Abstract. We provide sufficient conditions for a uniform $L^2(\Omega)$ bound to imply a uniform $L^\infty(\Omega)$ bound for positive classical solutions to a class of subcritical elliptic problems in bounded $C^2$ domains in $\mathbb{R}^N$. We also establish an equivalent result for sequences of boundary value problems.

1. Introduction

We consider the existence of $L^\infty(\Omega)$ a priori bounds for classical positive solutions to the boundary value problem

(1.1) \[-\Delta u = f(u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega,\]

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where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with $C^2$ boundary $\partial \Omega$. We provide sufficient conditions on $f$ for $L^{2^*}(\Omega)$ a priori bounds to imply $L^\infty(\Omega)$ a priori bounds, where $2^* = 2N/(N - 2)$ is the critical Sobolev exponent. The converse is obviously true without any additional hypotheses.

The existence of a priori bounds for (1.1) has a rich history. In chronological order, [18], [14], [17], [4], [15], [11], [10] and [2] are some of the main contributors to such a development. We refer the reader to [6] where their roles are discussed.

The ideas for the proof of our main Theorem are similar to those used in [6, Theorem 1.1]. In [6] we give sufficient conditions on the nonlinearity to have $L^\infty(\Omega)$ a priori bounds, while here we prove the equivalence between the existence of $L^\infty(\Omega)$ a priori bounds and the existence of $L^{2^*}(\Omega)$ a priori bounds for subcritical elliptic equations. Unlike the proof in [6], here we do not use Pohozaev or moving planes arguments.

Our main result is the following theorem.

**Theorem 1.1.** Assume that the nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitzian function that satisfies:

(H1) There exists a constant $C_0 > 0$ such that

$$\liminf_{s \to \infty} \frac{1}{f(s)} \min_{[s/2, s]} f \geq C_0.$$  

(H2) There exists a constant $C_1 > 0$ such that

$$\limsup_{s \to \infty} \frac{1}{f(s)} \max_{[0, s]} f \leq C_1.$$  

(F) $\lim_{s \to +\infty} \frac{f(s)}{s^{2^*-1}} = 0$; that is, $f$ is subcritical.

Then the following conditions are equivalent:

(a) there exists a uniform constant $C$ (depending only on $\Omega$ and $f$) such that, for every positive classical solution $u$ of (1.1),

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

(b) there exists a uniform constant $C$ (depending only on $\Omega$ and $f$) such that, for every positive classical solution $u$ of (1.1)

$$\int_{\Omega} |f(u)|^{2N/(N+2)} \, dx \leq C,$$

(c) there exists a uniform constant $C$ (depending only on $\Omega$ and $f$) such that, for every positive classical solution $u$ of (1.1),

$$\|u\|_{L^{2^*}(\Omega)} \leq C.$$  

In [7] and [8] the associated bifurcation problem for the nonlinearity $f(\lambda, s) = \lambda s + g(s)$ with $g$ subcritical is studied. Sufficient conditions guaranteeing that
either for any \( \lambda < \lambda_1 \) there exists at least a positive solution, or that there exists a \( \lambda^* < 0 \) and a continuum \((\lambda, u_\lambda), \lambda^* < \lambda < \lambda_1, \) of positive solutions such that

\[
\| \nabla u_\lambda \|_{L^2(\Omega)} \to \infty, \quad \text{as } \lambda \to \lambda^*,
\]

are provided. See [8, Theorem 2]. In the case \( \Omega \) is convex, for any \( \lambda < \lambda_1 \) there exists at least a positive solution, see [7, Theorem 1.2]. In [9] the concept of regions with convex-starlike boundary is introduced and sufficient conditions for the existence of \( \text{a priori} \) bounds in such regions are established. In [16] the existence of \( \text{a priori} \) bounds for elliptic systems is provided.

In this paper, we also provide sufficient conditions for the equivalence of the existence of \( L^2(\Omega) \) \( \text{a priori} \) bound with that of \( L^\infty(\Omega) \) \( \text{a priori} \) bound for sequences of boundary value problems. In fact, we prove the following theorem.

Theorem 1.2. Consider the following sequence of BVPs

\[
(1.3)_k \quad -\Delta v = g_k(v) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega,
\]

with \( g_k : \mathbb{R}^+ \to \mathbb{R} \) locally Lipschitzian. We assume that the following hypotheses are satisfied

\begin{align*}
\text{(H1)}_k & \quad \text{There exists a uniform constant } C_1 > 0, \text{ such that } \\
& \quad \liminf_{s \to +\infty} \frac{1}{g_k(s)} \min_{[s/2, s]} g_k \geq C_1.
\end{align*}

\begin{align*}
\text{(H2)}_k & \quad \text{There exists a uniform constant } C_2 > 0 \text{ such that } \\
& \quad \limsup_{s \to +\infty} \frac{1}{g_k(s)} \max_{[0, s]} g_k \leq C_2.
\end{align*}

Let \( \{v_k\} \) be a sequence of classical positive solutions to \((1.3)_k \) for \( k \in \mathbb{N} \). If

\[(F)_k \quad \lim_{k \to +\infty} g_k(\|v_k\|/\|v_k\|^{2^*-1}) = 0,
\]

then, the following two conditions are equivalent:

\begin{enumerate}
\item[(a)] there exists a uniform constant \( C \), depending only on \( \Omega \) and the sequence \( \{g_k\} \), but independent of \( k \), such that for every \( v_k > 0 \), classical solution to \((1.3)_k \)

\[
\limsup_{k \to \infty} \|v_k\|_{L^\infty(\Omega)} \leq C;
\]

\item[(b)] there exists a uniform constant \( C \), depending only on \( \Omega \) and the sequence \( \{g_k\} \), but independent of \( k \), such that for every \( v_k > 0 \), classical solution to \((1.3)_k \)

\[
\limsup_{k \to \infty} \int_\Omega |g_k(v_k)|^{2N/(N+2)} dx \leq C.
\]
\end{enumerate}

\begin{enumerate}
\item[(c)] there exists a uniform constant \( C \) (depending only on \( \Omega \) and the sequence \( \{g_k\} \)) such that for every positive classical solution \( v_k \) of \((1.3)_k \)

\[
\|v_k\|_{L^{2^*}(\Omega)} \leq C.
\]
\end{enumerate}
Hypothesis \((H1)_k\), and \((H2)_k\), are not sufficient for the existence of an \(L^\infty\) \emph{a priori} bound. Atkinson and Pelletier in [1] show that for \(f_\varepsilon(s) = s^{2^* - 1 - \varepsilon}\) and \(\Omega\) a ball in \(\mathbb{R}^3\), there exists \(x_0 \in \Omega\) and a sequence of solutions \(u_\varepsilon\) such that \(\lim_{\varepsilon \to 0} u_\varepsilon = 0\) in \(C^1(\Omega \setminus \{x_0\})\) and \(\lim_{\varepsilon \to 0} u_\varepsilon(x_0) = +\infty\). See also Han [13], for non-spherical domains.

Furthermore, hypotheses \((H1)_k\), \((H2)_k\), and \((F)_k\), are not sufficient for the existence of an \(L^\infty\) \emph{a priori} bound. In fact, in Section 4 we construct a sequence of BVP satisfying \((H1)_k\), \((H2)_k\), and \((F)_k\), and a sequence of solutions \(v_k\) such that \(\lim_{k \to \infty} \|v_k\|_\infty = +\infty\). Our example also shows the non-uniqueness of positive solutions.

2. Proof of Theorems 1.1 and 1.2

In this section, we state and prove our main results that hold for general bounded domains, including the non-convex case. We provide a sufficient condition for a uniform \(L^{2^*}(\Omega)\) bound to imply a uniform \(L^\infty(\Omega)\) bound for classical positive solutions of the subcritical elliptic equation (1.1). We also give sufficient conditions such that the \(L^\infty(\Omega)\) bound of a sequence of classical positive solutions of a sequence of BVPs (1.3)_k is equivalent to the uniform \(L^{2^*}(\Omega)\) bound of the sequence of reaction functions. The arguments rely on the estimation of the radius \(R\) of a ball where the function \(u\) exceeds half of its \(L^\infty\) bound, see Figure 1.

All throughout this paper, we assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain with \(C^2\) boundary, and \(C\) denotes several constants independent of \(u\), where \(u > 0\) is any classical solution to (1.1).

\[\text{Figure 1. A solution, its } L^\infty \text{ norm, and the estimate of the radius } R \text{ such that } u(x) \geq \|u\|_\infty / 2 \text{ for all } x \in B(x_0, R), \text{ where } x_0 \text{ is such that } u(x_0) = \|u\|_\infty.\]
Remark 2.1. By (1.2), elliptic regularity and the Sobolev embeddings imply that
\begin{equation}
\|u\|_{H^1_0(\Omega)} \leq \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \leq C.
\end{equation}
Hence, for any classical solutions to (1.1), we have
\begin{equation}
\int_\Omega uf(u) \, dx = \|u\|_{H^1_0(\Omega)}^2 \leq C.
\end{equation}

Proof of Theorem 1.1. Since $\Omega$ is bounded (a) implies (b) and (c). From elliptic regularity and condition (1.2), we deduce that
\begin{equation}
\|u\|_{W^{2,2N/(N+2)}(\Omega)} \leq C.
\end{equation}
It follows using twice the Sobolev embedding that a uniform bound in $W^{2,2N/(N+2)}(\Omega)$ implies a uniform bound in $L^2(\Omega)$ and a uniform bound in $L^{2^*}(\Omega)$, that is,
\begin{equation}
\|u\|_{L^{2^*}(\Omega)} \leq C,
\end{equation}
for all classical positive solution $u$ of equation (1.1). Therefore, (b) implies (c).

Now, assume that (c) holds. It follows from the subcriticality condition (F) that $|f(s)|^{2N/(N+2)} \leq s^{2^*}$ for all $s$ large enough. Thus, for any classical solution to (1.1), we have
\begin{equation}
\int_\Omega |f(u)|^{2N/(N+2)} \, dx \leq \int_\Omega |u|^{2N/(N-2)} \, dx + C < C.
\end{equation}
Thus (b) and (c) are equivalent.

Next, we concentrate our attention in proving that (b) implies (a). Since $2N/(N+2) = 1 + 1/(2^* - 1)$, the hypothesis (1.2) can be written
\begin{equation}
\int_\Omega |f(u)|^{1+1/(2^* - 1)} \, dx \leq C.
\end{equation}
Therefore,
\begin{equation}
\int_\Omega |f(u(x))|^q \, dx \leq \int_\Omega |f(u(x))|^{1+1/(2^* - 1)}|f(u(x))|^{q-1-1/(2^* - 1)} \, dx
\leq C\|f(u(\cdot))\|^{q-1-1/(2^* - 1)}_{\infty},
\end{equation}
for any $q > N/2$.

From the elliptic regularity (see [3] and [12, Lemma 9.17]), it follows that
\begin{equation}
\|u\|_{W^{2,q}(\Omega)} \leq C\|\Delta u\|_{L^q(\Omega)} \leq C\|f(u(\cdot))\|^{1-1/q-1/(2^* - 1)}_{\infty}.
\end{equation}
Let us restrict $q \in (N/2, N)$. From the Sobolev embeddings, for $1/q^* = 1-q-1/N$ with $q^* > N$ we can write
\begin{equation}
\|u\|_{W^{1,q^*}(\Omega)} \leq C\|u\|_{W^{2,q}(\Omega)} \leq C\|f(u(\cdot))\|^{1-1/q-1/(2^* - 1)}_{\infty}.
\end{equation}
From Morrey’s Theorem, (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant $C$ (depending only on $\Omega$, $q$ and $N$) such that, for all $x_1, x_2 \in \Omega$,
\begin{equation}
|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^{N/q^*}\|u\|_{W^{1,q^*}(\Omega)}.
\end{equation}
Therefore, for all \( x \in B(x_1, R) \subset \Omega \),

\[
|u(x) - u(x_1)| \leq CR^{2-N/q} \|u\|_{W^{2,q}(\Omega)}.
\]

(2.9)

Now, we shall argue by contradiction. Suppose that there exists a sequence \( \{u_k\} \) of classical positive solutions of (1.1) such that

\[
\lim_{k \to \infty} \|u_k\| = +\infty, \quad \text{where } \|u_k\| := \|u_k\|_\infty.
\]

(2.10)

Let \( x_k \in \Omega \) be such that \( u_k(x_k) = \max_{\Omega} u_k \). Let us choose \( R_k \) such that \( B_k = B(x_k, R_k) \subset \Omega \), and

\[
\forall x \in B_k, \quad u_k(x) \geq \frac{1}{2} \|u_k\|
\]

and there exists \( y_k \in \partial B(x_k, R_k) \) such that

\[
u_k(y_k) = \frac{1}{2} \|u_k\|.
\]

(2.11)

Let us denote by

\[
m_k := \min_{\|u_k\|/2, \|u_k\|} f, \quad M_k := \max_{\|u_k\|} f.
\]

\[
\text{Then, reasoning as in (2.5), we obtain}
\]

\[
\int_\Omega |f(u_k)|^q \, dx \leq C M_k^{q-1/(2^*-1)}.
\]

(2.13)

From the elliptic regularity, see (2.6), we deduce

\[
\|u_k\|_{W^{2,q}(\Omega)} \leq CM_k^{1-1/q-1/(2^*-1)q}.
\]

(2.14)

Therefore, from Morrey’s Theorem, see (2.9), for any \( x \in B(x_k, R_k) \)

\[
|u_k(x) - u_k(x_k)| \leq C(R_k)^{2-N/q} M_k^{1-1/q-1/(2^*-1)q}.
\]

(2.15)

Taking \( x = y_k \) in the above inequality and from (2.11) we obtain

\[
C(R_k)^{2-N/q} M_k^{1-1/q-1/(2^*-1)q} \geq |u_k(y_k) - u_k(x_k)| = \frac{1}{2} \|u_k\|,
\]

which implies

\[
(R_k)^{2-N/q} \geq \frac{1}{2C} \frac{\|u_k\|}{M_k^{1-1/q-1/(2^*-1)q}},
\]

(2.17)

or equivalently,

\[
R_k \geq \left( \frac{1}{2C} \frac{\|u_k\|}{M_k^{1-1/q-1/(2^*-1)q}} \right)^{1/(2-N/q)}.
\]

(2.18)
Finally, from (2.10) and the hypothesis (F) we deduce

\[ \limsup_{k \to \infty} \int_{B(x_k, R_k)} u_k^2 \geq \left( \frac{1}{2} \|u_k\| \right)^{2^*} \omega(R_k)^N, \]

where \( \omega = \omega_N \) is the volume of the unit ball in \( \mathbb{R}^N \).

Due to \( B(x_k, R_k) \subset \Omega \), substituting inequality (2.18), taking into account hypothesis (H2), and rearranging terms, we obtain

\[
\|u_k\|_{L^{2^*}(\Omega)}^2 = \int_{\Omega} u_k^2 \geq \left( \frac{1}{2} \|u_k\| \right)^{2^*} \omega \left( \frac{1}{2C} \frac{\|u_k\|}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{N/(2-N/q)} \\
\geq \left( \frac{1}{2} \|u_k\| \right)^{2^*} \omega \left( \frac{1}{2C} \frac{\|u_k\|}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{1/(2-N-1/q)} \\
= C \|u_k\|^{2^*-1} \left( \frac{\|u_k\|^{2N-1/q}}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{1/(2-N-1/q)} \\
= C \|u_k\|^{2^*-1} \left( \frac{\|u_k\|^{1+2N-1/q}}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{1/(2-N-1/q)} \\
\geq C \|u_k\|^{2^*-1} \left( \frac{\|u_k\|^{(N+2)[1/N-1/(N+2)q]}}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{1/(2-N-1/q)} \\
\geq C \|u_k\|^{2^*-1} \left( \frac{\|u_k\|^{(N+2)[1/N-1/(N+2)q]}}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{1/(2-N-1/q)}.
\]

Finally, from (2.10) and the hypothesis (F) we deduce

\[
\int_{\Omega} u_k^{2^*} \geq C \frac{\|u_k\|^{2^*-1}}{f(\|u_k\|)} \left( \frac{\|u_k\|^{2N-1/q}}{M_k^{1-1/q-1/((2^*-1)q)}} \right)^{(N-2)[1/N-1/(N+2)q](2-N-1/q)} \\
= \left( \frac{\|u_k\|^{2^*-1}}{f(\|u_k\|)} \right)^{1+(N-2)[1/N-1/(N+2)q](2-N-1/q)} \to \infty \quad \text{as} \quad k \to \infty,
\]

which contradicts (2.3). Thus (b) implies (a).

\[ \square \]

**Remark 2.2.** One can easily see that condition (1.4) implies that there exists a uniform constant \( C_4 > 0 \) such that

\[ (2.19) \quad \limsup_{k \to \infty} \int_{\Omega} v_k g_k(v_k) \, dx \leq C_4, \]

for all classical positive solutions \( \{v_k\} \) to (1.3)k.

**Proof of Theorem 1.2.** Clearly, condition (a) implies (b) and (c). By the elliptic regularity and condition (1.4), we have that \( \|v_k\|_{W^{2,2N/(N+2)}} \leq C \). Therefore, \( \|v_k\|_{H^1(\Omega)} \leq C \). Hence, by the Sobolev embedding, we deduce that

\[ (2.20) \quad \|v_k\|_{L^{2^*}(\Omega)} \leq C \quad \text{for all } k. \]

Using similar arguments as in Theorem 1.1 and condition (F)k, one can show that (b) and (c) are equivalent. We shall concentrate our attention in proving that (b) implies (a). All throughout this proof \( C \) denotes several constants independent of \( k \).
Observe that $1 + 1/(2^* - 1) = 2N/(N + 2)$. From hypothesis (b), see (1.4), there exists a fixed constant $C > 0$, (independent of $k$) such that
\begin{equation}
(2.21) \quad \int_{\Omega} |g_k(v_k(x))|^q \, dx \leq \int_{\Omega} |g_k(v_k(x))|^{1+1/(2^*-1)}|g_k(v_k(x))|^{q-1-1/(2^*-1)} \, dx \leq C\|g_k(v_k(\cdot))\|_{L^{2^*}}^{q-1-1/(2^*-1)},
\end{equation}
for $k$ big enough, and for any $q > N/2$. Therefore, from the elliptic regularity, see [12, Lemma 9.17]
\begin{equation}
(2.22) \quad \|v_k\|_{W^{2,q}(\Omega)} \leq C\|\Delta v_k\|_{L^q(\Omega)} \leq C \|g_k(v_k(\cdot))\|_{L^\infty}^{1-1/q-1/(2^*-1)q},
\end{equation}
for $k$ big enough.

Let us restrict $q \in (N/2, N)$. From Sobolev embeddings, for $1/q^* = 1/q-1/N$ with $q^* > N$ we can write
\begin{equation}
(2.23) \quad \|v_k\|_{W^{1,q^*}(\Omega)} \leq C\|v_k\|_{W^{2,q}(\Omega)} \leq C \|g_k(v_k(\cdot))\|_{L^\infty}^{1-1/q-1/(2^*-1)q},
\end{equation}
for $k$ big enough. From Morrey’s Theorem, (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant $C$ only dependent on $\Omega$, $q$, and $N$ such that
\begin{equation}
(2.24) \quad |v_k(x_1) - v_k(x_2)| \leq C|x_1 - x_2|^{1-q^*/q}\|v_k\|_{W^{1,q^*}(\Omega)},
\end{equation}
for all $x_1, x_2 \in \Omega$ and for any $k$. Therefore, for all $x \in B(x_1, R) \subset \Omega$
\begin{equation}
(2.25) \quad |v_k(x) - v_k(x_1)| \leq C R^{2-N/q}\|v_k\|_{W^{2,q}(\Omega)},
\end{equation}
for any $k$.

From now on, we argue by contradiction. Let $\{v_k\}$ be a sequence of classical positive solutions to (1.3) and assume that
\begin{equation}
(2.26) \quad \lim_{k \to \infty} \|v_k\| = +\infty, \quad \text{where } \|v_k\| := \|v_k\|_{L^\infty}.
\end{equation}
Let $x_k \in \Omega$ be such that $v_k(x_k) = \max_{\Omega} v_k$. Let us choose $R_k$ such that $B_k := B(x_k, R_k) \subset \Omega$, and
\begin{equation}
(2.27) \quad v_k(x) \geq \frac{1}{2}\|v_k\| \quad \text{for any } x \in B_k.
\end{equation}
and there exists $y_k \in \partial B_k$ such that
\begin{equation}
(2.28) \quad v_k(y_k) = \frac{1}{2}\|v_k\|.
\end{equation}
Let us denote by
\begin{equation}
\begin{aligned}
m_k := \min_{\|v_k\|/2,\|v_k\|} g_k, & \quad M_k := \max_{[0,\|v_k\|]} g_k,
\end{aligned}
\end{equation}
Therefore, we obtain
\begin{equation}
(2.29) \quad m_k \leq g_k(v_k(x)) \quad \text{if } x \in B_k, \quad g_k(v_k(x)) \leq M_k \quad \text{for all } x \in \Omega.
\end{equation}
Then, reasoning as in (2.21), we obtain

\[(2.29)\quad \int_{\Omega} |g_k(v_k)|^q \, dx \leq CM_k^{q-1-1/(2^*-1)}.\]

From the elliptic regularity, see (2.22), we deduce

\[(2.30)\quad \|v_k\|_{W^{2,q}({\Omega})} \leq CM_k^{1-1/q-1/(2^*-1)}.
\]

Therefore, from Morrey’s Theorem, see (2.25), for any \(x \in B_k\),

\[(2.31)\quad |v_k(x) - v_k(x_k)| \leq C(R_k)^{2-N/q}M_k^{1-1/q-1/(2^*-1)}.
\]

Particularizing \(x = y_k\) in the above inequality and from (2.27) we obtain

\[(2.32)\quad C(R_k)^{2-N/q}M_k^{1-1/q-1/(2^*-1)} \geq |v_k(y_k) - v_k(x_k)| = \frac{1}{2} \|v_k\|,
\]

which implies

\[(2.33)\quad (R_k)^{2-N/q} \geq \frac{1}{2C} \frac{\|v_k\|}{M_k^{1-1/q-1/(2^*-1)}},
\]

or equivalently

\[(2.34)\quad R_k \geq \left( \frac{1}{2C} \frac{\|v_k\|}{M_k^{1-1/q-1/(2^*-1)}} \right)^{1/(2-N/q)}.
\]

Consequently, taking into account (2.28),

\[\int_{B_k} v_k|g_k(v_k)| \, dx \geq \frac{1}{2} \|v_k\| m_k \omega(R_k)^N,
\]

where \(\omega = \omega_N\) is the volume of the unit ball in \(\mathbb{R}^N\), see Figure 2 (b).

Due to \(B_k \subset \Omega\), substituting inequality (2.34), and rearranging terms, we obtain

\[\int_{\Omega} v_k|g_k(v_k)| \, dx \geq \frac{1}{2} \|v_k\| m_k \omega \left( \frac{\|v_k\|}{2C} \frac{\|v_k\|}{M_k^{1-1/q-1/(2^*-1)}} \right)^{N/(2-N/q)} \]

\[= C m_k \left( \frac{\|v_k\|^{2/N-1/q}}{M_k^{1-1/q-1/(2^*-1)}} \right)^{1/(2-N-1/q)} \]

\[= C m_k \left( \frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-1/q-1/(2^*-1)}} \right)^{1/(2-N-1/q)} \]

\[= C m_k \left( \frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-2/(2^*-1)}} \right)^{1/(2-N-1/q)} \]

At this moment, let us observe that from hypothesis (H1)_k and (H2)_k

\[(2.35)\quad \frac{m_k}{M_k} \geq C, \quad \text{for all} \ k \ \text{big enough.}\]
Hence, taking again into account hypothesis \((H2)_k\), and rearranging exponents, we can assert that
\[
\int_{\Omega} v_k |g_k(v_k)| \, dx \geq C \left( \frac{\|v_k\|^{1+2/N-1/q}}{M_k^{1-2/N-1/((2^*−1)q)}} \right)^{1/(2/N-1/q)}.
\]
Finally, from hypothesis \((F)_k\) we deduce
\[
\int_{\Omega} v_k |g_k(v_k)| \, dx \geq C \left( \frac{\|v_k\|^{1+2/N-1/q}}{g_k(\|v_k\|)} \right)^{1/(2/N-1/q)} \to \infty,
\]
as \(k \to \infty\), which contradicts (2.19).

3. Radial problems with almost critical exponent

In this section, we build an example of a sequence of functions \(\{g_k\}\) growing subcritically, and satisfying the hypotheses \((H1)_k\), \((H2)_k\), and \((F)_k\), such that the corresponding sequence of BVP
\[
\begin{cases}
\Delta w_k + g_k(w_k) = 0 & \text{in } |x| \leq 1, \\
w_k(x) = 0 & \text{for } |x| = 1.
\end{cases}
\]
has an unbounded (in the \(L^\infty(\Omega)\)-norm) sequence \(\{w_k\}\) of positive solutions. As a consequence of Theorem 1.2, this sequence \(\{w_k\}\) is also unbounded in the \(L^{2^*} (\Omega)\)-norm.

Let \(N \geq 3\) be an integer. For each positive integer \(k > 2\) let
\[
g_k(s) = \begin{cases}
0 & \text{for } s < 0, \\
s^{(N+2)/(N−2)} & \text{for } s \in [0, k], \\
k^{(N+2)/(N−2)} & \text{for } s \in [k, k^{(N+2)/(N−2)}], \\
k^{(N+2)/(N−2)} + (s - k^{(N+2)/(N−2)})^{(N+1)/(N−2)} & \text{for all } s > k^{(N+2)/(N−2)}.
\end{cases}
\]
For the sake of simplicity in notation, we write \(g_k := g\).

Let \(u_k := u\) denote the solution to
\[
\begin{cases}
u'' + \frac{N−1}{r} u' + g(u) = 0 & \text{for } r \in (0, 1), \\
u(0) = k^{N/(N−2)} & \text{for } u'(0) = 0.
\end{cases}
\]
Let \( r_1 = \sup \{ r > 0 : u_k(s) \geq k \text{ on } [0, r] \} \). Since \( g \geq 0 \), \( u \) is decreasing, consequently for \( r \in [0, r_1] \), \( k \leq u(r) \leq k^{N/(N-2)} \), and

\[
(3.3) \quad -r^{N-1} u'(r) = \int_0^r s^{N-1} g(u(s)) \, ds
= \int_0^r s^{N-1} k^{(N+2)/(N-2)} \, ds = \frac{k^{(N+2)/(N-2)}}{N} r^N,
\]

so

\[
(3.4) \quad u'(r) = \frac{k^{(N+2)/(N-2)}}{N} r.
\]

Hence

\[
(3.5) \quad u(r) = k^{N/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N} r^2, \quad \text{for } r \in [0, r_1].
\]

Thus, \( u(r) \geq k^{N/(N-2)}/2 \), for all \( 0 \leq r \leq r_0 := \sqrt{N}/k^{1/(N-2)} \), and \( u(r_0) = k^{N/(N-2)}/2 \).

By well established arguments based on the Pohozaev identity, see [5], we have

\[
(3.6) \quad P(r) := r^N E(r) + \frac{N-2}{2} r^{N-1} u(r) u'(r) = \int_0^r s^{N-1} \Gamma(u(s)) \, ds,
\]

where

\[
E(r) = \frac{1}{2} (u'(r))^2 + G(u(r)), \quad \Gamma(s) = N G(s) - \frac{N-2}{2} s g(s), \quad G(s) = \int_0^s g(t) \, dt.
\]

For \( s \in [k, k^{N/(N-2)}] \),

\[
(3.7) \quad \Gamma(s) = -\frac{N+2}{2} k^{2N/(N-2)} + \frac{N+2}{2} k^{(N+2)/(N-2)} \geq 0.
\]

Hence

\[
\Gamma(u(r)) \geq \frac{N+2}{8} k^{(2N+2)/(N-2)} \quad \text{for all } r \leq r_0, \quad k \geq 4^{(N-2)/2}.
\]

Due to \( \Gamma(s) = 0 \) for all \( s \leq k \), (3.6) and (3.7), for \( r \geq r_0 \),

\[
P(r) \geq P(r_0) \geq \frac{N+2}{8N} k^{(2N+2)/(N-2)} r_0^N \geq \frac{N+2}{8} N^{N-2/2} k^{(N+2)/(N-2)}.
\]

Due to (3.7), for \( r \geq r_0 \), we have

\[
P(r) \geq P(r_0) \geq \frac{N+2}{8} N^{(N-2)/2} k^{(N+2)/(N-2)}.
\]

From (3.5) \( u(r_1) = k \) with

\[
r_1 = \left\lceil 2N \left( \frac{1}{k} \right)^{2/(N-2)} - \left( \frac{1}{k} \right)^{4/(N-2)} \right\rceil = \sqrt{2N} \left( \frac{1}{k} \right)^{1/(N-2)} + o \left( \left( \frac{1}{k} \right)^{1/(N-2)} \right).
\]
From the definition of \( g, -u'(r_1) = k^{(N+2)/(N-2)} r_1/N \) (see (3.4)), which implies
\[
P(r_1) \geq r_1^{N+2}O(k^{2(N+2)/(N-2)}) - r_1^{N}O(k^{2N/(N-2)})
\geq O(k^{(N+2)/(N-2)}) - O(k^{N/(N-2)}) \geq O(k^{(N+2)/(N-2)}).
\]
For \( r \geq r_1 \),
\[
(3.8) \quad \frac{N-2}{2} r^{N-1} u(r) u'(r) \geq \frac{(N-2)r^N}{2N} u(r) u(r)^{(N+2)/(N-2)}
= \frac{(N-2)r^N}{2N} u(r)^{2N/(N-2)} = r^N G(u(r)).
\]
This and Pohozaev’s identity imply
\[
[u'(r)]^2 \geq O(k^{(N+2)/(N-2)}) \frac{1}{r^N} \quad \text{or} \quad -u'(r) \geq O(k^{(N+2)/(2(N-2))}) \frac{1}{r^{N/2}}.
\]
Integrating on \([r_1, r]\) we have
\[
u(r) \leq k - O(k^{(N+2)/(2(N-2))}) \left( \frac{1}{r_1^{(N-2)/2}} - \frac{1}{r^{(N-2)/2}} \right),
\]
which implies that there exists \( k_0 \) such that if \( k \geq k_0 \) then \( u(r) = 0 \) for some \( r \in (r_1, 2r_1] \). Since (3.8), \( r_1 = r_1(k) \to 0 \) as \( k \to \infty \).

Let \( v := v_k \) denote the solution to
\[
(3.9) \quad \begin{cases} v'' + \frac{N-1}{r} v' + g(v) = 0, & r \in (0, 1], \\ v(0) = k^{(N+2)/(N-2)}, \quad v'(0) = 0. \end{cases}
\]

Let \( r_1 = \sup\{r > 0 : v_k(s) \geq k \text{ on } [0, r]\} \). For \( v(r) \geq k \), integrating (3.4), we deduce
\[
(3.10) \quad v(r) = k^{(N+2)/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N} r^2, \quad \text{for } r \in [0, r_1],
\]
\[
(3.11) \quad v(r_1) = k^{(N+2)/(N-2)} - \frac{k^{(N+2)/(N-2)}}{2N} r_1^2 = k,
\]
therefore
\[
(3.12) \quad r_1 = \sqrt{2N \left( 1 - \left( \frac{1}{k} \right)^{4/(N-2)} \right)} > 1,
\]
therefore \( v(r) \geq k \) for all \( r \in [0, 1] \). So, by continuous dependence on initial conditions, there exists \( d_k \in (k^{N/(N-2)}, k^{(N+2)/(N-2)}) \) such that the solution \( w = w_k \) to
\[
\begin{cases} w'' + \frac{N-1}{r} w' + g_{k}(w) = 0, & r \in (0, 1], \\ w(0) = d_k, \quad w'(0) = 0. \end{cases}
\]
satisfies \( w(r) \geq 0 \) for all \( r \in [0, 1] \), and \( w(1) = 0 \). Since \( k \) may be taken arbitrarily large, and as a consequence of Theorem 1.2, we have established the following result.
Corollary 3.1. There exists a sequence of functions \(g_k: \mathbb{R} \to \mathbb{R}\) and a sequence \(\{w_k\}\) of positive solutions to (3.1), such that each function \(g_k\) grows subcritically and satisfies the hypotheses \((H1)_k\), \((H2)_k\) and \((F)_k\) of Theorem 1.2, and the sequence \(\{w_k\}\) of positive solutions to (3.1), is unbounded in the \(L^\infty(\Omega)\)-norm. Moreover, this sequence \(\{w_k\}\) is also unbounded in the \(L^2(\Omega)\)-norm.

Let now \(v := v_k\) denote the solution to

\[
\begin{cases}
 v'' + \frac{N - 1}{r} v' + g(v) = 0, & r \in (0, 1],
 v(0) = k, & v'(0) = 0.
\end{cases}
\]

Since \(\Gamma(s) = 0\) for all \(s \leq k\), and the solution is decreasing, by Pohozaev’s identity

\[
r(v'(r))^2 + \frac{N - 2}{4N} r v(r)^{2N/(N-2)} + \frac{N - 2}{2} v(r)v'(r) = 0, \quad \text{for all } r \in [0, 1].
\]
Hence, if \(v(\hat{r}) = 0\) for some \(\hat{r} \in (0, 1]\), then \(v'(\hat{r}) = 0\) and the uniqueness of the solution of the IVP (3.13), implies \(v(r) = 0\) for all \(r \in [0, 1]\). Since this contradicts \(v(0) = k > 0\) we conclude that \(v(r) > 0\) for all \(r \in [0, 1]\). Therefore, by continuous dependence on initial conditions, there exists \(d'_k \in (k, k^{N/(N-2)})\) such that the solution \(z = z_k\) to

\[
\begin{cases}
 z'' + \frac{N - 1}{r} z' + g_k(z) = 0, & r \in (0, 1],
 z(0) = d'_k, & z'(0) = 0.
\end{cases}
\]
satisfies \(z(r) \geq 0\) for all \(r \in [0, 1]\), and \(z(1) = 0\).

Corollary 3.2. For any \(k \in \mathbb{N}\), the BVP (3.1) has at least two positive solutions.

References


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