Topological Methods in Nonlinear Analysis
Volume 55, No. 2, 2020, 429–449
DOI: 10.12775/TMNA.2019.088

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EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR DIFFERENTIAL VARIATIONAL INEQUALITIES WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In the paper we study a class of semilinear differential variational systems with nonlocal boundary conditions, which are obtained by mixing evolution equations and generalized variational inequalities. Firstly, we show the properties of the solution set for generalized variational inequalities. Then, the existence results are established and proved mainly by the topological degree theory and the Yosida approximations of the generator of C_0-semigroup.

1. Introduction

In 2008, differential variational inequalities (DVIs, for short) were formally introduced and systematically studied by Pang and Stewart [28]. DVIs are useful for representing models involving both dynamics and constraints in the form

2020 *Mathematics Subject Classification.* 37L05, 58E35, 35A01.

Key words and phrases. Semilinear differential variational inequalities; existence of mild solutions; topological degree theory; Yosida approximations; nonlocal boundary conditions.

This research was supported by National Natural Science Foundation of China Grant No. 11671101, Guangxi Natural Science Foundation under Grant Nos. 2018GXNSFDA138002, 2018GXNSFBA281180, 2017GXNSFBA198031, 2018GXNSFAA294010, Project of Guangxi Education Department Grant No. 2020KY16017, Hundred Talent Program for Introducing the Overseas High-Level Talents of Guangxi Colleges and Universities, Program for Innovative Team of Guangxi University of Finance and Economics. This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH.

of inequalities. It was found that DVI provides a powerful modeling paradigm for many applied problems. Since DVIs have captured special attention regarding significant applications, fruitful and significant results were obtained under different conditions in recent years (see [14], [15], [17], [19]–[21], [23], [24], [33] and the references therein).

Recently, the DVIs in infinite dimensional spaces has been attracting scholars' increasing interest. In [23], some basic results concerning the properties of mild solution set are presented for semilinear DVIs. Later, the existence and properties of solution set are obtained for a new kind DVI problems which may be regarded as feedback control problems [21], [35]. Furthermore, some general existence results are established for a DVI with nonlocal condition and the fractional DVIs in infinite dimensional spaces, we refer the reader to [14], [18], [20], [22], [25], [26] and the references therein.

Byszewski firstly studied the nonlocal problem for a semilinear differential equation with a C_0-semigroup generator [5]. The technique used in [5] is based on the Banach fixed point theorem for contraction mappings. Then others fixed point theorems such as the Leray–Schauder fixed point theorem for compact mappings, fixed point theorem for condensing mappings, the Kakutani fixed point theorem for multivalued mappings are used to study semilinear differential equations and inclusions in Banach spaces with various boundary conditions (see, e.g. [1], [7], [11]–[13], [16], [31], [32] and the references therein).

Motivated by above mentioned work, we will study a class of nonlocal problem for semilinear differential variational inequality problems composed of evolution equations and generalized variational inequalities, which can be stated as follows:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), u(t)) & \text{for a.e. } t \in [0, b], \\ u(t) \in \mathrm{SOL}(K, G(t, x(t), \cdot\,), h) & \text{for a.e. } t \in [0, b], \\ x(0) = \varphi(x), \end{cases} \tag{1.1}$$

where $A\colon D(A) \subset H \to H$ is the generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$. Here H is a separable Hilbert space and V is a separable reflexive Banach space, V^* the dual of V. Set K be a nonempty, closed and convex subset of V. The following functions: $f\colon [0, b] \times H \times V \to H$, $G\colon [0, b] \times H \times V \to P(V^*)$, $h\colon K \times K \to [-\infty, +\infty]$ and $\varphi\colon C([0, b], H) \to H$ will be specified later. $\mathrm{SOL}(K, G(t, x(t), \cdot\,), h)$ stands for the solution set of generalized mixed variational inequality (GMVI, for short): find $u\colon [0, b] \to K$ and $u^*\colon [0, b] \to V^*$ such that $u^*(t) \in G(t, x(t), u(t))$ for almost every $t \in [0, b]$ and

$$\langle u^*(t), v - u(t)\rangle + h(u(t), v) \geq 0, \tag{1.2}$$

for almost every $t \in [0, b]$ and for all $v \in K$.

According to [23], [29], [28], the mild solutions of (1.1) are understood in the following sense.

Definition 1.1. A pair of functions (x,u) with $x \in C([0,b],H)$ and $u\colon [0,b] \to K \subseteq V$ integrable, is said to be a mild solution of system (1.1) if

$$x(t) = T(t)\varphi(x) + \int_0^t T(t-s)f(s,x(s),u(s))\,ds, \quad \text{for } t \in [0,b],$$

where $u(t) \in \mathrm{SOL}(K, G(t,x(t),\,\cdot\,),h)$ for almost every $t \in [0,b]$. If (x,u) is a mild solution of problem (1.1), then x is called the mild trajectory and u the variational control trajectory for (1.1).

In this paper, we will introduce a new method for the study of problem (1.1) which is the combination of the the topology degree theory and Yosida approximations of the generator of C_0-semigroups. In fact, the DVI problem (1.1) is of much use in optimization, mechanical and electrical engineering. However, it seems that no attempt has been made to investigate (1.1). In the case $H = \mathbb{R}^n$, $V = \mathbb{R}^m$, $A = 0$ and $\varphi(x) = x_0$, the DVI problem (1.1) becomes a finite dimensional problem, which has been studied by Pang and Stewart [28], Li, Huang and O'Regan [17], Liu, Loi and Obukhovskiĭ [19], etc. Furthermore, it is worth to point out that, we also do not require the Lipschitz continuity of the nonlinear term, which are always necessary for DVI problems in most of the previous work (see [17], [20], [23], [28] and the references therein).

The structure of this paper can be summarized as follows. In Section 2, we first provide some preliminary facts. In Section 3, we introduce a related GMVI and show that the solution set of GMVI is nonempty, convex, closed and bounded. Next, the weak upper semicontinuity, the measurability of solution set for GMVI are shown and proved. In Section 4, by means of the topology degree theory and Yosida approximations, we infer the existence results of nonlocal DVI problem (1.1). Finally, we also obtain the weak compactness of a mild solution set of DVI problem (1.1).

2. Preliminaries

Throughout this paper, let $I = [0,b]$, assume that $(H, \|\cdot\|_H)$ is a separable Hilbert space, $(E, \|\cdot\|_E)$ and $(V, \|\cdot\|_V)$ are reflexive Banach spaces. The spaces H, E, V endowed with their weak topologies are denoted by H_w, E_w and V_w, respectively. We use $\rightharpoonup$ to denote the weak convergence. The notation $P(Y)$ denotes the collection of all nonempty subsets of a topological space Y. Now, we recall some prerequisites regarding multivalued mappings that can be found in [13], [27].

Definition 2.1. Let X, Y be two topological spaces and $F\colon X \to P(Y)$ be a multivalued map, we say that

(a) F is convex (bounded, closed, compact, weak compact) valued, if $F(x)$ is convex (bounded, closed, compact, weak compact) for each $x \in X$;
(b) F is upper semicontinuous (u.s.c. for short) at $x_0 \in X$, if for each open set $\mathcal{U}$ of Y such that $F(x_0) \subset \mathcal{U}$, there exists a neighbourhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$. We say that F is u.s.c. if F is u.s.c. at each $x_0 \in X$.
(c) F is locally compact if every point $x \in X$ has a neighbourhood $\mathcal{V}(x)$ such that the restriction of F to $\mathcal{V}(x)$ is compact. If the multivalued map F is closed and locally compact, then F is u.s.c.
(d) If $X \cap Y \neq \emptyset$, then $x \in X$ is a fixed point of F if $x \in F(x)$.

Definition 2.2. Let K be a nonempty, closed and convex subset of a reflexive Banach space E with its dual space denoted by E^*. The mapping $F\colon K \to P(E^*)$ is said to be

(a) lower semicontinuous at x, if for any $x^* \in F(x)$ and sequence $\{x_n\} \subset K$ with $x_n \to x$, we can find a sequence $x_n^* \in F(x_n)$ such that $x_n^* \to x^*$;
(b) lower hemicontinuous, if the restriction of F to every line segment of K is lower semicontinuous with respect to the weak topology in E^*.

Theorem 2.3 ([27, Theorem 3.13]). *Let $0 < b < \infty$ and F be an upper semicontinuous multifunction from a Hausdorff locally convex space X to the closed convex subsets of a Banach space E endowed with the weak topology. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of functions such that*

(a) *$x_n\colon I \to X$ and $y_n\colon I \to E$ are measurable functions for all $n \in \mathbb{N}$.*
(b) *For almost all $t \in I$ and for every neighbourhood $\mathcal{N}(0)$ of 0 in $X \times E$, there exists $n_0 \in \mathbb{N}$ such that $(x_n(t), y_n(t)) \in \mathrm{Gr}(F) + \mathcal{N}(0)$ for all $n \geq n_0$. Here $\mathrm{Gr}(F)$ denotes the graph of the multifunction F.*
(c) *$x_n(t) \to x(t)$ for almost every $t \in I$, where $x\colon I \to X$.*
(d) *$y_n \in L^1(I,E)$ and $y_n \rightharpoonup y$ in $L^1(I,E)$, where $y \in L^1(I,E)$.*

Then $(x(t), y(t)) \in \mathrm{Gr}(F)$, i.e. $y(t) \in F(x(t))$ for almost every $t \in I$.

Let $U \subset E$ be an open bounded subset, $F\colon \overline{U} \to P_{kv}(E) = \{D \in P(E) : D$ is compact and convex set$\}$ be a compact u.s.c. multivalued map such that $x \notin F(x)$ for all $x \in \partial U$, and denote by i the inclusion map. Then the topological degree $\deg(i - F, \overline{U})$ is well-defined (see e.g. [13]) and has the usual properties of Leray–Schauder topological degree.

Also, we need a classical result as follows (see e.g. [9], [10], [23]).

Lemma 2.4. *Let K be a nonempty subset of a Hausdorff topological vector space X and $G\colon K \to P(X)$ be a multivalued mapping with the properties:*

(a) *G is a* KKM *mapping, that is, for any $\{v_1, \ldots, v_n\} \subset K$, its convex hull $\mathrm{co}\{v_1, \ldots, v_n\}$ is contained in $\bigcup_{i=1}^{n} G(v_i)$;*

(b) *$G(v)$ is closed in X for every $v \in K$;*

(c) *$G(v_0)$ is compact in X for some $v_0 \in K$.*

Then it holds $\bigcap_{v\in K} G(v) \neq \emptyset$.

DEFINITION 2.5 ([8]). Let $S \subseteq \mathbb{R}$ be a measurable subset. A subset $D \subset L^1(S,E)$ is called uniformly integrable if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\Omega \subset S$ with $\mu(\Omega) < \delta$ implies

$$\bigg\| \int_\Omega f\,d\mu \bigg\| < \varepsilon, \quad f \in D,$$

where μ is the Lebesgue measure on $\mathbb{R}$.

The weak compactness criterion is contained in the following.

THEOREM 2.6 ([8, Corollary 2.6]). *Let $D \subset L^1(I,E)$ be a bounded, uniformly integrable subset such that $D(t) = \{f(t), f \in D\}$ is weakly relatively compact for almost every $t \in I$. Then D is weakly relatively compact in $L^1(I,E)$.*

THEOREM 2.7 ([4, Theorem 4.3]). *Let $\{x_n\} \subset C(I,E)$, then $x_n \rightharpoonup x \in C(I,E)$ if and only if*

(a) *there exists $N > 0$ such that $\|x_n(t)\| \leq N$ for every $n \in \mathbb{N}$ and $t \in I$;*

(b) *$x_n(t) \rightharpoonup x(t)$ for every $t \in I$.*

Now, we consider an orthonormal basis $\{e_i\}$ of H. H_m denotes the subspace of H with basis $\{e_1, \ldots, e_m\}$. Base on the natural projections $\mathbb{P}_m \colon H \to H_m$ ($m \in \mathbb{N}$), we can approximate the original problem by apply a family of auxiliary problems. Some properties of $\mathbb{P}_m$ are contained in the following.

LEMMA 2.8 ([2, Lemma 6]). *The projections $\mathbb{P}_m \colon H \to H_m$ satisfy the following properties*:

(a) *$\mathbb{P}_m \colon H_w \to H_m$ is continuous*;

(b) *if $x_m \rightharpoonup x$ in H, then $\mathbb{P}_m x_m \rightharpoonup x$ in H.*

LEMMA 2.9 ([3, Lemma 2.2]). *If $f_m \rightharpoonup f$ in $L^1(I,H)$, then $\mathbb{P}_m f_m \rightharpoonup f$ in $L^1(I,H)$.*

3. Results of generalized variational inequalities

Let K be a nonempty closed and convex subset of a reflexive Banach space V. Assume that $h \colon K \times K \to [-\infty, +\infty]$ satisfies the following conditions:

(h_1) $D_1(h) = \{u \in K : h(u,v) \neq -\infty,$ for all $v \in K\}$ is nonempty.

(h_2) h is an equilibrium bifunction on K, i.e. $h(u,u) = 0$ for all $u \in K$.

(h_3) $u \to h(u,v)$ is concave and weakly upper semicontinuous for all $v \in K$, that is,

$$h(\lambda u_1 + (1-\lambda)u_2, v) \geq \lambda h(u_1,v) + (1-\lambda)h(u_2,v)$$

for all $u_1, u_2 \in K$, $\lambda \in (0,1)$ and $v \in K$, and if every $u \in K$ and any $\{u_n\} \subset K$ such that $u_n \rightharpoonup u$ in V, we have

$$\limsup_{n\to\infty} h(u_n, v) \le h(u,v) \quad \text{for all } v \in K.$$

(h_4) $v \to h(u,v)$ is convex for all $u \in K$.

To prove the properties of solution set for (1.2), we give the following definition (see [30], [6] and the references therein).

Definition 3.1. Let $h\colon K \times K \to \mathbb{R}$ be a bifunction and $Q\colon V \to P(V^*)$ a multivalued mapping. Then Q is said to be h-pseudomonotone on K if for all $u, v \in K$ and $u^* \in Q(u)$ it holds

$$\langle u^*, v-u\rangle + h(u,v) \ge 0 \ \Rightarrow\ \langle v^*, v-u\rangle + h(u,v) \ge 0, \quad \text{for all } v^* \in Q(v).$$

Now, let us consider the following generalized mixed variational inequality (GMVI): find $u \in K$ and $u^* \in Q(u)$ such that

$$\langle u^*, v-u\rangle + h(u,v) \ge 0, \quad \text{for all } v \in K. \tag{3.1}$$

Lemma 3.2. *Assume that the following conditions hold*:

(I) *The bifunction* $h\colon K \times K \to [-\infty, +\infty]$ *satisfies* (h_1)–(h_4).

(II) $Q\colon V \to P(V^*)$ *is h-pseudomonotone and lower hemicontinuous on* K.

Then $u \in K$ is a solution of (3.1) *if and only if $u \in K$ is a solution of the following inequality*

$$\sup_{v^* \in Q(v)} \langle v^*, u-v\rangle \le h(u,v), \quad \textit{for all } v \in K. \tag{3.2}$$

Proof. The proof is the similar to [6, Theorem 1]. Let $u \in K$ be a solution of (3.1). Then there exists $u^* \in Q(u)$ such that

$$\langle u^*, v-u\rangle + h(u,v) \ge 0, \quad \text{for all } v \in K.$$

Since Q is h-pseudomonotone, it is easy to see that

$$\langle v^*, v-u\rangle + h(u,v) \ge 0, \quad \text{for all } v \in K,\ v^* \in Q(v).$$

Hence, by above inequality, we have

$$\sup_{v^* \in Q(v)} \langle v^*, u-v\rangle \le h(u,v), \quad \text{for all } v \in K,$$

which implies that $u \in K$ satisfies (3.2).

Conversely, assume that there exists $u \in K$ such that (3.2) hold. Then, for any $v \in K$,

$$\langle v^*, u-v\rangle \le h(u,v), \quad \text{for all } v^* \in Q(v).$$

The convexity of K means that for any $v \in K$, $u_n := v/n + (1-1/n)u \in K$ for $n \in \mathbb{N}$, thus

$$\langle u_n^*, u_n-u\rangle + h(u,u_n) \ge 0, \quad \text{for all } u_n^* \in Q(u_n).$$

From (h_2) and (h_4), we have

$$\langle u_n^*, v-u\rangle + h(u,v) \geq 0.$$

Applying the lower hemicontinuity of Q and $u_n \to u$, for any $u^* \in Q(u)$, we can find a sequence $u_n^* \in Q(u_n)$ such that $u_n^* \rightharpoonup u^*$ in E^*. Then we have

$$\langle u^*, v-u\rangle + h(u,v) \geq 0, \quad \text{for all } v \in V,$$

which implies that there exists $u^* \in Q(u)$ such that $u \in K$ fulfills (3.1).

Lemma 3.3. *Assume that the conditions* (I) *and* (II) *of Lemma* 3.2 *and the following condition hold*:

(III) *If K is unbounded, then there exist $u_0 \in K$ and $r > 0$ such that*

$$\inf_{u^* \in Q(u)} \langle u^*, u-u_0\rangle > h(u,u_0) \quad \textit{for all } \|u\| > r. \tag{3.3}$$

Then the solution set of (3.1) *is nonempty, convex and closed in V.*

Proof. The proof of nonemptiness is similar to [6, Theorems 2 and 3] and [34, Theorem 4.1]. Thus we omit it here.

Now, we prove that the solution set of (3.1) is convex and closed. Let $u_1, u_2 \in K$ be the solutions of (3.1) and $u_\lambda = \lambda u_1 + (1-\lambda)u_2 \in K$ for $\lambda \in (0,1)$, then from Lemma 3.2 and the fact that $h(\,\cdot\,, v)$ is concave for each $v \in K$, we easily obtain

$$\begin{aligned}\sup_{v^* \in Q(v)} \langle v^*, u_\lambda - v\rangle &\leq \lambda \sup_{v^* \in Q(v)} \langle v^*, u_1 - v\rangle + (1-\lambda) \sup_{v^* \in Q(v)} \langle v^*, u_2 - v\rangle \\ &\leq \lambda h(u_1, v) + (1-\lambda) h(u_2, v) \leq h(u_\lambda, v),\end{aligned}$$

for $v \in K$ and u_λ is a solution of (3.1) by use of Lemma 3.2 again. This implies the solution set of (3.1) is convex.

Moreover, let u_n be solutions of (3.1) such that $u_n \to u \in K$ in V. Then, for each $v \in K$, the upper semicontinuity of $h(\,\cdot\,, v)$ implies

$$\langle v^*, v-u\rangle = \lim_{n\to\infty} \langle v^*, v-u_n\rangle \leq \limsup_{n\to\infty} h(u_n, v) \leq h(u,v), \quad v^* \in Q(v),$$

which equivalent to

$$\sup_{v^* \in Q(v)} \langle v^*, u-v\rangle \leq h(u,v).$$

According to the equivalence between (3.1) and (3.2), it is easy to see that $u \in K$ is a solution of (3.1) and the solution set of (3.1) is closed. □

Next, we provide the following assumptions:

(H1) For $u \in K$, $G(\,\cdot\,, \,\cdot\,, u)\colon I \times E \to P(V^*)$ is lower semicontinuous.

(H2) For every $(t,x) \in I \times E$, $G(t,x,\,\cdot\,)\colon V \to P(V^*)$ is h-pseudomonotone, lower hemicontinuous on K. If K is unbounded, for each bounded set $\Omega \subset E$, there exist $u_0 \in K$ and $r_\Omega > 0$ such that

$$\inf_{u^* \in G(t,x,u)} \langle u^*, u - u_0\rangle > h(u,u_0) \quad \text{for all } \|u\| > r_\Omega \text{ and } x \in \Omega. \tag{3.4}$$

(H3) The bifunction $h\colon K \times K \to [-\infty,+\infty]$ satisfies (h_1)–(h_4).

THEOREM 3.4. *Let E and V be two separable and reflexive Banach spaces, K be a nonempty closed and convex subset of V. If* (H1)–(H3) *hold, then the multivalued mapping $U\colon I \times E \to 2^V$ defined by*

$$U(t,x) := \big\{u \in K : \exists\, u^* \in G(t,x,u) \text{ such that } \langle u^*, v-u\rangle + h(u,v) \geq 0, \text{ for all } v \in K\big\} \tag{3.5}$$

has the following properties:

(a) *for $(t,x) \in I \times E$, $U(t,x)$ is nonempty, convex, closed and bounded in K. Moreover, there exists a constant L_Ω such that*

$$\|U(t,x)\| := \sup\{\|u\|, u \in U(t,x)\} \leq L_\Omega \text{ for all } (t,x) \in I \times \Omega. \tag{3.6}$$

(b) *U is strongly-weakly upper semicontinuous from $I \times E$ to V, i.e. U is upper semicontinuous from $I \times E$ to V_w.*

(c) *for $x \in E$, $U(\,\cdot\,,x)\colon I \to P(V)$ is measurable.*

PROOF. *Step 1.* Firstly, we prove judgment (a). From (H1)–(H3) and Lemma 3.3, we know that the set $U(t,x)$ is nonempty, closed and convex for every $(t,x) \in I \times E$. Now we confirm that the range $U(I \times \Omega)$ is bounded in V for bounded set $\Omega \subset E$.

Arguing by contradiction, suppose that for any n we can find $u_n \in U(I \times \Omega)$ with $\|u_n\| \geq n$. Hence there exists $(t_n,x_n) \in I \times \Omega$ such that $u_n^* \in G(t_n,x_n,u_n)$ and

$$\langle u_n^*, v-u_n\rangle + h(u_n,v) \geq 0, \quad \text{for all } v \in K.$$

On the other hand, we know by (H3) that for n sufficiently large there holds

$$\langle u_n^*, u_n - u_0\rangle > h(u_n,u_0).$$

This contradiction establishes that $U(I \times \Omega)$ is bounded. Thus there exists a constant $L_\Omega > 0$ such that (3.6) holds. Therefore, $U(t,x)$ is bounded in K for $(t,x) \in I \times E$.

Step 2. To prove (b), we will show that $U^-(C) := \{(t,x) \in I \times E : U(t,x) \cap C \neq \emptyset\}$ is closed in $\mathbb{R} \times E$ for each weakly closed subset $C \subset V$. To this aim, suppose that $\{(t_n,x_n)\} \subset U^-(C)$ satisfies $(t_n,x_n) \to (t,x)$ in $I \times E$. Thus we can choose $u_n \in U(t_n,x_n) \cap C$. By (a) and the reflexivity of V, we conclude that

$\{u_n\}$ is weakly compact, without loss of generality we assume that $u_n \rightharpoonup u \in C$. Moreover, since $u_n \in U(t_n, x_n)$, there exists $u_n^* \in G(t_n, x_n, u_n)$ such that

$$\langle u_n^*, v - u_n \rangle + h(u_n, v) \geq 0, \quad \text{for all } v \in K. \tag{3.7}$$

By Lemma 3.2, we have for any $v \in K$

$$\sup_{z^* \in G(t_n, x_n, v)} \langle z^*, u_n - v \rangle \leq h(u_n, v),$$

which is equivalent to

$$\langle z^*, u_n - v \rangle \leq h(u_n, v), \quad \text{for all } z^* \in G(t_n, x_n, v).$$

As $G(\,\cdot\,,\,\cdot\,, v)$ is lower semicontinuous for $v \in K$, for each $v^* \in G(t, x, v)$, we can find $v_n^* \in G(t_n, x_n, v)$ such that $v_n^* \to v^*$. Then we obtain

$$\langle v^*, u - v \rangle = \limsup_{n\to\infty} \langle v_n^*, u_n - v \rangle \leq \limsup_{n\to\infty} h(u_n, v) \leq h(u, v), \tag{3.8}$$

— for all $v^* \in G(t, x, v)$, which implies that

$$\sup_{v^* \in G(t, x, v)} \langle v^*, v - u \rangle \leq h(u, v), \quad \text{for all } v \in K.$$

Using Lemma 3.2 again, we obtain $u \in U(t, x) \cap C$, i.e. U is strongly-weakly upper semicontinuous.

Step 3. According to [20, Theorem 3.4], we obtain the conclusion (c) by a similar way. □

From the proof of Theorem 3.4, we have the following result.

Corollary 3.5. *If* (H2)–(H3) *and the following condition*:

(H1′) *for $u \in K$, $G(\,\cdot\,,\,\cdot\,, u) \colon I \times E \to P(V^*)$ is weakly-strongly lower semicontinuous, i.e. $G(\,\cdot\,,\,\cdot\,, u)$ is lower semicontinuous from $I \times E_w$ to V^**

hold. Then the multivalued map U defined by (3.5) *has the properties* (a) *and* (b) *of Theorem* 3.4. *Moreover,*

(b′) *U is upper semicontinuous from $I \times E_w$ to V_w.*

Finally, we define an operator $\mathcal{P}_U \colon C(I, E) \to P\big(L^2(I, V)\big)$ as follows:

$$\mathcal{P}_U(x) := \big\{u \in L^2(I, V) : u(t) \in U(t, x(t)) \text{ for a.e. } t \in I\big\}, \quad x \in C(I, E),$$

where U is defined by (3.5). Then, we have the following lemma.

Lemma 3.6. *Let E and V be two separable and reflexive Banach spaces. Assume that* (H1′), (H2)–(H3) *hold. Then $\mathcal{P}_U(x)$ has nonempty, convex and weak compact values in $L^2(I, V)$ for each $x \in C(I, E)$. Moreover, the operator $\mathcal{P}_U$ is weakly u.s.c.*

Proof. From Theorem 3.4, we can obtain the first conclusion by use a same prove way of [27, Lemma 5.3]. From Lemma 2.7, $x_n \rightharpoonup x$ in $C(I,E)$ implies that $x_n(t) \rightharpoonup x(t)$ in E for all $t \in I$. Moreover, according the fact that $u_n \in \mathcal{P}_U(x_n)$ with $u_n \rightharpoonup u$ in $L^2(I,V)$, it is easy to obtain the conclusion that $u \in \mathcal{P}_U(x)$ by Theorem 3.4 (b) and Theorem 2.3. □

4. Existence results for DVI problems

In this section, we mainly consider the existence results of semilinear differential variational inequality (1.1). We need the following assumptions to obtain the existence results.

(A1) For $u \in K$, $G(\,\cdot\,,\,\cdot\,,u)\colon I \times H \to P(V^*)$ is weakly-strongly lower semicontinuous, i.e. $G(\,\cdot\,,\,\cdot\,,u)$ is lower semicontinuous from $I \times H_w$ to V^*.

(A2) For every $(t,x) \in I \times H$, $G(t,x,\,\cdot\,)\colon V \to P(V^*)$ is h-pseudomonotone, lower hemicontinuous on K. If K is unbounded, for each bounded set $\Omega \subset H$, there exist $u_0 \in K$ and $r_\Omega > 0$ such that

$$\inf_{u^* \in G(t,x,u)} \langle u^*, u - u_0 \rangle > h(u,u_0) \quad \text{for all } \|u\| > r_\Omega \text{ and } x \in \Omega. \tag{4.1}$$

(H4) $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$.

(H5) The nonlinearity $f\colon I \times H \times V \to H$ satisfies the conditions:

(a) $f(\,\cdot\,,x,u)\colon I \to H$ is measurable for each $(x,u) \in H \times V$;

(b) $f(t,\,\cdot\,,\,\cdot\,)\colon H_w \times V_w \to H_w$ is continuous for almost every $t \in I$;

(c) for every bounded subset $\Omega \subset H \times V$, there exists a function $\mu_\Omega \in L^1(I,\mathbb{R}^+)$ such that

$$\|f(t,x,u)\| \leq \mu_\Omega(t), \quad \text{for a.e. } t \in I \text{ and each } (x,u) \in \Omega;$$

(d) for almost every $t \in I$ and every $x \in H$, the set $f(t,x,D)$ is convex in H for every convex set $D \subset K$;

(e) there exist $R_0 > r_0 > 0$ such that for every $x \in B_H(0,r_0,R_0) = \{w \in H : r_0 \leq \|w\| \leq R_0\}$ and almost every $t \in I$, it holds

$$\langle x, f(t,x,z)\rangle \leq 0, \tag{4.2}$$

for all $z \in U(t,x)$, where U is introduced in (3.5).

(H6) $\varphi\colon C(I,H) \to H$ is a weakly continuous operator such that

$$\|\varphi(x)\| \leq \|x\|_C.$$

Remark 4.1. (a) If $f(t,x,\,\cdot\,)\colon K \to H$ is affine for all $(t,x) \in I \times H$. Then it is a particular case of (H5) (d). In works of Pang and Stewart [28] and Liu, Loi and Obukhovskiĭ [19], it is utilized as the form $f(t,x,u) = g(t,x) + B(t,x)u$.

(b) Usually, we will assume that the inequality (4.2) of (H5) (e) holds for all $z \in K$.

(c) The class of boundary value problems with (H6) is sufficiently large. Local boundary conditions such as initial-value, periodic, anti-periodic, and nonlocal boundary conditions such as mean value, multipoint and integral conditions are included in our investigation and their interest lies in many physical applications.

(d) The initial value problems with (H6) includes the classical nonlocal condition $x(0) = Mx$, where the linear operator M satisfies $\|M\| \le 1$.

Now, we provide the following existence result.

THEOREM 4.2. *Assume that the hypotheses* (A1)–(A2) *and* (H3)–(H6) *are satisfied. Then* (1.1) *admits at least one mild solution.*

PROOF. To our goal, we divide the proof into three steps.

Step 1. Make some preparations. Choose a constant $r^* \in (r_0, R_0)$ and, for $m \in \mathbb{N}$, we denote

$$\mathcal{Q}_m = \big\{q \in C(I, H_m) : \|q(t)\| \le r^*,\ t \in I\big\},$$

where $r_0 > 0$ and $R_0 > 0$ are given by (H5) (e). Obviously, $\mathcal{Q}_m$ is is a closed, convex set with nonempty interior.

For each $n \in \mathbb{N}$, let $A_n := n^2 R(n, A) - n\mathcal{I}$ be the Yosida approximation of A, where $R(n, A) = (n\mathcal{I} - A)^{-1}$ and $\mathcal{I}$ is the identity operator. It is well known (see, e.g. [31, Lemma 3.2.1 and 3.2.2]) that $A_n \in \mathcal{L}(H)$ and $\{e^{tA_n}\}$ is a semigroup of contractions such that

$$\lim_{n\to\infty} e^{tA_n} z = T(t)z \quad \text{for } t \in I \text{ and } z \in H. \tag{4.3}$$

Now, for each $n \in \mathbb{N}$, we consider the auxiliary problem:

$$\begin{cases} x'(t) = A_n x(t) + f(t, x(t), u(t)) & \text{for a.e. } t \in I, \\ u(t) \in \mathrm{SOL}(K, G(t, x(t), \,\cdot\,), h) & \text{for a.e. } t \in I, \\ x(0) = \varphi(x). \end{cases} \tag{4.4}$$

To prove the existence solutions of the problem (4.4), we will use the topological degree theory and approximation solvability method.

Step 2. Solvability of auxiliary problem (4.4). Firstly, for every $m \in \mathbb{N}$, given $q \in \mathcal{Q}_m$ and $\lambda \in [0, 1]$, there exists a function $u \in \mathcal{P}_U(q) \subset L^2(I, V)$ such that the linear problem

$$\begin{cases} x'(t) = \lambda \mathbb{P}_m A_n q(t) + \lambda \mathbb{P}_m f(t, q(t), u(t)) & \text{for a.e. } t \in I, \\ x(0) = \lambda \mathbb{P}_m \varphi(q), \end{cases} \tag{4.5}$$

has a unique solution $x \in C(I, H_m)$ given by

$$x(t) = \lambda \mathbb{P}_m \varphi(q) + \int_0^t \lambda[\mathbb{P}_m A_n q(s) + \mathbb{P}_m f(s, q(s), u(s))]\, ds. \tag{4.6}$$

We denote by $\mathcal{S}(u,\lambda) \in C(I, H_m)$ the solution of (4.5). Now, define a multivalued map $\mathcal{T}\colon \mathcal{Q}_m \times [0,1] \to P(C(I,H_m))$ as

$$\mathcal{T}(q,\lambda) = \{\mathcal{S}(u,\lambda) : u \in \mathcal{P}_U(q)\}.$$

Next, we show that $\mathcal{T}(q,\lambda)$ is convex for each $(q,\lambda) \in \mathcal{Q}_m \times [0,1]$. Fix $(q,\lambda) \in \mathcal{Q}_m \times [0,1]$ and $x_1, x_2 \in \mathcal{T}(q,\lambda)$. Then there exist $u_1, u_2 \in \mathcal{P}_U(q)$ such that $u_1(t), u_2(t) \in U(t, q(t))$ for almost every $t \in I$, and

$$x_i(t) = \lambda \mathbb{P}_m \varphi(q) + \int_0^t \lambda[\mathbb{P}_m A_n q(s) + \mathbb{P}_m f(s, q(s), u_i(s))]\, ds$$

for almost every $t \in I$ and $i = 1,2$. According to (H5) (d) and the fact that $U(t,q(t))$ is convex, we know $f(t,q(t),U(t,q(t)))$ is convex for almost every $t \in I$. Hence, for $\theta \in (0,1)$,

$$\theta f(t,q(t),u_1(t)) + (1-\theta) f(t,q(t),u_2(t)) \in f(t,q(t),U(t,q(t))) \quad \text{for a.e. } t \in I,$$

which imply that there exists a function $u_\theta : I \to K$ such that $u_\theta(t) \in U(t,q(t))$ and

$$\theta f(t,q(t),u_1(t)) + (1-\theta) f(t,q(t),u_2(t)) = f(t,q(t),u_\theta(t)) \quad \text{for a.e. } t \in I.$$

Moreover, by Corollary 3.5, there exists a constant $L_{\mathcal{Q}_m}$ such that

$$\|u_\theta(t)\| \le \|U(t,q(t))\| \le L_{\mathcal{Q}_m} \quad \text{for a.e. } t \in I,$$

which implies that $u_\theta \in L^\infty(I,V) \subset L^2(I,V)$ and $u_\theta \in \mathcal{P}_U(q)$. Furthermore, we have

$$\begin{aligned} x_\theta(t) &= \theta x_1(t) + (1-\theta)x_2(t) = \lambda \mathbb{P}_m \varphi(q) \\ &\quad + \int_0^t \lambda\big[\mathbb{P}_m A_n q(s) + \theta \mathbb{P}_m f(t,q(t),u_1(t)) + (1-\theta)\mathbb{P}_m f(t,q(t),u_2(t))\big]\, ds \\ &= \lambda \mathbb{P}_m \varphi(q) + \int_0^t \lambda\big[\mathbb{P}_m A_n q(s) + \mathbb{P}_m f(t,q(t),u_\theta(t))\big]\, ds. \end{aligned}$$

Therefore, $x_\theta \in \mathcal{T}(q,\lambda)$. Then $\mathcal{T}(q,\lambda)$ is convex for each $(q,\lambda) \in \mathcal{Q}_m \times [0,1]$.

After, we will apply the relative topological degree to prove the existence of fixed points of $\mathcal{T}(\,\cdot\,,1)$. In the following, we prove it by several claims.

Claim (i). *$\mathcal{T}$ has a closed graph in $\mathcal{Q}_m \times [0,1] \times C(I,H_m)$.*

Assume that $(q_j,\lambda_j,x_j) \to (q_0,\lambda_0,x_0)$ in $\mathcal{Q}_m \times [0,1] \times C(I,H_m)$, where $x_j \in \mathcal{T}(q_j,\lambda_j)$. Moreover, $x_j \in \mathcal{T}(q_j,\lambda_j)$ means that there exists a sequence $\{u_j\}$ with $u_j \in \mathcal{P}_U(q_j)$ and such that

$$x_j(t) = \lambda_j \mathbb{P}_m \varphi(q_j) + \int_0^t \big[\lambda_j \mathbb{P}_m A_n q(s) + \lambda_j \mathbb{P}_m f(s,q_j(s),u_j(s))\big]\, ds, \quad t \in I.$$

From (H6) and Lemma 2.8, we get $\lambda_j \mathbb{P}_m \varphi(q_j) \to \lambda_0 \mathbb{P}_m \varphi(q_0)$ in H_m as $j \to \infty$. Indeed, Corollary 3.5 ensures that there exists a constant $L_{\mathcal{Q}_m}$ such that

$$\|u_j(t)\| \le \|U(t, q_j(t))\| \le L_{\mathcal{Q}_m}, \tag{4.7}$$

which implies that the set $\{u_j\}$ is bounded in $L^2(I, V)$. By the reflexivity of $L^2(I, V)$, we may assume $u_j \rightharpoonup u_0$ in $L^2(I, V)$.

Next, we show that $u_0 \in \mathcal{P}_U(q_0)$. The convergence $q_j \to q_0$ in $C(I, H_m)$ implies

$$q_j(t) \to q_0(t) \quad \text{in } H_m \text{ for all } t \in I.$$

Combining with $u_j \rightharpoonup u_0 \in L^2(I, V)$, $u_j \in \mathcal{P}_U(u_j)$ with the inclusion

$$u_j(t) \in U(t, q_j(t)) \quad \text{for a.e. } t \in I,$$

and the convergence result stated in Theorem 2.3, we deduce that

$$u_0(t) \in U(t, q_0(t)) \quad \text{for a.e. } t \in I,$$

which implies $u_0 \in \mathcal{P}_U(q_0)$. Furthermore, from the condition (H5) (b) and the fact that $u_j(t) \rightharpoonup u_0(t)$ in V for almost every $t \in I$, we have

$$f(s, q_j(s), u_j(s)) \to f(s, q_0(s), u_0(s)) \text{ weakly in } H \text{ for a.e. } s \in I.$$

Since A_n is linear and bounded, by (4.7) and (H5) (c), there exist a constant $L_{\mathcal{Q}_m} > 0$ and a function $\mu_{\mathcal{Q}_m} \in L^1(I, \mathbb{R}^+)$ such that,

$$\|\lambda_j \mathbb{P}_m A_n q_j(t) + \lambda_j \mathbb{P}_m f(s, q_j(t), u_j(t))\| \le r^* \|A_n\|_{\mathcal{L}} + \mu_{\mathcal{Q}_m}(t) \text{ for a.e. } t \in I.$$

Applying the Lebesgue dominated convergence theorem, we can confirm quickly that

$$x_j(t) \to \lambda_0 \mathbb{P}_m \varphi(q_0) + \int_0^t \lambda_0 \big[\mathbb{P}_m A_n q_0(s) + \mathbb{P}_m f(s, q_0(s), u_0(s))\big]\, ds =: \gamma(t)$$

in H_m for $t \in I$. The uniqueness of the limit in H_m implies that $x_0(t) = \gamma(t)$ for all $t \in I$. Therefore, $x_0 \in \mathcal{T}(q_0, \lambda_0)$ and $\mathcal{T}$ has a closed graph.

Claim (ii). *$\mathcal{T}$ is a compact operator, i.e. that $\mathcal{T}(\mathcal{Q}_m \times [0,1])$ is relatively compact in $C(I, H_m)$.*

In fact, for each $x \in \mathcal{T}(q, \lambda)$, according to (4.6), (4.7), (H5) (c) and (H6), there exist a constant $L_{\mathcal{Q}_m} > 0$ and a function $\mu_{\mathcal{Q}_m} \in L^1(I, \mathbb{R}^+)$ such that

$$\begin{aligned} \|x(t)\| &\le \lambda \|\mathbb{P}_m \varphi(q)\| + \int_0^t \lambda \|\mathbb{P}_m A_n q(s) + \mathbb{P}_m f(s, q(s), u(s))\|\, ds \\ &\le r^* + b r^* \|A_n\|_{\mathcal{L}} + \|\mu_{\mathcal{Q}_m}\|_{L^1(I, \mathbb{R}^+)}, \end{aligned} \tag{4.8}$$

for $t \in I$. Hence $\mathcal{T}(\mathcal{Q}_m \times [0,1])$ is bounded in $C(I, H_m)$. Since

$$\|x'(t)\| \le \lambda \|\mathbb{P}_m A_n q(s)\| + \|\mathbb{P}_m f(s, q(s), u(s))\| \le r^* \|A_n\|_{\mathcal{L}} + \mu_{\mathcal{Q}_m}(t)$$

for almost every $t \in I$, it follows that $\{x' : x \in \mathcal{T}(\mathcal{Q}_m \times [0,1])\}$ is bounded in $L^1(I, H_m)$. Then $\{x : x \in \mathcal{T}(\mathcal{Q}_m \times [0,1])\}$ is equicontinuous. By the Ascoli–Arzelá theorem, we obtain the conclusion. Furthermore, as any closed and compact multivalued maps are u.s.c., then $\mathcal{T}$ is a compact u.s.c. multivalued map.

Claim (iii). *$\mathcal{T}(\,\cdot\,,\lambda)$ is fixed point free on $\partial\mathcal{Q}_m$ for all $\lambda \in [0,1)$.*

It is easy to see that $\mathcal{T}(\,\cdot\,,0) = \{0\}$ has no fixed points on $\partial\mathcal{Q}_m$, so it remains to prove this property for $\mathcal{T}(\,\cdot\,,\lambda)$ with $\lambda \in (0,1)$.

To the contrary, assume that there exists $(q,\lambda) \in \partial\mathcal{Q}_m \times (0,1)$ such that $q \in \mathcal{T}(q,\lambda)$. Then $q \in \partial\mathcal{Q}_m$ means that there exists an $t_0 \in I = [0,b]$ such that $\|q(t_0)\| = r^*$. If $t_0 = 0$, then

$$r^* = \|q(t_0)\| = \|\lambda\varphi(q)\| < \|q\|_C = r^*, \tag{4.9}$$

it is a contradiction. So, $t_0 \in (0,b]$. According to $\|q(0)\| < r^* = \|q(t_0)\|$, we can take a sufficiently small $\varepsilon > 0$ such that $r_0 < \|q(t)\| \le r^*$ for all $t \in (t_0-\varepsilon, t_0)$ and $\|q(t_0-\varepsilon)\| < r^*$. Then the condition (H5) (e) implies that for $u(t) \in U(t,q(t))$, it holds

$$\langle q(t), \mathbb{P}_m f(t,q(t),u(t))\rangle \le 0 \quad \text{for a.e. } t \in (t_0-\varepsilon, t_0). \tag{4.10}$$

On the other hand, since $\|R(n,A)\|_{\mathcal{L}} \le n^{-1}$ (see [31, Theorem 3.1.1]), we have

$$\begin{aligned}\langle q(t), \mathbb{P}_m A_n q(t)\rangle &= \langle q(t), n^2 R(n,A) q(t)\rangle - n\langle q(t), q(t)\rangle \\ &\le n^2\|R(n,A)\|_{\mathcal{L}}\|q(t)\|^2 - n\|q(t)\|^2 \le 0\end{aligned}$$

for all $t \in [0,b]$. Consequently,

$$\int_{t_0-\varepsilon}^{t_0} \langle q(t), \mathbb{P}_m A_n q(t) + \mathbb{P}_m f(t,q(t),u(t))\rangle\,dt \le 0.$$

However,

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0} \langle q(t), \mathbb{P}_m A_n q(t) + \mathbb{P}_m f(t,q(t),u(t))\rangle\,dt &= \int_{t_0-\varepsilon}^{t_0} \langle q(t), q'(t)\rangle\,dt \\ = \frac{1}{2}\int_{t_0-\varepsilon}^{t_0} \frac{d}{dt}\big(\|q(t)\|^2\big)\,dt &= \frac{1}{2}\big(\|q(t_0)\|^2 - \|q(t_0-\varepsilon)\|^2\big) > 0,\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{T}(\cdot,\lambda)$ has no fixed point on $\partial\mathcal{Q}_m$ for all $\lambda \in [0,1)$.

Claim (iv). *$\mathcal{T}(\,\cdot\,,1)$ has a fixed point in $\mathcal{Q}_m$.*

If there is a $q \in \partial\mathcal{Q}_m$ such that $q \in \mathcal{T}(q,1)$, it completes the proof. If $q \notin \mathcal{T}(q,1)$ for all $q \in \partial\mathcal{Q}_m$, then the multivalued map $\mathcal{T}$ is a homotopy connecting multivalued maps $\mathcal{T}(\,\cdot\,,0)$ and $\mathcal{T}(\,\cdot\,,1)$. By applying normalization property and homotopy invariance of the topological degree (see, e.g. [13, Section 3.1]), we have

$$\deg(i - \mathcal{T}(\,\cdot\,, 1), \mathcal{Q}_m) = \deg(i - \mathcal{T}(\,\cdot\,, 0), \mathcal{Q}_m) = 1.$$

From [13, Theorem 3.1.3], there exists at least one $q \in \mathcal{Q}_m$ such that $q \in \mathcal{T}(q, 1)$. Therefore, for each $m \in \mathbb{N}$, there exists a (strong) solution $x_m \in \mathcal{Q}_m$ and $u_m \in \mathcal{P}_U(x_m)$ satisfying the following equation:

$$\begin{cases} x_m'(t) = \mathbb{P}_m A_n x_m(t) + \mathbb{P}_m f(t, x_m(t), u_m(t)) & \text{for a.e. } t \in I, \\ x_m(0) = \mathbb{P}_m \varphi(x_m). \end{cases} \tag{4.11}$$

Now, we prove that, for each $n \in \mathbb{N}$, the auxiliary problem (4.4) has at least one solution. For convenience, let $\xi_m(t) = A_n x_m(t) + f(t, x_m(t), u_m(t))$. According to the assumption (H5) (c), the reflexivity of H and Theorem 2.6, we know that $\{\xi_m\}$ is weakly relatively compact in $L^1(I, H)$. Without loss of generality, assume that $\xi_m \rightharpoonup \xi \in L^1(I, H)$. Moreover, by Lemma 2.9,

$$x_m' = \mathbb{P}_m \xi_m \rightharpoonup \xi \in L^1(I, H).$$

The condition (H6) means that the set $\{x_m(0) = \mathbb{P}_m \varphi(x_m)\}$ is a bounded set in a reflexive space H. Without loss of generality, we may assume that $x_m(0) = \mathbb{P}_m \varphi(x_m) \rightharpoonup \eta$. Consider the function

$$y(t) := \eta + \int_0^t \xi(s)\, ds, \quad \text{for } t \in I,$$

then

$$x_m(t) = x_m(0) + \int_0^t x_m'(s)\, ds \rightharpoonup y(t), \quad \text{for } t \in I.$$

Therefore, by Theorem 2.7, $x_m \rightharpoonup y \in C(I, H)$ and $y'(t) = \xi(t)$, for almost every $t \in I$.

Furthermore, from Theorem 3.4, the sequence $\{u_m\}$ is bounded in $L^2(I, V)$. Then $\{u_m\}$ is weakly relatively compact. So there exists a subsequence (still denote by $\{u_m\}$) such that $u_m \rightharpoonup u \in L^2(I, V)$. Then $u_m(t) \rightharpoonup u(t)$ in V for a.e. $t \in I$. By (H5), Lemma 2.8 and the continuity of A_n, we have

$$\mathbb{P}_m A_n x_m(t) + \mathbb{P}_m f(t, x_m(t), u_m(t)) \rightharpoonup A_n y(t) + f(t, y(t), u(t)) \tag{4.12}$$

for almost every $t \in I$. From (H6) and the Lebesgue dominated convergence theorem, we get that

$$\begin{aligned} x_m(t) = {} & \mathbb{P}_m \varphi(x_m) + \int_0^t \big[\mathbb{P}_m A_n x_m(s) + \mathbb{P}_m f(s, q_m(s), u_m(s))\big]\, ds \\ & \rightharpoonup \varphi(y) + \int_0^t [A_n y(s) + f(s, y(s), u(s))]\, ds = y(t) \end{aligned}$$

as $m \to \infty$. Then $y(0) = \varphi(y)$ and $y'(t) = A_n y(t) + f(t, y(t), u(t))$ a.e. $t \in I$. Finally, from $x_m \rightharpoonup y$ in $C(I, H)$, $u_m \rightharpoonup u \in L^2(I, V)$ and Lemma 3.6, we can deduce that $u \in \mathcal{P}_U(y)$. Therefore, the function pair (y,u) is a solution of the auxiliary problem (4.4).

Step 3. We show that the problem (1.1) has a mild solution. For each $n \in \mathbb{N}$, assume that $(x_n,u_n) \in C(I,H) \times L^2(I,V)$ is a solution of the problem (4.4), then $u_n \in \mathcal{P}_U(x_n)$ and

$$x_n(t) = e^{tA_n}\varphi(x_n) + \int_0^t e^{(t-s)A_n} f(s,x_n(s),u_n(s))\,ds.$$

According to Step 2, we know $x_n \in \mathcal{R}^* = \{q \in C(I,H_m) : \|q(t)\| \le r^* + 1,\ t \in I\}$. From (H6) and the reflexivity of the space H, we get that there exists $\overline{x} \in H$ such that, up to subsequence, $\varphi(x_n) \rightharpoonup \overline{x}$ in H. Moreover, denoting $f_n(t) = f(t,x_n(t),u_n(t))$ and from $\Omega = \mathcal{R}^* \times U(I,\mathcal{R}^*)$, it follows that there exist a constant $L_{\mathcal{R}^*} > 0$ and a function $\mu_{\mathcal{R}^*} \in L^1(I,\mathbb{R}^+)$ such that

$$\begin{aligned} &\|u_n(t)\| \le \|U(t,x_n(t))\| \le L_{\mathcal{R}^*} &&\text{for a.e. } t \in I,\\ &\|f_n(t)\| \le \mu_{\mathcal{R}^*}(t) &&\text{for a.e } t \in I. \end{aligned}$$

Hence, the set $\{f_n(t)\}$ is bounded in H for almot every $t \in I$, and the set $\{f_n\}$ is bounded and uniformly integrable in $L^1(I,H)$. By the reflexivity of H and Theorem 2.6, $\{f_n\}$ is weakly relatively compact in $L^1(I,H)$. Without loss of generality, assume that $f_n \rightharpoonup f_0$ in $L^1(I,H)$. Moreover, for every $t \in I$, we have

$$\int_0^t \|e^{(t-s)A_n} f_n(s)\|\,ds \le \int_0^t \|f_n(s)\|\,ds \le \|\mu_{\mathcal{R}^*}\|_{L^1(I,\mathbb{R}^+)}$$

and similarly

$$\int_0^t \|T(t-s)f_0(s)\|\,ds \le \|\mu_{\mathcal{R}^*}\|_{L^1(I,\mathbb{R}^+)}.$$

Consequently, the maps $e^{(t-\,\cdot\,)A_n} f_n(\,\cdot\,)$ and $T(t-\,\cdot\,)f_0(\,\cdot\,)$ belong to the space $L^1([0,t],H)$ for every $t \in I$. Now let us show that

$$e^{(t-\,\cdot\,)A_n} f_n(\,\cdot\,) \rightharpoonup T(t-\,\cdot\,)f_0(\,\cdot\,) \quad \text{for each } t \in I.$$

To this goal, assume that $\Phi\colon L^1([0,t],H) \to \mathbb{R}$ is a linear and bounded functional. Hence, we can find an $\omega \in L^\infty([0,t],H)$ such that

$$\Phi(g) = \int_0^t \langle g(s),\omega(s)\rangle\,ds \quad \text{for all } g \in L^1([0,t],H).$$

We can get

$$\begin{aligned} &\Phi\big(e^{(t-\,\cdot\,)A_n} f_n(\,\cdot\,) - T(t-\,\cdot\,)f_0(\,\cdot\,)\big)\\ &\qquad\qquad = \int_0^t \langle e^{(t-s)A_n} f_n(s) - T(t-s)f_0(s),\omega(s)\rangle\,ds \end{aligned}$$

$$= \int_0^t \langle e^{(t-s)A_n} f_n(s) - e^{(t-s)A_n} f_0(s), \omega(s)\rangle\, ds$$
$$+ \int_0^t \langle e^{(t-s)A_n} f_0(s) - T(t-s) f_0(s), \omega(s)\rangle\, ds.$$

By (4.3) and the Lebesgue dominated convergence theorem, we have that

$$\int_0^t \langle e^{(t-s)A_n} f_0(s) - T(t-s) f_0(s), \omega(s)\rangle\, ds \to 0 \quad \text{for every } t \in I.$$

Further, let $A^*\colon D(A^*) \subset H \to H$ be the adjoint operator of A. It is well known (see, e.g. [31, Theorem 3.7.1]) that A^* is the generator of the C_0 semigroup of contractions $\{T^*(t)\}_{t\geq 0}$, where $T^*(t)$ is the adjoint of the operator $T(t)$. Moreover,

$$\lim_{n\to\infty} e^{tA_n^*} z^* = T^*(t) z^* \quad \text{for } t \in I \text{ and } z^* \in H, \tag{4.13}$$

where $A_n^* = n^2 R(n, A^*) - n\mathcal{I}$. Consequently,

$$\int_0^t \langle e^{(t-s)A_n} f_n(s) - e^{(t-s)A_n} f_0(s), \omega(s)\rangle\, ds$$
$$= \int_0^t \langle f_n(s) - f_0(s), e^{(t-s)A_n^*}\omega(s)\rangle\, ds$$
$$= \int_0^t \langle f_n(s) - f_0(s), e^{(t-s)A_n^*}\omega(s) - T^*(t-s)\varphi(s)\rangle\, ds$$
$$+ \int_0^t \langle f_n(s) - f_0(s), T^*(t-s)\omega(s)\rangle\, ds.$$

It is obvious that $T^*(t-\,\cdot\,)\omega(\,\cdot\,) \in L^\infty([0,t],H)$, thus

$$\int_0^t \langle f_n(s) - f_0(s), T^*(t-s)\omega(s)\rangle\, ds \to 0 \quad \text{as } n \to \infty.$$

Notice that, from (4.13) we get that $e^{(t-s)A_n^*}\omega(s) \to T^*(t-s)\omega(s)$ for almost every $s \in [0,t]$. Moreover, the convergence is dominated, thus

$$e^{(t-\,\cdot\,)A_n^*}\omega(\,\cdot\,) \to T^*(t-\,\cdot\,)\omega(\,\cdot\,) \quad \text{in } L^\infty([0,t],H)$$

and we get that

$$\int_0^t \big\langle f_n(s) - f_0(s), e^{(t-s)A_n^*}(s) - T^*(t-s)\omega(s)\big\rangle\, ds \to 0 \quad \text{as } n \to \infty.$$

So $\Phi(e^{(t-\,\cdot\,)A_n} f_n(\,\cdot\,)) \to \Phi(T(t-\,\cdot\,)f_0(\,\cdot\,))$ for every $t \in I$.

Now let us show that if $\{z_n\} \subset H$, $z_n \rightharpoonup z \in H$, then $e^{tA_n} z_n \rightharpoonup T(t)z$ for each $t \in I$. In fact, for every $g \in H$, we have

$$\langle g, e^{tA_n} z_n - T(t)z\rangle = \langle g, e^{tA_n} z - T(t)z\rangle + \langle g, e^{tA_n}(z_n - z)\rangle.$$

By virtue of (4.13) it follows that $\langle g, e^{tA_n}z - T(t)z\rangle \to 0$ as $n \to \infty$. Further,

$$\begin{aligned}\langle g, e^{tA_n}(z_n - z)\rangle &= \big\langle \big(e^{tA_n}\big)^* g, z_n - z\big\rangle = \big\langle e^{tA_n^*} g, z_n - z\big\rangle \\ &= \big\langle e^{tA_n^*} g - T^*(t)g, z_n - z\big\rangle + \big\langle T(t)^* g, z_n - z\big\rangle \to 0\end{aligned}$$

as $n \to \infty$. Moreover, for $t \in I$,

$$\begin{aligned}\|x_n(t)\| &\le \|T(t)\varphi(x_n)\| + \int_0^t \|T(t-s)f(t, x_n(s), u_n(s))\|\,ds \\ &\le r^* + \|\mu_{\mathcal{R}^*}\|_{L^1(I,\mathbb{R}^+)} := N,\end{aligned}$$

thus the Theorem 2.7 verifies that $x_n \rightharpoonup x$ in $C(I,H)$.

On the other hand, Theorem 3.4 ensures that the sequence $\{u_n\}$ is bounded in $L^2(I,V)$. Similar to Claim (i), $\{u_n\}$ is weakly relatively compact in $L^2(I,V)$. Hence, we may assume $u_n \rightharpoonup u$ in $L^2(I,V)$. Then $u_n(t) \rightharpoonup u(t)$ in V for almost every $t \in I$. Thus, according to (H4)–(H6), we get

$$\begin{aligned}e^{tA_n}\varphi(x_n) &\rightharpoonup T(t)\varphi(x), \\ \int_0^t e^{(t-s)A_n} f_n(s)\,ds &\rightharpoonup \int_0^t T(t-s)f(s, x(s), u(s))\,ds, \quad \text{for } t \in I.\end{aligned}$$

Therefore, by the uniqueness of the weak limit we have

$$x(t) = T(t)\varphi(x) + \int_0^t T(t-s)f(s, x(s), u(s))ds, \ t \in I.$$

Finally, from $x_n \to x$ in $C(I,H)$, $u_n \rightharpoonup u \in L^2(I,V)$ and Lemma 3.6, we deduce that $u \in \mathcal{P}_U(x)$. Therefore, (x,u) is a mild solution of the problem (1.1). □

Theorem 4.3. *Assume that the hypotheses* (A1)–(A2) *and* (H3)–(H6) *are satisfied. Then the set of mild solutions of* (1.1) *is weakly compact.*

Proof. From Theorem 4.2, we know that the set of mild solutions of (1.1) is nonempty. Assume that $\{(x_m, u_m)\} \subset \mathcal{R}^* \times L^2(I,V)$ is a sequence of mild solutions of the problem (1.1). Hence, $u_m \in \mathcal{P}_U(x_m)$ and

$$x_m(t) = T(t)\varphi(x_m) + \int_0^t T(t-s)f(s, x_m(s), u_m(s))\,ds, \quad t \in I.$$

In a similar way to Theorem 4.2, we know that the sequences $\{u_m\}$ and $\{f(\,\cdot\,, x_m(\,\cdot\,), u_m(\,\cdot\,))\}$ are weakly relatively compact in $L^2(I,V)$ and $L^1(I,H)$ respectively. Moreover, from (H6), $\{\varphi(x_m)\}$ is bounded in the reflexive space H. Thus $\{\varphi(x_m)\}$ is weakly compact in H too. Without loss of generality, assume that

$$(\varphi(x_m), u_m, f(\,\cdot\,, x_m(\,\cdot\,), u_m(\,\cdot\,))) \to \big(\overline{x}, u, \overline{f}\big) \quad \text{weakly in } H \times L^2(I,V) \times L^1(I,H).$$

Hence,

$$x_m(t) \rightharpoonup T(t)\overline{x} + \int_0^t T(t-s)\,\overline{f}(s)\,ds := x(t), \quad \text{for } t \in I.$$

Moreover, by Theorem 2.7 and the fact

$$\|x_m(t)\| \le \|T(t)\varphi(x_m)\| + \int_0^t \|T(t-s)\|\,ds \le r^* + 1 + \|\widetilde{\mu}_\Omega\|_{L^1} := N,$$

we get $x_m \rightharpoonup x$ in $C(I,H)$. According to (H5), we have

$$\begin{aligned} x_m(t) = T(t)\varphi(x_m) + \int_0^t T(t-s)f(t, x_m(s), u_m(s))\,ds \\ \rightharpoonup T(t)\varphi(x) + \int_0^t T(t-s)f(s, x(s), u(s))\,ds \end{aligned}$$

in H for $t \in I$. Therefore, by the uniqueness of the weak limit, we have

$$x(t) = T(t)\varphi(x) + \int_0^t T(t-s)f(s, x(s), u(s))\,ds, \quad \text{for } t \in I,$$

Similarly to the Step 3 of Theorem 4.2, we also obtain that $u \in \mathcal{P}_U(x)$. Therefore, (x,u) is a mild solution of the problem (1.1). Consequently, the set of all mild solutions of problem (1.1) is weakly compact. □

Acknowledgments. The authors would like to thank anonymous referees for their careful reading and constructive suggestions that improved the manuscript substantially.

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Manuscript received August 22, 2018

accepted June 9, 2019

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