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EXISTENCE AND STABILITY OF STANDING WAVES FOR THE CHOQUARD EQUATION WITH PARTIAL CONFINEMENT

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Abstract. In this paper we study the existence and orbital stability of the Choquard equation with partial confinement. This type equation originates from Fröhlich and Pekar's model of the polaron, where free electrons in an ionic lattice interact with phonons associated with deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole). On the one hand, we prove the existence of global minimizer of the associate energy functional subject to the L^2-constraint. On the other hand, we discuss the orbital stability and asymptotic behavior of the global minimizer.

1. Introduction and main results

In this paper we are concerned with the existence, stability, qualitative and symmetry properties of standing waves associated with the following Cauchy problem

$$\begin{cases} -i\partial_t u - \Delta u + (x_1^2 + \ldots + x_{N-1}^2)u = (J_\alpha * |u|^p)\,|u|^{p-2}u, \\ u(0,x) = \phi(x), \quad (t, x_1, \ldots, x_N) \in \mathbb{R}\times\mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $J_\alpha\colon \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined at each point $x \in \mathbb{R}^N \setminus \{0\}$ by

$$J_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{and} \quad A_\alpha = \frac{\Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)\pi^{N/2}\, 2^\alpha},$$

$\alpha \in (p+N-4, N)$, $N \geq 3$ and $(N-2)/(N+\alpha) < 1/p \leq N/(N+\alpha+2)$.

In general, the Cauchy problem (1.1) is a model large system of non-relativistic bosonic atoms and molecules under an attractive interaction that is weaker and has a longer range than that of the nonlinear Schrödinger equation (where the interaction potential J_α is formally Dirac's delta at the origin) [19]. The problem (1.1) arises as a mean-field limit of a bosonic system with attractive two-body interactions, this limit can be taken rigorously in many cases, see [19] and [29]. In showing that his polaron model arises as an asymptotic limit of the Fröhlich polaron, Pekar had conjectured that the groundstate level of the Pekar polaron problem should be characterised in terms of Brownian motion. This conjecture was proved by Donsker and Varadhan [16], [17]. Another mathematical analysis of the asymptotics of the Fröhlich polaron was provided by Lieb and Thomas [33]. On the other hand, the Choquard equation is also known as the Schrödinger–Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity, for instance, see [15], [26], [27], [45], [46] (also see [48], [49] for relativistic versions). Such a model has been also proposed for boson stars [51] and for the collapse of galaxy fluctuations of scalar field dark matter [23], [24]. Further models have been developed including a gravitomagnetic potential [38] and self-field coupling [18].

In the past few years, many mathematicians are devoted to study the existence of stationary solutions of (1.1), i.e.

$$-\Delta u + V(x)u = (J_\alpha * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

where $\alpha \in (0, N)$ and $(N-2)/(N+\alpha) < 1/p \leq N/(N+\alpha+2)$. For example, one can see the papers [10]–[13], [20]–[22], [40]–[42], [2], [50], [8], [52], [32], [34], [36] and the references therein. Particularly, for $N = 3$, $\alpha = 2$ and $p = 2$ in the equation (1.2) is the Choquard–Pekar equation which goes back to the 1954's work by Pekar on quantum theory of a Polaron at rest [14], [44] and to 1976's model of Choquard of an electron trapped in its own hole, in an approximation to Hartree–Fock theory of one-component plasma [30]. In the 1990s the same equation reemerged as a model of self-gravitating matter [27], [39] and is known in that context as the Schrodinger–Newton equation. The paper [37] proved the unique positive solution of (1.2). Recently, the papers [22], [40] proved the existence of positive and nodal solution of (1.2). For more results on this direction one can refer to [41], [42] and the survey paper [43].

As far as we known, the existence and stability of standing states for (1.1) in the presence of a partial confinement has not been studied in the literature.

In the present paper we shall prove the existence of global minimizer of the associate energy functional subject to the L^2-constraint and orbital stability of global minimizer for the system (1.1).

In order to state the main results we first need some definitions that will be used in the rest of the paper. Let

$$X=\left\{u\in H^1(\mathbb{R}^N,\mathbb{C}):\int_{\mathbb{R}^N}(x_1^2+\ldots+x_{N-1}^2)u^2\,dx<\infty\right\}. \tag{1.3}$$

The corresponding norm is denoted by

$$\|u\|_X^2=\int_{\mathbb{R}^N}\left(|\nabla u|^2+(x_1^2+\ldots+x_{N-1}^2)|u|^2\right)dx. \tag{1.4}$$

Clearly, the energy is defined by

$$\Phi(u)=\frac{1}{2}\|u\|_X^2-\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \tag{1.5}$$

Let us first recall the following Gagliardo–Nirenberg inequality (see [53, (2.13)])

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}&\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\ &\le S\left(\int_{\mathbb{R}^N}|u|^2\right)^{((N+\alpha)-p(N-2))/2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{(pN-(N+\alpha))/2},\end{aligned} \tag{1.6}$$

where $\alpha\in(0,N)$, $(N-2)/(N+\alpha)\le 1/p\le N/(N+\alpha)$ and $S>0$ is a constant. From this we know that the functional Φ is well-defined in Y when $\alpha\in(0,N)$ and $(N-2)/(N+\alpha)\le 1/p\le N/(N+\alpha)$. Here the space Y is defined by

$$Y=\left\{u\in X:\int_{\mathbb{R}^N}|u|^2<\infty\right\}, \tag{1.7}$$

and the norm $\|\cdot\|_Y$ is defined by

$$\|u\|_Y^2=\|u\|_X^2+|u|_2^2. \tag{1.8}$$

We define the following sets which will be used later.

$$\mathbb{S}_r=\left\{u\in X:\int_{\mathbb{R}^N}u^2\,dx=r^2\right\}\quad\text{and}\quad\mathbb{B}_R=\left\{u\in X:\|u\|_X^2\le R^2\right\}. \tag{1.9}$$

Set

$$\mathbb{I}_r=\inf_{u\in\mathbb{S}_r}\Phi(u). \tag{1.10}$$

If $N/(N+\alpha+2)<1/p\le N/(N+\alpha)$ (subcritical case), by the Gagliardo–Nirenberg inequality, we find that the energy functional $\Phi(u)$ restricted to $\mathbb{S}_r$ is bounded from below, that is $\mathbb{I}_r>-\infty$. Therefore, we consider the minimizing problem (1.10). It is easy to see that the function $r\mapsto\mathbb{I}_r$ is continuous for any $r\ge 0$.

But if we assume that $(N-2)/(N+\alpha)<1/p\le N/(N+\alpha+2)$ (supercritical case), then one has that $\mathbb{I}_r=-\infty$. We shall borrow an idea of [3] to construct orbitally stable solutions by considering a suitable localized version of the minimization problems above. More precisely, for every given $M > 0$ we consider the following localized minimization problems:

$$\mathbb{J}_r^M = \inf_{u \in \mathbb{S}_r \cap \mathbb{B}_M} \Phi(u). \tag{1.11}$$

The aim of this work is to show that for every $M > 0$ there exists $\sigma_0 > 0$ such that $\mathbb{S}_r \cap \mathbb{B}_M \neq \emptyset$ for $r < \sigma_0$ and moreover all minimizing sequences to $\mathbb{J}_r^M$ are compact, up to the action of translations with respect to x_3, provided that $r < \sigma_0$. In order to guarantee that the minimizers of (1.11) are critical points of $\Phi(u)$ restricted on $\mathbb{S}_r$ it is also necessary to show that they do not belong to the boundary of $\mathbb{S}_r \cap \mathbb{B}_M$. Then it is classical, see for example [28, Proposition 14.3] (or [55]), that for any minimizer u there exists $\gamma = \gamma(u) \in \mathbb{R}$ such that the Euler–Lagrange equation

$$-\Delta u + \big(x_1^2 + \ldots + x_{N-1}^2\big)u = \gamma u + \bigg(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\,dy\bigg)|u|^{p-2}u \tag{1.12}$$

holds. The associated standing wave is then given by $e^{-i\gamma t}u(x)$. For the case $N = 3, \alpha = 2$ and $p = 2$, the paper [9] establishes the orbital stability of some standing waves of the Choquard equation (1.1). For the sake of completeness, we recall the notion of stability of a set $\mathbb{M} \subset X$ under the flow associated with (1.1) namely

$$\forall\, \varepsilon > 0 \quad \exists\, \eta > 0 \quad \inf_{v \in \mathbb{M}} \|\phi - v\|_Y < \eta \Rightarrow \sup_{t \in \mathbb{R}} \inf_{v \in \mathbb{M}} \|u(t) - v\|_Y < \varepsilon. \tag{1.13}$$

First, we consider the subcritical case and have the following main results for this case.

Theorem 1.1. *Assume that* $\alpha \in (p+N-4, N)$, $N \geq 3$ *and* $N/(N+\alpha+2) < 1/p \leq N/(N+\alpha)$. *Then any minimizing sequence to* (1.10), *up to translation, is compact in* X.

Remark 1.2. Throughout the paper we assume that $N \geq 3$. If $N = 2$, we infer from [4] that the Riesz potential defined at each point $x \in \mathbb{R}^2 \setminus \{0\}$ by $J_\alpha(x) = \ln|x|/(2\pi)$. Then (1.1) becomes the logarithmic Choquard equation. We shall back on this problem in the future.

Next we state the main results for the supercritical case.

Theorem 1.3. *Assume that* $\alpha \in (N+p-4, N)$, $N \geq 3$ *and*

$$\max\left\{\frac{N-2}{N+\alpha}, \frac{1}{4}\right\} < \frac{1}{p} \leq \frac{N}{N+\alpha+2}.$$

For each $M > 0$ *there exists* $\sigma_0 = \sigma_0(M) > 0$ *such that the following conclusions hold*:

(a) $\mathbb{S}_r \cap \mathbb{B}_M \neq \emptyset$, *for all* $r < \sigma_0$.

(b) $\emptyset \neq \mathbb{M}_r^M \subset \mathbb{B}_{rM}$ (*for all* $r < \sigma_0$), *where* $\mathbb{M}_r^M = \{u \in \mathbb{S}_r \cap \mathbb{B}_M : \Phi(u) = \mathbb{J}_r^M\}$.

(c) *The set* $\mathbb{M}_r^M$ *is stable under the flow associated with* (1.1) *for any* $r < \sigma_0$.

Remark 1.4. (a) We adopt the similar argument of [9] to consider the proof of the stability. If for all $r < r_0$, all minimizing sequence $(u_n) \in \mathbb{S}_r \cap \mathbb{B}_M$ such that $\lim_{n\to\infty} \Phi(u_n) = \mathbb{J}_r^M$, there exists $(h_n) \in \mathbb{R}$ such that $(u_n(x_1, \dots, x_{N-1}, x_N - h_n))$ is compact in X. Notice that once this is established then we will obtain the existence of minimizers.

(b) The main difficulty in Theorem 1.1 and Theorem 1.3 is the lack of compactness, due to the translation invariance w.r.t. x_N. In fact, if one replace $x_1^2 + \dots + x_{N-1}^2$ by trapping potential $x_1^2 + \dots + x_N^2$, then one benefits from the compactness of the inclusion of X into $L^q(\mathbb{R}^N)$ where $q \in [2, 2N/(N-2))$, see [56, Lemma 3.1] and the proof of Theorems 1.1 and 1.3 would be rather simple. We overcome this lack of compactness by using a concentration-compactness arguments, then we have to adapt in a suitable minimization problem.

(c) It is worth to point out that our main results can be easily generalized. On the one hand, we can replace the confinement $(x_1^2 + \dots + x_{N-1}^2)$ by the potential $V\left(\sqrt{x_1^2 + \dots + x_{N-1}^2}\right)$, where $V\colon (0,\infty) \to \mathbb{R}$ is strictly increasing and unbounded. On the other hand, by a direct adaptation of our proofs we can extend our results as to cover the general situation where

$$\begin{cases} -i\partial_t u - \Delta u + (x_1^2 + \dots + x_{N-d}^2)u = (J_\alpha * |u|^p)|u|^{p-2}u, \\ u(0,x) = \phi(x), \quad (t, x_1, \dots, x_N) \in \mathbb{R} \times \mathbb{R}^N, \quad 1 \le d < N, \end{cases}$$

and $\max\{(N-2)/(N+\alpha), 1/(3+d)\} < 1/p \le N/(N+\alpha+2)$. One can prove the same results as in Theorem 1.3.

Next, we provide some results for the properties of the minimizers in Theorem 1.3.

Theorem 1.5. *Every minimizer obtained in Theorem* 1.3 (*that is in principle* $\mathbb{C}$*-valued*) *is of the form* $e^{i\theta} w(x_1, \dots, x_N)$ *where* w *is a positive real valued minimizer and* $\theta \in \mathbb{R}$. *Moreover, for some* $h \in \mathbb{R}$, $w(x_1, \dots, x_{N-1}, x_N - h)$ *is radially symmetric and decreasing with respect to* $(x_1, \dots, x_{N-1})$. *In addition, for every fixed* $M > 0$, *for every* $r < \sigma_0(M)$ *and for every* $u \in \mathbb{M}_r^M$ *there exists* $\gamma = \gamma(w) > 0$ *such that*

$$-\Delta u + (x_1^2 + \dots + x_{N-1}^2)u = \gamma u + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\, dy\right)|u|^{p-2}u \tag{1.14}$$

with the estimates

$$(1 - Cr^{2p-2})\Gamma_0 \le \gamma < \Gamma_0 \tag{1.15}$$

where $C > 0$ is an universal constant and

$$\Gamma_0 = \inf\bigg(\operatorname{spec}\bigg(-\sum_{i=1}^{N}\partial_{x_i}^2 + (x_1^2 + \ldots + x_{N-1}^2)\bigg)\bigg).$$

Moreover, we have

$$\sup_{\mathbb{M}_r^M} \|u(x_1, \ldots, x_N) - \psi_0(x_N)\phi_0(x_1, \ldots, x_{N-1})\|_X = o(r) \tag{1.16}$$

where $\phi(x_1, \ldots, x_{N-1})$ is the unique normalized positive eigenvector of the quantum harmonic oscillator

$$\sum_{i=1}^{N-1}\partial_{x_i}^2 + x_1^2 + \ldots + x_{N-1}^2$$

and

$$\psi_0(x_N) = \int_{\mathbb{R}^N} u(x_1, \ldots, x_N)\,\phi_0(x_1, \ldots, x_{N-1})\,dx_1, \ldots, dx_{N-1}. \tag{1.17}$$

Finally, we can show that our solutions are ground state in the following sense with the properties given in Theorem 1.5.

Definition 1.6. Let $r > 0$ be arbitrary, we say that $u \in \mathbb{S}_r$ is a ground state if

$$\Phi'|_{\mathbb{S}_r}(u) = 0 \quad \text{and} \quad \Phi(u) = \inf\{\Phi(u) : u \in \mathbb{S}_r,\ \Phi'|_{\mathbb{S}_r}(u) = 0\}$$

Theorem 1.7. *Let $(N-2)/(N+\alpha) < 1/p \le N/(N+\alpha+2)$. Then, for any fixed $M > 0$ and sufficiently small $r > 0$, the local minimizers $u \in \mathbb{M}_r^M$ are ground states.*

2. Proof of Theorem 1.1

In this section we main focus on the subcritical case and give the proof of Theorem 1.1. To accomplish this we first recall the spectral conclusion. Let

$$\Gamma_0 = \inf_{\int_{\mathbb{R}^N}|u|^2=1}\bigg(\int_{\mathbb{R}^N}\big(|\nabla_x u|^2 + \big(x_1^2 + \ldots + x_{N-1}^2\big)u^2\big)\,dx\bigg) \tag{2.1}$$

and

$$\gamma_0 = \inf_{\int_{\mathbb{R}^{N-1}}|\psi|^2\,d\widetilde{x}=1}\bigg(\int_{\mathbb{R}^N}\big(|\nabla_{\widetilde{x}}\psi|^2 + \big(x_1^2 + \ldots + x_{N-1}^2\big)\psi^2\big)\,d\widetilde{x}\bigg), \tag{2.2}$$

where $\widetilde{x} = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. From [3, Lemma 2.1], we know that

$$\gamma_0 = \Gamma_0. \tag{2.3}$$

For the notation convenience, we define $\mu := r^2$. Correspondingly, it follows that

$$\widetilde{\mathbb{S}}_\mu := \mathbb{S}_r = \left\{ u \in X : \int_{\mathbb{R}^N} u^2 \, dx = r^2 = \mu > 0 \right\} \tag{2.4}$$
$$\text{and} \quad \widetilde{\mathbb{I}}_\mu := \mathbb{I}_r = \inf_{u \in \widetilde{\mathbb{S}}_\mu} \Phi(u).$$

To accomplish our results we shall apply the concentration compactness principle [35] to the minimizing sequence. That is, we should exclude the possibilities of the vanishing and dichotomy cases. In order to eliminate vanishing case we shall borrow an idea of [3, Lemma 2.1]. Finally, we rule out the dichotomy case by proving the following strict subadditivity inequality

$$\widetilde{\mathbb{I}}_\mu < \widetilde{\mathbb{I}}_{\mu-\mu_0} + \widetilde{\mathbb{I}}_{\mu_0} \quad \text{for } 0 < \mu_0 < \mu.$$

The next lemma state that any minimizing sequence to (1.10) is nonvanishing.

LEMMA 2.1. *Assume that* $\alpha \in (p+N-4, N)$, $N \geq 3$ *and* $N/(N+\alpha+2) < 1/p \leq N/(N+\alpha)$. *Let* $\{u_n\} \subset \widetilde{\mathbb{S}}_\mu$ *be a minimizing sequence to* (2.4). *Then there exists* $\delta > 0$ *such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \geq \delta.$$

PROOF. Arguing by contradiction, without restriction we assume that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy = o_n(1).$$

Thus, we have that

$$\widetilde{\mathbb{I}}_\mu = \Phi(u) + o_n(1) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \big(x_1^2 + \ldots + x_{N-1}^2\big)|u_n|^2 \, dx + o_n(1) \tag{2.5}$$
$$\geq \frac{\mu \Gamma_0}{2} + o_n(1).$$

where Γ_0 is defined by (2.1). It is clear that γ_0 can be achieved by $w \in H^1(\mathbb{R}^{N-1})$. From the definition (2.2), let

$$\int_{\mathbb{R}} |\varphi|^2 \, dx_N = \mu, \quad \varphi \in H^1(\mathbb{R}).$$

Then, for $\lambda > 0$ we set, $u(x) := w(\widetilde{x})\varphi_\lambda(x_N) \in \widetilde{\mathbb{S}}_\mu$, where $\varphi_\lambda(x_N) = \lambda\varphi(\lambda^2 x_N)$. Thus, it follows that

$$\begin{aligned} \Phi(u) = {} & \frac{\mu\gamma_0}{2} + \frac{1}{2} \int_{\mathbb{R}} |\partial_{x_N} \varphi_\lambda|^2 \, dx \\ & - \frac{\mu}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(\widetilde{x})\varphi_\lambda(x_N)|^p |w(y')\varphi_\lambda(y_N)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \\ = {} & \frac{\mu\Gamma_0}{2} + \frac{\lambda^4}{2} \int_{\mathbb{R}} |\partial_{x_N} \varphi|^2 \, dx \end{aligned} \tag{2.6}$$

$$-\frac{\lambda^{2(p+N-\alpha-2)}}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(|w(\widetilde{x})\varphi(x_N)|^p|w(\widetilde{y})\varphi(y_N)|^p \\ /\big(|\lambda^\beta x_1-\lambda^\beta y_1|^2+\ldots+|\lambda^\beta x_{N-1}-\lambda^\beta y_{N-1}|^2 \\ +|x_N-y_N|^2\big)^{(N-\alpha)/2}\big)\,dx\,dy.$$

Since $N/(N+\alpha+2)<1/p\leq N/(N+\alpha)$ and $\alpha>p+N-4$, it follows from (2.6) that $\widetilde{\mathbb{I}}_\mu\leq\Phi(u)<\mu\Gamma_0/2$ whenever $\lambda>0$ small enough. However, it contradicts with (2.5), which completes the proof. □

Lemma 2.2. *Assume that $\alpha\in(p+N-4,N)$, $N\geq 3$ and $N/(N+\alpha+2)<1/p\leq N/(N+\alpha)$. Let $\{u_n\}\subset\widetilde{\mathbb{S}}_\mu$ be a minimizing sequence to (2.4). Then there exist a sequence $\{h_n\}\subset\mathbb{R}$ and $u\in X\setminus\{0\}$ such that $u_n(\widetilde{x},x_N-h_n)\rightharpoonup u$ in X as $n\to\infty$.*

Proof. By Lemma 2.1 we know that u_n is bounded in X. Furthermore, we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\geq\delta_1, \tag{2.7}$$

where $\delta_1>0$. To accomplish the proof, we first prove that the Gagliardo–Nirenberg inequality (1.6) still holds on $T_h:=\mathbb{R}^{N-1}\times[h,h+1)$ for $h\in\mathbb{N}$. Let $\chi_h(x)=1$ for $x_N\in[h,h+1)$ and $\chi_h(x)=0$ for $x_N\notin[h,h+1)$. We define $\widehat{u}(x)=\chi_h(x)u(x)$. Then $\widehat{u}\in\mathbb{R}^N$. From the classical Hardy–Littlewood–Sobolev inequality (see [31, Theorem 4.3]), one infers that there exists a constant $C_0>0$ such that

$$\int_{T_h}\int_{T_h}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ =\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widehat{u}^p(x)\widehat{u}^p(y)}{|x-y|^{N-\alpha}}\,dx\,dy\leq C_0\bigg(\int_{\mathbb{R}^N}|\widehat{u}|^{2pN/(N+\alpha)}\bigg)^{(N+\alpha)/N}, \tag{2.8}$$

where $\alpha\in(0,N)$ and $(N-2)/(N+\alpha)\leq 1/p\leq N/(N+\alpha)$. On the other hand, by interpolation inequality, we have that

$$\begin{aligned}\bigg(\int_{\mathbb{R}^N}|\widehat{u}|^{4N/(N+2)}\bigg)^{(N+2)/(4N)}&\leq\bigg(\int_{\mathbb{R}^N}|\widehat{u}|^2\bigg)^{\theta/2}\bigg(\int_{\mathbb{R}^N}|\widehat{u}|^{2^*}\bigg)^{(1-\theta)/2^*}\\&\leq\bigg(\int_{T_h}|u|^2\bigg)^{\theta/2}\bigg(\int_{\mathbb{R}^N}|u|^{2^*}\bigg)^{(1-\theta)/2^*}\\&\leq C_1\bigg(\int_{T_h}|u|^2\bigg)^{\theta/2}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\bigg)^{(1-\theta)/2},\end{aligned} \tag{2.9}$$

where $2^*=2N/(N-2)$ $(N\geq 3)$, $\theta=(N+\alpha-p(N-2))/(2p)$ and $C_1>0$ is a constant. Combining (2.8) and (2.9), we know that

$$\int_{T_h}\int_{T_h}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\leq C\bigg(\int_{T_h}|u_n|^2\,dx\bigg)^{((N+\alpha)-p(N-2))/2}\|u_n\|_X^{pN-(N+\alpha)}.$$

Note that $\mathbb{R}^N = \bigcup_{-\infty}^{\infty} \mathbb{R}^{N-1} \times [h, h+1)$. Summing the above inequality with respect to $h \in \mathbb{N}$ and by (2.7), one sees that

$$\delta_1 \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \leq C\bigg(\sup_{h\in\mathbb{Z}}\int_{T_h}|u_n|^2\,dx\bigg)^{((N+\alpha)-p(N-2))/2}\|u_n\|_X^{pN-(N+\alpha)}. \tag{2.10}$$

We infer from the boundedness of u_n in X and (2.10) that for any $n \in \mathbb{N}$ there admit $h_n \in \mathbb{N}$ and $\delta_2 > 0$ such that

$$\int_{T_{h_n}} |u_n|^2\,dx \geq \delta_2.$$

Set $w_n(x) := u_n(\widetilde{x}, x_N - h_n)$. Then one sees that $\int_{T_0} |w_n|^2\,dx \geq \delta_2$. Since the embedding $H^1(T_0) \hookrightarrow L^2(T_0)$ is compact and $w_n(x)$ is bounded in X, there exists $u \in X \setminus \{0\}$ such that $w_n \rightharpoonup u$ in X as $n \to \infty$. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume $\{u_n\} \subset \widetilde{\mathbb{S}}_\mu$ be a minimizing sequence to (2.4). Recalling Lemma 2.2, there is a sequence $\{h_n\} \subset \mathbb{R}$ such that $w_n(x) := u_n(\widetilde{x}, x_N - h_n)$ has a nontrivial weak limit u in X. Note that $\{w_n\} \subset \widetilde{\mathbb{S}}_\mu$ is also a minimizing sequence to (2.4). At this point, to check the compactness of $\{w_n\}$ it suffices to prove $u \in \widetilde{\mathbb{S}}_\mu$. To do this, we use the contradiction argument. Assume that $0 < \mu_0 := \|u\|_2^2 < \mu$. In order to obtain the contradiction, we need prove the subadditivity inequality.

By the Brezis–Lieb Lemma (see [40, Lemma 2.4] and [6] is the classical Brezis–Lieb Lemma) for the functional with the nonlocal term, we get that

$$\|w_n\|_X^2 = \|w_n - u\|_X^2 + \|u\|_X^2 + o_n(1),$$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)-u(x)|^p|w_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy + o_n(1),\end{aligned}$$

for $1 \leq p < \infty$. Therefore, one sees that

$$\Phi(w_n) = \Phi(w_n - u) + \Phi(u) + o_n(1).$$

Thus, we prove that

$$\widetilde{\mathbb{I}}_\mu \geq \widetilde{\mathbb{I}}_{\mu-\mu_0} + \widetilde{\mathbb{I}}_{\mu_0}, \tag{2.11}$$

where we employed the continuity of $\widetilde{\mathbb{I}}_\mu$ in regard to $\mu > 0$. Next we claim that

$$\widetilde{\mathbb{I}}_{\theta\mu} < \theta\,\widetilde{\mathbb{I}}_\mu \quad \text{for } \theta > 1. \tag{2.12}$$

Indeed, a direct computation shows that

$$\begin{aligned}\widetilde{\mathbb{I}}_{\theta\mu} \le \Phi(v_n) &= \frac{\theta}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 + \big(x_1^2 + \ldots + x_{N-1}^2\big)|w_n|^2\,dx \\ &\quad - \frac{\mu\theta^p}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &= \theta\Phi(u_n) + \frac{\mu}{2p}(\theta - \theta^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy,\end{aligned}$$

where $v_n := \theta^{1/2}w_n \in \widetilde{\mathbb{S}}_{\theta\mu}$. As a result of $\theta > 1$, $p > 1$, then it follows from Lemma 2.1 that (2.12) necessarily holds.

Now we are ready to find the contradiction. We infer from (2.12) that

$$\widetilde{\mathbb{I}}_\mu = \frac{\mu - \mu_0}{\mu}\,\widetilde{\mathbb{I}}_\mu + \frac{\mu_0}{\mu}\,\widetilde{\mathbb{I}}_\mu < \widetilde{\mathbb{I}}_{\mu-\mu_0} + \widetilde{\mathbb{I}}_{\mu_0}.$$

This contradicts the inequality (2.11). Then $u \in \widetilde{\mathbb{S}}_\mu$ is holds. Hence, we know that $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $q \in (2, (N+\alpha)/(N-2))$ as $n \to \infty$. By the weak lower semicontinuity of the norm, we then have that $\Phi(u) \le \widetilde{\mathbb{I}}_\mu$. Since $\widetilde{\mathbb{I}}_\mu \le \Phi(u)$, it follows that $\Phi(u) = \widetilde{\mathbb{I}}_\mu = \Phi(u) + o_n(1)$. Hence we get that $w_n \to u$ in X as $n \to \infty$, and the proof is finished. □

3. Proof of Theorem 1.3

In this section, we will give the proof of Theorem 1.3. To accomplish this we borrow an idea of [3]. The proof is divide into several steps. The main point is to proving the compactness. That is, for all $r < r_0$, for any minimizing sequence $\{u_n\} \subset \mathbb{S}_r \cap \mathbb{B}_M$ such that $\lim\limits_{n\to\infty} \Phi(u_n) = \mathbb{J}_r^M$ there exists $\{h_n\} \subset \mathbb{R}$ such that $(u_n(x_1, \ldots, x_{N-1}, x_N - h_n))$ is compact in Y.

We first prove the local minima structure for $\Phi(u)$ on $\mathbb{S}_r$. That is, the conclusion (a) of Theorem 1.3.

Lemma 3.1. *Assume that $\alpha \in (0, N)$ and $(N-2)/(N+\alpha) < 1/p \le N/(N+\alpha+2)$. For each $M > 0$ there exists $r_0 = r_0(M) > 0$ such that, for all $r < r_0$,*

$$\mathbb{S}_r \cap \mathbb{B}_M \ne \emptyset \quad \text{and} \quad \inf_{\mathbb{S}_r \cap \mathbb{B}_{Mr/2}} \Phi(u) < \inf_{\mathbb{S}_r \cap (\mathbb{B}_M \setminus \mathbb{B}_{Mr})} \Phi(u). \tag{3.1}$$

Proof. We choose some $u_0 \in Y$ such that $\|u_0\|_X = M$ and let $r_0 = |u_0|_2$. For each $0 < r \le r_0$, we know that the element $w = ru_0/r_0 \in (\mathbb{S}_r \cap \mathbb{B}_M)$. That is, $\mathbb{S}_r \cap \mathbb{B}_M \ne \emptyset$. Next we prove the latter conclusion of the lemma. We infer from the Gagliardo–Nirenberg inequality (1.6) for $u \in \mathbb{S}_r$ that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \le Cr^{(N+\alpha)-p(N-2)}\|u\|_X^{Np-(N+\alpha)}. \tag{3.2}$$

then

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|_X^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\ &\geq \frac{1}{2}\|u\|_X^2 - Cr^{(N+\alpha)-p(N-2)}\|u\|_X^{Np-(N+\alpha)} = G_r(\|u\|_X).\end{aligned}$$

Similarly, we infer that

$$\Phi(u) \leq \frac{1}{2}\|u\|_X^2 = H_r(\|u\|_X), \quad \text{for all } u \in \mathbb{S}_r. \tag{3.3}$$

Since $(N-2)/(N+\alpha) < 1/p \leq N/(N+\alpha+2)$, it follows that

$$(N+\alpha) > p(N-2) \quad \text{and} \quad Np \geq N+\alpha+2. \tag{3.4}$$

Hence we deduce that, for $s \in (0, M)$ and $r < r_0$ small,

$$G_r(s) = \frac{s^2}{2}\Big(1 - 2Cr^{(N+\alpha)-p(N-2)}\|u\|_X^{Np-(N+\alpha)}\Big) > \frac{2s^2}{5}. \tag{3.5}$$

This implies that

$$\inf_{s\in(rM,M)} G_r(s) > \frac{2}{5}\, r^2M^2. \tag{3.6}$$

On the other hand, we infer that

$$H_r\bigg(\frac{rM}{2}\bigg) = \frac{1}{8}\, r^2M^2. \tag{3.7}$$

Combining (3.2)–(3.7) we know that for $r < r_0$ small enough

$$\inf_{\mathbb{S}_r\cap\mathbb{B}_{Mr/2}} \Phi(u) \leq H_r\bigg(\frac{rM}{2}\bigg) < \inf_{s\in(rM,M)} G_r(s) \leq \inf_{\mathbb{S}_r\cap(\mathbb{B}_M\setminus\mathbb{B}_{Mr})} \Phi(u). \tag{3.8}$$

This finishes the proof. □

REMARK 3.2. Here Φ has a geometry of local minima. When $x_1^2+\ldots+x_{N-1}^2$ in (1.1) is replaced by a general potential $V(x)$, this geometry still holds. Actually, this geometry only depends on the assumption that $1/p \leq N/(N+\alpha+2)$. Now the fact that the geometry implies the existence of a local minimizer depends on a balance between the potential and the strength of the nonlinear term. If $V \equiv 0$, then it is well known that, for $(N-2)/(N+\alpha) < 1/p \leq N/(N+\alpha+2)$, no local minimizer exists (in practise any minimizing sequence is vanishing). On the contrary when $V(x) = x_1^2+\ldots+x_N^2$, as already indicated, a local minimizer exists for any $(N-2)/(N+\alpha) < 1/p \leq N/(N+\alpha+2)$.

Next we are going to study the nonvanishing property of the minimizing sequence of Φ.

LEMMA 3.3. *Assume that $\alpha \in (N+p-4, N)$, $N \geq 3$ and*

$$\max\bigg\{\frac{N-2}{N+\alpha}, \frac{1}{4}\bigg\} < \frac{1}{p} \leq \frac{N}{N+\alpha+2}.$$

Let $M > 0$ *and* $r_0 = r_0(M) > 0$ *be given as Lemma* 3.1. *If* $\{u_n\} \subset \mathbb{S}_r \cap \mathbb{B}_M$ *such that*

$$\lim_{n\to\infty} \Phi(u_n) = \inf_{u\in\mathbb{S}_r\cap\mathbb{B}_M} \Phi(u) = \mathbb{J}_r^M, \tag{3.9}$$

then one sees

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy > 0. \tag{3.10}$$

Proof. We use the contradiction arguments. Assume that (3.10) is not satisfied. Then we get

$$\begin{aligned}\mathbb{J}_r^M = \lim_{n\to\infty}\bigg[&\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + (x_1^2+\ldots+x_{N-1}^2)|u_n|^2\big) \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\bigg] \\ = \lim_{n\to\infty}&\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + \big(x_1^2+\ldots+x_{N-1}^2\big)|u_n|^2\big) \geq \frac{r^2}{2}\,\Gamma_0.\end{aligned} \tag{3.11}$$

So, it suffices to show that

$$\mathbb{J}_r^M < \frac{r^2}{2}\,\Gamma_0. \tag{3.12}$$

In fact, it is clear that the problem

$$-\Delta_{\widetilde{x}}\phi + \big(x_1^2+\ldots+x_{N-1}^2\big)\phi = \gamma\phi, \qquad \int_{\mathbb{R}^{N-1}} |\phi(\widetilde{x})|^2\,d\widetilde{x} = 1 \tag{3.13}$$

has a sequences of eigenvalue $\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \ldots$, and the corresponding eigenfunctions denoted by $\{\phi_j\}$ (see [1, Proposition 2.1 and Corollary 2.2] or [47]). Obviously, that $\{\phi_j\}$ is a Hilbert basis for $L^2(\mathbb{R}^{N-1})$. Let

$$v(x) = \phi_0(\widetilde{x})\psi(x_N) \quad \text{and} \quad \int_{\mathbb{R}} |\psi(x_N)|^2\,dx_N = r^2, \tag{3.14}$$

where $\psi(x_N)$ to be chosen later. A direct computation shows that

$$\begin{aligned}\Phi(v) =& \frac{1}{2}\int_{\mathbb{R}} |\partial_{x_N}\psi(x_N)|^2\,dx_N + \frac{\Gamma_0}{2}\int_{\mathbb{R}} |\psi(x_N)|^2\,dx_N \\ &- \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(\widetilde{x})|^p|\phi_0(\widetilde{y})|^p\,\frac{|\psi(x_N)|^p|\psi(y_N)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ =& \frac{\Gamma_0 r^2}{2} + \frac{1}{2}\int_{\mathbb{R}} |\partial_{x_N}\psi(x_N)|^2\,dx_N \\ &- \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(\widetilde{x})|^p|\phi_0(\widetilde{y})|^p\,\frac{|\psi(x_N)|^p|\psi(y_N)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.\end{aligned} \tag{3.15}$$

To prove (3.12) it suffices to show that $\|v\|_X^2 \le M^2$ and

$$\begin{aligned}(3.16)\quad & \frac{1}{2}\int_{\mathbb{R}} |\partial_{x_N}\psi(x_N)|^2\,dx_N \\ &\quad - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(\widetilde{x})|^p|\phi_0(\widetilde{y})|^p\,\frac{|\psi(x_N)|^p|\psi(y_N)|^p}{|x-y|^{N-\alpha}}\,dx\,dy < 0.\end{aligned}$$

Fix ψ_0 such that

$$\int_{\mathbb{R}^N} |\psi_0(x_N)|^2\,dx_N = r^2.$$

We take β_1, β_2 such that $1 \le 2\beta_2/\beta_1 < (3+\alpha-N)/(p-1)$ and let

$$(3.17)\qquad v(x) = \phi_0(\widetilde{x})\psi_\lambda(x_N) \quad\text{and}\quad \psi_\lambda(x_N) = \lambda^{\beta_2}\psi_0(\lambda^{\beta_1}x).$$

Then we infer that

$$\begin{aligned}(3.18)\quad & \frac{1}{2}\int_{\mathbb{R}} |\partial_{x_N}\psi_\lambda(x_N)|^2\,dx_N \\ &\quad - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(\widetilde{x})|^p|\phi_0(\widetilde{y})|^p\,\frac{|\psi_\lambda(x_N)|^p|\psi_\lambda(y_N)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &= \frac{\lambda^{2\beta_2+\beta_1}}{2}\int_{\mathbb{R}} |\partial_{x_N}\psi_0(x_N)|^2\,dx_N \\ &\quad - \frac{\lambda^{2\beta_2 p+\beta_1(N-\alpha-2)}}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(\widetilde{x})|^p|\phi_0(\widetilde{y})|^p \\ &\quad \times |\psi_0(x_N)|^p|\psi_0(y_N)|^p/\big(|\lambda^{\beta_1}(x_1-y_2)|^2+\dots \\ &\quad + |\lambda^{\beta_1}(x_{N-1}-y_{N-1})|^2 + |(x_N-y_N)|^2\big)^{(N-\alpha)/2}\,dx\,dy.\end{aligned}$$

Since $p<4$, we know that (3.16) holds for $\lambda>0$ small. Furthermore, a direct computation shows that

$$\begin{aligned}(3.19)\quad & \|\phi_0(\widetilde{x})\psi_\lambda(x_N)\|_X^2 \\ &= \lambda^{2\beta_2+\beta_1}\int_{\mathbb{R}} |\partial_{x_N}\psi_0(x_N)|^2\,dx_N + \Gamma_0\int_{\mathbb{R}}|\psi_\lambda(x_N)|^2\,dx_N \\ &= \lambda^{2\beta_2+\beta_1}\int_{\mathbb{R}} |\partial_{x_N}\psi_0(x_N)|^2\,dx_N + \Gamma_0\lambda^{2\beta_2-\beta_1}r^2.\end{aligned}$$

Hence we prove the conclusion by choosing λ small enough and $r_0 = r_0(M)$ such that $2\Gamma_0 r_0^2 < M^2$. □

The next lemma studies the further property of the minimizing sequences $\{u_n\}$ in Lemma 3.3.

LEMMA 3.4. *Assume that* $\|u_n\|_Y^2 < \infty$ *and there exists* $\sigma_0 > 0$ *such that*

$$(3.20)\qquad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \ge \sigma_0 > 0.$$

Moreover, there exist $h_n\in\mathbb{R}$ *such that*

$$(3.21)\qquad u_n(x_1,\dots,x_{N-1},x_N-h_n) \rightharpoonup v \ne 0 \quad \text{in } Y.$$

Proof. We infer from the Gagliardo–Nirenberg inequality (1.6) that

$$\begin{aligned}\int_{\Lambda_h}\int_{\Lambda_h}&\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\&\leq S\bigg(\int_{\Lambda_h}|u|^2\bigg)^{((N+\alpha)-p(N-2))/2}\bigg(\int_{\Lambda_h}|\nabla u|^2\bigg)^{(pN-(N+\alpha))/2}\\&\leq C|u_n|_{L^2(\Lambda_h)}^{(N+\alpha)-p(N-2)}\|u_n\|_{H^1(\Lambda_h)}^{pN-(N+\alpha)},\end{aligned}$$

where $\Lambda_h=\mathbb{R}^{N-1}\times(h,h+1)$ for $h\in\mathbb{Z}$. By taking a sum over $h\in\mathbb{Z}$, we deduce that

$$|u_n|_{L^2(\mathbb{R}^N)}\leq C\Big(\sup_h|u_n|_{L^2(\Lambda_h)}\Big)^{(N+\alpha)-p(N-2)}\|u_n\|_{H^1(\mathbb{R}^N)}^{pN-(N+\alpha)}.$$

Moreover, by the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$, we infer that there exist $h_n\in\mathbb{R}$ such that $\inf_n|u_n|_{L^2(\Lambda_{h_n})}\geq\varrho_0>0$, where ϱ_0 is a constant.

Set $v_n=u_n(x_1,\ldots,x_{N-1},x_N-h_n)$. Then we know that

$$\sup_n\bigg(\int_{\Lambda_1}\big(|\nabla_{\widetilde{x}}v_n|^2+\big(x_1^2+\ldots+x_{N-1}^2\big)|v_n|^2+|v_n|^2\big)\,dx\bigg)<\infty$$

and $|v_n|_{L^2(\Lambda_1)}>0$. Since the embedding $H^1(\Lambda_1)\hookrightarrow L^2(\Lambda_1)$ is compact, it follows that there exists $v\in Y\setminus\{0\}$ such that $v_n\rightharpoonup v\neq0$ in Y. □

Now we are ready to give the proof of Theorem 1.3

Proof of Theorem 1.3. Let $\{u_n\}\subset\mathbb{S}_r\cap\mathbb{B}_M$ be the minimizing sequences of $\mathbb{J}_r^M$. It is clear that $\{u_n\}$ is bounded in Y. By Lemma 3.4, we know that $v_n=u_n(x_1,\ldots,x_{N-1},x_N-h_n)\rightharpoonup v\neq0$ in Y. To accomplish the Theorem 1.3, it suffices to prove that $v_n\to v\neq0$ in Y. That is we only need to prove that $|v|_{L^2(\mathbb{R}^N)}=r^2$. In fact, if $|v|_{L^2(\mathbb{R}^N)}=r^2$, we know that $v_n\to v$ in $L^2(\mathbb{R}^N)$. Furthermore, we infer from the Gagliardo–Nirenberg inequality

$$|u|_p\leq C(N,p)|\nabla u|_2^{\alpha}|u|_2^{1-\alpha},\quad u\in H^1(\mathbb{R}^N),\ \alpha=\frac{N(p-2)}{2p}\tag{3.22}$$

that $v_n\to v$ in $L^t(\mathbb{R}^N)$ (for all $t\in(2,2^*)$). Furthermore, one sees from Hardy–Littewood–Sobolev inequality that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)|^p|v_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\to\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)|^p|v(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy,$$

as $n\to\infty$. Therefore, from the weak convergence in Y we see that $\Phi(v)\leq\lim_{n\to\infty}\Phi(v_n)=\mathbb{J}_r^M$. If $\|v\|_Y^2<\lim_{n\to\infty}\|v_n\|_Y^2$, then $\Phi(v)<\mathbb{J}_r^M$. This is contradiction. So, $\lim_{n\to\infty}\|v_n\|_Y^2=\|v\|_Y^2$. That is, $v_n\to v$ in Y.

In the following we still need to prove $|v|_{L^2(\mathbb{R}^N)}=r^2$. Assume by contradiction that $|v|_{L^2(\mathbb{R}^N)}=r_0^2<r^2$. We define $v_n=(v_n-v)+v$. Then one infers from

the Brezis–Lieb Lemma [6] that

$$\|v_n\|_X^2 = \|v_n - v\|_X^2 + \|v\|_X^2 + o(1),$$
$$\|v_n\|_{L^2(\mathbb{R}^N)}^2 = \|v_n - v\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_{L^2(\mathbb{R}^N)}^2 + o(1).$$

Moreover, by Brezis–Lieb type l emma for the nonlocal problem (see [40, Lemma 2.4]), we know that

$$\begin{aligned}(3.23)\qquad &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^p|v_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v(x)|^p|v_n(y)-v(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^p|v(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy + o(1).\end{aligned}$$

Hence we infer that

$$\Phi(u_n) = \Phi(v_n) = \Phi(v_n - v) + \Phi(v) + o(1) \ge \mathbb{J}_{r_n}^M + \mathbb{J}_{r_0}^M + o(1), \tag{3.24}$$

where $r_n = |v_n - v|_{L^2(\mathbb{R}^N)}$ and $r_0 = |v|_{L^2(\mathbb{R}^N)}$. Hence, we get $r_n^2 + r_0^2 = r^2 + o(1)$. Assume that $r_n \to \ell$ and hence $\ell^2 + r_0^2 = r^2$. By (3.24), we arrive at

$$\mathbb{J}_r^M \ge \mathbb{J}_\ell^M + \mathbb{J}_{r_0}^M. \tag{3.25}$$

Next we need to show that

$$\mathbb{J}_r^M < \mathbb{J}_\ell^M + \mathbb{J}_{r_0}^M. \tag{3.26}$$

Then we get the contradiction. To prove (3.26), we first claim that

$$r^2\mathbb{J}_s^M < s^2\mathbb{J}_r^M, \quad \text{for } 0 < r < s < \min\{1, r_0\}. \tag{3.27}$$

Indeed, let $\{w_n\} \subset \mathbb{S}_r \cap \mathbb{B}_M$ be such that $\lim_{n\to\infty} \Phi(w_n) = \mathbb{J}_r^M$. By Lemma 3.1 we know that $w_n \in \mathbb{B}_{Mr}$ when $r < r_0(M)$ and n large enough. Particularly, for $s < 1$, we have that $sw_n/r \in \mathbb{S}_s \cap \mathbb{B}_{sM} \subset \mathbb{S}_s \cap \mathbb{B}_M$. Hence we infer from Lemma 3.3 that

$$\begin{aligned}\mathbb{J}_s^M \le \Phi\Big(\frac{s}{r}w_n\Big) &= \frac{s^2}{r^2}\bigg(\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla w_n|^2 + \big(x_1^2 + \ldots + x_{N-1}\big)|w_n|^2\big)\bigg)\\ &\quad - \frac{1}{2p}\frac{s^{2p}}{r^{2p}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\ &= \frac{s^2}{r^2}\bigg(\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla w_n|^2 + (x_1^2 + \ldots + x_{N-1})|w_n|^2\big)\\ &\quad - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dxdy\bigg)\\ &\quad + \frac{1}{2p}\Big(\frac{s^2}{r^2} - \frac{s^{2p}}{r^{2p}}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\end{aligned}$$

$$\leq \frac{s^2}{r^2}\,\Phi(w_n) + o(1) + \bigg(\frac{s^2}{r^2} - \frac{s^{2p}}{r^{2p}}\bigg)\delta_0$$
$$\leq \frac{s^2}{r^2}\,\mathbb{J}_r^M + o(1) + \bigg(\frac{s^2}{r^2} - \frac{s^{2p}}{r^{2p}}\bigg)\delta_0 < \frac{s^2}{r^2}\,\mathbb{J}_r^M.$$

Here we have used the fact that

$$\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \geq \delta_0$$

for n large enough. So, the claim (3.27) holds. Since $\ell^2 + r_0^2 = r^2$, it follows from (3.27) that

$$\mathbb{J}_\ell^M > \frac{\ell^2}{r^2}\,\mathbb{J}_r^M \quad \text{and} \quad \mathbb{J}_{r_0}^M > \frac{r_0^2}{r^2}\,\mathbb{J}_r^M$$

Thus, we arrive at

$$\mathbb{J}_\ell^M + \mathbb{J}_{r_0}^M > \mathbb{J}_r^M.$$

Finally, it is well-known that the orbital stability of the set $\mathbb{M}_r^M$ is equivalent to the fact that any minimizing sequence $\{u_n\} \subset \mathbb{S}_r \cap \mathbb{B}_M$ is compact up to translation (see [9]). □

4. Properties of the minimizer

In this section we are devoted to study the properties of the minimizer which obtained in Theorems 1.3. We first make the characterization of the $\mathbb{C}$-value minimizer. To accomplish this we need the below theorem and the proof was given in [3, Theorem 5].

Theorem 4.1. *Let $w \in C^1\big(\mathbb{R}^N;\mathbb{C}\setminus\{0\}\big)$ be such that*

$$\int_{\mathbb{R}^N}\sum_j |\partial_{x_j}|w||^2\,dx = \int_{\mathbb{R}^N}|\partial_{x_j}w|^2\,dx, \tag{4.1}$$

then we have $w(x) = e^{i\theta}\rho(x)$, where $\theta \in \mathbb{R}$ is a constant and $\rho(x) \in \mathbb{R}$ for every $x \in \mathbb{R}^N$.

Now we are ready to make the characterization of the $\mathbb{C}$-value minimizer.

Lemma 4.2. *For each minimizer obtained in Theorem 1.3, then it has the form $e^{i\theta}w(x_1,\ldots,x_N)$, where w is a positive real valued minimizer and $\theta \in \mathbb{R}$.*

Proof. Here we assume that $w \in H^1\big(\mathbb{R}^N;\mathbb{C}\big)$ be a complex valued minimizer. And w is of class C^1 by a standard elliptic regularity bootstrap. It is easy to see that, by the diamagnetic inequality, $|w| \in C^1\big(\mathbb{R}^N;\mathbb{C}\big)$ is a minimizer. Moreover by the Euler–Lagrange equation and the strong maximum principle we have $|w| > 0$, Hence $w \in C^1\big(\mathbb{R}^N;C\setminus\{0\}\big)$. We know that w and $|w|$ are minimizers, and all the terms involved in the energy (that we are minimizing) are unchanged by replacing w by $|w|$ except in principle the kinetic term, then we get that if

$$\int_{\mathbb{R}^N} \big|\nabla_x |w|\big|^2\,dx = \int_{\mathbb{R}^N} |\nabla_x w|^2\,dx,$$

then w and $|w|$ are both minimizers. We conclude this lemma by Theorem 4.1. □

Now we focus on the symmetry of the minimizers. We first give the result for Schwartz symmetrization and the proof was given in [3, Theorem 4].

THEOREM 4.3. *Let* $V\colon \mathbb{R}^N \to [0,\infty)$ *be a measurable function, radially symmetric satisfying* $V(|x|) \le V(|y|)$ *for* $|x| \le |y|$, *then we have*:

$$\int_{\mathbb{R}^N} V(|x|)|u^*|^2\,dx \le \int_{\mathbb{R}^N} V(|y|)|u|^2\,dx. \tag{4.2}$$

If, in addition, $V(|x|) < V(|y|)$ *for* $|x| < |y|$, *then*

$$\int_{\mathbb{R}^N} V(|x|)|u^*|^2\,dx = \int_{\mathbb{R}^N} V(|y|)|u|^2\,dx \;\Rightarrow\; u(x) = u^*(|x|). \tag{4.3}$$

The next lemma prove the radially symmetric and decreasing property of minimizer of (1.1).

LEMMA 4.4. *For each minimizer obtained in Theorem* 1.3, *then* $w(x_1,\ldots,x_{N-1},x_N-h)$ *is radially symmetric and decreasing with respect to* $(x_1,\ldots,x_{N-1})$, *where* w *is a positive real valued minimizer and* $h\in\mathbb{R}$.

PROOF. We shall use the Schwartz symmetrization and reflexion type arguments to prove results. Assume that $u(x_1,\ldots,x_N)$ is a real minimizer. The following is the partial symmetrization with respect to the variables $(x_1,\ldots,x_{N-1})$

$$u^\star(x_1,\ldots,x_N) = u^*_{x_N}(x_1,\ldots,x_{N-1}),$$

where $u^*_{x_N}(x_1,\ldots,x_{N-1}) = u(x_1,\ldots,x_N)$ and $*$ denotes the Schwartz rearrangement with respect to $w(x_1,\ldots,x_{N-1})$ (see [31], [54]). We infer from [7, Theorem 8.2] and [31] that

$$\int_{\mathbb{R}^N} |\nabla_x u^\star|^2\,dx \le \int_{\mathbb{R}^N} |\nabla_x u|^2\,dx, \qquad \int_{\mathbb{R}^N} |u^\star|^2\,dx = \int_{\mathbb{R}^N} |u|^2\,dx,$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^\star(x)|^p|u^\star(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.$$

Also, from (4.2) we deduce that

$$\int_{\mathbb{R}^{N-1}} \big(x_1^2+\ldots+x_{N-1}^2\big)|u^\star|^2\,d\widetilde{x} \le \int_{\mathbb{R}^{N-1}} \big(x_1^2+\ldots+x_{N-1}^2\big)|u|^2\,d\widetilde{x}, \tag{4.4}$$

for all $x_N\in\mathbb{R}$. Integration with respect to dx_N in (4.4) gives

$$\int_{\mathbb{R}^N} \big(x_1^2+\ldots+x_{N-1}^2\big)|u^\star|^2\,dx \le \int_{\mathbb{R}^N} \big(x_1^2+\ldots+x_{N-1}^2\big)|u|^2\,dx.$$

Thus we get that $u^\star$ is also a minimizer. Moreover, since u is a minimizer, then one has that

$$\int_{\mathbb{R}^N} (x_1^2+\ldots+x_{N-1}^2)|u^\star|^2\,dx = \int_{\mathbb{R}^N} (x_1^2+\ldots+x_{N-1}^2)|u|^2\,dx.$$

By this fact and (4.4) we get

$$\int_{\mathbb{R}^{N-1}} (x_1^2+\ldots+x_{N-1}^2)|u^\star|^2\,d\widetilde{x} = \int_{\mathbb{R}^{N-1}} (x_1^2+\ldots+x_{N-1}^2)|u|^2\,d\widetilde{x},$$

for almost every $x_N\in\mathbb{R}$. Hence, from Theorem 4.3 (with $V(x_1,\ldots,x_{N-1}) = x_1^2+\ldots+x_{N-1}^2$) we get that $u(x_1,\ldots,x_N)=u^\star(x_1,\ldots,x_N)$ for almost every $x_N\in\mathbb{R}$.

Summarizing, $u(x_1,\ldots,x_N)$ is radially symmetric and decreasing with respect to $(x_1,\ldots,x_{N-1})$ for x_N in a set with full measure. On the other hand $u(x_1,\ldots,x_N)$ is continuous and hence $u(x_1,\ldots,x_N)$ is radially symmetric and decreasing with respect to $(x_1,\ldots,x_{N-1})$ for every x_N. □

Next we shall give the estimates for the Lagrange multiplier.

Lemma 4.5. *For every fixed $M>0$, for every $r<\sigma_0(M)$ and for every $u\in\mathbb{M}_r^M$ there exists $\gamma=\gamma(w)>0$ such that* (1.14) *holds. Moreover, we have*

$$\big(1-Cr^{2p-2}\big)\Gamma_0\le\gamma<\Gamma_0,$$

where $C>0$ is an universal constant and

$$\Gamma_0=\inf\bigg(\operatorname{spec}\bigg(-\sum_{i=1}^N\partial_{x_i}^2+\big(x_1^2+\ldots+x_{N-1}^2\big)\bigg)\bigg).$$

Proof. For $u\in\mathbb{M}_r^M$, then we have $u\in\mathbb{B}_{rM}$ by Theorem 1.3, which implies that u is a critical point of $\Phi(u)$ on $\mathbb{S}_r$ and hence there exists $\gamma=\gamma(u)\in\mathbb{R}$ such that

$$-\Delta u+\big(x_1^2+\ldots+x_{N-1}^2\big)u-\bigg(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^{N-\alpha}}\,dy\bigg)|u|^{p-2}u=\gamma u.$$

Multiplying by u and integrating by parts we have that

$$\gamma=\frac{1}{\displaystyle\int_{\mathbb{R}^N}|u|^2\,dx}\bigg(\|u\|_X^2-\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\bigg)\bigg)<\frac{2\Phi(u)}{r^2}, \tag{4.5}$$

where we have used $p>1$. Moreover, by Lemma 3.3, we get $\Phi(u)<r^2\Gamma_0/2$. Thus, one sees that

$$\gamma<\frac{2\Phi(u)}{r^2}<\Gamma_0. \tag{4.6}$$

This give the upper bounded of γ. Next we prove the low bounded of γ. From (3.2), (4.5) and $u \in \mathbb{B}_{rM}$, we deduce that

$$\begin{aligned}\gamma &\geq \frac{1}{r^2}\big(\|u\|_X^2 - Cr^{(N+\alpha)-p(N-2)}\|u\|_X^{Np-(N+\alpha)}\big)\\ &\geq \frac{\|u\|_X^2}{r^2}\big(1 - Cr^{(N+\alpha)-p(N-2)}\|u\|_X^{Np-(N+\alpha+2)}\big)\\ &\geq \frac{\|u\|_X^2}{r^2}\big(1 - Cr^{(N+\alpha)-p(N-2)}(M^2r^2)^{(Np-(N+\alpha+2))}/2\big)\\ &\geq \frac{\|u\|_X^2}{r^2}\big(1 - Cr^{2p-2}\big),\end{aligned}$$

where the value $C > 0$ is a constant. Now, by definition of Γ_0, we see that

$$\frac{\|u\|_X^2}{r^2} \geq \Gamma_0.$$

Therefore, we obtain that $\gamma \geq \Gamma_0\big(1 - Cr^{2p-2}\big)$. As a consequence, for some constant $C > 0$,

$$\Gamma_0 > \gamma \geq \Gamma_0\big(1 - Cr^{2p-2}\big).$$

In particular, we have $\gamma \to \Gamma_0$, as $r \to 0$. □

We are now in position to establish the asymptotic profile of the minimizer as $r \to 0$.

Lemma 4.6. *We have*

$$\sup_{\mathbb{M}_r^M} \big\|u(x_1,\ldots,x_N) - \psi_0(x_N)\phi_0(x_1,\ldots,x_{N-1})\big\|_X = o(r)$$

where $\phi_0(x_1,\ldots,x_{N-1})$ is the unique normalized positive eigenvector of the quantum harmonic oscillator $\sum_{i=1}^{N-1}\partial_{x_i}^2 + x_1^2 + \ldots + x_{N-1}^2$ and

$$\psi_0(x_N) = \int_{\mathbb{R}^N} u(x_1,\ldots,x_N)\phi_0(x_1,\ldots,x_{N-1})\,dx_1\ldots dx_{N-1}.$$

Proof. Recall that from Theorem 1.3, for every fixed $M > 0$ and for every $r < r_0(M)$,

$$\mathbb{M}_r^M = \big\{u \in \mathbb{S}_r \cap \mathbb{B}_M : \Phi(u) = \mathbb{J}_r^M\big\} \neq \emptyset.$$

Since $\mathbb{M}_r^M \subset \mathbb{B}_{Mr}$, and by (3.2), we get that for any $u \in \mathbb{M}_r^M$ there exists $C > 0$, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \leq Cr^{2p}. \tag{4.7}$$

Now we use the Hilbert basis $\{\phi_j\}$ of $L^2\big(\mathbb{R}^{N-1}\big)$ introduced in the proof of Lemma 3.3, then for any $u \in \mathbb{M}_r^M$

$$u(x_1,\ldots,x_N) = \sum_{j\in\mathbb{N}\cup\{0\}} \psi_j(x_N)\phi_j(x_1,\ldots,x_{N-1}),$$

where

$$\psi_j(x_N) = \int_{\mathbb{R}^N} u(x_1, \ldots, x_N)\phi_j(x_1, \ldots, x_{N-1})\, dx_1 \ldots dx_{N-1}.$$

Then, by $\psi_j \in L^2(\mathbb{R})$ and orthogonality, the following equation holds

$$\sum_{j\in\mathbb{N}\cup\{0\}} \|\psi_j\|_{L^2}^2 = \sum_{j\in\mathbb{N}\cup\{0\}} \|\psi_j\|_{L^2}^2 \|\phi_j\|_{L^2}^2 = \|u\|_{L^2}^2 = r^2.$$

In addition, by using the similar argument with in the proof of Lemma 3.3 and (4.7), one sees that

$$\Phi(u) \geq \frac{1}{2} \sum_{j\in\mathbb{N}\cup\{0\}} \|\partial_{x_N}\psi_j\|_{L^2}^2 + \gamma_j\|\psi_j\|_{L^2}^2 - Cr^{2p}. \tag{4.8}$$

By Lemma 3.3, we deduce that

$$\Phi(u) = \inf_{\mathbb{S}_r\cap\mathbb{B}_M} \Phi(u) < \frac{r^2\gamma_0}{2}, \quad \text{for all } r < r_0.$$

Therefore, one sees from (4.8) that

$$\frac{1}{r^2} \sum_{j\in\mathbb{N}\cup\{0\}} \|\partial_{x_N}\psi_j\|_{L^2}^2 + \gamma_j\|\psi_j\|_{L^2}^2 < \gamma_0 + 2Cr^{2(p-1)}. \tag{4.9}$$

A direct computation shows that

$$\frac{\gamma_0}{r^2} \sum_{j\in\mathbb{N}\cup\{0\}} \|\psi_j\|_{L^2}^2 = \frac{\gamma_0}{r^2}\, \|u\|_{L^2}^2 = \gamma_0.$$

Since $\gamma_0 < \gamma_j$ for all $j \in N$, we infer from the estimate above that

$$\begin{gathered} \sum_{j\in\mathbb{N}\cup\{0\}} \frac{1}{r^2}\, \|\partial_{x_N}\psi_j\|_{L^2}^2 < 2Cr^{2(p-1)}, \\ \sum_{j\in\mathbb{N}} \frac{1}{r^2}\, \|\psi_j\|_{L^2}^2 < \frac{2Cr^{2(p-1)}}{\inf\limits_{j\in N}(\gamma_j - \gamma_0)}, \end{gathered} \tag{4.10}$$

or identically,

$$\left| \frac{\|\psi_0\|_{L^2}^2}{r^2} - 1 \right| < \frac{2Cr^{2(p-1)}}{\inf\limits_{j\in N}(\gamma_j - \gamma_0)}. \tag{4.11}$$

From (4.9)–(4.11) it follows that there exists $C > 0$ such that, for any $u \in \mathbb{M}_r^M$,

$$\left\| \frac{u}{r} - \frac{\psi_0}{r}\phi_0 \right\|_X = \left(\frac{1}{r^2} \sum_{j\in\mathbb{N}\cup\{0\}} \|\partial_{x_N}\psi_j\|_{L^2}^2 + \gamma_j\|\psi_j\|_{L^2}^2 \right)^{1/2} \leq Cr^{p-1}.$$

Then we finish the proof. □

Proof of Theorem 1.5. It is easy to that Theorem 1.5 follows from Lemmas 4.2, 4.4–4.6. □

5. Proof of Theorem 1.7

Finally, we give the proof of Theorem 1.7. Fix $M > 0$, from Theorem 1.3 (b) and (3.2), we get

$$\lim_{r\to 0} \mathbb{J}_r^M = 0. \tag{5.1}$$

We use the contradiction arguments. Assume that there exists a critical point $\widetilde{u}$ for $\Phi(u)$ on $\mathbb{S}_r$ satisfying $\Phi(\widetilde{u}) < \mathbb{J}_r^M$. Using the Pohozaev's type identity as in [5], [25], [53], then we know that if u is a solution of (1.1), then u such that $P(\widetilde{u}) = 0$, where

$$P(\widetilde{u}) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \big(x_1^2 + \ldots + x_{N-1}^2\big)|u|^2\, dx - \frac{N(p-1)-\alpha}{2p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\, dx\, dy.$$

Therefore, we have

$$\begin{aligned}\Phi(\widetilde{u}) &= \Phi(\widetilde{u}) - \frac{1}{N(p-1)-\alpha} P(\widetilde{u}) \\ &= \frac{N(p-1)-\alpha-2}{N(p-1)-\alpha} \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2\, dx \\ &\quad + \frac{N(p-1)-\alpha+2}{N(p-1)-\alpha} \int_{\mathbb{R}^N} \big(x_1^2 + \ldots + x_{N-1}^2\big)|\widetilde{u}|^2\, dx \\ &> \frac{N(p-1)-\alpha-2}{N(p-1)-\alpha} \|\widetilde{u}\|_X^2.\end{aligned} \tag{5.2}$$

By (5.1) and (5.2), one can deduce that

$$\frac{N(p-1)-\alpha-2}{N(p-1)-\alpha} \|\widetilde{u}\|_X^2 < \Phi(\widetilde{u}) < \mathbb{J}_r^M = o(1).$$

This is a contradiction, since this implies that, for $r > 0$ small enough, $\|\widetilde{u}\|_X^2 < M^2$ and $\mathbb{J}_r^M$ is the infimum of the energy on $\mathbb{S}_r \cap \mathbb{B}_M$. □

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