)$, for each $m \in \mathbb{N}$ and for any $p > M(\varepsilon) + b_k - a_1$. Using (3.3), (3.4) and the definition of the product metric $\widetilde{d}$, we get $\widetilde{d}(h_1^j(x, y), h_1^j(x_i, y_i)) < \varepsilon$, for each $a_i \leq j \leq b_i$, $i = 1, \ldots, k$. Thus, $(X \times Y, f_{1,\infty} \times g_{1,\infty})$ has SSP.

Conversely, suppose $h_{1,\infty} = f_{1,\infty} \times g_{1,\infty}$ has SSP. Let $\delta > 0$ be arbitrary and consider a sequence of non-negative integers $a_1 \leq b_1 < \ldots < a_k \leq b_k$ with $a_j - b_{j-1} > M(\delta)$, where $M(\delta)$ is a positive integer as in the definition of SSP for $h_{1,\infty}$. By SSP of $h_{1,\infty}$, for any $(x_i, y_i) \in X \times Y$, $1 \leq i \leq k$, there exists $(x, y) \in X \times Y$ such that $\widetilde{d}(h_1^j(x, y), h_1^j(x_i, y_i)) < \delta$, for $a_i \leq j \leq b_i$ and $h_1^{pm}(x, y) = (x, y)$, for $p > M(\delta) + b_k - a_1$ and for each $m \in \mathbb{N}$. Now,

$$\widetilde{d}\big(h_1^j(x, y), h_1^j(x_i, y_i)\big) = \max\big\{d_1\big(f_1^j(x), f_1^j(x_i)\big), d_2\big(g_1^j(y), g_1^j(y_i)\big)\big\} < \delta,$$

which implies that

$$d_1\big(f_1^j(x), f_1^j(x_i)\big) < \delta \quad \text{and} \quad d_2\big(g_1^j(y), g_1^j(y_i)\big) < \delta,$$

for $a_i \leq j \leq b_i$, $1 \leq i \leq k$. Also,

$$(f_1^{pm} \times g_1^{pm})(x, y) = (f_1^{pm}(x), g_1^{pm}(y)) = (x, y),$$

implying that $f_1^{pm}(x) = x$ and $g_1^{pm}(y) = y$, for any $p > M(\delta) + b_k - a_1$ and for each $m \in \mathbb{N}$. Thus, both $f_{1,\infty}$ and $g_{1,\infty}$ have SSP. □

Remark 3.8. We have the following conclusions.

(a) Above result is true for any finite product by induction.

(b) By similar arguments, Theorems 3.4–3.7 and Corollary 3.6 are also true for WSP and QSP.

4. Specification properties and chaos

In this section, we first show that for a non-autonomous system, QSP is equivalent to topological mixing on a compact metric space. A counterexample is given to justify that result is not true when the family is not surjective. It is shown that a k-periodic non-autonomous system on intervals having WSP or QSP is Devaney chaotic and if the system has SSP, then this result is true in general.

Theorem 4.1. *A non-autonomous system* $(X, f_{1,\infty})$ *has* QSP *if and only if* $(X, f_{1,\infty})$ *is topologically mixing.*

Proof. Suppose $(X, f_{1,\infty})$ has QSP. Let U, $V \subseteq X$ be any two non-empty open subsets, then we can find $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$ and $B_d(y, \varepsilon) \subseteq V$, for any $x \in U$, $y \in V$. Let $M(\varepsilon)$ be as in the definition of QSP. For every $n \geq M(\varepsilon)$, using surjectivity of $(X, f_{1,\infty})$, for $y \in V$, there exists $z \in X$ such that $f_1^n(z) = y$. By QSP of $f_{1,\infty}$, we have existence of a $w \in X$ such that $d(x, w) < \varepsilon$ and $d(f_1^n(w), f_1^n(z)) < \varepsilon$. Therefore, we get $w \in B_d(x, \varepsilon) \subseteq U$ and $f_1^n(w) \in B_d(y, \varepsilon) \subseteq V$ and hence $f_1^n(U) \cap V \neq \emptyset$, for all $n \geq M(\varepsilon)$. Thus, $f_{1,\infty}$ is topologically mixing.

Conversely, let $f_{1,\infty}$ be topologically mixing. Since X is compact, therefore for any open cover $\{x \in X : B_d(x, \varepsilon/2)\}$ of X, there exists $\{x_i\}_{i=1}^k$ such that $X = \bigcup_{i=1}^{k} B_d(x_i, \varepsilon/2)$, for any $\varepsilon > 0$. Now, by topological mixing of $f_{1,\infty}$, there exists a positive integer $M(\varepsilon/2)$ such that

$$f_1^n(B_d(x_i, \varepsilon/2)) \cap B_d(x_j, \varepsilon/2) \neq \emptyset, \tag{4.1}$$

for each $n \geq M(\varepsilon/2)$ and for all $1 \leq i, j \leq k$.

Let y_1, $y_2 \in X$ be arbitrary, then for every $n \geq M(\varepsilon/2)$, y_1, $f_1^n(y_2) \in X$, there exist $1 \leq r, s \leq k$ such that $y_1 \in B_d(x_r, \varepsilon/2)$ and $f_1^n(y_2) \in B_d(x_s, \varepsilon/2)$. Also, by (4.1), there exists $z \in B_d(x_r, \varepsilon/2) \subseteq X$ such that $f_1^n(z) \in B_d(x_s, \varepsilon/2)$. Thus, using triangle inequality we get that $d(z, y_1) < \varepsilon$ and $d(f_1^n(z), f_1^n(y_2)) < \varepsilon$, for all $n \geq M(\varepsilon/2)$ implying that $(X, f_{1,\infty})$ has QSP. □

If we remove the condition of surjection of the family $f_{1,\infty}$ from the hypothesis of the above theorem, then QSP need not be equivalent to topological mixing as justified by the following example.

Example 4.2. Consider the non-autonomous system $([0, 1], f_{1,\infty})$, where $f_n(x) = 1/2^n$, for each $n \in \mathbb{N}$ and $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$. Then $f_{1,\infty}$ is not surjective. Now note that for every $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that $\left|f_1^k(z) - 0\right| < \varepsilon/2$, for all $z \in [0, 1]$ and $k \geq M(\varepsilon)$. For any set of non-negative integers $a_1 \leq b_1 < a_2 \leq b_2$ with $a_2 - b_1 > M(\varepsilon)$ and each pair of points x_1, $x_2 \in [0, 1]$, choosing $z = x_1$, we get that $|f_1^j(z) - f_1^j(x_1)| = 0 < \varepsilon$, for each $a_1 \leq j \leq b_1$ and for each $a_2 \leq j \leq b_2$, we have $\left|f_1^j(z) - f_1^j(x_2)\right| \leq \left|f_1^j(z)\right| + \left|f_1^j(x_2)\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, $f_{1,\infty}$ satisfies all the conditions of WSP for $s = 2$ and hence of QSP but $([0, 1], f_{1,\infty})$ is not topologically transitive and hence cannot be topologically mixing.

Next, we provide an example showing that QSP need not imply WSP.

Example 4.3. Let S^1 be the unit circle, $f_n \colon S^1 \to S^1$ be given by $f_n(e^{i\theta}) = e^{i(n+1)\theta/n}$, for each $n \in \mathbb{N}$ and $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$. Then $(S^1, f_{1,\infty})$ is topologically mixing on compact metric space S^1 with each f_n being surjective and hence, by Theorem 4.1, the given system has QSP. In [12, Example 2.2], authors have proved that $(S^1, f_{1,\infty})$ does not possess WSP. Thus, $(S^1, f_{1,\infty})$ has QSP but does not have WSP.

By [20, Lemma 13] and Theorem 4.1, we have the following result on QSP.

Corollary 4.4. *Let $(X, f_{1,\infty})$ be a k-periodic non-autonomous system and $g = f_k \circ \ldots \circ f_1$. Then $(X, f_{1,\infty})$ has* QSP *if and only if the corresponding autonomous system (X, g) has* QSP.

Remark 4.5. We know that every topologically mixing non-autonomous system has sensitive dependence on initial conditions, so if a non-autonomous system has SSP or WSP or the QSP, then it has sensitive dependence on initial conditions.

Corollary 4.6. *If a non-autonomous system $(X, f_{1,\infty})$ has* WSP (QSP)*, then it is Wiggins chaotic.*

We know that for autonomous dynamical systems, WSP on intervals implies it is Devaney chaotic. In [17], authors have shown that on unit intervals topological transitivity need not imply Devaney chaos for non-autonomous discrete systems. Therefore, WSP on intervals may not imply Devaney chaos in non-autonomous systems. We have following result giving a condition under which WSP (QSP) implies Devaney chaos.

Theorem 4.7. *Let $(I, f_{1,\infty})$ be a k-periodic non-autonomous system, where I is any interval. If the system $(I, f_{1,\infty})$ has* WSP (QSP)*, then it is Devaney chaotic.*

Proof. Since $(I, f_{1,\infty})$ has WSP (QSP), therefore by Remark 3.8, the corresponding autonomous system $(I, f_k \circ \ldots \circ f_1)$ has WSP (QSP) and hence is topologically mixing. Now in autonomous systems, since topological mixing on intervals implies Devaney chaos, therefore $(I, f_k \circ \ldots \circ f_1)$ has dense set of periodic points. By [20, Lemma 1], we get that $(I, f_{1,\infty})$ has dense set of periodic periodic points. Thus, $(I, f_{1,\infty})$ is Devaney chaotic. □

By Theorem 4.1 and Theorem 4.7, we have the following result related to topological mixing for non-autonomous systems.

Corollary 4.8. *Let $(I, f_{1,\infty})$ be a k-periodic non-autonomous system, where I is any interval. If the system $(I, f_{1,\infty})$ is topologically mixing, then it is Devaney chaotic.*

Next, we show that if a non-autonomous system has SSP, then it directly implies the system is Devaney chaotic.

THEOREM 4.9. *If a non-autonomous system* $(X, f_{1,\infty})$ *has* SSP*, then it has dense set of periodic points.*

PROOF. Let $x \in X$ and U be an open neighbourhood of x. We need to show that there is a periodic point in X intersecting U. We have $x \in U$, so there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Let $M(\varepsilon)$ be as in the definition of SSP for $f_{1,\infty}$ and any sequence of non-negative integers $a_1 \leq b_1 < \ldots < a_k \leq b_k$ with $a_j - b_{j-1} > M(\varepsilon)$ and take any $p > M(\varepsilon) + b_k - a_1$. Therefore, by SSP of $(X, f_{1,\infty})$, there exists a periodic point $z \in X$, of period p, that is, $f_1^{pm}(z) = z$, for every $m \in \mathbb{N}$ such that $d\big(f_1^j(z), f_1^j(x_i)\big) < \varepsilon$, for every $x_1, \ldots, x_k \in X$ and for all $a_i \leq j \leq b_i$, $1 \leq i \leq k$. In particular, for $x \in X$ and $a_1 = b_1 = 0$ in the above sequence, we get $d\big(f_1^0(z), f_1^0(x)\big) < \varepsilon$, that is, $d(z, x) < \varepsilon$ which implies that $z \in B(x, \varepsilon) \subseteq U$. Hence, $(X, f_{1,\infty})$ has dense set of periodic points. □

The following example justifies that the converse of the above theorem is not true in general.

EXAMPLE 4.10. Let f be any bijective continuous self map on X. Consider the non-autonomous system $(X, f_{1,\infty})$, where $f_{1,\infty} = \big\{f, f^{-1}, f, f^{-1}, f, f^{-1}, \ldots\big\}$. Since $f_1^{2k}(x) = x$, for each $k \in \mathbb{N}$ and for each $x \in X$, therefore every point of X is periodic and hence $(X, f_{1,\infty})$ has dense set of periodic points. But as $\mathcal{O}_{f_{1,\infty}}(x) = \{x, f(x)\}$, so $(X, f_{1,\infty})$ can never be topologically transitive and hence cannot possess SSP.

COROLLARY 4.11. *If a non-autonomous system* $(X, f_{1,\infty})$ *has* SSP*, then it Devaney chaotic.*

Converse of the above result is not true in general as shown in the following example.

EXAMPLE 4.12. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}} = \big\{\big(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, \ldots\big) : x_i \in \{0,1\}, \text{ for every } i \in \mathbb{Z}\big\}$ with metric

$$\rho(x, y) = \sum_{j=-\infty}^{\infty} \frac{|x_j - y_j|}{2^{|j|}}$$

for any pair $x = (\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, \ldots)$; $y = \big(\ldots, y_{-2}, y_{-1}, \boxed{y_0}, y_1, \ldots\big) \in \Sigma_2$. Define $\sigma : \Sigma_2 \to \Sigma_2$ by $\sigma(x) = (\ldots, x_{-2}, x_{-1}, x_0, \boxed{x_1}, x_2, \ldots)$, where $x = (\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2 \ldots) \in \Sigma_2$, then σ is a homeomorphism and is called the *shift map* on Σ_2. Consider the non-autonomous system $(\Sigma_2, f_{1,\infty})$, where

$$f_{1,\infty} = \big\{\sigma, \sigma^{-1}, \sigma^2, \sigma^{-2}, \sigma^3, \sigma^{-3}, \ldots\big\}.$$

Let U and V be any two non-empty open subsets of Σ_2. Since σ is topologically mixing, therefore there exists $k \in \mathbb{N}$ such that $\sigma^n(U) \cap V \neq \emptyset$, for all $n \geq k$. Now, $f_1^{2k-1} = \sigma^k$, which implies that $f_1^{2k-1}(U) \cap V \neq \emptyset$ and hence $f_{1,\infty}$ is topologically transitive. Note that $N_{f_{1,\infty}}(U,V)=\{2k-1,2k+1,2k+3,\ldots\}$, for non-empty open disjoint subsets U and V of Σ_2, so $f_{1,\infty}$ cannot be topologically mixing. Similarly, using sensitivity of σ, it can be shown that $f_{1,\infty}$ is sensitive. Also, $f_1^{2m}(x)=x$, for each $m\in\mathbb{N}$ and for each $x\in\Sigma_2$, implying that every point of Σ_2 is periodic. Thus, $(\Sigma_2,f_{1,\infty})$ is Devaney chaotic but it is not topologically mixing and hence cannot have any kind of above discussed specification properties.

5. Specification properties of induced systems

In this section, we study specification properties of systems induced on hyperspaces and probability measures spaces. It is proved that if a non-autonomous system has SSP, then the corresponding system induced on the hyperspaces also has SSP and this result holds both ways for WSP (QSP). It is proved that if a non-autonomous system has SSP, then the corresponding system induced on the probability measures space also has SSP and this result holds both ways for QSP.

Theorem 5.1. *If a non-autonomous system* $(X,f_{1,\infty})$ *has* SSP*, then* $(\mathcal{K}(X),\overline{f}_{1,\infty})$ *also has* SSP*.*

Proof. Let $\varepsilon>0$ be arbitrary and $M(\varepsilon/2)$ be a positive integer as in the definition of SSP. Let $A_1,\ldots,A_k\in\mathcal{K}(X)$ and $a_1\le b_1<\ldots<a_k\le b_k$, be any sequence of non-negative integers with $a_j-b_{j-1}>M(\varepsilon/2)$, $2\le j\le k$ and take any $p>M(\varepsilon/2)+b_k-a_1$. We have $\overline{f}_1^j$ is continuous on compact metric space $\mathcal{K}(X)$, for every $j\ge 0$, so for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathcal{H}(A,B)<\delta\ \Rightarrow\ \mathcal{H}\big(\overline{f}_1^j(A),\overline{f}_1^j(B)\big)<\varepsilon/2,\quad \text{for all } A,B\in\mathcal{K}(X). \tag{5.1}$$

Now, $\mathcal{F}(X)$ is dense in $\mathcal{K}(X)$, therefore there exist $B_1,\ldots,B_k\in\mathcal{F}(X)$ such that $\mathcal{H}(A_i,B_i)<\delta$ and hence by (5.1), we get that

$$\mathcal{H}\big(\overline{f}_1^j(A_i),\overline{f}_1^j(B_i)\big)<\varepsilon/2,\quad \text{for } i=1,\ldots,k. \tag{5.2}$$

We can take $n>0$ and points $x_i^l\in X$, $l=1,\ldots,k$, $i=1,\ldots,n$, such that $B_l=\{x_i^l\}_{i=1}^n$ (note that the points x_i^l for a given l may repeat). By SSP of $(X,f_{1,\infty})$, there exists $z_i\in X$ for each x_i^l such that $d\big(f_1^j(z_i),f_1^j(x_i^l)\big)<\varepsilon/2$, for each $a_l\le j\le b_l$, $l=1,\ldots,k$ and $f_1^{mp}(z_i)=z_i$, for each $m\in\mathbb{N}$ and $i=1,\ldots,n$. Let $C=\{z_i\}_{i=1}^n$, then

$$\mathcal{H}\big(\overline{f}_1^j(C),\overline{f}_1^j(B_l)\big)=d\big(f_1^j(z_i),f_1^j(x_i^l)\big)<\varepsilon/2,\quad a_l\le j\le b_l,\ l=1,\ldots,k, \tag{5.3}$$

for each $i=1,\ldots,n$. Thus, using (5.2), (5.3) and triangle inequality, we get that $\mathcal{H}\big(\overline{f}_1^j(C),\overline{f}_1^j(A_l)\big)<\varepsilon$, for each $a_l\le j\le b_l$, $l=1,\ldots,k$ and $\overline{f}_1^{mp}(C)=C$, for each $m\in\mathbb{N}$. Therefore, $\big(\mathcal{K}(X),\overline{f}_{1,\infty}\big)$ has SSP. □

For WSP, we have the following result.

Theorem 5.2. *A non-autonomous system* $(X, f_{1,\infty})$ *has* WSP *if and only if* $\big(\mathcal{K}(X), \overline{f}_{1,\infty}\big)$ *has* WSP.

Proof. Suppose that $(\mathcal{K}(X), \overline{f}_{1,\infty})$ has WSP. Let $\varepsilon > 0$ be arbitrary and $M(\varepsilon)$ be the positive integer as in the definition of WSP. Let $x_1, x_2, \ldots, x_k \in X$ and $a_1 \leq b_1 < \ldots < a_k \leq b_k$, be any sequence of non-negative integers with $a_j - b_{j-1} > M(\varepsilon)$, $2 \leq j \leq k$. Since $(\mathcal{K}(X), \overline{f}_{1,\infty})$ has WSP, therefore for all $A_1, \ldots, A_k \in \mathcal{K}(X)$, there exists $B \in \mathcal{K}(X)$ such that

$$\mathcal{H}\big(\overline{f}_1^j(B), \overline{f}_1^j(A_l)\big) < \varepsilon, \quad a_l \leq j \leq b_l,\ l = 1, \ldots, k. \tag{5.4}$$

Taking $A_i = \{x_i\}$, for each $i = 1, \ldots, k$, we get

$$\mathcal{H}\big(\overline{f}_1^j(B), \overline{f}_1^j(\{x_l\})\big) < \varepsilon, \quad a_l \leq j \leq b_l,\ l = 1, \ldots, k,$$

where $\overline{f}_1^j\big(\{x_l\}\big) = \big\{f_1^j(x_l)\big\}$.

Assume if possible that $d\big(f_1^j(b), f_1^j(x_l)\big) \geq \varepsilon$, for each $b \in B$ and each $a_l \leq j \leq b_l$, $l = 1, \ldots, k$. Therefore, we get $f_1^j(B) \cap B_{\mathcal{H}}\big(\overline{f}_1^j(x_l), \varepsilon\big) = \emptyset$, for each j, that is,

$$f_1^j(B) \cap N\big(\{f_1^j(x_l)\}, \varepsilon\big) = \emptyset \quad \text{and} \quad f_1^j(x_l) \notin N\big(f_1^j(B), \varepsilon\big),$$

which implies that

$$\mathcal{H}\big(\overline{f}_1^j(B), \overline{f}_1^j(\{x_l\})\big) \geq \varepsilon, \quad a_l \leq j \leq b_l,\ l = 1, \ldots, k,$$

which is a contradiction to (5.4). Thus, there exists $b \in B \subseteq X$ such that $d\big(f_1^j(b), f_1^j(x_l)\big) < \varepsilon$, for all $a_l \leq j \leq b_l$, $l = 1, \ldots, k$ implying that $(X, f_{1,\infty})$ has WSP. Converse follows by similar arguments as given in the proof of Theorem 5.1. $\square$

Corollary 5.3. *The non-autonomous system* $\big(\mathcal{K}(X \times Y), \overline{f}_{1,\infty} \times \overline{g}_{1,\infty}\big)$ *has* WSP *if and only if each of* $\big(\mathcal{K}(X), \overline{f}_{1,\infty}\big)$ *and* $\big(\mathcal{K}(Y), \overline{g}_{1,\infty}\big)$ *has* WSP.

Remark 5.4. By similar arguments, Theorem 5.2 and Corollary 5.3 are also true for a non-autonomous system having QSP.

Theorem 5.5. *If a non-autonomous system* $(X, f_{1,\infty})$ *has SSP, then* $(\mathcal{M}(X), \widetilde{f}_{1,\infty})$ *also has* SSP.

Proof. Let $\varepsilon > 0$ be arbitrary and $M(\varepsilon/2)$ be the positive integer as in the definition of SSP. Let $\mu_1, \ldots, \mu_k \in \mathcal{M}(X)$ be given and $a_1 \leq b_1 < \ldots < a_k \leq b_k$, be any sequence of non-negative integers with $a_j - b_{j-1} > M(\varepsilon/2)$, $2 \leq j \leq k$ and take any $p > M(\varepsilon/2) + b_k - a_1$. Now, each $\widetilde{f}_i$ is continuous from $\mathcal{M}(X)$ to itself and $\mathcal{M}(X)$ is compact, therefore for given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\mathcal{D}(\mu, \nu) < \eta \ \Rightarrow\ \mathcal{D}\big(\widetilde{f}_1^j(\mu), \widetilde{f}_1^j(\nu)\big) < \varepsilon/2, \tag{5.5}$$

for all $\mu,\nu \in \mathcal{M}(X)$ and for each $a_l \le j \le b_l$, where $1 \le l \le k$. Now, $\mu_i \in \mathcal{M}(X)$ and $\mathcal{M}_\infty(X)$ is dense in $\mathcal{M}(X)$, therefore there exist $\nu_1,\dots,\nu_k \in \mathcal{M}_n(X)$ such that $\mathcal{D}(\mu_i,\nu_i) < \eta$ and hence by (5.5), we get

$$\mathcal{D}\big(\widetilde{f}_1^j(\mu_i),\widetilde{f}_1^j(\nu_i)\big) < \varepsilon/2, \quad \text{for } i=1,\dots,k. \tag{5.6}$$

Let $\nu_i = \Big(\sum_{l=1}^{n}\delta_{x_l^i}\Big)\Big/n$, for $i=1,\dots,k$. Since $f_{1,\infty}$ satisfies SSP, therefore there exist $z_l \in X$ such that $f_1^{pm}(z_l)=z_l$, for every $m\in\mathbb{N}$ and $d\big(f_1^j(z_l),f_1^j(x_l^i)\big) < \varepsilon/2$, for each $a_i \le j \le b_i$, $i=1,\dots,k$ and $l=1,\dots,n$. Let $\rho = \Big(\sum_{l=1}^{n}\delta_{z_l}\Big)\Big/n$ and A be any Borel measurable set, then

$$\begin{aligned}\widetilde{f}_1^{pm}(\rho)(A) &= \rho\big(f_1^{-pm}(A)\big) = \frac{1}{n}\big(\delta_{z_1}+\dots+\delta_{z_n}\big)\big(f_1^{-pm}(A)\big)\\ &= \frac{1}{n}\big(\delta_{z_1}\big(f_1^{-pm}(A)\big)+\dots+\delta_{z_n}\big(f_1^{-pm}(A)\big)\big) = \frac{1}{n}\sum_{l=1}^{n}\delta_{z_l}(A),\end{aligned}$$

because $\delta_{z_l}\big(f_1^{-pm}(A)\big) = \delta_{z_l}(A)$ using $f_1^{pm}(z_l)=z_l$ and therefore $\widetilde{f}_1^{pm}(\rho)(A) = \rho(A)$, for each $A\in\mathcal{B}(X)$. Hence, $\widetilde{f}_1^{pm}(\rho)=\rho$, for every $m\in\mathbb{N}$ and we have

$$\mathcal{D}\big(\widetilde{f}_1^j(\rho),\widetilde{f}_1^j(\nu_i)\big) < \varepsilon/2, \quad a_i \le j \le b_i,\ i=1,\dots,k. \tag{5.7}$$

Thus, using (5.6), (5.7) and triangle inequality, we get that $\mathcal{D}\big(\widetilde{f}_1^j(\rho),\widetilde{f}_1^j(\mu_i)\big) < \varepsilon$, for each $a_i \le j \le b_i$, $i=1,\dots,k$ implying that $(\mathcal{M}(X),\widetilde{f}_{1,\infty})$ has SSP. □

Remark 5.6. By similar arguments, Theorem 5.5 also holds for weak specification property.

Next, we show that for the systems having QSP above result holds both ways.

Theorem 5.7. *A non-autonomous system $(X,f_{1,\infty})$ is topological mixing if and only if $\big(\mathcal{M}(X),\widetilde{f}_{1,\infty}\big)$ is topological mixing.*

Proof. First suppose that $(X,f_{1,\infty})$ is topological mixing. Let W_1 and W_2 be any two non-empty open subsets of $\mathcal{M}(X)$. Since $\mathcal{M}_\infty(X)$ is dense in $\mathcal{M}(X)$, therefore there exists $\mu_j\in\mathcal{M}_\infty(X)$ such that $\mu_j = \Big(\sum_{i=1}^{m}\delta_{x_i^j}\Big)\Big/m \in W_j$, for each $j=1,2$. We can choose open neighbourhood U_i^j of x_i^j in such a manner that if $y_i^j \in U_i^j$, then $\Big(\sum_{i=1}^{m}\delta_{y_i^j}\Big)\Big/m \in W_j$, for each $j=1,2$. It is easy to see that if $f_{1,\infty}$ is topological mixing, then $\underbrace{f_{1,\infty}\times\dots\times f_{1,\infty}}_{m\text{-times}}$ is also topological mixing. Therefore, there exists $k\in\mathbb{N}$ such that $f_1^n(U_i^1)\cap U_i^2 \ne \emptyset$, for all $n\ge k$ and for every $i=1,\dots,m$. Let $z_i^2 \in f_1^n(U_i^1)$ and $z_i^2\in U_i^2$, that is, $f_1^{-n}(z_i^2)\in U_i^1$ and $z_i^2\in U_i^2$, for all $n\ge k$ and for every $i=1,\dots,m$ and hence we get that $\nu = \Big(\sum_{i=1}^{m} \delta_{z_i^2}\Big)\Big/ m \in W_2$ and $\widetilde{f}_1^{-n}(\nu) = \Big(\sum_{i=1}^{r} \delta_{f_1^{-n}(z_i^2)}\Big)\Big/ m \in W_1$. Thus, $\widetilde{f}_1^n(W_1) \cap W_2 \neq \emptyset$, for all $n \geq k$, giving that $(\mathcal{M}(X), \widetilde{f}_{1,\infty})$ is topological mixing.

Conversely, let U, V be non-empty open subsets of X. Let

$$W_1 = \{\mu \in \mathcal{M}(X) : \mu(U) > 4/5\} \quad \text{and} \quad W_2 = \{\mu \in \mathcal{M}(X) : \mu(V) > 4/5\},$$

then W_1 and W_2 are non-empty open subsets of $\mathcal{M}(X)$. If $\mu_n \to \mu$ such that $\mu_n \in \mathcal{M}(X) \setminus W_1$, then $\mu_n(U) \leq 4/5$ implying that $\mu(U) \leq 4/5$ and hence $\mu \in \mathcal{M}(X) \setminus W_1$. Thus, W_1 is open in $\mathcal{M}(X)$ and similarly W_2 is also open in $\mathcal{M}(X)$. Now, since the system $(\mathcal{M}(X), \widetilde{f}_{1,\infty})$ is topologically mixing, therefore there exists $k \in \mathbb{N}$ such that $\widetilde{f}_1^n(W_1) \cap W_2 \neq \emptyset$, for all $n \geq k$. This implies that there exists $\nu \in W_1$ with $\widetilde{f}_1^n(\nu) \in W_2$ and hence

$$\widetilde{f}_1^n(\nu)(V) = \nu(f_1^{-n}(V)) > 4/5, \quad \text{for all } n \geq k.$$

Also, as $\nu(U) > 4/5$, so $f_1^n(U) \cap V \neq \emptyset$, for all $n \geq k$. Thus, $(X, f_{1,\infty})$ is topologically mixing. □

By above theorem and Theorem 4.1, we have the following result.

Corollary 5.8. *A non-autonomous system $(X, f_{1,\infty})$ has QSP if and only if $(\mathcal{M}(X), \widetilde{f}_{1,\infty})$ has QSP.*

Corollary 5.9. *The non-autonomous system $\big(\mathcal{M}(X \times Y), \widetilde{f}_{1,\infty} \times \widetilde{g}_{1,\infty}\big)$ has QSP if and only if each of $(\mathcal{M}(X), \widetilde{f}_{1,\infty})$ and $(\mathcal{M}(Y), \widetilde{g}_{1,\infty})$ has* QSP.

Now, we give an example supporting results of this section.

Example 5.10. Consider a 3-periodic non-autonomous system $(X, f_{1,\infty})$, where $f_1 = \sigma$, $f_2 = \sigma^{-2}$, $f_3 = \sigma^2$ and $X = \Sigma_2$, that is, $f_{1,\infty} = \{\sigma, \sigma^{-2}, \sigma^2, \sigma, \sigma^{-2}, \sigma^2, \ldots\}$, where σ is the shift map as defined in Example 4.12. Since, $f_3 \circ f_2 \circ f_1 = \sigma$ and σ has QSP, therefore the induced autonomous system $(X, f_3 \circ f_2 \circ f_1)$ has QSP and hence by Corollary 4.4, $(\Sigma_2, f_{1,\infty})$ has QSP. Thus, by Remark 5.4 and Corollary 5.8, both the systems $(\mathcal{K}(\Sigma_2), \overline{f}_{1,\infty})$ and $(\mathcal{M}(\Sigma_2), \widetilde{f}_{1,\infty})$ have QSP. Also, if $g_1 \colon I \to I$ is a continuous map on a closed unit interval I, given by $g_1 = 4x(1-x)$ and $g_2 \colon I \to I$ is the identity map and $(I, g_{1,\infty})$ is the corresponding 2-periodic non-autonomous system, then $(I, g_{1,\infty})$ has QSP. Therefore, by Remark 3.8, we have $(\Sigma_2 \times I, f_{1,\infty} \times g_{1,\infty})$ has QSP and by Remark 5.4 and Corollary 5.9 respectively, the systems $(\mathcal{K}(\Sigma_2 \times I), \overline{f}_{1,\infty} \times \overline{g}_{1,\infty})$ and $(\mathcal{M}(\Sigma_2 \times I), \widetilde{f}_{1,\infty} \times \widetilde{g}_{1,\infty})$ have QSP.

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