ON CRITICAL PSEUDO-RELATIVISTIC HARTREE EQUATION WITH POTENTIAL WELL

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ABSTRACT. The aim of this paper is to investigate the existence and asymptotic behavior of the solutions for the critical pseudo-relativistic Hartree equation

$$\sqrt{-\Delta + m^2} u + (\beta V(x) - \lambda) u = \left( \int_{\mathbb{R}^N} \frac{|u(z)|^{2^*_\mu}}{|x-z|^\mu} \, dz \right) |u|^{2^*_\mu - 2} u$$

for $\mathbb{R}^N$, where $m, \lambda, \beta \in \mathbb{R}^+$, $0 < \mu < N$, $N \geq 3$, $2^*_\mu = (2N - \mu)/(N - 1)$ plays the role of critical exponent due to the Hardy–Littlewood–Sobolev inequality. By transforming the nonlocal problem into a local one via the Dirichlet-to-Neumann map, we are able to obtain the existence of the solutions by variational methods. Suppose that $0 < \lambda < \lambda_1(\Omega)$ with $\lambda_1(\Omega)$ the first eigenvalue and the parameter $\beta$ is large enough, we can prove the existence of ground state solutions. Furthermore, for any sequences $\beta_n \to \infty$, we can show that the ground state solutions $\{u_n\}$ converges to a solution of

$$\sqrt{-\Delta + m^2} u - \lambda u = \left( \int_{\Omega} \frac{|u(z)|^{2^*_\mu}}{|x-z|^\mu} \, dz \right) |u|^{2^*_\mu - 2} u \quad \text{in } \Omega,$$

where $\Omega := \text{int } V^{-1}(0)$ is a nonempty bounded set with smooth boundary. By the way we also establish the existence and nonexistence results for the ground state solutions of the problems set on bounded domain.

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1. Introduction

To describe the boson stars in mean-field theory [18], [27], a reasonable model is to study the nonlinear mean-field equation called the pseudo-relativistic Hartree equation given by

\[
i\partial_t \psi = \left( \sqrt{-\Delta + m^2} - m \right) \psi + W(x) \psi - \left( \frac{1}{|x|} * |\psi|^2 \right) \psi \quad \text{in} \ R^3,
\]

where \( \psi(t, x) \) is a complex-valued wave field, and the symbol \( * \) stands for the convolution in \( R^3 \). The operator \( \sqrt{-\Delta + m^2} - m \) is the kinetic energy operator of a relativistic particle of mass \( m > 0 \) and the convolution kernel, \( |x|^{-1} \), represents the Newtonian gravitational potential in appropriate physical units (see [18]). It has recently been proved such an equation is locally well posed and the solution is global in time for small initial data in \( L^2 \) (see [26]). Blow-up has been proved in [21].

The study of solitary wave type solutions \( \psi(t, x) = e^{it\lambda} u(x), \lambda > 0 \) for equation (1.1) leads to investigating the non-local single equation

\[
\sqrt{-\Delta + m^2} u + V(x) u = \left( \frac{1}{|x|} * |u|^2 \right) u \quad \text{in} \ R^3.
\]

More generally, the generalized pseudo-relativistic Hartree equation is given by

\[
\sqrt{-\Delta + m^2} u + V(x) u = (I_\alpha * |u|^p)|u|^{p-2} u \quad \text{in} \ R^N,
\]

where \( m > 0, \)

\[
I_\alpha = \frac{c^{N,\alpha}}{|x|^{N-\alpha}} \quad (x \neq 0), \ \alpha \in (0, N)
\]

is a convolution kernel and \( c^{N,\alpha} \) is a positive constant. Sometimes we may set \( c^{N,\alpha} = 1 \) for convenience. If \( N = 3, \alpha = p = 2 \), equation (1.3) goes back to the pseudo-relativistic Hartree equation (1.2) with Coulomb kernel which has attracted a great deal of attention in theoretical and numerical astrophysics over the past years. Lenzmann [25], [26] proved the uniqueness of ground states for pseudo-relativistic Hartree equations and he also obtained local and global well-posedness for semi-relativistic Hartree equations of critical type. Cingolani and Secchi [13], [14] investigated the existence of positive semiclassical states for the pseudo-relativistic Hartree equations and ground states for the pseudo-relativistic Hartree equation with external potential. Coti Zelati and Nolasco [17] obtained the existence of ground states for some fractional Schrödinger equation involving the operator \( \sqrt{-\Delta + m^2} \) with \( m > 0 \). Mugnai [30] established several existence and non existence results of solitary waves for a class of nonlinear pseudo-relativistic Hartree equations with general nonlinearities. For recent progress in this field, we may refer to [11], [16], [21], [22], [25], [26] and the references therein. The pseudo-relativistic Hartree equation describing the dynamics of boson stars is closely related to the so-called Choquard or nonlinear...
Schrödinger–Newton equation, see [1]–[3], [7], [9], [24], [29] and the references therein.

In the present paper we are going to consider the existence of solutions for the following pseudo-relativistic Hartree equation with potential well:

\[
\begin{cases}
\sqrt{-\Delta + m^2} u + (\beta V(x) - \lambda) u = \left( \int_{\mathbb{R}^N} \frac{|u(z)|^{2^*_\mu}}{|x - z|^\mu} \, dz \right) |u|^{2^*_\mu - 2} u & \text{in } \mathbb{R}^N, \\
u \in H^{1/2}(\mathbb{R}^N),
\end{cases}
\]

where \(\lambda, \beta \in \mathbb{R}^+, 0 < \mu < N, \ N \geq 4, \ 2^*_\mu = \frac{2N - \mu}{N - 1}\) and the potential \(V\) satisfies the assumptions:

1. \((V_1)\) \(V \in C(\mathbb{R}^N, \mathbb{R}), \ V \geq 0\) and \(\Omega := \int V^{-1}(0)\) is a nonempty bounded set with smooth boundary, \(0\) is in interior of \(\Omega\) and \(\overline{\Omega} = V^{-1}(0)\).
2. \((V_2)\) There exists \(M_0 > 0\) such that \(L\{x \in \mathbb{R}^N : V(x) \leq M_0\} < \infty\), where \(L\) denotes the Lebesgue measure in \(\mathbb{R}^N\).

Problem (1.4) is driven by a nonlocal pseudo-differential operator closely related to the square root of the Laplacian. The kinetic energy operator \(\sqrt{-\Delta + m^2}\) is appropriate for describing relativistic quantum particles of mass \(m > 0\) and it is a well studied operator in the whole space. In fact this operator is a nonlocal operator in \(\mathbb{R}^N\) can be realized through a local problem in \(\mathbb{R}^N \times (0, \infty)\). For any function \(u \in H^{1/2}(\mathbb{R}^N)\), there is a unique function \(v \in H^1(\mathbb{R}^{N+1}_+)\) (here \(\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} \mid y > 0\}\)) such that

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
v(x,0) = u(x) & \text{on } \partial \mathbb{R}^{N+1}_+.
\end{cases}
\]

Setting

\[Tu(x) = -\frac{\partial v}{\partial y}(x,0),\]

we have that the equation

\[
\begin{cases}
-\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
w(x,0) = Tu(x) = -\frac{\partial v}{\partial y}(x,0) & \text{for } x \in \mathbb{R}^N
\end{cases}
\]

with the solution \(w(x,y) = -\frac{\partial v}{\partial y}(x,y)\). From (1.5) we have that

\[T(Tu)(x) = -\frac{\partial w}{\partial y}(x,0) = \frac{\partial^2 v}{\partial y^2}(x,0) = \left(-\Delta_x v + m^2 v\right)(x,0)\]

and hence \(T^2 = (-\Delta_x + m^2)\). Thus the operator \(T\) maps the Dirichlet-type data \(u\) to the Neumann-type data \(-\frac{\partial v}{\partial y}(x,0)\) is actually \(\sqrt{-\Delta + m^2}\). In this way: for
equation (1.4), we will study the following mixed value boundary problem:

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+,
\frac{\partial v}{\partial \nu} = \left( \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_p}}{|x-z|^\mu} \, dz \right) |v|^{2^*_p - 2} v - (\beta V(x) - \lambda) v & \text{on } \partial \mathbb{R}^{N+1}_+,
\end{cases}
\]

where \( \nu \) is the unit outer normal to \( \mathbb{R}^N \).

To state the main results, we need to study a nonlocal critical problem on bounded domain which plays the role of limit problem for equation (1.4), that is

\[
\begin{cases}
\sqrt{-\Delta + m^2} u = \left( \int_{\Omega} \frac{|u(z)|^{2^*_p}}{|x-z|^\mu} \, dz \right) |u|^{2^*_p - 2} u + \lambda u & \text{in } \Omega,
\end{cases}
\]

Let

\[
E := \left\{ v \in H^1(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy < \infty, \, v(\cdot,0) \in L^2(\mathbb{R}^N) \right\},
\]

equipped with norm

\[
\|v\|_E = \left( \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + \int_{\Omega} v^2(x,0) \, dx \right)^{1/2}.
\]

We now define \( E_0 \) which is a closed linear subspace of \( E \) as follows.

\[
E_0 := \left\{ v(x,y) \in E : v(x,0) = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}
\]

equipped with norm

\[
\|v\| = \left( \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \right)^{1/2}.
\]

We also define by \( \text{tr}_\Omega E_0 \) the trace operator on \( \Omega \) for functions in \( E_0 \):

\[
\text{tr}_\Omega E_0 := \left\{ v(x,0) = u(x) \text{ for } v(x,y) \in E_0 \right\}.
\]

Thus we call \( v \in E_0 \) the harmonic extension of \( u \) in \( \mathbb{R}^{N+1}_+ \) which vanishes on \( \mathbb{R}^N \setminus \Omega \) and will be denoted by \( v := h\text{-ext}(u) \). It is easy to see that for every \( \eta \in C^\infty \) and \( \eta \equiv 0 \) on \( \mathbb{R}^N \setminus \Omega \),

\[
\int_{\mathbb{R}^{N+1}_+} \nabla v \nabla \eta \, dx \, dy = \int_{\Omega} \frac{\partial v}{\partial \nu} \eta \, dx.
\]

Now define the operator \( \sqrt{-\Delta + m^2} : \text{tr}_\Omega E_0 \to (\text{tr}_\Omega E_0)^* \) by

\[
\sqrt{-\Delta + m^2} u := \left. \frac{\partial v}{\partial \nu} \right|_{\Omega},
\]

\[
\frac{\partial v}{\partial \nu} = \left( \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_p}}{|x-z|^\mu} \, dz \right) |v|^{2^*_p - 2} v - (\beta V(x) - \lambda) v \quad \text{on } \partial \mathbb{R}^{N+1}_+,
\]

\[\frac{\partial v}{\partial \nu} = \left( \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_p}}{|x-z|^\mu} \, dz \right) |v|^{2^*_p - 2} v - (\beta V(x) - \lambda) v \quad \text{on } \partial \mathbb{R}^{N+1}_+,
\]
ON CRITICAL PSEUDO-RELATIVISTIC HARTREE EQUATION WITH POTENTIAL WELL

189

where \((\text{tr}_\Omega E_0)^*\) denotes the dual space of \(\text{tr}_\Omega E_0\) and \(v = h \text{-ext}(u) \in E_0\) and \(v(\cdot, 0)\) vanishes on \(\mathbb{R}^N \setminus \Omega\). In other words, \(v\) is found by solving the problem:

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\
u(x) = v(x, 0) & \text{on } \Omega.
\end{cases}
\]

Instead of (1.7), we are lead to look for a function \(v\) with \(v(\cdot, 0) = u\) in \(\Omega\) satisfying the following boundary value problem:

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\
\frac{\partial v}{\partial \nu} = (\int_{\Omega} |v(z, 0)|^{2^*_{\mu}} |x - z|^{\mu} dz) |v|_{2^*_{\mu}}^{-2} v + \lambda v & \text{on } \Omega,
\end{cases}
\]

where \(v\) satisfies (1.11), then the trace \(u\) on \(\Omega\) of the function \(v\) will be a solution of problem (1.7). By studying (1.11), we establish the results for (1.7). To treat the convolution part, we need to recall the Hardy–Littlewood–Sobolev inequality, see for example [28].

**Proposition 1.1.** Let \(t, r > 1\) and \(0 < \mu < N\) with \(1/t + \mu/N + 1/r = 2\), \(f \in L^t(\mathbb{R}^N)\) and \(h \in L^r(\mathbb{R}^N)\). There exists a sharp constant \(C(t, N, \mu, r)\), independent of \(f, h\), such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\mu} \, dx \, dy \leq C(t, N, \mu, r)|f|_t|h|_r.
\]

If \(t = r = 2N/(2N - \mu)\), then

\[
C(t, N, \mu, r) = C(N, \mu) = \pi^{\mu/2} \frac{\Gamma(N/2 - \mu/2)}{\Gamma(N - \mu/2)} \left\{ \frac{\Gamma(N/2)}{\Gamma(N)} \right\}^{-1 + \mu/N}.
\]

In this case there is equality in (1.12) if and only if \(f \equiv Ch\) and

\[
h(x) = A(\gamma^2 + |x - a|^2)^{-(2N - \mu)/2}
\]

for some \(A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}\) and \(a \in \mathbb{R}^N\).

From the Hardy–Littlewood–Sobolev inequality, the integral

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x, 0)|^q |v(z, 0)|^q}{|x - z|\mu} \, dx \, dz
\]

is well defined if \(|v(x, 0)|^q \in L^t(\mathbb{R}^N)\) for some \(t > 1\) satisfying \(2/t + \mu/N = 2\) and

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x, 0)|^q |v(z, 0)|^q}{|x - z|\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)} \leq C(N, \mu)^{(N-1)/(2N-\mu)} \left( \int_{\mathbb{R}^N} |v(x, 0)|^{tq} \, dx \right)^{2/(tq)}.
\]
By [16], we know that, for any $v \in H^1 (\mathbb{R}^{N+1})$

$$\left( \int_{\mathbb{R}^N} |v(x,0)|^s \, dx \right)^{1/s} \leq C \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \right)^{1/2},$$

where $2 \leq s \leq 2N/(N-1)$ and so $2 \leq tq \leq 2N/(N-1)$, that is

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-1}.$$And so $(2N-\mu)/N$ will be called the lower critical exponent and $2^*_\mu = (2N-\mu) / (N-1)$ the upper critical exponent.

For $v \in E_0$, its extension by zero in $\mathbb{R}^N \setminus \Omega$ can be approximated by functions compactly supported in $\mathbb{R}^{N+1}$. Thus the Sobolev trace inequality (1.13) and the Hölder inequality lead to the imbedding lemma [31].

**Lemma 1.2.** Let $1 \leq s < 2^*_\mu = 2N/(N-1)$ for $N \geq 3$. Then $\text{tr}_\Omega (E_0)$ is compactly embedded in $L^s(\Omega)$.

For the eigenvalues and corresponding eigenfunctions of operator $\sqrt{-\Delta + m^2}$, a similar proof can be found in [31]. For the reader’s convenience, we give the details here. Thus we have

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy : v \in E_0, \int_{\Omega} v(x,0)^2 \, dx = 1 \right\}.$$The main results of this paper are the following theorems.

**Theorem 1.3.** Suppose that assumptions $(V_1)$ and $(V_2)$ hold, $0 < \mu < N$, $N \geq 4$. Then, for any $\lambda \in (0, \lambda_1)$ there exists $\beta_\lambda > 0$ such that, for each $\beta \geq \beta_\lambda$ equation (1.4) has at least one ground state solution $u$, where $\lambda_1$ is the first eigenvalue of $\sqrt{-\Delta + m^2}$ on $\Omega$ with boundary condition $u = 0$. Furthermore, for any sequences $\beta_n \to \infty$, then every sequence of solutions $\{u_n\}$ of (1.4) satisfying

$$I_{\beta,\lambda}(u_n) \to c_{\beta,\lambda} \left[ \frac{N+1-\mu}{4N-2\mu} S_C (2N-\mu)/(N-1) \right] \text{ as } n \to \infty$$

(where $S_C$ will be defined as (2.4)), converges to a solution of

$$\begin{cases}
\sqrt{-\Delta + m^2} u = \left( \int_{\Omega} \frac{|u(z)|^{2^*}}{|x-z|^\mu} \, dz \right) |u|^{2^* - 2} u + \lambda u \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}$$

where $\Omega$ is defined as in $(V_1)$.

As we all know, the local nonlinear Schrödinger equation with deepening potential well has also been widely investigated. Consider

$$-\Delta u + (\beta V(x) - \lambda) u = |u|^p u \quad \text{in } \mathbb{R}^N,$$

where the potential $V(x)$ satisfies $(V_1)$ and $(V_2)$, $\lambda, \beta \in \mathbb{R}^+$, $N \geq 2$. Bartsch, Pankov and Wang [7] studied the subcritical case and proved the existence
of a least energy solution of (1.15) for large $\beta$. They also showed that the sequence of least energy solutions converges strongly to a least energy solution for a problem in bounded domain. The critical case was considered in [15], there Clapp and Ding proved the existence and multiplicity of positive solutions which localize near the potential well for $\lambda$ small and $\beta$ large. We would also like to mention the related nonlocal problem in [34] Shen, Gao and Yang obtained the existence of solutions of the critical Choquard type equation with potential well. It is quite natural to ask how the appearance of the potential well will affect the existence of solutions of the critical Hartree equation (1.4) as the parameter $\beta$ goes to infinity.

Involving the problem set on bounded domain, we have the following results.

**Theorem 1.4.** Let $\lambda < 0$, $N \geq 3$, $0 < \mu < N$ and $2_\mu^* = (2N - \mu)/(N - 1)$. Assume $\Omega$ is star-shaped with respect to a point in $\mathbb{R}^N$, then solution of (1.7) is trivial.

**Theorem 1.5.** Let $N \geq 3$, $0 < \mu < N$ and $2_\mu^* = (2N - \mu)/(N - 1)$. Assume $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ and $\lambda_1$ is the first eigenvalue of $\sqrt{-\Delta + m^2}$. Then, for every $\lambda \in (0, \lambda_1)$, the problem (1.7) has a nontrivial solution.

**Theorem 1.6.** Let $\{m_n\}$ be a positive sequence with $m_n \to 0$ as $n \to \infty$ and $0 < \lambda < \sqrt{\lambda'_1}$ (where $\lambda'_1$ is the first eigenvalue of $-\Delta$ on $\Omega$ with Dirichlet boundary condition). Then, the solution sequence $\{u_m\}$ of (1.7) has a subsequence converging to a solution $\bar{u}$ of the following equation

$$
\sqrt{-\Delta} u = \left( \int_{\Omega} \frac{|u(z)|^{2_\mu^*}}{|x - z|^\mu} \, dz \right) |u|^{2_\mu^* - 2} - \mu + \lambda u.
$$

An analogue problem to (1.7) for the Laplacian operator has been investigated by Gao and Yang [24]. Tan [35] discussed the the square root of the Laplacian, when $\lambda < 0$ he proved that there is no positive solution and $\lambda \in (0, \sqrt{\lambda'_1})$ there exists at least one positive solution. Recently, many people studied the Brézis–Nirenberg type results for the equations with fractional Laplacian, for details and recent works we refer to [32], [35] and the references therein for a recent progress.

We need to point out that all the papers we mentioned above were about the nonlinear Hartree equation with superlinear subcritical nonlinearities. As far as we know there seems no result for the nonlinear Hartree equation with upper critical exponent with respect to the Hardy–Littlewood–Sobolev inequality.

An outline of the paper is as follows: In Sections 2 and 3, we use two approaches to prove that the problem (1.7) has a nontrivial solution and also give the proof of Theorem 1.6. In Section 4, we prove regularity of solutions for (1.7). In Section 5, we prove a Pohozaev identity for (1.7) and use it to prove the nonexistence of solutions. In Section 6, we give the proof of Theorem 1.3.
2. A Brézis–Nirenberg type result

First we are going to study the existence of solutions for problem (1.7). In fact, we can establish a Brézis–Nirenberg type result in Theorem 1.5 for fractional Laplacian case. Equivalently, we consider the following problem:

\[
\begin{align*}
-\Delta v + m^2 v &= 0 \quad \text{in } \mathbb{R}^{N+1}, \\
v &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega, \\
\frac{\partial v}{\partial \nu} &= \left( \int_{\Omega} \frac{|v(z,0)|^{2^*_0}}{|x-z|^\mu} \, dz \right) |v|^{2^*_0-2} v + \lambda v \quad \text{on } \Omega.
\end{align*}
\]

In order to prove Theorem 1.5, we need to estimate the functional:

\[
Q_\lambda(v) = \frac{\int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy - \lambda \int_{\Omega} v(x,0)^2 \, dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^{2^*_0} |v(z,0)|^{2^*_0}}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)}}.
\]

Denote by $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)$ the closure of the set of smooth functions compactly supported in $\mathbb{R}^{N+1}_+$ with respect to the norm

\[
\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)} = \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy \right)^{1/2},
\]

we recall the well known Sobolev trace inequality. For all $v \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)$, we have

\[
\left( \int_{\mathbb{R}^N} |v(x,0)|^{2N/(N-1)} \, dx \right)^{(N-1)/(2N)} \leq C \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy \right)^{1/2}. \tag{2.2}
\]

From the Hardy–Littlewood–Sobolev inequality, we know

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_0} |v(z,0)|^{2^*_0}}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)} \leq C(N,\mu)^{(N-1)/(2N-\mu)} \left( \int_{\mathbb{R}^N} |v(x,0)|^{2^*_0} \, dx \right)^{2^{*_0}},
\]

where $C(N,\mu)$ is defined as in the Proposition 1.1 and $2^*_0 = 2N/(N-1)$. Consequently,

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_0} |v(z,0)|^{2^*_0}}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)} \leq C \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy. \tag{2.3}
\]
Thus, we can define the best constant $S_C$ by

$$\tag{2.4} S_C := \inf_{v \in D^{1,2}(\mathbb{R}^{N+1}_+ \setminus \{0\})} \left( \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx \, dy}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^2 |v(z,0)|^2 \, dx \, dz}{|x-z|^\mu} \right)^{(N-1)/(2N-\mu)}} \right).$$

We recall

$$S_0 = \inf_{v \in D^{1,2}(\mathbb{R}^{N+1}_+ \setminus \{0\})} \left( \int_{\mathbb{R}^N} |\nabla v|^2 dx \, dy \left( \int_{\mathbb{R}^N} |v(x,0)|^{2^*-2} \, dx \right)^{2/2^*} \right)^{1/2^*}.$$

By [19] we know that $S_0$ is achieved by the extremal functions

$$U_\varepsilon(x, y) = \frac{\varepsilon^{(N-1)/2}}{|(x, y + \varepsilon)|^{N-1}},$$

where $\varepsilon > 0$ is arbitrary.

**Lemma 2.1.** The constant $S_C$ defined in (2.4) is achieved by

$$\tag{2.5} U_\varepsilon(x, y) = \frac{\varepsilon^{(N-1)/2}}{|(x, y + \varepsilon)|^{N-1}},$$

where $\varepsilon > 0$ is arbitrary. What’s more,

$$S_C = \frac{C(N, \mu)^{(N-1)/(2N-\mu)}}{S_0}.$$  

**Proof.** On one hand, by the Hardy–Littlewood–Sobolev inequality, we can see

$$S_C \geq \frac{1}{C(N, \mu)^{(N-1)/(2N-\mu)}} \inf_{v \in D^{1,2}(\mathbb{R}^{N+1}_+ \setminus \{0\})} \left( \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx \, dy}{\left( \int_{\mathbb{R}^N} |v(x,0)|^{2^*-2} \, dx \right)^{2/2^*}} \right)^{1/2^*} \int_{\mathbb{R}^{N+1}} |\nabla U_\varepsilon(x, y)|^2 dx \, dy$$

$$= \frac{1}{C(N, \mu)^{(N-1)/(2N-\mu)}} \left( \int_{\mathbb{R}^N} |U_\varepsilon(x,0)|^{2^*} \, dx \right)^{2/2^*} \int_{\mathbb{R}^{N+1}} |\nabla U_\varepsilon(x, y)|^2 dx \, dy$$

$$= \frac{1}{C(N, \mu)^{(N-1)/(2N-\mu)}} \left( \int_{\mathbb{R}^N} |U_\varepsilon(x,0)|^{2^*} \, dx \right)^{2/2^*} S_0 = \frac{C(N, \mu)^{(N-1)/(2N-\mu)}}{S_0}.$$
So, by the definition of $S_C$, we have $S_C = S_0/C(N, \mu)^{(N-1)/(2N-\mu)}$ and $S_C$ is achieved at $U_\varepsilon(x,y)$. \(\square\)

We can obtain the following important estimate.

**Proposition 2.2.** Let $\lambda \in (0, \lambda_1)$ and denote by $S_C$ the best constant for the Sobolev trace inequality defined by (2.4). Then we have $S_\lambda = S_\lambda(\Omega) := \inf\{Q_\lambda(v)|v \in E_0\} < S_C$.

**Proof.** By Lemma 2.1, it is known that $S_C$ is achieved by the extremal functions $U_\varepsilon$. We see that

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x,0)|^{2\mu} |U_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx \, dz
$$

$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)^{(2N-\mu)/2}} \frac{|x-z|^\mu}{(|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} dx \, dz
$$

$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon^{2N-\mu}}{(|x|^2 + 1)^{(2N-\mu)/2}} \frac{|x-z|^\mu}{(\frac{|z|^2}{\varepsilon} + 1)^{(2N-\mu)/2}} dx \, dz
$$

$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{(|x|^2 + 1)^{(2N-\mu)/2}} \frac{|x-z|^\mu}{(\frac{|z|^2}{\varepsilon} + 1)^{(2N-\mu)/2}} dx \, dz := K_1.
$$

Without loss of generality, we can denote that $B_0^+ = \{(x,y)|(x,y) < \delta \text{ and } y > 0\}$. Let $\psi \in C^\infty(\overline{\mathbb{R}_+^{N+1}})$, $0 \leq \psi \leq 1$ and for small fixed $\delta$,

$$
\psi(x,y) = \begin{cases} 
1 & \text{if } (x,y) \in B_0^+/2, \\
0 & \text{if } (x,y) \notin B_0^+/4,
\end{cases}
$$

$$
0 \leq \psi(x,y) \leq 1 \quad \text{for all } (x,y) \in \overline{\mathbb{R}_+^{N+1}},
$$

$$
|D\psi(x,y)| \leq C = \text{const.} \quad \text{for all } (x,y) \in \overline{\mathbb{R}_+^{N+1}}.
$$

We take $\delta$ small enough so that $\overline{B_0^+} \subset \mathbb{R}_+^{N+1} \cup \Omega$. Thus the function $\psi U_\varepsilon \in E_0$ and we will use it as test function $v$ in the expression for $Q_\lambda$. We have

$$
(2.6) \int_{\Omega} \int_{\Omega} \frac{|U_\varepsilon(x,0)|^{2\mu} |\psi U_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx \, dz
$$

$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^{2\mu} |\psi(z,0)|^{2\mu}}{(|x|^2 + \varepsilon^2)^{(2N-\mu)/2}} \frac{\varepsilon^{2N-\mu}}{(|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} dx \, dz
$$

$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^{2\mu} - 1) (|\psi(z,0)|^{2\mu} - 1) \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)^{(2N-\mu)/2} |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} dx \, dz
$$

$$
+ 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^{2\mu} \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)^{(2N-\mu)/2} |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} dx \, dz - K_1
$$

$$
= A + 2B - K_1,
$$
where

\[ A = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^{2^*_\mu} - 1)(|\psi(z,0)|^{2^*_\mu} - 1) \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \]

and

\[ B = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^{2^*_\mu} \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz. \]

By a direct computation, we know

\[ (2.7) \quad A = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^{2^*_\mu} - 1)(|\psi(z,0)|^{2^*_\mu} - 1) \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \\
= O(\varepsilon^{2N-\mu}) \times \int_{R^N/B^+} \int_{R^N/B^+} \frac{1}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \\
= O(\varepsilon^{2N-\mu}) \times \int_{R^N/B^+} \int_{R^N/B^+} \frac{1}{|x-z|^\mu |z|^{2N-\mu}} \, dx \, dz = O(\varepsilon^{2N-\mu}) \]

and

\[ (2.8) \quad B = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^{2^*_\mu} \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \\
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^{2^*_\mu} - 1) \varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \\
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon^{2N-\mu}}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz \\
= -O(\varepsilon^{2N-\mu}) \times \int_{\mathbb{R}^N/B^+} \int_{\mathbb{R}^N/B^+} \frac{1}{(|x|^2 + \varepsilon^2)(2N-\mu)/2 |x-z|^\mu (|z|^2 + \varepsilon^2)^{(2N-\mu)/2}} \, dx \, dz + K_1 \\
= -O(\varepsilon^{2N-\mu}) \left( \int_{\mathbb{R}^N/B^+} \frac{1}{(|x|^2 + \varepsilon^2)^{(2N-\mu)/(2N)}} dx \right)^{(2N-\mu)/(2N)} \\
\times \left( \int_{\mathbb{R}^N} \frac{1}{(|z|^2 + \varepsilon^2)^{(2N-\mu)/(2N)}} \, dz \right)^{(2N-\mu)/(2N)} + K_1 \\
= -O(\varepsilon^{(2N-\mu)/2}) \left( \int_{\mathbb{R}^N/B^+} \frac{1}{|x|^{2N}} \, dx \right)^{(2N-\mu)/(2N)} \\
\times \left( \int_0^\infty \frac{r^{N-1}}{(1 + r^2)^N} \, dr \right)^{(2N-\mu)/(2N)} + K_1 = -O(\varepsilon^{(2N-\mu)/2}) + K_1. \]
It follows from (2.6) to (2.8) that

$$\int_{\Omega} \int_{\Omega} \frac{|\psi U_{\epsilon}(x,0)|^2 + |\psi U_{\epsilon}(z,0)|^2}{|x-z|^\mu} \, dx \, dz$$

$$= O(\varepsilon^{2N-\mu}) - O(\varepsilon^{(2N-\mu)/2}) + 2K_1 - K_1 = K_1 - O(\varepsilon^{(2N-\mu)/2}).$$

Hence we see

$$\left( \int_{\Omega} \int_{\Omega} \frac{|\psi U_{\epsilon}(x,0)|^2 + |\psi U_{\epsilon}(z,0)|^2}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)}$$

$$= K_1^{(N-1)/(2N-\mu)} - O(\varepsilon^{(N-1)/2}).$$

Let now

$$K_2 := \int_{\mathbb{R}^{N+1}_+} |\nabla U_{\epsilon}|^2 \, dx \, dy.$$

Since $U_{\epsilon}$ are minimizers for $S_\lambda$, we have that $S_\lambda = K_2 / K_1^{(N-1)/(2N-\mu)}$. We compute the following term, when $\varepsilon < \delta/2$,

$$\int_{\mathbb{R}^{N+1}_+} |\psi U_{\epsilon}|^2 \, dx \, dy = \int_{\mathbb{R}^{N+1}_+} \frac{\varepsilon^{N-1} \psi^2(x,y)}{|x|^2 + |y + \varepsilon|^2} \, dx \, dy$$

$$\geq \int_{|x|^2 + |y|^2 < e^2} \frac{\varepsilon^{N-1}}{|x|^2 + |y + \varepsilon|^2} \, dx \, dy$$

$$+ \int_{e^2 < |x|^2 + |y|^2 < \delta^2/4} \frac{\varepsilon^{N-1}}{|x|^2 + |y + \varepsilon|^2} \, dx \, dy$$

$$\geq C \int_{|x|^2 + |y|^2 < e^2} \frac{1}{\varepsilon^{N-1}} \, dx \, dy + \int_{e^2 < |x|^2 + |y|^2 < \delta^2/4} \frac{\varepsilon^{N-1}}{2(|x|^2 + |y|^2)} \, dx \, dy$$

$$= C\varepsilon^2 + O(\varepsilon^{N-1}).$$

By a direct calculation

$$\int_{\mathbb{R}^{N+1}_+} |\nabla (\psi U_{\epsilon})|^2 \, dx \, dy = \int_{\mathbb{R}^{N+1}_+} |\nabla U_{\epsilon}|^2 \, dx \, dy + O(\varepsilon^{N-1}) = K_2 + O(\varepsilon^{N-1})$$

and

$$\int_{\Omega} (\psi U_{\epsilon})(x,0)^2 \, dx = C\varepsilon + O(\varepsilon^{N-1}).$$

By using the facts above, we can compute that

$$Q_{\lambda}(\psi U_{\epsilon}) = \int_{\mathbb{R}^{N+1}_+} \left( |\nabla (\psi U_{\epsilon})|^2 + m^2 (\psi U_{\epsilon})^2 \right) \, dx \, dy - \lambda \int_{\Omega} (\psi U_{\epsilon})(x,0)^2 \, dx$$

$$\left( \int_{\Omega} \int \frac{(|\psi U_{\epsilon})(x,0)|^2 + (\psi U_{\epsilon})(z,0)|^2}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)}.$$
Lemma 2.3 (see Lemma 2.3 in [24]). If we take $\varepsilon > 0$ small enough.

Therefore

$$Q_\lambda(v) = S_C + Cm^2 K_1^{(1-N)/(2N-\mu)} \varepsilon^2 - \lambda C K_1^{(1-N)/(2N-\mu)} \varepsilon$$

if we take $\varepsilon > 0$ small enough.

Next we will prove that $\inf\{Q_\lambda(v) | v \in E_0\}$ is achieved.

**Lemma 2.3** (see Lemma 2.3 in [24]). Assume $N \geq 3$ and $0 < \mu < N$. Let

$$\| \cdot \|_{NL^2^\mu} := \left( \int_{\Omega} \int_{\Omega} \frac{|\cdot|^2}{|x-z|^\mu} dx dz \right)^{1/(2^\mu)}$$

and

$$X_{NL^2^\mu} := \{ v : \mathbb{R}^{N+1} \rightarrow \mathbb{R}; \| v \|_{NL^2^\mu} < +\infty \}.$$  

Then $\| \cdot \|_{NL^2^\mu}$ is a norm in $X_{NL^2^\mu}$. Moreover, under the norm $\| \cdot \|_{NL^2^\mu}$, $X_{NL^2^\mu}$ is a Banach space.

**Proof of Theorem 1.5.** For $0 < \lambda < \lambda_1$, let $\{v_n\} \subset E_0$ be a minimizing sequence for $S_\lambda$ satisfying $\|v_n\|_{NL^2^\mu(\Omega)} = 1$. Since the $\|v_n\|_{NL^2^\mu(\Omega)}$ is bounded, the minimizing property leads to $\|v_n\|$ is bounded. Replacing $v_n$, by $\|v_n\|$, we may assume $v_n \geq 0$. Then, by Lemma 1.2, we extract a subsequence, still denoted by $\{v_n\}$, such that, as $n \rightarrow \infty$,

$$v_n \rightharpoonup v$$  weakly in $E_0$,

$$v_n(\cdot,0) \rightarrow v(\cdot,0)$$  strongly in $L^s(\Omega)$, $2 \leq s < 2^*$,

$$v_n(x,0) \rightarrow v(x,0)$$  a.e. in $\Omega$.

Using the Brézis–Lieb Lemma and weak convergence, we have

$$\int_{\mathbb{R}^{N+1}} |\nabla v_n|^2 dx dy = \int_{\mathbb{R}^{N+1}} |\nabla v_n|^2 dx dy - \int_{\mathbb{R}^{N+1}} |\nabla v|^2 dx dy + o(1).$$

On the other hand, from Lemma 2.2 in [24], it follows that

$$\|v_n(\cdot,0) - v(\cdot,0)\|_{NL^2^\mu(\Omega)}^2 = \|v_n(\cdot,0)\|_{NL^2^\mu(\Omega)}^2 - \|v(\cdot,0)\|_{NL^2^\mu(\Omega)}^2.$$
Therefore, we see

\[ Q_\lambda(v_{\varepsilon_n}) = \int_{\mathbb{R}^N_{+}^{1}} (|\nabla v_{\varepsilon_n}|^2 + m^2 v_{\varepsilon_n}^2) \, dx \, dy - \lambda \int_{\Omega} v(x, 0)^2 \, dx + o(1) \]

\[ \geq \int_{\mathbb{R}^N_{+}^{1}} |\nabla (v_{\varepsilon_n} - v)|^2 \, dx \, dy + \int_{\mathbb{R}^N_{+}^{1}} |\nabla v|^2 \, dx \, dy \]

\[ + m^2 \int_{\mathbb{R}^N_{+}^{1}} v^2 \, dx \, dy - \lambda \int_{\Omega} v(x, 0)^2 \, dx + o(1) \]

\[ \geq S_C \|v_{\varepsilon_n} - v\|^2_{\mathcal{NL}^2_2(\Omega)} + S_\lambda \|v\|^2_{\mathcal{NL}^2_2(\Omega)} + o(1) \]

\[ \geq S_C \|v_{\varepsilon_n} - v\|^2_{\mathcal{NL}^2_2(\Omega)} + S_\lambda \|v_{\varepsilon_n}\|^2_{\mathcal{NL}^2_2(\Omega)} + o(1) \]

\[ = (S_C - S_\lambda) \|v_{\varepsilon_n} - v\|^2_{\mathcal{NL}^2_2(\Omega)} + S_\lambda \|v_{\varepsilon_n}\|^2_{\mathcal{NL}^2_2(\Omega)} + o(1) \]

Hence we have

\[ S_\lambda \geq (S_C - S_\lambda) \|v_{\varepsilon_n} - v\|^2_{\mathcal{NL}^2_2(\Omega)} + S_\lambda + o(1) \]

This implies, by Proposition 2.2, we have \( S_C - S_\lambda > 0 \), that

\[ v_{\varepsilon_n}(\cdot, 0) \to v(\cdot, 0) \text{ in } \mathcal{NL}^2_2(\Omega) \]

Hence, by lower semi-continuity, we see that \( v \geq 0 \) is a minimizer for \( Q_\lambda \). Now compute the first variation of \( Q_\lambda \), we see that a positive multiple of \( v \) is a solution of (2.1). The regularity of the solution follows from Proposition 4.3. \( \square \)

3. The mountain-pass arguments

The results in Theorem 1.5 can also be obtained by Mountain Pass arguments, we introduce the energy functional associated to equation (1.11) by

\[ I_{m,\lambda}(v) = \frac{1}{2} \left( \int_{\mathbb{R}^N_{+}^{1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \right) - \frac{\lambda}{2} \int_{\Omega} v(x, 0)^2 \, dx \]

\[ - \frac{1}{2 \cdot 2^*_{\mu}} \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^{2^*_{\mu}} |v(z, 0)|^{2^*_{\mu}}}{|x - z|^{\mu}} \, dx \, dz. \]

Let us recall the Mountain Pass Theorem [5].

LEMMMA 3.1. Let \( M \) be a real Banach space with its dual space \( M^* \) and suppose that \( I_{m,\lambda} \in C^1(M, \mathbb{R}) \) satisfies the condition

\[ \max\{I_{m,\lambda}(0), I_{m,\lambda}(v_1)\} \leq \alpha < \kappa \leq \inf_{\|v_1\| = \rho} I_{m,\lambda}(v) \]

for some \( \kappa > \alpha, \rho > 0 \) and \( v_1 \in M \) with \( \|v_1\| > \rho \). Let \( c_{m,\lambda} \geq \kappa \) be characterized by

\[ c_{m,\lambda} = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I_{m,\lambda}(\gamma(\tau)), \]
where $\Gamma = \{ \gamma \in C([0,1], M) \mid \gamma(0) = 0, \, \gamma(1) = v_1 \}$ is the set of continuous paths joining 0 and $v_1$. Then, there exists a sequence $\{v_{\epsilon_n}\} \subset M$ such that, as $n \to \infty$,

$$I_{m,\lambda}(v_{\epsilon_n}) \to c_{m,\lambda} \geq \kappa \quad \text{and} \quad I'_{m,\lambda}(v_{\epsilon_n})|_{M^*} \to 0.$$  

Let

(3.2) $\Sigma = \{ v \in E_0 \setminus \{0\} \mid \langle I'_{m,\lambda} v, v \rangle = 0 \}$.

By defining the critical values for the functional as follows:

$$c_{m,\lambda}^* = \inf_{v \in \Sigma} I_{m,\lambda}(v), \quad c_{m,\lambda} = \inf_{v \in \Gamma} \sup_{t \in [0,1]} I_{m,\lambda}(v(t)), \quad c_{m,\lambda}^{**} = \inf_{v \in E_0 \setminus \{0\}} \sup_{t \geq 0} I_{m,\lambda}(tv),$$

where $\Gamma := \{ \gamma \in C([0,1], E_0) \mid \gamma(0) = 0, \, I_{m,\lambda}(\gamma(1)) < 0 \}$. We have the following relations, whose proofs are standard.

**Lemma 3.2.** $c_{m,\lambda} = c_{m,\lambda}^* = c_{m,\lambda}^{**}$.

Then, the following lemma indicates that the compactness was recovered below some level.

**Lemma 3.3.** Assume that the energy functional $I_{m,\lambda}(v)$ is defined as in (3.1) and $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$. Then, for every $\lambda \in (0, \lambda_1)$ and $\mu \in (0, N)$, every sequence $v_{\epsilon_n}$ in $E_0$ such that, as $n \to \infty$,

(3.3) $I_{m,\lambda}(v_{\epsilon_n}) \to c_{m,\lambda} < \frac{N - \mu + 1}{4N - 2\mu} \frac{S^{(2N-\mu)/(N-\mu+1))}}{C} \cdot o(1)(1 + \|v_{\epsilon_n}\|) + \frac{1}{2\mu} \langle I'_{m,\lambda}(v_{\epsilon_n}), v_{\epsilon_n} \rangle \geq \frac{1}{2\mu} \langle I'_{m,\lambda}(v_{\epsilon_n}), v_{\epsilon_n} \rangle \geq C\|v_{\epsilon_n}\|^2.$

Thus it follows that $\{v_{\epsilon_n}\}$ is bounded in $E_0$. By Lemma 1.2, we extract a subsequence, still denoted by $\{v_{\epsilon_n}\}$, as $n \to \infty$,

$$v_{\epsilon_n} \rightharpoonup v \quad \text{weakly in } E_0,$$

$$v_{\epsilon_n}(\cdot, 0) \to v(\cdot, 0) \quad \text{strongly in } L^s(\Omega), \quad 2 \leq s < 2^*,$$

$$v_{\epsilon_n}(x, 0) \to v(x, 0) \quad \text{in } L^{2s}(\Omega),$$

$$v_{\epsilon_n}(x, 0) \to v(x, 0) \quad \text{a.e. in } \Omega.$$

On the other hand,

$$|v_{\epsilon_n}|_{2s}^2 \to |v|_{2s}^2 \quad \text{in } L^{2N/(2N-\mu)}(\Omega) \quad \text{as } n \to +\infty.$$
Moreover, the Brézis–Lieb Lemma and Lemma 2.2 in [24] leads to
\[ |x|^{-\mu} * |v_n(z,0)|^{2^{*}} \to |x|^{-\mu} * |v(z,0)|^{2^{*}} \quad \text{in} \ L^{2^{N}/\mu}(\Omega) \]
as \( n \to +\infty \). Combining with the fact that
\[ |v_n|^{2^{*} - 2}v_n \to |v|^{2^{*} - 2}v \quad \text{in} \ L^{2^{N}/(N-\mu+1)}(\Omega) \]
as \( n \to +\infty \), we have
\[ (|x|^{-\mu} * |v_n(z,0)|^{2^{*}})|v_n|^{2^{*} - 2}v_n \to (|x|^{-\mu} * |v(z,0)|^{2^{*}})|v|^{2^{*} - 2}v \]
in \( L^{2^{N}/(N+1)}(\Omega) \) as \( n \to +\infty \). Then, for every \( \varphi \in E_0 \), as \( n \to \infty \), we have,
\[
\lim_{n \to \infty} \langle I'_{m,\lambda}(v_n), \varphi \rangle = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^{N+1}_+} (\nabla v_n \cdot \nabla \varphi + m^2 v_n \varphi) \, dx \, dy - \lambda \int_{\Omega} v_n(x,0)\varphi(x,0) \, dx \right\} = \int_{\mathbb{R}^{N+1}_+} (\nabla v \cdot \nabla \varphi + m^2 v \varphi) \, dx \, dy - \lambda \int_{\Omega} v(x,0)\varphi(x,0) \, dx = (I'_{m,\lambda}(v), \varphi).
\]
By hypothesis \( I'_{m,\lambda}(v_n) \to 0 \), we deduce \( \langle I'_{m,\lambda}(v), \varphi \rangle = 0 \), by choosing \( \varphi = v \),
\[
0 = (I'_{m,\lambda}(v), v) = \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy - \lambda \int_{\Omega} v(x,0)^2 \, dx \right\} = \int_{\mathbb{R}^{N+1}_+} \frac{|v(x,0)|^{2^{*}} |v(z,0)|^{2^{*}}}{|x-z|^{\mu}} \, dx \, dz.
\]
Hence, we see
\[
I_{m,\lambda}(v) = \frac{1}{2} \left( \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \right) - \frac{\lambda}{2} \int_{\Omega} v(x,0)^2 \, dx - \frac{1}{2} \frac{\mu}{2^{*}} \int_{\Omega} \frac{|v(x,0)|^{2^{*}} |v(z,0)|^{2^{*}}}{|x-z|^{\mu}} \, dx \, dz = \frac{N - \mu + 1}{4N - 2\mu} \int_{\Omega} \frac{|v(x,0)|^{2^{*}} |v(z,0)|^{2^{*}}}{|x-z|^{\mu}} \, dx \, dz \geq 0.
\]
Moreover, the Brézis–Lieb Lemma and Lemma 2.2 in [24] leads to
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (v_n - v)|^2 \, dx \, dy = \int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2 \, dx \, dy - \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy + o(1),
\]
Furthermore,

\[
\int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0)|^{2\nu} |v_{\epsilon_n}(z,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz = \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0) - v(x,0)|^{2\nu} |v_{\epsilon_n}(z,0) - v(x,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz \\
+ \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^{2\nu} |v(x,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz + o(1).
\]

Thus, we have

\[
(3.4) \quad I_{m,\lambda}(v_{\epsilon_n}) = I_{m,\lambda}(v) + I_{m,0}(v_{\epsilon_n} - v) + o(1).
\]

Furthermore,

\[
o(1) = (I'_{m,\lambda}(v_{\epsilon_n}), v_{\epsilon_n} - v) = (I'_{m,\lambda}(v_{\epsilon_n}) - I'_{m,\lambda}(v), v_{\epsilon_n} - v)
\]

\[
= \int_{\mathbb{R}^{N+1}} |\nabla(v_{\epsilon_n} - v)|^2 \, dx \\
- \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0)|^{2\nu} |v_{\epsilon_n}(z,0)|^{2\nu-2} v_{\epsilon_n}(z,0)(v_{\epsilon_n}(z,0) - v(z,0))}{|x-z|^\mu} \, dx \, dz \\
+ \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^{2\nu} |v(z,0)|^{2\nu-2} v(z,0)(v_{\epsilon_n}(z,0) - v(z,0))}{|x-z|^\mu} \, dx \, dz.
\]

Set

\[
D = \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0)|^{2\nu} |v_{\epsilon_n}(z,0)|^{2\nu-2} v_{\epsilon_n}(z,0)(v_{\epsilon_n}(z,0) - v(z,0))}{|x-z|^\mu} \, dx \, dz \\
- \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^{2\nu} |v(z,0)|^{2\nu-2} v(z,0)(v_{\epsilon_n}(z,0) - v(z,0))}{|x-z|^\mu} \, dx \, dz
\]

\[
= \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0) - v(x,0)|^{2\nu} |v_{\epsilon_n}(z,0) - v(z,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz.
\]

It gives

\[
o(1) = \int_{\mathbb{R}^{N+1}} |\nabla(v_{\epsilon_n} - v)|^2 \, dx \\
- \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0) - v(x,0)|^{2\nu} |v_{\epsilon_n}(z,0) - v(z,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz.
\]

Then we obtain

\[
I_{m,0}(v_{\epsilon_n} - v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} |\nabla(v_{\epsilon_n} - v)|^2 \, dx \\
- \frac{1}{2 \cdot 2^\nu} \int_{\Omega} \int_{\Omega} \frac{|v_{\epsilon_n}(x,0) - v(x,0)|^{2\nu} |v_{\epsilon_n}(z,0) - v(z,0)|^{2\nu}}{|x-z|^\mu} \, dx \, dz
\]

\[
= \left( \frac{1}{2} - \frac{1}{2 \cdot 2^\nu} \right) \int_{\mathbb{R}^{N+1}} |\nabla(v_{\epsilon_n} - v)|^2 \, dx \, dy + o(1).
\]
On the other hand, by (3.4) and since 
\[ I_{m, \lambda}(v) \geq 0, \]
we see that there is a large \( \varepsilon_{n_0} > 0 \) such that, for \( \varepsilon_n \geq \varepsilon_{n_0} \),
\[
I_{m, \lambda}(v_{\varepsilon_n}) - I_{m, \lambda}(v) + o(1) 
\leq I_{m, \lambda}(v_{\varepsilon_n}) + o(1) < \frac{N - \mu + 1}{4N - 2\mu} S_C^{(2N-\mu)/(N-\mu+1)}.
\]

Therefore, we have the following inequality
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy < S_C^{(2N-\mu)/(N-\mu+1)}.
\]

This implies
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy < S_C^{(2N-\mu)/(N-\mu+1)}
\]
\[
\leq \left( \int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy \right)^{(N-1)/(2N-\mu)(N-\mu+1)}
\]
\[
\left( \int_{\mathbb{R}^{N+1}_+} |v_{\varepsilon_n}(x,0) - v(x,0)|^{2\mu} |v_{\varepsilon_n}(z,0) - v(z,0)|^{2\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)(N-\mu+1)}.
\]

Thus
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy 
\leq C < 1,
\]
for all \( \varepsilon_n \geq \varepsilon_{n_0} \). Then we obtain that, as \( n \to \infty \),
\[
(1 - C) \int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy 
\leq \left( \int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy \right)^{(N-1)/(2N-\mu)(N-\mu+1)}
\]
\[
\times \left( 1 - \int_{\Omega} \int_{\Omega} |v_{\varepsilon_n}(x,0) - v(x,0)|^{2\mu} |v_{\varepsilon_n}(z,0) - v(z,0)|^{2\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)(N-\mu+1)}
\]
\[
\leq \int_{\mathbb{R}^{N+1}_+} |\nabla(v_{\varepsilon_n} - v)|^2 \, dx \, dy 
- \int_{\Omega} \int_{\Omega} |v_{\varepsilon_n}(x,0) - v(x,0)|^{2\mu} |v_{\varepsilon_n}(z,0) - v(z,0)|^{2\mu} \, dx \, dz = o(1),
\]
establishing that \( v_{\varepsilon_n} \to v \) strongly in \( E_0 \).
\[ \square \]
Proof of Theorem 1.5. It is sufficient to prove that $I_{m, \lambda}$ satisfies the Lemma 3.3. Considering the functional $I_{m, \lambda}$, for every $v \in E_0$ and $t \geq 0$, we have

$$I_{m, \lambda}(tv) = \frac{t^2}{2} \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \right) - \frac{\lambda t^2}{2} \int_{\Omega} v(x, 0)^2 \, dx$$

$$- \frac{t^2 2^*}{2 \cdot 2^* \mu} \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^{2^*} |v(z, 0)|^{2^*}}{|x - z|^{\mu}} \, dx \, dz = \frac{t^2}{2} G_1 - \frac{t^2 2^*}{2 \cdot 2^* \mu} G_2,$$

where

$$G_1 = \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy - \lambda \int_{\Omega} v(x, 0)^2 \, dx,$$

$$G_2 = \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^{2^*} |v(z, 0)|^{2^*}}{|x - z|^{\mu}} \, dx \, dz.$$

We see that $I_{m, \lambda}(tv)$ has its maximum at

$$t_0 = \left( \frac{G_1}{G_2} \right)^{1/(2^* - 2)} = \left( \frac{G_1}{G_2} \right)^{(N-1)/(2(N-\mu+1))}.$$

Hence, we obtain

$$\sup_{t \geq 0} I_{m, \lambda}(tv) = \max_{t \geq 0} I_{m, \lambda}(tv) = I_{m, \lambda}(tv) |_{t = t_0}$$

$$= \frac{N - \mu + 1}{4N - 2\mu} \left( \frac{G_1}{G_2} \right)^{(2N-\mu)/(N-\mu+1)}.$$

This implies, for $\lambda \in (0, \lambda_1)$, there exists $v$ such that $Q_\lambda(v) = S_\lambda < S_C$, we get

$$\inf_{0 \neq v \in E_0} \sup_{t \geq 0} I_{m, \lambda}(tv) \leq \frac{N - \mu + 1}{4N - 2\mu} \left( \inf_{0 \neq v \in E_0} Q_\lambda(v) \right)^{(2N-\mu)/(N-\mu+1)}$$

$$< \frac{N - \mu + 1}{4N - 2\mu} S_C^{(2N-\mu)/(N-\mu+1)}.$$

Then by using Lemmas 3.1–3.3, we obtain

$$c^{**}_{m, \lambda} = \inf_{0 \neq v \in E_0} \sup_{t \geq 0} I_{m, \lambda}(tv)$$

is a critical value of $I_{m, \lambda}$. Finally we complete the proof of regularity of the solution by Proposition 4.3. □

Proof of Theorem 1.6. Firstly we shall show that $c^{**}_{m, \lambda}$ is strictly increasing for $m > 0$. For any $0 < m_1 < m_2$, we observe the fact that, for any $v \in E_0$ with...
we have

\[ v \neq 0, \text{there exists a unique } t_{m_1, m_2} > 0 \text{ such that} \]

\[
\sup_{t \geq 0} I_{m, \lambda}(tv) = I_{m, \lambda}(t_{m_1, m_2}v)
\]

\[
= \frac{N - \mu + 1}{4N - 2\mu} t_{m_1, m_2}^2 \left( \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m_1^2 v^2) \, dx \, dy - \lambda \int_{\Omega} (v(0))^2 \, dx \right)
\]

\[
= \frac{N - \mu + 1}{4N - 2\mu} t_{m_1, m_2}^2 \int_{\Omega} \frac{|v(x, 0)|^2 |v(z, 0)|^2}{|x - z|^\mu} \, dx \, dz,
\]

we have

\[
t_{m_1, m_2}^{2\nu - 2} = \frac{\int_{\mathbb{R}^{N+1}_+} (|\nabla v_{m_2}|^2 + m_2^2 v_{m_2}^2) \, dx \, dy - \lambda \int_{\Omega} v_{m_2}(x, 0)^2 \, dx}{\int_{\Omega} \frac{|v_{m_2}(x, 0)|^2 |v_{m_2}(z, 0)|^2}{|x - z|^\mu} \, dx \, dz}
\]

\[
< \frac{\int_{\mathbb{R}^{N+1}_+} (|\nabla v_{m_2}|^2 + m_2^2 v_{m_2}^2) \, dx \, dy - \lambda \int_{\Omega} v_{m_2}(x, 0)^2 \, dx}{\int_{\Omega} \frac{|v_{m_2}(x, 0)|^2 |v_{m_2}(z, 0)|^2}{|x - z|^\mu} \, dx \, dz} = 1,
\]

and one has that

\[
c_{m_1, \lambda} = \max_{t \geq 0} I_{m_1, \lambda}(tv_{m_2})
\]

\[
= \frac{N - \mu + 1}{4N - 2\mu} t_{m_1, m_2}^2 \int_{\Omega} \frac{|v_{m_2}(x, 0)|^2 |v_{m_2}(z, 0)|^2}{|x - z|^\mu} \, dx \, dz
\]

\[
= \frac{2^{2\nu} - 2}{m_1, m_2} c_{m_2, \lambda} < \nu_{m_2, \lambda}.
\]

Hence, \( c_{m_1, \lambda} \) is strictly increasing for \( m > 0 \).

Now, let \( m_n \in (0, +\infty) \), be a sequence with \( m_n \to 0 \) as \( n \to \infty \), for \( 0 < \lambda < \sqrt{N} \), let \( \{ v_{m_n} \} \) be a sequence of the ground state solutions of (1.11) and

\[
I_{m_n, \lambda}(v_{m_n}) \to d < \frac{N - \mu + 1}{4N - 2\mu} S_{C}^{(2N-\mu)/(N-\mu+1)}.
\]

we have

\[
I_{m_n, \lambda}(v_{m_n}) = \frac{N - \mu + 1}{4N - 2\mu} \int_{\Omega} \frac{|v_{m_n}(x, 0)|^2 |v_{m_n}(z, 0)|^2}{|x - z|^\mu} \, dx \, dz,
\]

and so

\[
\lim_{n \to \infty} \int_{\Omega} \frac{|v_{m_n}(x, 0)|^2 |v_{m_n}(z, 0)|^2}{|x - z|^\mu} \, dx \, dz < S_{C}^{(2N-\mu)/(N-\mu+1)}.
\]
We can deduce that
\[
\frac{N - \mu + 1}{4N - 2\mu} C \left[ \frac{(2N - \mu)/(N - \mu + 1)}{\min(\mu, N - \mu + 1)} \right] \geq \frac{N - \mu + 1}{4N - 2\mu} C \left[ \min(\mu, N - \mu + 1) \right],
\]
we have \( v_m \) is bounded in \( E_0 \). Then, passing to a subsequence, we may assume \( v_m \rightharpoonup v \) in \( E_0 \), due to the similar as the proof of Lemma 3.3. Then \( v \) satisfies
\[
\begin{align*}
-\Delta v &= 0 & \text{in } \mathbb{R}^{N+1}, \\
v &= 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\
\frac{\partial v}{\partial \nu} &= \left( \int_{\Omega} \frac{|v(z, 0)|^{2^*_\mu}}{|x - z|^{\mu}} dz \right)^{\frac{1}{2^*_\mu} - 2} v + \lambda v & \text{on } \Omega.
\end{align*}
\]
Noting that \( I_{0,\lambda}'(\bar{v}) = 0 \) and \( \lim_{n \to \infty} I_{0,\lambda}(v_m) = \lim_{n \to \infty} I_{m,\lambda}(v_m)\), we see that \( I_{0,\lambda}'(\bar{v}) = 0 \) and \( \lim_{n \to \infty} I_{0,\lambda}(v_m) = \lim_{n \to \infty} c_{m,\lambda} > 0. \) By Lemma 3.1, we have
\[
0 < \lim_{n \to \infty} c_{m,\lambda} \leq \lim_{n \to \infty} I_{0,\lambda}(v_m)
\leq c_{0,\lambda} \leq c_{m,\lambda} < \frac{N - \mu + 1}{4N - 2\mu} C \left[ \frac{(2N - \mu)/(N - \mu + 1)}{\min(\mu, N - \mu + 1)} \right].
\]
Using the similar arguments as the proof of Lemma 3.3. One can easily get
\( v_m \to \bar{v} \) in \( E_0 \).
Combing this and (3.7), we can deduce that \( \bar{v} \) a solution of (1.16). \( \square \)

4. Regularity

The following lemma is taken from [29], we put it here for the convenience of the readers.

LEMMA 4.1 (see [29]). Let \( q, r, l, t \in [1, \infty) \) and \( \lambda \in [0, 2] \) such that
\[
1 + \frac{\mu}{N} - \frac{1}{l} = \frac{1}{t} = \frac{\lambda}{q} + \frac{2 - \lambda}{r}.
\]
If \( \theta \in (0, 2) \) satisfies
\[
\max(q, r) \left( \frac{\mu}{N} - \frac{1}{t} \right) < \theta < \max(q, r) \left( 1 - \frac{1}{l} \right),
\]
\[
\max(q, r) \left( \frac{\mu}{N} - \frac{1}{t} \right) < 2 - \theta < \max(q, r) \left( 1 - \frac{1}{l} \right),
\]
then, for every $H \in L^t(\mathbb{R}^N)$, $K \in L^t(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^*(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} (|x|^{-\mu} * (H|u|^\theta))K|u|^{2-\theta} \, dx \leq C \left( \int_{\mathbb{R}^N} |H|^t \, dx \right)^{1/t} \times \left( \int_{\mathbb{R}^N} |K|^\ell \, dx \right)^{1/\ell} \times \left( \int_{\mathbb{R}^N} |u|^q \, dx \right)^{\lambda/q} \times \left( \int_{\mathbb{R}^N} |v|^r \, dx \right)^{(2-\lambda)/r}.
$$

Next we will adapt the method of Brézis and Kato [9] as in [29] and obtain the regularity of the weak solutions by using the Morse iteration technique. One can find that the appearance of operator $\sqrt{-\Delta + m^2}$ makes the proof much more complicated. We will use the following estimate lemma for the convolution part.

**Lemma 4.2.** Let $N \geq 3$, $\mu \in (0, N)$ and $\theta \in (0, N)$. If $H, K \in L^{2N/(N-\mu)}(\Omega) + L^{2N/(N+1-\mu)}(\Omega)$, $(1-\mu/N) < \theta < (1+\mu/N)$, then for any $\varepsilon > 0$, there exists $C_{\varepsilon, \theta} \in \mathbb{R}$ such that, for every $v \in E_0$,

$$
\int_{\Omega} (|x|^{-\mu} * (H|v(x,0)|^{\theta}))K|v(x,0)|^{2-\theta} \, dx \\
\leq \varepsilon^2 \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + C_{\varepsilon, \theta} \int_{\Omega} |v(x,0)|^2 \, dx.
$$

**Proof.** Let $(1-\mu/N) < \theta < (1+\mu/N)$ and $v \in E_0$. Since $0 < \mu < N, m > 0$, we may assume that $H = H^* + H_*$ and $K = K^* + K_*$ with $H^*, K^* \in L^{2N/(N-\mu)}(\Omega)$ and $H_*, K_* \in L^{2N/(N+1-\mu)}(\Omega)$. Similar to [29], we can take

$$
\begin{align*}
q &= r = \frac{2N}{N-1}, & l &= t = \frac{2N}{N-\mu+1}, & \lambda &= 0, \\
q &= r = 2, & l &= t = \frac{2N}{N-\mu}, & \lambda &= 2, \\
q &= 2, \quad r = \frac{2N}{N-1}, & l &= t = \frac{2N}{N-\mu+1}, & \lambda &= 1, \\
q &= 2, \quad r = \frac{2N}{N-1}, & l &= t = \frac{2N}{N-\mu}, & \lambda &= 1.
\end{align*}
$$

We obtain

$$
(4.1) \quad \int_{\Omega} (|x|^{-\mu} * (H_*|v(x,0)|^{\theta}))K_*|v(x,0)|^{2-\theta} \, dx \\
\leq C \left( \int_{\Omega} |H_*|^{2N/(N-\mu+1)} \, dx \right)^{(N-\mu+1)/(2N)} \times \left( \int_{\Omega} |K_*|^{2N/(N-\mu+1)} \, dx \right)^{(N-\mu+1)/(2N)} \times \left( \int_{\Omega} |v(x,0)|^{2N/(N-1)} \, dx \right)^{(N-1)/N}.
$$
Then, applying (1.13) and the above inequalities (4.1)–(4.4), we have, for every $v \in E_0$,

$$\int_{\Omega} \left( |x|^{-\mu} \ast (H^{|v(x, 0)|}) |K^*|v(x, 0) \right)^2 dx \leq C \left( \int_{\Omega} |H^*|^{2N/(N-\mu)} dx \right)^{(N-\mu)/(2N)} \times \left( \int_{\Omega} |K^*|^{2N/(N-\mu)} dx \right)^{(N-\mu)/(2N)} \times \left( \int_{\Omega} v(x, 0)^2 dx \right)^{1/2} \left( \int_{\Omega} |v(x, 0)|^{2N/(N-1)} dx \right)^{(N-1)/(2N)},$$

for $\varepsilon > 0$, we choose $H^*$ and $K^*$ such that

$$C \left( \int_{\Omega} |H^*|^{2N/(N-\mu+1)} dx \int_{\Omega} |K^*|^{2N/(N-\mu+1)} dx \right)^{(N-\mu+1)/(2N)} \leq \varepsilon^2,$$
then there exists $C_{\epsilon, \theta} \in \mathbb{R}$ such that
\[
\int_{\Omega} (|x|^{-\mu} \ast (H|v(x, 0)|^\theta)) K|v(x, 0)|^{2-\theta} \, dx \\
\leq \epsilon^2 \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + C_{\epsilon, \theta} \int_{\Omega} v(x, 0)^2 \, dx. \quad \square
\]

The regularity property can be stated as:

**Proposition 4.3.** Let $u \in H^{1/2}(\Omega)$ be a solution to the problem

\[
\begin{cases}
\sqrt{-\Delta + m^2} u = g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

and assume that
\[
g(u) = \left( \int_{\Omega} \frac{|u(z)|^{2 \mu}}{|x-z|^{\mu}} \, dz \right) |u|^{2\mu-2} u + \lambda u, \quad \lambda > 0,
\]

then $u \in L^\infty(\Omega)$ and $u \in C^2(\Omega)$.

**Proof.** Notice that $v \in E_0$, $v(x, 0) = u(x)$, and $v$ is a weak solution of

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\
\frac{\partial v}{\partial \nu} = g(v(\cdot, 0)) & \text{on } \Omega.
\end{cases}
\]

So, in order to prove that $u \in L^\infty(\Omega) \cap C^2(\Omega)$, we only need to show that $v \in L^\infty(\Omega) \cap C^2(\mathbb{R}^{N+1}_+)$. Let us define the truncation $v_r: \mathbb{R}^{N+1}_+ \to \mathbb{R}$, for $r > 0$ large,

\[
v_r(x, y) = \begin{cases}
-\tau & \text{if } v \leq -\tau, \\
v(x, y) & \text{if } -\tau < v < \tau, \\
\tau & \text{if } v \geq \tau.
\end{cases}
\]

Since $|v_r|^{s-2} v_r \in E_0$ for $s \geq 2$, we take $|v_r|^{s-2} v_r$ as a test function in (4.6), we obtain
\[
\frac{4(s-1)}{s^2} \int_{\mathbb{R}^{N+1}_+} (|\nabla (v_r(x, y))|^{s/2})^2 + m^2 |v_r(x, y)|^{s/2}) \, dx \, dy \\
= \int_{\mathbb{R}^{N+1}_+} (\nabla (v_r(x, y)) \nabla (|v_r(x, y)|^{s-2} v_r(x, y)) + m^2 |v_r(x, y)|^s) \, dx \, dy \\
\leq \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^{2 \mu}}{|x-z|^{\mu}} \, dz \, |v_r(x, 0)|^{2\mu-1} |v_r(x, 0)|^{s-1} \, dx \\
+ \lambda \int_{\Omega} |v_r(x, 0)|^{s-2} v_r \, dx
\]
\[ \leq \int_{\Omega} \int_{\Omega} \frac{|v(z,0)|^{2s}}{|x-z|^\mu} \, dz \, |v_r(x,0)|^{2s-1} |v_r(x,0)|^{s-1} \, dx + C \int_{\Omega} (1 + v^2) |v_r(x,0)|^{s-2} \, dx. \]

If \( 2 \leq s < 2N/(N-\mu) \), using Lemma 4.2 with \( \theta = 2/s \), there exists \( C > 0 \) such that
\[
\int_{\Omega} \int_{\Omega} \frac{|v_r(z,0)|^{2s}}{|x-z|^\mu} \, dz \, |v_r(x,0)|^{2s-1} |v_r(x,0)|^{s-1} \, dx \\
\leq \frac{2(s-1)}{s^2} \int_{\mathbb{R}^{N+1}} (|\nabla (v_r(x,y))|^{s/2} + m^2 |v_r(x,y)|^{s/2}) \, dx \, dy \\
\leq C \int_{\Omega} |v(x,0)|^s \, dx + \int_{A_r} \int_{\Omega} \frac{|v(z,0)|^{2s-1} |v(z,0)|^{s-1}}{|x-z|^\mu} \, dz \, |v(x,0)|^{2s} \, dx \\
+ C \int_{\Omega} (1 + v^2) |v_r(x,0)|^{s-2} \, dx,
\]
where \( A_r = \{ x \in \Omega : |v| > \tau \} \). We have
\[
\frac{2(s-1)}{s^2} \int_{\mathbb{R}^{N+1}} (|\nabla (v_r(x,y))|^{s/2} + m^2 |v_r(x,y)|^{s/2}) \, dx \, dy \\
\leq C \int_{\Omega} |v(x,0)|^s \, dx + \int_{A_r} \int_{\Omega} \frac{|v(z,0)|^{2s-1} |v(z,0)|^{s-1}}{|x-z|^\mu} \, dz \, |v(x,0)|^{2s} \, dx.
\]
Since \( 2 \leq s < 2N/(N-\mu) \), applying the Hardy–Littlewood–Sobolev inequality again,
\[
\int_{A_r} \int_{\Omega} \frac{|v(z,0)|^{2s-1} |v(z,0)|^{s-1}}{|x-z|^\mu} \, dz \, |v(x,0)|^{2s} \, dx \\
\leq C \left( \int_{\Omega} |v(x,0)|^{2s-1} |v(x,0)|^{s-1} \, dx \right)^{1/r} \left( \int_{A_r} |v(x,0)|^{2s} \, dx \right)^{1/l},
\]
with
\[
\frac{1}{r} = 1 + \frac{N-\mu}{2N} - \frac{1}{s} \quad \text{and} \quad \frac{1}{l} = \frac{N-\mu}{2N} + \frac{1}{s}.
\]
By Hölder’s inequality, if \( v(x,0) \in L^s(\Omega) \), then
\[
|v(x,0)|^{2s} \in L^1(\Omega) \quad \text{and} \quad |v(x,0)|^{2s-1} |v(x,0)|^{s-1} \in L^1(\Omega),
\]
whence by Lebesgue’s dominated convergence theorem
\[
\lim_{\tau \to \infty} \int_A \int_\Omega \frac{|v(z,0)|^{2^*_s-1} |v(z,0)|^{s-1}}{|x-y|^\mu} dz |v(x,0)|^{2^*_s} dx = 0.
\]
On the other hand,
\[
\int_\Omega |v_\tau(x,0)|^{s-2} dx + \int_\Omega v^2 |v_\tau(x,0)|^{s-2} dx \leq C_{\tau_0},
\]
here \( C \) can be taken independent of \( \tau \). Hence, by Sobolev embedding theory, we obtain that there exists a constant \( C_{\tau_0} \), independent of \( \tau \), for which it holds
\[
\left( \int_\Omega |v_\tau(x,0)|^{sN/(N-1)} dx \right)^{(N-1)/N} \leq C \int_\Omega |v(x,0)|^s dx + C_{\tau_0}.
\]
Letting \( \tau \to \infty \) we conclude that \( v(x,0) \in L^{sN/(N-1)}(\Omega) \). By iterating over \( s \) a finite number of times we cover the range \( s \in [2, 2N/(N - \mu)] \). So we can get weak solution \( v(x,0) \in L^s(\Omega) \) of (1.11) for every \( s \in [2, 2N^2/((N - \mu)(N - 1))] \).

Thus,
\[
|v(x,0)|^{2^*_s} \in L^s(\Omega) \quad \text{for every } s \in \left[ \frac{2(N-1)}{2N-\mu}, \frac{2N^2}{(N-\mu)(2N-\mu)} \right).
\]
Since
\[
\frac{2(N-1)}{2N-\mu} < \frac{N}{N-\mu} < \frac{2N^2}{(N-\mu)(2N-\mu)},
\]
we have
\[
\int_\Omega |v(z,0)|^{2^*_s} dz \in L^\infty(\Omega)
\]
and so \( g(v(\cdot,0)) \leq C(1 + |v|^{2^*_s-1}) \). Similar to the proof of Theorem 1.16 of [4], we have the weak solution \( v \in L^\infty(\Omega) \) and \( v \in C^2(\mathbb{R}^N_+) \). \qed

5. Pohožaev formula and nonexistence of solutions

Next we prove a Pohožaev type formula for the problem
\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\
\frac{\partial v}{\partial \nu} = \left( \int_\Omega \frac{|v(z,0)|^{2^*_s}}{|x-z|^\mu} dz \right) |v|^{2^*_s-2} v + \lambda v & \text{on } \Omega.
\end{cases}
\] (5.1)

**Lemma 5.1.** If \( v \) is a weak solution of (5.1) in \( L^\infty(\Omega) \cap C^2(\overline{\mathbb{R}^{N+1}_+}) \), then \( v \) satisfies a Pohožaev type identity:
\[
\begin{align*}
\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2(x,\nu) d\sigma + \frac{N-1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 dx dy & \\
+ \frac{m^2(N+1)}{2} \int_{\mathbb{R}^{N+1}_+} v^2 dx dy
\end{align*}
\] (5.2)
\[
\frac{N-1}{2} \int_{\Omega \times \{y=0\}} \int_{\Omega \times \{y=0\}} \frac{|v(x,0)\nabla v(x,0)\nabla v(z,0)|^2}{|x-z|^\mu} \, dx \, dz \\
+ \frac{\lambda N}{2} \int_{\Omega \times \{y=0\}} v(x,0)^2 \, dx.
\]

**Proof.** Let \( v \) be a weak bounded solution of (5.1) and \( X = (x, y) \), then we know by Proposition 4.3 that \( v \in C^2(\mathbb{R}^{N+1}_+) \). The following identity is known:

\[
\text{div} \left\{ (X, \nabla v) \nabla v - X \frac{|
abla v|^2}{2} \right\} + \left( \frac{N+1}{2} - 1 \right) |
\nabla v|^2 = (X, \nabla v) \Delta v.
\]

Thus, by (5.1), we know that in \( \mathbb{R}^{N+1}_+ \),

\[
\text{div} \left\{ (X, \nabla v) \nabla v - X \frac{|
abla v|^2}{2} \right\} + \left( \frac{N+1}{2} - 1 \right) |
\nabla v|^2 - m^2 v(X, \nabla v) = 0.
\]

Integrating the above equation over \( \mathbb{R}^N \times (0, R) \), we see

\[
\frac{1}{2} \int_{\partial \mathbb{R}^{N+1}_+} |
\nabla v|^2 (x, \nu) \, d\sigma \\
+ \int_{\mathbb{R}^N \times \{y=R\}} \left\{ (x, \nabla_x v) + R \partial_y v - R \frac{|
abla v|^2}{2} \right\} \, dx \\
+ \int_{\mathbb{R}^N \times \{y=0\}} (x, \nabla_x v)(\nabla v, \nu) \, dx \\
+ \frac{N-1}{2} \int_{\mathbb{R}^N \times (0, R)} |\nabla v|^2 \, dx \, dy - m^2 \int_{\mathbb{R}^N \times (0, R)} v(X, \nabla v) \, dx \, dy = 0.
\]

Form Proposition 6.2 of [24], we get

\[
\int_{\Omega \times \{y=0\}} (x \cdot \nabla_x v(x,0)) \left( \int_{\Omega \times \{y=0\}} \frac{|v(z,0)|^{2^*}_\mu}{|x-z|^\mu} \, dz \right) |v(x,0)|^{2^*_\mu-1} \, dx \\
= - \int_{\Omega \times \{y=0\}} v(x,0) \nabla \left( x \int_{\Omega \times \{y=0\}} \frac{|v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dz \right) |v(x,0)|^{2^*_\mu-1} \, dx \\
= - \int_{\Omega \times \{y=0\}} v(x,0) \left( N \int_{\Omega \times \{y=0\}} \frac{|v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dz \right) |v(x,0)|^{2^*_\mu-1} \, dx \\
+ (2^*_\mu - 1) |v(x,0)|^{2^*_\mu-2} x \cdot \nabla_x v(x,0) \int_{\Omega \times \{y=0\}} \frac{|v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dz \\
+ |v(x,0)|^{2^*_\mu-1} \int_{\Omega \times \{y=0\}} (-\mu) x \cdot (x-z) \frac{|v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dz \, dx \\
= - N \int_{\Omega \times \{y=0\}} \int_{\Omega \times \{y=0\}} \frac{|v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz.
\]
\[- (2^\mu - 1) \int_{\Omega \times \{y = 0\}} x \cdot \nabla_x v(x, 0) \]
\[
\times \int_{\Omega \times \{y = 0\}} \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu} \, dz \, |v(x, 0)|^{2^\mu - 1} \, dx 
\]
\[
+ \mu \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} x \cdot (x - z) \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu + 2} v(x, 0)^{2^\mu} \, dz \, dx.
\]

This implies that
\[
2^\mu \int_{\Omega \times \{y = 0\}} (x \cdot \nabla_x v(x, 0)) \left( \int_{\Omega \times \{y = 0\}} \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu} \, dz \right) |v(x, 0)|^{2^\mu - 1} \, dx 
\]
\[
= - N \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} \frac{|v(x, 0)|^{2^\mu}}{|x - z|^\mu} v(z, 0)^{2^\mu} \, dx \, dz 
\]
\[
+ \mu \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} x \cdot (x - z) \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu + 2} v(x, 0)^{2^\mu} \, dz \, dx.
\]

Similarly,
\[
2^\mu \int_{\Omega \times \{y = 0\}} (z \cdot \nabla_z v(z, 0)) \left( \int_{\Omega \times \{y = 0\}} \frac{|v(x, 0)|^{2^\mu}}{|x - z|^\mu} \, dx \right) |v(z, 0)|^{2^\mu - 1} \, dz 
\]
\[
= - N \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu} v(x, 0)^{2^\mu} \, dz \, dx 
\]
\[
+ \mu \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} z \cdot (z - x) \frac{|v(x, 0)|^{2^\mu}}{|x - z|^\mu + 2} v(z, 0)^{2^\mu} \, dx \, dz.
\]

Consequently, we get
\[
\int_{\Omega \times \{y = 0\}} (x \cdot \nabla_x v(x, 0)) \left( \int_{\Omega \times \{y = 0\}} \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu} \, dz \right) |v(x, 0)|^{2^\mu - 1} \, dx 
\]
\[
= \frac{\mu - 2N}{2 \cdot 2^\mu} \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} \frac{|v(x, 0)|^{2^\mu}}{|x - z|^\mu} v(z, 0)^{2^\mu} \, dx \, dz.
\]

Then, using the third equation in (5.1), we know
\[
(5.4) \quad \int_{\Omega \times \{y = 0\}} (x, \nabla_x v)(\nabla v, v) \, dx 
\]
\[
= \int_{\Omega \times \{y = 0\}} (x, \nabla_x v) \left( \left( \int_{\Omega \times \{0\}} \frac{|v(z, 0)|^{2^\mu}}{|x - z|^\mu} \, dz \right) |v(x, 0)|^{2^\mu - 2} v(x, 0) \right. 
\]
\[
\left. + \lambda v(x, 0) \right) \, dx 
\]
\[
= \frac{N - 1}{2} \int_{\Omega \times \{y = 0\}} \int_{\Omega \times \{y = 0\}} \frac{|v(x, 0)|^{2^\mu}}{|x - z|^\mu} v(z, 0)^{2^\mu} \, dx \, dz 
\]
\[
- \frac{\lambda N}{2} \int_{\Omega \times \{y = 0\}} v(x, 0)^2 \, dx.
\]
Next, by Green’s formula we get

\[ (5.5) \quad \int_{\mathbb{R}^N \times (0, R)} v(X, \nabla v) \, dx \, dy = - \frac{N+1}{2} \int_{\mathbb{R}^N \times (0, R)} v^2 \, dx \, dy + \frac{R}{2} \int_{\mathbb{R}^N \times \{y=R\}} v^2 \, dx. \]

From Lemma 3.1 of [35], there exists a sequence \( R_m \to \infty \) such that

\[ (5.6) \quad \lim_{m \to \infty} \int_{\mathbb{R}^N \times \{y=R\}} (x, \nabla_x v) + R_m \partial_y v - \frac{R_m}{2} \frac{|\nabla v|^2}{2} \, dx = 0. \]

Thus, by taking \( R = R_m \to \infty \), we can get

\[ (5.7) \quad \int_{\mathbb{R}^N \times \{y=R\}} v^2 \, dx = 0. \]

By (5.3)–(5.7), we have

\[ \frac{1}{2} \int_{\partial \mathbb{R}^{N+1}_+} |\nabla v|^2(x, \nu) \, d\sigma + \frac{N-1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy \]
\[ + \frac{m^2 (N+1)}{2} \int_{\mathbb{R}^{N+1}_+} v^2 \, dx \, dy \]
\[ = \left( \frac{2N-\mu}{2 \cdot 2^*_\mu} \right) \int_{\Omega \times \{y=0\}} \int_{\Omega \times \{y=0\}} \frac{|v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \]
\[ + \frac{\lambda N}{2} \int_{\Omega \times \{y=0\}} v(x,0)^2 \, dx. \]

**Proof of Theorem 1.4.** We assume that \( v \) is a nontrivial solution of (5.1). By Green’s formula

\[ (5.8) \quad \int_{\mathbb{R}^N \times (0, R)} |\nabla v|^2 \, dx \, dy = - \int_{\mathbb{R}^N \times (0, R)} v \Delta v \, dx \, dy 
+ \int_{\mathbb{R}^N \times \{y=0\}} v \frac{\partial v}{\partial \nu} \, d\sigma + \int_{\mathbb{R}^N \times \{y=R\}} v \partial_y v \, dx 
= - \int_{\mathbb{R}^N \times (0, R)} v \Delta v \, dx \, dy 
+ \int_{\Omega \times \{y=0\}} \int_{\Omega \times \{y=0\}} \frac{|v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz 
+ \lambda \int_{\Omega \times \{y=0\}} v(x,0)^2 \, dx. \]

From Lemma 5.1, we can obtain

\[ \frac{1}{2} \int_{\partial \mathbb{R}^{N+1}_+} |\nabla v|^2(x, \nu) \, d\sigma + m^2 \int_{\mathbb{R}^{N+1}_+} v^2 \, dx \, dy = \frac{\lambda}{2} \int_{\Omega \times \{y=0\}} v(x,0)^2 \, dx. \]
Since $\lambda < 0$, we have the right hand side is non-positive, but the left hand side is positive if after a translation we take $\Omega$ star-shaped with respect to the origin. This gives a contradiction. □

6. The potential well case

In this section we denote by
$$F = \left\{ v \in H^1(\mathbb{R}^{N+1}_+): \int_{\mathbb{R}^N} V(x)v(x,0)^2\,dx < +\infty \right\},$$
the Hilbert space equipped with norm
$$\|v\|_1 = \left( \|v\|_2^2 + \int_{\mathbb{R}^N} V(x)v(x,0)^2\,dx \right)^{1/2}.$$If $\beta > 0$, then it is equivalent to the norms
$$\|v\|_{\beta} = \left( \|v\|_2^2 + \int_{\mathbb{R}^N} \beta V(x)v(x,0)^2\,dx \right)^{1/2}.$$
Obviously, $E_0 \subset F$, where $\Omega$ is defined as in (V1). We are going to study the energy functional for equation (1.6) by
$$I_{\beta,\lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2)\,dx\,dy + \frac{1}{2} \int_{\mathbb{R}^N} (\beta V(x) - \lambda)v(x,0)^2\,dx$$
$$- \frac{1}{2} \cdot 2^* \mu \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*} |v(z,0)|^{2^*}}{|x-z|^\mu} \,dx\,dz.$$
Then we see $v$ is a weak solution of (1.6) if and only if $v$ is a critical point of functional $I_{\beta,\lambda}$. Furthermore, a function $v_0$ is called a ground state of (1.6) if $v_0$ is a critical point of (1.6) and $I_{\beta,\lambda}(v_0) \leq I_{\beta,\lambda}(v)$ holds for any critical point $v$ of (1.6), i.e.
$$I_{\beta,\lambda}(v_0) = c_{\beta,\lambda} := \inf \{ I_{\beta,\lambda}(v) : v \in E \setminus \{0\} \text{ is a critical point of (1.6)} \}.$$We denote the operator $L_{\beta,\lambda} := \sqrt{-\Delta + m^2 + \beta V(x) - \lambda}$ and particularly, $L_{\beta,0} := \sqrt{-\Delta + m^2 + \beta V(x)}$ and $L_{0,\lambda} := \sqrt{-\Delta + m^2 - \lambda}$. Observe that
$$0 \leq a_\beta = \inf \left\{ (L_{\beta,0}v,v) : v \in F, \int_{\mathbb{R}^N} v(x,0)^2\,dx = 1 \right\}$$
and that $a_\beta$ is nondecreasing in $\beta$.

The following two lemmas are similar to the ones obtained in [15].

**Lemma 6.1.** If $v_n \in F$ be such that $\beta_n \to \infty$ and $\|v_n\|_{\beta_n}^2 < C$. Then, there is a $v \in F$ such that $v(x,0) \in E_0$, up to a subsequence,
$$v_n \to v \quad \text{in } F,$n(\cdot,0) \to v(\cdot,0) \quad \text{in } L^2(\mathbb{R}^N).$$
ON CRITICAL PSEUDO-RELATIVISTIC HARTREE EQUATION WITH POTENTIAL WELL

215

**Proof.** Since $\|v_n\|^2 \leq \|v_n\|_{\Delta_0}^2 < C$, we may assume
\[
v_n \to v \quad \text{in } F,\]
\[
v_n(\cdot, 0) \to v(\cdot, 0) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N).
\]

We claim that $v(\cdot, 0)|_{\Omega^c} = 0$ and $v(x, 0) \in E_0$, where $\Omega^c := \mathbb{R}^N \setminus \Omega$. Indeed, if $v(\cdot, 0)|_{\Omega^c} \neq 0$. There exists a compact subset $G \subset \Omega^c$ with $\text{dist}(G, \Omega) > 0$ such that $v(x, 0)|_G \neq 0$ and
\[
\int_G v_n(x, 0)^2 \, dx \to \int_G v(x, 0)^2 \, dx > 0.
\]

From the fact that $\Omega = \text{int } V^{-1}(0)$, there exists $a_0 > 0$ such that $V(x) \geq a_0 > 0$ for all $x \in G$, as $n \to \infty$, which implies that
\[
I_{\beta_n}(v_n) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} \left( |\nabla v_n|^2 + m^2 v_n^2 \right) \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} (\beta_n V(x) - \lambda) v_n(x, 0)^2 \, dx
\]
\[
- \frac{1}{2} \cdot 2^* \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x, 0)|^2 |v_n(z, 0)|^{2^*}}{|x - z|^\mu} \, dx \, dz
\]
\[
= \left( \frac{N - \mu + 1}{4N - 2\mu} \right) \int_{\mathbb{R}^{N+1}} (|\nabla v_n|^2 + m^2 v_n^2) \, dx \, dy
\]
\[
+ \left( \frac{N - \mu + 1}{4N - 2\mu} \right) \int_{\mathbb{R}^N} (\beta_n V(x) - \lambda) v_n(x, 0)^2 \, dx
\]
\[
\geq \left( \frac{N - \mu + 1}{4N - 2\mu} \right) \int_{\mathbb{R}^N} (\beta_n a_0 - \lambda) v_n(x, 0)^2 \, dx \to \infty.
\]

This is a contradiction and thus $v(x, 0) = 0$ for almost every $x \in \mathbb{R}^N \setminus \Omega$.

We now show that $v_n(\cdot, 0) \to v(\cdot, 0)$ in $L^2(\mathbb{R}^N)$. Let $H = \{ x \in \mathbb{R}^N : V(x) \leq M_0 \}$ with $M_0$ as in $(V_2)$, and let $H^c = \mathbb{R}^N \setminus H$. Then
\[
\int_{H^c} v_n(x, 0)^2 \, dx \leq \frac{1}{\beta_n M_0} \int_{H^c} \beta_n V(x) v_n(x, 0)^2 \, dx \leq \frac{C}{\beta_n M_0} \to 0 \quad \text{as } n \to \infty.
\]

Setting $B_R^c = \mathbb{R}^N \setminus B_R$, where $B_R = \{ x \in \mathbb{R}^N : |x| \leq R \}$, and choosing $r \in (1, N/(N - 1))$, $r' = r/(r - 1)$, we have
\[
\int_{B_R^c \cap H} (v_n(x, 0) - v(x, 0))^2 \, dx \leq \|v_n(x, 0) - v(x, 0)\|_{L^2(B_R^c \cap H)}^2
\]
\[
\leq C\|v_n(x, 0) - v(x, 0)\|_{L^1(B_R^c \cap H)} \to 0 \quad \text{as } R \to \infty, \text{ due to } (V_2).
\]

Finally, since $v_n(x, 0) \to v(x, 0)$ in $L^2_{\text{loc}}(\mathbb{R}^N)$,
\[
\int_{B_R} (v_n(x, 0) - v(x, 0))^2 \, dx \to 0 \quad \text{as } n \to \infty. \quad \square
LEMMA 6.2. For every \(0 < \lambda < \lambda_1\), there exists \(\beta_\lambda > 0\) such that \(a_\beta \geq (\lambda + \lambda_1)/2\) for \(\beta \geq \beta_\lambda\). Consequently,

\[
C\|v\|_2^2 \leq \langle L_{\lambda_\beta} v, v \rangle \quad \text{for all } v \in F, \beta \geq \beta_\lambda,
\]

where \(C > 0\) is a constant.

PROOF. Assume, by contradiction, there exists a sequence \(\beta_n \to \infty\) such that \(a_{\beta_n} < (\lambda + \lambda_1)/2\) for all \(n\) and \(a_{\beta_n} \to \tau \leq (\lambda + \lambda_1)/2\). Let \(v_n \in F\) be such that

\[
\int_{\mathbb{R}^N} v_n(x, 0)^2 \, dx = 1 \quad \text{and} \quad \langle (L_{\lambda_{\beta_n}} - a_{\beta_n}) v_n, v_n \rangle \to 0.
\]

Then

\[
\|v_n\|_{\beta_n}^2 = \int_{\mathbb{R}^{N+1}_+} (|\nabla v_n|^2 + m^2 v_n^2) \, dx \, dy + (1 + \beta_n V(x)) \int_{\mathbb{R}^N} v_n(x, 0)^2 \, dx
\]

\[
= \langle (L_{\lambda_{\beta_n}} - a_{\beta_n}) v_n, v_n \rangle + (1 + a_{\beta_n}) \int_{\mathbb{R}^N} v(x, 0)^2 \, dx \leq 2(1 + \lambda_1)
\]

for all \(n\) large. By Lemma 6.1 there is a \(v \in E_0\) such that \(v_n \rightharpoonup v\) weakly in \(F\) and \(v_n(\cdot, 0) \to v(\cdot, 0)\) in \(L^2(\mathbb{R}^N)\). Therefore

\[
\int_{\mathbb{R}^N} v(x, 0)^2 \, dx = 1
\]

and

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_n|^2 + m^2 v_n^2) \, dx \, dy \geq \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy.
\]

It follows that

\[
\int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy - \tau \int_{\Omega} v(x, 0)^2 \, dx
\]

\[
\leq \liminf_{n \to \infty} \int_{\mathbb{R}^{N+1}_+} (|\nabla v_n|^2 + m^2 v_n^2) \, dx \, dy - \liminf_{n \to \infty} \int_{\mathbb{R}^N} a_{\beta_n} v_n(x, 0)^2 \, dx
\]

\[
\leq \liminf_{n \to \infty} \langle (L_{\lambda_{\beta_n}} - a_{\beta_n}) v_n, v_n \rangle = 0.
\]

Hence we have

\[
\int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \leq \tau \leq \frac{\lambda + \lambda_1}{2} < \lambda_1.
\]

This is a contradiction, because by definition,

\[
\lambda_1 \leq \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) \, dx \, dy \quad \text{for all } v \in E_0 \text{ with } \int_{\mathbb{R}^N} v(x, 0)^2 \, dx = 1. \quad \square
\]
It follows from Lemma 6.2 that the operator $L_{\beta,\lambda}$ is positive if $\beta \geq \beta_{\lambda}$ and thus we can introduce on $F$ a new inner product
\[
(v_1, v_2) = \int_{\mathbb{R}^{N+1}} \left( |\nabla v_1| |\nabla v_2| + m^2 v_1 v_2 \right) \, dx \, dy
+ \beta \int_{\mathbb{R}^{N}} V(x)v_1(x,0)v_2(x,0) \, dx - \lambda \int_{\mathbb{R}^{N}} v_1(x,0)v_2(x,0) \, dx \, dy
\]
with the norm $\|v\|_{L_{\beta,\lambda}}^2 = (v, v)$.

Moreover, noting that for $\lambda > 0$, $\|v\|_{L_{\beta,\lambda}} \leq \|v\|_{\beta}$, for all $v \in F$, we know $\|v\|_{L_{\beta,\lambda}}$ in fact is equivalent to the norm $\|v\|_{\beta}$ on $F$ if $\beta \geq \beta_{\lambda}$. For future use, enlarging $\beta_{\lambda}$ if necessary, we may assume that $\beta_{\lambda} \geq \lambda/M_0$, thus
\[
(6.1) \quad \beta M_0 - \lambda \geq 0 \quad \text{for all } \beta \geq \beta_{\lambda},
\]
where $M_0$ is given in (V$_2$).

Since we consider the critical case, we need to show where the compactness condition is recovered.

**Proposition 6.3.** For each $0 < \lambda < \lambda_1$ and $\beta \geq \beta_{\lambda}$, $I_{\beta,\lambda}$ satisfies the (PS)$_c$ condition for all $c_{\beta,\lambda} < ((N - \mu + 1)/(4N - 2\mu))S_c^{(2N-\mu)/(N-\mu+1)}$.

**Proof.** Let $\{v_j\}$ be a (PS)$_c$ sequence, i.e.
\[
(6.2) \quad I_{\beta,\lambda}(v_j) \to c_{\beta,\lambda},
\]
\[
(6.3) \quad \sup \left\{ \left| \left\langle I'_{\beta,\lambda}(v_j), \varphi \right\rangle \right| : \varphi \in F, \|\varphi\|_{L_{\beta,\lambda}} = 1 \right\} \to 0
\]
as $j \to +\infty$. By (6.2) and (6.3), for any $j \in \mathbb{N}$, it easily follows that there exists $C > 0$ such that
\[
(6.4) \quad |I_{\beta,\lambda}(v_j)| \leq C,
\]
\[
(6.5) \quad \left| \left\langle I'_{\beta,\lambda}(v_j), \frac{v_j}{\|v_j\|_{L_{\beta,\lambda}}} \right\rangle \right| \leq C.
\]
Consequently, we have
\[
(6.6) \quad \frac{N - \mu + 1}{4N - 2\mu} \|v_j\|_{L_{\beta,\lambda}}^2 = I_{\beta,\lambda}(v_j) - \frac{1}{2} \left\langle I'_{\beta,\lambda}(v_j), v_j \right\rangle \leq C(1 + \|v_j\|_{L_{\beta,\lambda}}),
\]
which means $\{v_j\}$ is bounded in $F$.

Now, up to a subsequence, still denoted by $\{v_j\}$, we may assume that there exists $v \in F$ such that $v_j \rightharpoonup v$ in $F$ and
\[
(6.7) \quad v_j(x,0) \to v(x,0) \quad \text{a.e. in } \mathbb{R}^N, \text{ as } j \to +\infty.
\]
The analogous to the proof of Lemma 3.3, we have

\begin{equation}
(6.8) \quad \int_{\mathbb{R}^N} \frac{|v_j(z,0)|^{2^*_N}}{|x-z|^\mu} \, dz \, |w_j(x,0)|^{2^*_N-2} v_j(x,0) \to \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_N}}{|x-z|^\mu} \, dz \, |v(x,0)|^{2^*_N-2} v(x,0)
\end{equation}

in $L^{2N/(N+1)}(\mathbb{R}^N)$ as $j \to +\infty$. Since, for any $\varphi \in F$, $\langle I_{\beta,\lambda}(v_j), \varphi \rangle \to 0$, passing to the limit as $j \to +\infty$ and taking into account (6.8) we get

\[
\int_{\mathbb{R}^{N+1}} (\nabla v \nabla \varphi + m^2 v \varphi) \, dx \, dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda) v(x,0) \varphi(x,0) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_N} |v(z,0)|^{2^*_N-2} v(z,0) \varphi(z,0)}{|x-z|^\mu} \, dx \, dz
\]

for any $\varphi \in F$, that means $v$ is a solution of problem (1.6). Moreover, taking $\varphi = v$ in $F$ as a test function in (1.6), we have

\[
\int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 v^2) \, dx \, dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda) v(x,0)^2 \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_N} |v(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz,
\]

thus

\[
I_{\beta,\lambda}(v) = \frac{N - \mu + 1}{4N - 2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_N} |v(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz \geq 0.
\]

Now, we write $w_j := v_j - v$, then, $w_j \to 0$ in $F$ and $w_j(x,0) \to 0$ almost everywhere in $\mathbb{R}^N$. By the Brézis–Lieb type splitting result for nonlocal term in [24] which says

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_j(x,0)|^{2^*_N} |v_j(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_j(x,0)|^{2^*_N} |w_j(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_N} |v(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz + o_j(1)
\]

as $j \to +\infty$, we know that

\begin{equation}
(6.9) \quad c_{\beta,\lambda} \leftarrow I_{\beta,\lambda}(v_j) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla w_j|^2 + m^2 w_j^2) \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} (\beta V(x) - \lambda) w_j(x,0)^2 \, dx
\end{equation}

\[
- \frac{1}{2} \cdot 2^*_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_j(x,0)|^{2^*_N} |w_j(z,0)|^{2^*_N}}{|x-z|^\mu} \, dx \, dz
\]
Analogously, we have
\[
\langle I'_{\beta,\lambda}(v_j), v_j \rangle = \langle I'_{\beta,\lambda}(w_j), w_j \rangle + \langle I'_{\beta,\lambda}(v), v \rangle + o_j(1).
\]
It follows from \( \langle I'_{\beta,\lambda}(v), v \rangle = 0 \) and \( \langle I'_{\beta,\lambda}(v_j), v_j \rangle \to 0 \) that
\[
\int_{\mathbb{R}^N_+} (|\nabla w_j|^2 + m^2 w_j^2) dx dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda) w_j(x,0)^2 dx \to b,
\]
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_j(x,0)|^{2^*_\mu} |w_j(z,0)|^{2^*_\mu}}{|x-z|^\mu} dx dz \to b.
\]
Since \( I_{\beta,\lambda}(v) \geq 0 \) and \( (6.9) \), we obtain,
\[
(6.10) \quad c_{\beta,\lambda} \geq \frac{N - \mu + 1}{4N - 2\mu} b.
\]
By Lemma 6.1 one knows that, as \( j \to \infty \),
\[
\int_H w_j(x,0)^2 dx \to 0,
\]
where \( H = \{ x \in \mathbb{R}^N : V(x) \leq M_0 \} \). Let \( H^c = \mathbb{R}^N \setminus H \). Then, from the definition of \( S_C \) and \( (6.1) \), we have
\[
S_C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_j(x,0)|^{2^*_\mu} |w_j(z,0)|^{2^*_\mu}}{|x-z|^\mu} dx dz \right)^{(N-1)/(2N-\mu)}
\]
\[
\leq \int_{\mathbb{R}^N_+} |\nabla w_j|^2 dx dy
\]
\[
\leq \int_{\mathbb{R}^N_+} (|\nabla w_j|^2 + m^2 w_j^2) dx dy + \int_{H^c} (\beta V(x) - \lambda) w_j(x,0)^2 dx
\]
\[
\leq \int_{\mathbb{R}^N_+} (|\nabla w_j|^2 + m^2 w_j^2)
\]
\[
+ \int_{\mathbb{R}^N} (\beta V(x) - \lambda) w_j(x,0)^2 dx + \lambda \int_{H} w_j(x,0)^2 dx
\]
\[
= \int_{\mathbb{R}^N_+} (|\nabla w_j|^2 + m^2 w_j^2) dx dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda) w_j(x,0)^2 dx + o_j(1),
\]
passing to the limit, it yields that \( b \geq S_C b^{(N-1)/(2N-\mu)} \). Then we have either \( b = 0 \) or \( b \geq S_C^{(2N-\mu)/(N-\mu+1)} \). If \( b = 0 \), the proof is complete. Otherwise
Sobolev inequality, for all $v$

**Proof.**

6.4 **Lemma**

$b > 0$ large enough, the functional $I_{\beta, \lambda}$ satisfies the Mountain-Pass geometry.

**Lemma 6.4.** For any $0 < \lambda < \lambda_1$, $\beta > 0$ large enough, the functional $I_{\beta, \lambda}$ satisfies the following conditions:

(a) There exist $\alpha, \rho > 0$ such that $I_{\beta, \lambda}(v) \geq \alpha$ for $\|v\|_{L_{\beta, \lambda}} = \rho$.

(b) There exists $w_1 \in F$ with $\|w_1\|_{L_{\beta, \lambda}} > \rho$ such that $I_{\beta, \lambda}(w_1) < 0$.

**Proof.** (a) By $0 < \lambda < \lambda_1$, the Sobolev embedding and Hardy–Littlewood–Sobolev inequality, for all $v \in F \setminus \{0\}$, we have

$$I_{\beta, \lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 + m^2 v^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} (\beta V(x) - \lambda)v(x, 0)^2 \, dx$$

$$- \frac{1}{2} \cdot \frac{2}{\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x, 0)|^{2^*_\mu} |v(z, 0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 + m^2 v^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} (\beta V(x) - \lambda)v(x, 0)^2 \, dx$$

$$- \frac{1}{2} \cdot \frac{2}{\mu} C \left( \int_{\mathbb{R}^N} v(x, 0)^2 \, dx \right)^{2^*_\mu/2^*_\mu} \geq C \|v\|^2_{L_{\beta, \lambda}} - C \|v\|^2_{L_{0, \lambda}}.$$ 

Since $2 < 2 \cdot 2^*_\mu$, we can choose some $\alpha, \rho > 0$ such that

$$I_{\beta, \lambda}(v) \geq \alpha \quad \text{for } \|v\|_{L_{\beta, \lambda}} = \rho.$$

(b) For any $v_1 \in F \setminus \{0\}$, we have

$$I_{\beta, \lambda}(tv_1) = \frac{t^2}{2} \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v_1|^2 + m^2 v_1^2 \, dx \, dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda)v_1(x, 0)^2 \, dx \right)$$

$$- \frac{t^2}{2} \cdot \frac{2}{\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_1(x, 0)|^{2^*_\mu} |v_1(z, 0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz < 0$$

for $t > 0$ large enough. Hence, we can take a $w_1 := t_1 v_1$ for some $t_1 > 0$ and (b) follows. \qed
Applying the mountain pass theorem without (PS) condition (see [36]), there exists a (PS) sequence \( \{v_n\} \) such that \( I_{\beta,\lambda}(v_n) \to c \) and \( I'_{\beta,\lambda}(v_n) \to 0 \) in \( F^* \) (\( F^* \) is the dual of \( F \)) at the minimax level
\[
c_{\beta,\lambda} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\beta,\lambda}(\gamma(t)) > 0,
\]
where \( \Gamma := \{ \gamma \in C([0,1], F) : \gamma(0) = 0, \ I_{\beta,\lambda}(\gamma(1)) < 0 \} \).

If we denote the Nehari manifold of \( I_{\beta,\lambda} \) by
\[
M_{\beta,\lambda} = \{ v \in F \setminus \{0\} : \langle I'_{\beta,\lambda}(v), v \rangle = 0 \},
\]
then \( 0 < \lambda < \lambda_1 \) and \( 2 < 2 \cdot 2^n_{\mu} \), the function \( t \in \mathbb{R} \to I_{\beta,\lambda}(tv) \) has an unique maximum point \( t(v) > 0 \) and \( t(v)v \in M_{\beta,\lambda} \). Then \( c_{\beta,\lambda} \) has an equivalent minimax characterization, that is
\[
(6.11) \quad c_{\beta,\lambda} := \inf_{v \in M_{\beta,\lambda}} I_{\beta,\lambda}(v) = \inf_{v \in F, v \neq 0} \max_{t \geq 0} I_{\beta,\lambda}(tv).
\]

Next we denote by \( I_{m,\lambda} \) the restriction of \( I_{\beta,\lambda} \) on \( E_0 \), that is
\[
I_{m,\lambda}(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} (|v|^2 + m^2 v^2) \, dx \, dy - \frac{\lambda}{2} \int_{\Omega} v(x,0)^2 \, dx
\]
\[
- \frac{1}{2 \cdot 2^n_{\mu}} \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^2^* |v(z,0)|^{2^*_n}}{|x-z|^\mu} \, dx \, dz,
\]
where \( \Omega \) is defined as in (V1). By Lemma 3.3, we have the following proposition.

**Proposition 6.5.** Let \( \lambda > 0, \lambda \neq \lambda_j \) for any \( j \geq 1 \). If \( \beta \geq \beta_\lambda \) then
\[
0 < c_{\beta,\lambda} \leq c_{m,\lambda} < \frac{N - \mu + 1}{4N - 2\mu} S_{C}^{(2N-\mu)/(N-\mu+1)}.
\]

**Proof.** Lemma 6.4 implies \( c_{\beta,\lambda} > 0 \). Since
\[
\left\{ v \in E_0 : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^2^* |v(z,0)|^{2^*_n}}{|x-z|^\mu} \, dx \, dy = 1 \right\}
\]
\[
\subset \left\{ v \in F : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^2^* |v(z,0)|^{2^*_n}}{|x-z|^\mu} \, dx \, dy = 1 \right\}
\]
and \( (L_{0,\lambda} v, v) = (L_0 v, v) \) for
\[
v \in \left\{ v \in E_0 : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^2^* |v(z,0)|^{2^*_n}}{|x-z|^\mu} \, dx \, dy = 1 \right\},
\]
it follows that \( c_{\beta,\lambda} \leq c_{m,\lambda} \). By Lemmas 3.3 and (6.11), we know
\[
c_{m,\lambda} < \frac{N - \mu + 1}{4N - 2\mu} S_{C}^{(2N-\mu)/(N-\mu+1)}.
\]
Hence, the conclusion is proved. \( \square \)

**Proof of Theorem 1.3.** Applying the Mountain-Pass Theorem without (PS) condition, we know there exists a (PS)\( c_{\beta,\lambda} \) sequence \( \{v_n\} \). Then we obtain from Propositions 6.3 and 6.5, (1.6) has at least one ground state solution \( v \). \( \square \)
In the following, we come to give the asymptotic behavior of the solutions of (1.4) as $\beta$ goes to infinity. For $0 < \lambda < \lambda_1$, let $\{v_n\}$ be a sequence of solutions of (1.6) such that

$$\beta_n \to \infty \quad \text{and} \quad I_{\beta_n, \lambda}(v_n) \to c_{\beta, \lambda} < \frac{N - \mu + 1}{4N - 2\mu} S_C^{2(N - \mu)/(N - \mu + 1)},$$

we have

$$I_{\beta_n, \lambda}(v_n) = \frac{N - \mu + 1}{4N - 2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x, 0)|^{2\mu} |v_n(z, 0)|^{2\mu}}{|x - z|^{\mu}} \, dx \, dz$$

and so,

$$(6.13) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x, 0)|^{2\mu} |v_n(z, 0)|^{2\mu}}{|x - z|^{\mu}} \, dx \, dz < S_C^{2(N - \mu)/(N - \mu + 1)}.$$  

By Lemma 6.2, we can deduce that

$$\frac{N - \mu + 1}{4N - 2\mu} S_C^{2(N - \mu)/(N - \mu + 1)} > I_{\beta_n, \lambda}(v_n)$$

$$= \frac{N - \mu + 1}{4N - 2\mu} \langle L_{\beta_n, \lambda}(v_n), v_n \rangle \geq \frac{N - \mu + 1}{4N - 2\mu} C \|v_n\|_{L_{\beta_n, \lambda}}^2$$

and so $\{v_n\}$ is bounded in $F$.

By Lemma 6.1, there is a $v \in E_0$ such that, up to a subsequence, $v_n \rightharpoonup v$ in $F$ and $v_n(\cdot, 0) \to v(\cdot, 0)$ in $L^2(\mathbb{R}^N)$. From the fact that $v_n$ is a solution of (1.6), we have

$$\int_{\mathbb{R}^{N+1}} (\nabla v_n \nabla \varphi + m^2 v_n \varphi) \, dx \, dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda)v_n(x, 0)\varphi(x, 0) \, dx$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x, 0)|^{2\mu} |v_n(z, 0)|^{2\mu - 2} v_n(z, 0)\varphi(z, 0)}{|x - z|^{\mu}} \, dx \, dz$$

for any $\varphi \in F$. If $\varphi \in E_0$ then

$$\beta_n \int_{\mathbb{R}^N} V v_n(x, 0)\varphi(x, 0) \, dx = 0 \quad \text{for all } n.$$  

Letting $n \to \infty$ we obtain

$$\int_{\mathbb{R}^{N+1}} (\nabla v \nabla \varphi + m^2 v \varphi) \, dx \, dy - \lambda \int_{\mathbb{R}^N} v(x, 0)\varphi(x, 0) \, dx$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x, 0)|^{2\mu} |v(z, 0)|^{2\mu - 2} v(z, 0)\varphi(z, 0)}{|x - z|^{\mu}} \, dx \, dz$$

for any $\varphi \in E_0$. So, $v$ is a solution of (1.6).

Define $w_n := v_n - v$, then $w_n(\cdot, 0) \to 0$ in $L^2(\mathbb{R}^N)$ and $w_n(x, 0) \to 0$ almost everywhere in $\mathbb{R}^N$ as $n \to +\infty$. Since $V(x) = 0$ for $x \in \Omega$, we get

$$(6.14) \quad \langle L_{\beta_n, \lambda} v_n, v_n \rangle = \langle L_{0, \lambda} v, v \rangle + \langle L_{\beta_n, \lambda} w_n, w_n \rangle.$$

Y. Zheng — M. Yang — Z. Shen
Since \( \{v_n\} \) is a sequence of solutions of (1.6) and \( v \) is a solution of (1.11), by the Brézis–Lieb Lemma, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x,0)|^{2^*_\mu} |v_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz + o(1),
\]
we can get
\[
(6.15) \quad \langle L_{\beta_n,\lambda} w_n, w_n \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz = o(1).
\]
We claim that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \to 0.
\]
Assume by contrary that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \to b > 0.
\]
Then, thanks to (2.4), (6.1) and (6.15), we have
\[
S_C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \right)^{(N-1)/(2N-\mu)}
\leq \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx \, dy
\leq \int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + m^2 w_n^2 \right) \, dx \, dy + \int_{H^c} (\beta V(x) - \lambda) |w_n(x,0)|^2 \, dx
\leq \int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + m^2 w_n^2 \right) \, dx \, dy
+ \int_{\mathbb{R}^N} (\beta V(x) - \lambda) |w_n(x,0)|^2 \, dx + \int_{H} |w_n(x,0)|^2 \, dx
= \int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + m^2 w_n^2 \right) \, dx \, dy + \int_{\mathbb{R}^N} (\beta V(x) - \lambda) |w_n(x,0)|^2 \, dx + o(1)
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz + o(1).
\]
It follows that
\[
S_C \leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^{2^*_\mu} |w_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \right)^{(N-\mu+1)/(2N-\mu)} + o(1)
\leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x,0)|^{2^*_\mu} |v_n(z,0)|^{2^*_\mu}}{|x-z|^\mu} \, dx \, dz \right)^{(N-\mu+1)/(2N-\mu)} + o(1),
\]
and so, by (6.13),

\[ S_C^{(2N-\mu)/(N-\mu+1)} \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x,0)|^2 |v_n(z,0)|^2}{|x-z|^\mu} \, dx \, dz \]

This is a contradiction and consequently

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x,0)|^2 |w_n(z,0)|^2}{|x-z|^\mu} \, dx \, dz \to 0. \]

From (6.15) we get

(6.16) \[ \langle L_{\beta_n,\lambda} w_n, w_n \rangle \to 0. \]

Hence, by (6.14)

(6.17) \[ \lim_{n \to \infty} \langle L_{\beta_n,\lambda} v_n, v_n \rangle = \langle L_{0,\lambda} v, v \rangle. \]

Recall that

\[ \int_{\mathbb{R}^{N+1}} (|\nabla v_n|^2 + m^2 v_n^2) \, dx \, dy \geq \lambda \int_{\mathbb{R}^N} v_n(x,0)^2 \, dx, \]

we know

\[ \int_{\mathbb{R}^N} V v_n(x,0)^2 \, dx \leq \int_{\mathbb{R}^N} \beta_n V v_n(x,0)^2 \, dx \]

\[ = \int_{\mathbb{R}^N} \beta_n V w_n(x,0)^2 \, dx \leq \langle L_{\beta_n,\lambda} (w_n), w_n \rangle, \]

since \( v_n = w_n \) in \( \mathbb{R}^N \setminus \Omega \) and \( V = 0 \) for \( x \in \Omega \). Combining this with (6.16), we know

\[ \int_{\mathbb{R}^N} V v_n(x,0)^2 \, dx \to 0 \]

and obtain from (6.17) that \( v_n \to v \) in \( F \).

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