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# Selected topics in non-Archimedean Banach spaces

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Toruń, 2018

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ISSN 2082–4335
ISBN 977-2082-433-80-9

I edition. 150 copies.

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# Preface

Non-Archimedean functional analysis was started by dutch mathematician Antoine Monna (1909–1995) in the 40s and 50s of the last century. In his pioneering papers, published in Indagationes Mathematicae, Monna developed fundamentals for the theory of Banach spaces over non-Archimedean valued fields. Over the years, the state of knowledge of this discipline was determined by monographs of Monna [38], Bachman, Beckenstein and Narici [6], van Rooij [57], Prolla [50], Bosch, Güntzer and Remmert [7], Robert [55], Schneider [68], Schikhof and Perez-Garcia [60] and [47].

The study of this topic is partially motivated by the Ostrowski's theorem, which asserts that every complete valued field which is not isomorphic (algebraically and topologically) to either $\mathbb{R}$ or $\mathbb{C}$ is non-Archimedean.

Some application of non-Archimedean analysis in mathematical physics and quantum mechanics may be another motivation. According to the Archimedean axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. So, we can measure distances as small as we want. However in quantum mechanics measurements of distances smaller than the Planck constant are impossible. This leads to the search for such geometries that do not satisfy the Archimedean axiom at very small distances. The non-Archimedean geometry is a one of the possible alternatives (see [1], [11], [26] and [70]).

This work collects some recent results concerning a few selected topics. Chapter 1 has an introductory character, it gathers some basic notions and concepts related to the theory of non-Archimedean Banach spaces. Chapter 2, the most extensive, covers several aspects of the existence of orthocomplemented linear subspaces in non-Archimedean Banach spaces. It presents results obtained mainly by A. Kubzdela, C. Perez-Garcia, A. van Rooij and W. Schikhof. Chapter 3 deals with applications, due to J. Kąkol and A. Kubzdela, of measures of noncompactness to study non-Archimedean Banach spaces equipped with the weak topology. Chapter 4 contains some results of W. Schikhof and A. Kubzdela concerning isometric maps and the distance preserving mappings defined on finite-dimensional non-Archimedean normed spaces.

I would like to express my heartfelt thanks to Cristina Perez-Garcia and Jerzy Kąkol. Collaboration with Them has lead to many interesting results, partially presented in Section 2.3 and Chapter 3. I am extremely grateful to Wim Schikhof (1937–2014), having in mind numerous discussions with Wim about non-Archimedean analysis, thanks to which getting some of the results presented in Chapter 2 was possible.

Also, I would like to thank the Reviewers for thorough reading and valuable remarks and comments which improved the text.

# 1 Preliminaries

This chapter, essential to the sequel, presents some basic classical notions and properties related to the theory of non-Archimedean Banach spaces. It is not a comprehensive treatment of the subject, but it shows only these concepts that will be used in next chapters. For more background on normed spaces over non-Archimedean valued fields we refer the reader to the magnificent books [47] and [57], among some others.

## 1.1 Basics

Let $\mathbb{K}$ be a field. A *valuation* defined on $\mathbb{K}$ is a map $|\cdot|\colon \mathbb{K} \to [0,\infty)$ satisfying the following conditions:

$$\begin{gathered} |\lambda| = 0 \quad \text{if and only if} \quad \lambda = 0, \\ |\lambda\mu| = |\lambda| \cdot |\mu|, \\ |\lambda + \mu| \leqslant |\lambda| + |\mu|, \end{gathered}$$

for all $\lambda, \mu \in \mathbb{K}$. The pair $(\mathbb{K}, |\cdot|)$ is said to be a *valued field*. The valuation $|\cdot|$ is called *non-Archimedean*, and $\mathbb{K}$ is called a *non-Archimedean valued field* if, additionally, the valuation satisfies *the strong triangle inequality*:

$$|\lambda + \mu| \leqslant \max\{|\lambda|, |\mu|\} \quad \text{for all } \lambda, \mu \in \mathbb{K}.$$

Let $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$ and $|\mathbb{K}^\times| := |\mathbb{K}| \cap (0,\infty)$. The set $|\mathbb{K}^\times|$, called the *value group of* $\mathbb{K}$, is a subgroup of the multiplicative group of the positive real numbers. $\mathbb{K}$ is said to be *trivially valued* if $|\mathbb{K}^\times| = \{1\}$. A non-trivial valuation is called *discrete* if 0 is the only accumulation point of $|\mathbb{K}^\times|$; otherwise, $|\mathbb{K}^\times|$ is a dense subset of $(0, \infty)$ and the valuation is called *dense*. If $\mathbb{K}$ is discretely valued, then there exists an *uniformizing element* $\rho \in \mathbb{K}$, $|\rho| < 1$ such that $|\mathbb{K}^\times| = \{|\rho|^n : n \in \mathbb{Z}\}$.

Let $B_{\mathbb{K}} := \{\lambda \in \mathbb{K} : |\lambda| \leqslant 1\}$ and $B_{\mathbb{K}}^- := \{\lambda \in \mathbb{K} : |\lambda| < 1\}$. Then $B_{\mathbb{K}}$ is a commutative ring with identity and $B_{\mathbb{K}}^-$ is a maximal ideal of $B_{\mathbb{K}}$. Therefore, $\mathbb{k} := B_{\mathbb{K}}/B_{\mathbb{K}}^-$ is a field, called the *residue class field* of $\mathbb{K}$. A non-Archimedean complete valued field $\mathbb{K}$ is locally compact if and only if it is discretely valued and its residue class field is finite ([47, Theorem 1.2.8]).

By $B_{\mathbb{K},r_n}(\lambda_n) := \{\mu \in \mathbb{K} : |\mu - \lambda_n| \leqslant r_n\}$ we will denote a closed ball in $\mathbb{K}$. We say that a sequence of closed balls $(B_{\mathbb{K},r_n}(\lambda_n))_n$ in $\mathbb{K}$ is *centered* if $B_{\mathbb{K},r_{n+1}}(\lambda_{n+1}) \subset B_{\mathbb{K},r_n}(\lambda_n)$ for every $n \in \mathbb{N}$. A non-Archimedean valued field $\mathbb{K}$ is called *spherically complete* if every centered sequence of closed balls $(B_{\mathbb{K},r_n}(\lambda_n))_n$ in $\mathbb{K}$ has a nonempty intersection. Every complete, non-Archimedean discretely valued field $\mathbb{K}$ (in particular, every locally compact field) is spherically complete, but the converse is not true.

Among all non-Archimedean valued fields, it is worth mentioning two examples. The first one is the *field of $p$-adic numbers* $\mathbb{Q}_p$ (for a given prime number $p$), which is a completion of the field of rational numbers $\mathbb{Q}$ under the metric generated by the $p$-adic valuation. $\mathbb{Q}_p$ is locally compact, thus, discretely valued and spherically complete. The other one is the field $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, which is algebraically closed and non-spherically complete; thus, it is not locally compact. Both valued fields, $\mathbb{Q}_p$ and $\mathbb{C}_p$ are separable (see [47, Examples 1.2.5 and 1.2.11, Definition 1.2.7 and Theorem 1.2.12]).

Let $(X, d)$ be a metric space. Then, $X$ is called an *ultrametric space*, and $d$ is called an *ultrametric* if $d$ satisfies the strong triangle inequality, i.e. $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

A normed linear space $E$ over a non-Archimedean valued field $\mathbb{K}$ is called a *non-Archimedean space* if its norm satisfies *the strong triangle inequality*, i.e. $\|x + y\| \leqslant \max\{\|x\|, \|y\|\}$ for all $x, y \in E$.

Let $\|E\| := \{\|x\| : x \in E\}$ and $\|E^\times\| := \|E\| \cap (0, \infty)$. Let $X$ be a subset of a linear space $E$. Then, $[X]$ means a linear span of $X$ in $E$. For $X = \{x_1, \dots, x_m\}$ we will write shortly $[x_1, \dots, x_m]$ instead of $[\{x_1, \dots, x_m\}]$.

Throughout, $\mathbb{K}$ will denote a non-Archimedean valued field which is complete with the metric induced by a non-trivial valuation and $E$ will denote a non-Archimedean linear space (over $\mathbb{K}$). In addition, unless otherwise declared, we will assume $|\mathbb{K}^\times| \subset \|E^\times\|$ (i.e. there exists $x \in E$ such that $\|x\| = 1$).

Note that, there exist normed linear spaces over $\mathbb{K}$ which are not non-Archimedean, even those that have no equivalent non-Archimedean norm, e.g. $l^p(\mathbb{K})$, $p \geqslant 1$, the linear space of sequences $(x_n)_n$ in $\mathbb{K}$ such that $\sum\limits_n |x_n|^p < \infty$ equipped with the norm

$$\|x\|_p = \left( \sum_n |x_n|^p \right)^{1/p}.$$

The set $B_{E,r}(x) := \{y \in E : \|x - y\| \leqslant r\}$ ($r > 0$, $x \in E$) is called a *closed ball* in $E$ and the set $B^-_{E,r}(x) := \{y \in E : \|x - y\| < r\}$ ($r > 0$, $x \in E$) is called an *open ball* in $E$, respectively. Note that both balls are closed and open (clopen). The topology induced on $E$ by a non-Archimedean norm is always zero-dimensional. It follows directly from the strong triangle inequality that every point of any ball is its center and any two balls in $E$ are either disjoint, or one is contained in the other. We will write shortly $B_{E,r}$ ($B^-_{E,r}$) instead of $B_{E,r}(0)$ ($B^-_{E,r}(0)$) and $B_E$ ($B^-_E$) instead of $B_{E,1}$ ($B^-_{E,1}$).

Simple consequences of the strong triangle inequality are the following lemmas.

**1.1.1. Lemma.** *Let* $x, y \in E$. *Then,*

$$\|x\| \neq \|y\| \implies \|x + y\| = \max\{\|x\|, \|y\|\}.$$

*Proof.* If $\|x\| < \|y\|$ then $\|y\| = \|x+y-x\| \leqslant \max\{\|x + y\|, \|x\|\}$. Hence, $\|y\| \leqslant \|x + y\|$. On the other hand, $\|x + y\| \leqslant \max\{\|x\|, \|y\|\} = \|y\|$ and we are done. □

**1.1.2. Lemma.** $B_{E,r_1} + B_{E,r_2} = B_{E,\max\{r_1,r_2\}}$ *for each* $r_1, r_2 > 0$.

*Proof.* Let $x \in B_{E,r_1}$ and $y \in B_{E,r_2}$. Then, $\|x+y\| \leqslant \max\{\|x\|, \|y\|\} \leqslant \max\{r_1, r_2\}$. If $z \in B_{E,\max\{r_1,r_2\}}$ then, assuming $r_1 \leqslant r_2$, we imply that $z \in B_{E,r_2}$. □

The concept of orthogonal sets is the one of the most important tools to study structural properties of non-Archimedean normed spaces. Let $t \in (0,1]$. For any nonempty set (not necessary countable) $I$, the set $\{x_i\}_{i\in I} \subset E$, $x_i \neq 0$, is called $t$-*orthogonal* (*orthogonal* for $t=1$) if

$$\left\| \sum_{j\in J} \lambda_j x_j \right\| \geqslant t \cdot \max_{j\in J} \left\{ \|\lambda_j x_j\| \right\}$$

for every finite subset $J \subset I$ and all $\lambda_j \subset \mathbb{K}$ $(j \in J)$. If, additionally $\overline{[\{x_i\}_{i\in I}]} = E$ (i.e. the closure of the linear space $[\{x_i\}_{i\in I}]$ spanned by $\{x_i\}_{i\in I}$ is equal to $E$), then $\{x_i\}_{i\in I}$ is said to be a $t$-*orthogonal base* of $E$. Then, every $x \in E$ has an unequivocal expansion

$$x = \sum_{i\in I} \lambda_i x_i \quad (\lambda_i \in \mathbb{K},\ i \in I).$$

We will say that a sequence $(x_n)_n$ is t-orthogonal if the set $\{x_1, x_2, \ldots\}$ is a t-orthogonal set. An orthogonal subset $X$ of $E$ is said to be *maximal*, if for every $z \in E$, $z \neq 0$, the set $\{z\} \cup X$ is not orthogonal. Every orthogonal set can be extended to a maximal orthogonal set. Clearly, every orthogonal base is a maximal orthogonal set, but the converse is not true, see [57, 5.B]. Any two maximal orthogonal sets in a given $E$ have the same cardinality ([57, Theorem 5.2]).

It is worthwhile to remark that perturbing the elements of an orthogonal set a little does not disturb orthogonality.

**1.1.3. Theorem** (see [47, Theorem 2.2.9]). *Let* $\{x_i : i \in I\}$ *be an orthogonal set in* $E$. *If* $Y_0 = \{y_i : i \in I\}$ *is a subset of* $E$ *such that* $\|x_i - y_i\| < \|x_i\|$ *for each* $i \in I$, *then* $Y_0$ *is an orthogonal set, either.*

**1.1.4. Theorem** (Gruson, see [57, Theorem 5.9]). *Let* $E$ *be a non-Archimedean Banach space with an orthogonal base. Then every closed linear subspace of* $E$ *has an orthogonal base, either.*

We say that $E$ is of *countable type* if it contains a countable set whose linear span is dense in $E$. If $\mathbb{K}$ is separable, then $E$ is of countable type if and only if it is separable.

Recall that a sequence of closed balls $(B_{E,r_n}(x_n))_n$ in $E$ is called *centered* if $B_{E,r_{n+1}}(x_{n+1}) \subset B_{E,r_n}(x_n)$ for each $n \in \mathbb{N}$. A normed linear space $E$ is called *spherically complete* if every centered sequence of closed balls in $E$ has a nonempty intersection. Every spherically complete normed linear space $E$ is complete, but the converse is not true. If $\mathbb{K}$ is spherically complete, then every finite-dimensional normed linear space over such $\mathbb{K}$ is spherically complete, either (see [57, Corollary 4.6]).

**1.1.5. Theorem** ([47, Theorems 2.3.7 and 2.3.25]). *Every non-Archimedean normed space of countable type has a $t$-orthogonal base for each $t \in (0,1)$. If $\mathbb{K}$ is spherically complete, then every non-Archimedean normed space of countable type has an orthogonal base. Every non-Archimedean normed space contains a $t$-orthogonal sequence for each $t \in (0,1)$.*

*Proof.* Let $E$ be an infinite-dimensional non-Archimedean normed space of countable type (for finite-dimensional $E$ the inductive construction below breaks off). Find $X = \{x_1, x_2, \ldots\}$, a subset of $E$ consisting of linearly independent nonzero elements such that $[X]$ is dense in $E$. Set $F_n := [x_1, \ldots, x_n]$, $n \in \mathbb{N}$. Let $t \in (0,1)$. Then, set $e_1 := x_1$ and select $t_2, t_3, \ldots \in (0,1)$ such that $\prod_{n=2}^{\infty} t_n \geqslant t$. For every $n \in \mathbb{N}$, $\mathrm{dist}(x_{n+1}, F_n) > 0$ since $F_n$ is closed and $x_{n+1} \notin F_n$. Hence, we can find $z_n \in F_n$ for which

$$t_{n+1} \cdot \|x_{n+1} - z_n\| \leqslant \mathrm{dist}(x_{n+1}, F_n).$$

Now, we set $e_{n+1} := x_{n+1} - z_n$. Then, clearly $F_n = [e_1, \ldots, e_n]$, so $[e_1, \ldots, e_n, \ldots]$ is dense in $E$, and

$$\mathrm{dist}(e_{n+1}, F_n) = \mathrm{dist}(x_{n+1}, F_n) \geqslant t_{n+1} \cdot \|x_{n+1} - z_n\| = t_{n+1} \cdot \|e_{n+1}\|.$$

So, by [47, Theorem 2.2.16], $\{e_1, e_2, \ldots\}$ is $t$-orthogonal and by [47, Theorem 2.3.6] it is a $t$-orthogonal base of $E$.

If $\mathbb{K}$ is spherically complete then, since $F_n$ is spherically complete for each $n \in \mathbb{N}$ by [57, Corollary 4.6], we can find $z_n \in F_n$ for which $\|x_{n+1} - z_n\| = \mathtt{dist}(x_{n+1}, F_n)$. Indeed, fix $n \in \mathbb{N}$. For every $r > \mathtt{dist}(x_{n+1}, F_n)$, $V_r := B_r(x_{n+1}) \cap F_n$ is a ball in $F_n$. Thus, $W_n := \bigcap_{r > \mathtt{dist}(x_{n+1}, F_n)} V_r$ is nonempty. Hence, there exists $z_n \in W_n \subset F_n$ such that $\|x_{n+1} - z_n\| \leqslant \inf\{r > \mathtt{dist}(x_{n+1}, F_n)$. Clearly, $\|x_{n+1} - z_n\| \geqslant \mathtt{dist}(x_{n+1}, F_n)$. Thus, $\|x_{n+1} - z_n\| = \mathtt{dist}(x_{n+1}, F_n)$.

If $E$ is a non-Archimedean normed space, then it contains a linear subspace of countable type, hence, by above, it contains a t-orthogonal sequence for each $t \in (0, 1)$. □

**1.1.6. Remark.** If $\mathbb{K}$ is non-spherically complete, then there are examples of non-Archimedean normed spaces of countable type without an orthogonal base, see Remark 1.2.13 and [47, Example 2.3.26 and Remark 2.3.27].

Let $D_1$, $D_2$ be closed linear subspaces of $E$. $D_1$ and $D_2$ are called t-*orthogonal* (relative to each other) if

$$\|x + y\| \geqslant t \cdot \max\{\|x\|, \|y\|\}$$

for all $x \in D_1$ and $y \in D_2$. If the above inequality holds for $t = 1$, we will say that $D_1$ and $D_2$ are *orthogonal*; then, we will write $D_1 \perp D_2$. In particular, if $D_1 = [x]$ for some nonzero $x \in E$ we will write $x \perp D_2$.

$D_1$ is said to be a t-*orthocomplement* (an *orthocomplement* for $t = 1$) of $D_2$ ($D_1$ and $D_2$ are t-*orthocomplemented* in $E$ ) if $D_1$ and $D_2$ are t-orthogonal and $E = D_1 + D_2$. Observe that if $D_1$ and $D_2$ are t-*orthocomplemented*, the sum $D_1 + D_2$ is direct.

An *operator* $T$ of $E$ to a normed linear space $F$ is a linear map $T\colon E \to F$. If $F$ is a Banach space, the set $L(E, F)$ of all bounded operators $E \to F$ is a non-Archimedean Banach space with the norm

$$\|T\| := \inf\{M > 0 : \|Tx\| \leqslant M \cdot \|x\| \text{ for all } x \in E\}.$$

We will say that $E$ is *isomorphic* to $F$ if there exists a bijective linear homeomorphism $T\colon E \to F$. If, additionally $T$ is isometric (i.e. $\|Tx\| =$

$\|x\|$ for every $x \in E$), we will write $E \simeq F$. A bounded operator $P\colon E \to E$ is called a *projection* if $P^2 = P$.

We can easily deduce that a linear subspace $D$ of $E$ is orthocomplemented in $E$ if and only if there exists a surjective projection $P\colon E \to D$ with $\|P\| \leqslant 1$ (called an *orthoprojection*).

**1.1.7. Proposition** (see [47, Lemma 2.3.20] and [57, Lemma 4.35])**.** *If every one-dimensional linear subspace of $E$ is orthocomplemented in $E$, then, every finite-dimensional linear subspace of $E$ is orthocomplemented in $E$.*

*Proof.* Let $D, D_0$ be linear subspaces of $E$ such that $D_0 \subset D$, $D_0$ has the codimension 1 in $D$ and $D_0$ is orthocomplemented in $E$. The proof will be complete if we show that $D$ is orthocomplemented in $E$. Since $D_0$ is orthocomplemented, there is an orthoprojection $P_0\colon E \to D_0$. Choose $x \in D$, $x \neq 0$, such that $P_0(x) = 0$ (then $D = D_0 + [x]$). Then, by assumption, there is an orthoprojection $P_x\colon E \to [x]$. But then $P = P_0 + P_x - P_x \circ P_0$ is a required orthoprojection $E \to D$. □

Let $E^* := L(E, \mathbb{K})$ and $E^{**} := L(E^*, \mathbb{K})$ be the *topological dual* and *bidual* of $E$, respectively. For $x \in E$ and $z^* \in E^*$ the formula $j_E(x)(z^*) := z^*(x)$ defines the *evaluation map* $j_E\colon E \to E^{**}$. In general, $\|j_E(x)\|_{E^{**}} \leqslant \|x\|_E$; thus $j_E$ is continuous linear map and, in fact, $\|j_E\| \leqslant 1$. But $j_E$ need not be isometric (for non-spherically complete $\mathbb{K}$ we can construct an infinite-dimensional Banach space $E$ for which $E^{**} = \{0\}$; then, clearly $j_E$ cannot be isometric). Considering the case when $j_E$ is an isometric embedding, using the natural identification, we will usually identify $E$ with $j_E(E) \subset E^{**}$ and for $x \in E$ we will write $x \in E^{**}$ instead of $j_E(x) \in E^{**}$. Recall that a non-Archimedean Banach space $E$ is *reflexive* if $j_E$ is a surjective isometry.

As usual, we define the weak topology and the weak star topology. The *weak topology* $\sigma(E, E^*)$ on $E$ is defined to be the weakest topology (that is, the topology with the fewest open sets) under which each element of $E^*$ remains continuous on $(E, \sigma(E, E^*))$. A base of zero-neigborhoods for the weak topology $\sigma(E, E^*)$ consists of sets of the form $\{x : |x^*(x)| < \varepsilon,\ x^* \in S\}$, where $\varepsilon > 0$ and $S$ is a finite subset of $E^*$. The *weak star topology* $\sigma(E^*, E)$ on $E^*$ is the weak topology on $E^*$ induced by the image of $j_E(E) \subset E^{**}$. We say that $E$ is *weakly sequentially complete* if every weakly Cauchy sequence in $E$ is weakly convergent in $E$.

Let $T \in L(E, F)$. Recall that the *adjoint* of $T$ is the linear map $T^*: F^* \to E^*$, $f \mapsto f \circ T$.

Let $A \subset E$ be a set. We define the *polar* of $A$ as the set $A^0 := \{f \in E^* : |f(x)| \leqslant 1 \text{ for all } x \in A\}$ and the *bipolar* of $A$ as $A^{00} := \{x \in E : |f(x)| \leqslant 1$ for all $f \in A^0\}$. The set $A$ is called *polar set* if $A = A^{00}$.

We say that $E$ is *normpolar* if $j_E$ is a homeomorphism of $E$ onto its image; then, for every finite-dimensional linear subspace $F \subset E$, every $\varepsilon > 0$ and every $f \in F^*$ there is an extension $f_0 \in E^*$ for which $\|f_0\| \leqslant (1 + \varepsilon)\|f\|$. $E$ is normpolar if and only if $B_E$ is a polar set, see [47, Corollary 4.4.11]. In this context, we recall the following facts.

**1.1.8. Proposition** ([57, Exercise 3.Q])**.** *All finite dimensional non-Archimedean normed spaces over any $\mathbb{K}$ are reflexive.*

**1.1.9. Proposition** ([57, Theorem 4.16])**.** *If $\mathbb{K}$ is spherically complete, then no infinite-dimensional normed space over $\mathbb{K}$ is reflexive.*

**1.1.10. Proposition** ([57, Corollary 4.18 ])**.** *If $\mathbb{K}$ is non-spherically complete, then every non-Archimedean Banach space of countable type over $\mathbb{K}$ is reflexive.*

We say that a non-Archimedean Banach space $F$ is *injective* if, for any $E$, every bounded operator from a linear subspace $D$ of $E$ into $F$ has a preserving norm, linear extension on the whole of $E$. Ingleton's theorem (see [18, Theorem 4.2] or [57, Theorem 4.10]) characterizes injective spaces as follows.

**1.1.11. Theorem** (Ingleton)**.** *A non-Archimedean Banach space $F$ is injective if and only if it is spherically complete.*

*Proof.* ($\Leftarrow$) Assume that $F$ is spherically complete. Applying Zorn's Lemma, it is enough to prove that if $D \subset E$ is a linear subspace of $E$, $T_0: D \to F$ is any bounded operator with the norm $\|T_0\|$, then for every

$y \in E \setminus D$ we can find a linear, preserving norm, extension of $T_0$ on $D + [y]$.

Take $y \in E \setminus D$. Consider the collection $\mathcal{F}$ of all closed balls of the form $\{B_{F,r_x}(T_0(x))\}_{x \in D}$, where $r_x = \|T_0\| \cdot \|y - x\|$. Let $B_{F,r_1}(T_0(x_1))$, $B_{F,r_2}(T_0(x_2))$ be any two elements of $\mathcal{F}$. Then

$$\begin{aligned}\|T_0(x_1) - T_0(x_2)\| &\leqslant \|T_0\| \cdot \|x_1 - x_2\| = \|T_0\| \cdot \|x_1 - y + y - x_2\| \\ &\leqslant \|T_0\| \cdot \max\{\|x_1 - y\|, \|x_2 - y\|\} = \max\{r_1, r_2\}.\end{aligned}$$

Thus, $T_0(x_1) \in B_{F,r_2}(T_0(x_2))$ or $T_0(x_2) \in B_{F,r_1}(T_0(x_1))$. Consequently, the collection $\mathcal{F}$ has the binary intersection property. But $F$ is spherically complete, thus, there exists

$$u_0 \in \bigcap_{x \in D} B_{F,r_x}(T_0(x)).$$

Now, define the operator $T\colon D + [y] \to F$, setting $T(x + \lambda y) := T_0(x) + \lambda u_0$, where $x \in D$, $\lambda \in \mathbb{K}$. Clearly, $T$ extends $T_0$. Let $\lambda \neq 0$. Then, since $u_0 \in B_{F,r}(-T_0(x)/\lambda)$, where $r = \|T_0\| \cdot \| -x/\lambda - y\|$, we get

$$\|T(x+\lambda y)\| = |\lambda| \cdot \left\| \frac{1}{\lambda}\, T_0(x) + u_0 \right\| \leqslant |\lambda| \cdot \|T_0\| \cdot \left\| -\frac{1}{\lambda}\, x - y \right\| = \|T_0\| \cdot \|x + \lambda y\|.$$

Hence,

$$\frac{\|T(x + \lambda y)\|}{\|x + \lambda y\|} \leqslant \|T_0\|$$

for all $x + \lambda y \in D + [y]$, $x + \lambda y \neq 0$. This shows $\|T\| = \|T_0\|$.

($\Rightarrow$) Assume for a contradiction that there exists a centered sequence of closed balls $(B_{F,r_n}(u_n))_n$ with an empty intersection. Then, for every $u \in F$ there exists $n_0 \in \mathbb{N}$ such that $u \notin B_{F,r_{n_0}}(u_{n_0})$. Hence, for any $m > n_0$ one gets

$$\|u - u_m\| = \|u - u_{n_0} + u_{n_0} - u_m\| = \|u - u_{n_0}\|;$$

thus, for every $u \in F$, $\lim\limits_{n\to\infty} \|u - u_n\|$ exists. Therefore, one can define unequivocally the function $\phi\colon F \to (0, \infty)$ by setting

$$\phi(u) := \lim_{n\to\infty} \|u - u_n\|.$$

Let $E := F \times \mathbb{K}$ be a normed space with the norm defined by

$$\|(x, \mu)\| = \begin{cases} |\mu| \cdot \phi\left(\dfrac{1}{\mu}\right) & \text{if } \mu \neq 0, \\ \|x\| & \text{if } \mu = 0. \end{cases}$$

Let us check the norm conditions:

(a) $\|\lambda z\| = |\lambda| \cdot \|z\|$ for all $\lambda \in \mathbb{K}$, $z \in E$: If $\lambda = 0$ then $\|\lambda z\| = |\lambda| = 0$ and we are done. Take $\lambda \neq 0$. Let $z = (x, \mu) \in E$. Then $\lambda z = (\lambda x, \lambda\mu)$. If $\lambda\mu = 0$, then $\mu = 0$ and $\|\lambda z\| = \|\lambda x\| = |\lambda| \cdot \|x\| = |\lambda| \cdot \|z\|$. If $\lambda\mu \neq 0$ then $\mu \neq 0$, and

$$\|\lambda z\| = |\lambda\mu| \cdot \phi\left(\frac{1}{\lambda\mu}\lambda x\right) = |\lambda| \cdot |\mu| \cdot \phi\left(\frac{1}{\mu}x\right) = |\lambda| \cdot \|z\|;$$

(b) $\|z_1 + z_2\| \leqslant \max\{\|z_1\|, \|z_2\|\}$ for all $z_1, z_2 \in E$: Take $z_1 = (x_1, \lambda_1)$ and $z_2 = (x_2, \lambda_2)$, elements of $E$. Since the sequence $(B_{F,r_n}(u_n))_n$ has an empty intersection, there exists $u_m$ for which

$$\|z_1\| = \|x_1 - \lambda_1 u_m\|, \ \|z_2\| = \|x_2 - \lambda_2 u_m\|$$

and

$$\|z_1 + z_2\| = \|(x_1 + x_2) - (\lambda_1 + \lambda_2)u_m\|$$

(if $\lambda = 0$, then $\|(x, \lambda)\| = \|x\| = \|x - \lambda u_n\|$ for all $n \in \mathbb{N}$). Thus,

$$\begin{aligned} \|z_1 + z_2\| &= \|(x_1 + x_2) - (\lambda_1 + \lambda_2)u_m\| \\ &\leqslant \max\{\|x_1 - \lambda_1 u_m\|, \|x_2 - \lambda_2 u_m\|\} = \max\{\|z_1\|, \|z_2\|\}. \end{aligned}$$

Now, let $D := F \times \{0\}$ be a linear subspace of $E$. Consider the operator $i\colon D \to F$ defined as $i(x, 0) := x$. Clearly, $\|i\| = 1$. Suppose, by a way of contradiction, that $i$ can be extended to a preserving norm linear operator $j\colon E \to F$. Let $j(0, -1) = x_0$. Then, for any nonzero $x \in F$,

$$j(x, 1) = j((x, 0) - (0, -1)) = x - x_0.$$

Hence,

$$\|x - x_0\| \leqslant \|j\| \cdot \|(x, 1)\| = \|(x, 1)\| = \phi(x).$$

In particular, for every $n \in \mathbb{N}$, we obtain $\|u_n - x_0\| = \phi(u_n) = r_n$ and conclude that $x_0 \in \bigcap\limits_{n=1}^{\infty} B_{F,r_n}(u_n)$, a contradiction. $\square$

As a simple consequence of Theorem 1.1.11 we obtain the following Hahn–Banach type theorem for linear functionals.

**1.1.12. Theorem.** *If $\mathbb{K}$ is spherically complete, then for every linear subspace $D$ of $E$ and every $f \in D^*$ there exists an extension $f_0 \in E^*$ such that $\|f\| = \|f_0\|$.*

Next result, due to van Rooij (see [56, Theorem 5.1 ]) extends Theorem 1.1.12.

**1.1.13. Theorem** (van Rooij). *Suppose there exists an infinite-dimensional Banach space $E$ with the following property: for every closed linear subspace $D$ of $E$ which is of countable type and every $f \in D^*$ there is an extension $f_0 \in E^*$ with $\|f\| = \|f_0\|$. Then, $\mathbb{K}$ is spherically complete.*

*Proof.* Assume the contrary and suppose that $\mathbb{K}$ is non-spherically complete. Then, there exists a centered sequence of closed balls $(B_{\mathbb{K},r_n}(\alpha_n))_n$ with an empty intersection. We can assume that $r_i > r_{i+1}$ for each $i \in \mathbb{N}$. Take $a \in E$, $a \neq 0$, and define the linear functional $f\colon [a] \to \mathbb{K}$ by $f(\lambda a) := \lambda$, $\lambda \in \mathbb{K}$.

Next, extend $f$ to $\overline{f} \in E^*$ with $\|\overline{f}\| = \|f\| = 1/\|a\|$. Let $j\colon \mathbb{K} \to [a]$ be the isomorphism defined by $j(\lambda) := \lambda \cdot a$. Then, $P = j \circ \overline{f}\colon E \to [a]$ is an orthoprojection. Thus, we deduce that every one-dimensional linear subspace of $E$ is orthocomplemented in $E$. Thus, by Proposition 1.1.7, every finite-dimensional linear subspace of $E$ is orthocomplemented in $E$. Applying this fact, we can choose inductively an infinite sequence $(e_n)_n$ of non-zero elements of $E$ such that $e_n \perp \sum\limits_{i<n} [e_i]$ for every $n \in \mathbb{N}$. $\mathbb{K}$, as non-spherically complete, is densely valued; thus, without loss of generality we can assume that $r_{n+1} < \|e_{n+1}\| < r_n$ $(n \in \mathbb{N})$. Now, we form a sequence $(d_n)_n$ setting $d_n := e_n - e_{n+1}$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we obtain

$$\|d_n\| = \|e_n - e_{n+1}\| = \max\{\|e_n\|, \|e_{n+1}\|\} = \|e_n\|.$$

Let us check that $\{d_1, d_2, \ldots\}$ is orthogonal. To do it, take $n_0 \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_{n_0} \in \mathbb{K}$.

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Then

$$\left\|\sum_{i=1}^{n_0}(\lambda_i d_i - \lambda_i e_i)\right\| = \left\|\sum_{i=1}^{n_0}\lambda_i e_{i+1}\right\| \leqslant \max_{i=1,\dots,n_0}\|\lambda_i e_{i+1}\|$$
$$< \max_{i=1,\dots,n_0}\|\lambda_i e_i\| = \left\|\sum_{i=1}^{n_0}\lambda_i e_i\right\|.$$

Hence,

$$\left\|\sum_{i=1}^{n_0}\lambda_i d_i\right\| = \left\|\sum_{i=1}^{n_0}(\lambda_i d_i - \lambda_i e_i) + \sum_{i=1}^{n_0}\lambda_i e_i\right\|$$
$$= \left\|\sum_{i=1}^{n_0}\lambda_i e_i\right\| = \max_{i=1,\dots,n_0}\|\lambda_i e_i\| = \max_{i=1,\dots,n_0}\|\lambda_i d_i\|.$$

Let $D := [d_1, d_2, \dots]$ be a linear subspace of $E$. There exists an unique linear functional $f\colon D \to \mathbb{K}$ with $f(d_n) = \alpha_n - \alpha_{n+1}$, $n \in \mathbb{N}$. Having in mind that $\{d_1, d_2, \dots\}$ is orthogonal, applying inequalities

$$|\alpha_n - \alpha_{n+1}| \leqslant r_n < \|e_n\| = \|d_n\|,$$

we imply $\|f\|, \leqslant 1$. Then, by assumption, there exists an extension $\bar{f} \in E^*$ of $f$ with $\|\bar{f}\| \leqslant 1$. Set $\alpha := \alpha_1 - \bar{f}(e_1)$. For each $n \in \mathbb{N}$ we have $-e_{n+1} = -e_1 + d_1 + d_2 + \ldots + d_n$. Thus,

$$-\bar{f}(e_{n+1}) = -\bar{f}(e_1) + (\alpha_1 - \alpha_2) + \ldots + (\alpha_n - \alpha_{n+1}) = \alpha - \alpha_{n+1}.$$

Therefore, $|\alpha - \alpha_{n+1}| \leqslant \|e_{n+1}\| < r_n$ and

$$|\alpha - \alpha_n| \leqslant \max\{|\alpha - \alpha_{n+1}|, |\alpha_{n+1} - \alpha_n|\} \leqslant r_n$$

for every $n \in \mathbb{N}$. Hence, $\alpha \in \bigcap_n B_{\mathbb{K},r_n}(\alpha_n)$, a contradiction. □

Let us recall one more fact related to this topic (note that we will not assume that every $f \in D^*$ has a preserving norm linear extension).

**1.1.14. Theorem** (see [56, Theorem 5.2])**.** *Let $E$ be an infinite-dimensional Banach space with the following property: for every closed linear subspace of countable type $D \subset E$, every $f \in D^*$ has an extension $\widehat{f} \in E^*$. Then,*

(1) *every closed linear subspace of* $E$ *which is of countable type is weakly closed, i.e. closed with respect to the weak topology* $\sigma(E, E^*)$;

(2) $E$ *has the Schur property, i.e. every weakly convergent sequence in* $E$ *is convergent*;

(3) *every weakly compact set in* $E$ *is compact*;

(4) $E$ *is weakly sequentially complete.*

*Proof.* (1) Let $D_0 \subset E$ be a closed linear subspace of countable type. Take any $x_0 \in E \setminus D_0$. Then, there exists a continuous linear functional $f \colon D_0 + [x_0] \to \mathbb{K}$ such that $D_0 \subset \ker f$ and $f(x_0) = 1$. By assumption, $f$ can be extended to $\widehat{f} \in E^*$. Hence, $D_0 \subset \ker \widehat{f}$ but $x_0 \notin \ker \widehat{f}$ and we conclude that $D_0$ is weakly closed.

(2) Assume for a contradiction that there exists a sequence $(x_n)_n \subset E$ weakly convergent to zero, which contains a subsequence $(x_{n_k})_k$ such that $\inf_k \|x_{n_k}\| > \varepsilon$ for some $\varepsilon > 0$. Let $D_0 := \overline{[x_1, x_2, \dots]}$. Since, by assumption, $\sigma(E, E^*)|_{D_0} = \sigma(D_0, D_0^*)$, applying [47, Corollary 2.3.9], without loss of generality, we can assume that $E = c_0$. Write $x_n = (x_n^1, x_n^2, \dots)$, $n \in \mathbb{N}$. For each $k \in \mathbb{N}$ the set $J_k := \{m : |x_{n_k}^m| > \varepsilon\}$ is nonempty and finite. Since $(x_n)_n$, as weakly convergent, tends to 0 coordinatewise, we can find a subsequence $(k_n)_n$ for which the sets $J_{k_n}$ $(n \in \mathbb{N})$ are pairwise disjoint. Now, select a sequence $(m_n)_n \subset \mathbb{N}$ such that $m_n \in J_{k_n}$ for each $n \in \mathbb{N}$ and define $f \in E^*$ setting

$$f((z^1, z^2, \dots)) := \sum_{n=1}^{\infty} z^{m_n}, \quad (z^1, z^2, \dots) \in E.$$

But then $|f(x_{n_k})| > \varepsilon$ for every $k \in \mathbb{N}$, a contradiction.

(3) Let $M$ be a weakly compact subset of $E$ and let $(x_n)_n$ be any sequence contained in $M$. By p-adic Eberlein–Šmulian theorem, which works in this context (see [27, Corollary 2.2 and Theorem 2.3]), $(x_n)_n$ contains a subsequence $(x_{n_k})_k$ which is weakly convergent to some $x_0 \in M$. But, by (2), $(x_{n_k})_k$ converges to $x$ in norm as well. Therefore $M$ is compact for the norm topology.

(4) Let $(x_n)_n$ be a weakly Cauchy sequence in $E$. Then, the sequence $(z_n)_n$, where $z_n := x_n - x_{n+1}$ $(n \in \mathbb{N})$ tends weakly to zero. Thus, by (2) it tends to 0 in the norm topology. Hence, $(x_n)_n$ is norm-Cauchy, thus, norm-convergent, therefore, weakly convergent. $\square$

Theorems 1.1.12 and 1.1.14 imply the following conclusion.

**1.1.15. Corollary.** *Every non-Archimedean normed space over spherically complete $\mathbb{K}$ has the Schur property.*

## 1.2 Immediate extensions

Let $D$ be a closed linear subspace of $E$ and let $x \in E \setminus D$. We say that the distance $\mathrm{dist}(x, D) := \inf_{d \in D} \|x - d\|$ is *attained* if there exists $d_0 \in D$ such that $\mathrm{dist}(x, D) = \|x - d_0\|$; otherwise, we will say that the distance $\mathrm{dist}(x, D)$ is *not attained*. Let $D$ and $E_0$ be linear subspaces of $E$. We will say that $E_0$ is an *immediate extension* of $D$ if $D \subsetneq E_0$ and there is no nonzero element of $E_0$ that is orthogonal to $D$. An immediate extension $E_0$ of $D$ is said to be *maximal* in $E$, if there is no linear subspace $G \subset E$ such that $E_0 \subsetneq G$ and $G$ is an immediate extension of $D$.

It turns out that immediate extensions of linear subspaces and their properties are powerful tools to solve problems considered in Chapter 2. Therefore we pay more attention to them.

We start this section with a few simple observations.

**1.2.1. Lemma.** *Let $x, y$ be nonzero elements of $E$. Then, a two-dimensional linear subspace $[x, y]$ of $E$ has an orthogonal base if and only if $\mathrm{dist}(x, [y])$ is attained.*

*Proof.* Assume that $\mathrm{dist}(x, [y])$ is attained. Then there exists a $\lambda \in \mathbb{K}$ such that $\mathrm{dist}(x, [y]) = \|x - \lambda y\|$. But then, we can easily check that $\{x, x - \lambda y\}$ is an orthogonal base of $[x, y]$. The converse is obvious. □

**1.2.2. Lemma.** *Let $D$ be a linear subspace of $E$. Then $E$ is an immediate extension of $D$ if and only if $\mathrm{dist}(x, D)$ is not attained for every $x$ in $E \setminus D$.*

*Proof.* Assume that there is $x_0 \in E \setminus D$ such that $\|x_0 + d_0\| = \mathrm{dist}(x, D)$ for some $d_0 \in D$. Then, $\|x_0 + d_0\| \leqslant \|(x_0 + d_0) + d\|$ for all $d \in D$; hence, $(x_0 + d_0) \perp D$, a contradiction. Conversely, if $\mathrm{dist}(x, D)$ is not attained, then for every $x \in E \setminus D$, there is no nonzero element of $E$ that is orthogonal to $D$. □

**1.2.3. Proposition.** *If* $D$ *is a spherically complete linear subspace of* $E$*, then for every* $x \in E \setminus D$, $\mathrm{dist}(x, D)$ *is attained. Thus,* $D$ *has no proper immediate extension in* $E$.

*Proof.* Fix $x \in E \setminus D$. Then, for every $r > \mathrm{dist}(x, D)$ we can find $y_r \in B_{E,r}(x) \cap D$. Thus, as $D$ is a spherically complete,

$$V := \bigcap_{r > \mathrm{dist}(x,D)} B_{D,r}(y_r) \neq \emptyset.$$

Let $y_0 \in V$. Then, $\|y_0 - x\| \leqslant \inf\{r : r > \mathrm{dist}(x, D)\}$ and $\|y_0 - x\| \geqslant \mathrm{dist}(x, D)$. Thus, we finally conclude $\|y_0 - x\| = \mathrm{dist}(x, D)$. □

If $\mathbb{K}$ is non-spherically complete, then we can construct a two-dimensional normed space without two non-zero orthogonal elements, thus, being an immediate extension of its every one-dimensional linear subspace. The construction relies heavily on the existence in $\mathbb{K}$, a centered sequence of closed balls with an empty intersection.

**1.2.4. Example.** (see [57, p. 68] and [47, Example 2.3.26]) Let $\mathbb{K}$ be non-spherically complete an let $(B_{\mathbb{K},r_n}(c_n))_n$ be a centered sequence of closed balls with an empty intersection such that $r_{n+1} < r_n$ $(n \in \mathbb{N})$. Then, for any $\lambda \in \mathbb{K}$ there exists $n_0 \in \mathbb{N}$ such that $\lambda \in B_{\mathbb{K},r_n}(c_{n_0}) \setminus B_{\mathbb{K},r_{n_0+1}}(c_{n_0+1})$. Hence, if $n > n_0 + 1$ then

$$|\lambda - c_n| = |\lambda - c_{n_0+1} + c_{n_0+1} - c_n| = |\lambda - c_{n_0+1}|.$$

Thus, $\lim\limits_{n\to\infty} |\lambda - c_n| = |\lambda - c_{n_0+1}|$. Therefore, the formula

$$\|(x_1, x_2)\|_\nu := \lim_{n\to\infty} |x_1 - x_2 c_n|, \quad (x_1, x_2) \in \mathbb{K}^2,$$

defines a non-Archimedean norm on the linear space $\mathbb{K}^2$. The normed space $\mathbb{K}^2_\nu = (\mathbb{K}^2, \|\cdot\|_\nu)$ is an immediate extension of the one-dimensional linear subspace $L := \mathbb{K} \times \{0\} \simeq \mathbb{K}$. Indeed, assume for a contradiction that there is $x \in \mathbb{K}^2_\nu \setminus L$, say $x = \lambda_1 e_1 + \lambda_2 e_2$, $\lambda_2 \neq 0$, such that $x \perp L$. For every $\mu \in \mathbb{K}$ one gets

$$\|x + \mu e_1\|_\nu \geqslant \max\{\|x\|_\nu, |\mu|_\nu\}. \tag{1.1}$$

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Since the sequence $(B_{\mathbb{K},r_n}(c_n))_n$ has an empty intersection, there is $m \in \mathbb{N}$ such that $\lambda_1/\lambda_2 \notin B_{\mathbb{K},r_m}(c_m)$. Then

$$\|x\|_\nu = \|\lambda_1 e_1 + \lambda_2 e_2\|_\nu = \lim_{n\to\infty} |\lambda_1 - \lambda_2 c_n| = \lim_{n\to\infty} |\lambda_2| \left|\frac{\lambda_1}{\lambda_2} - c_n\right|.$$

Set $\mu := \lambda_2 c_{m+1} - \lambda_1$. Then,

$$\|\mu e_1\|_\nu = |\lambda_2 c_{m+1} - \lambda_1| = |\lambda_2| \left|\frac{\lambda_1}{\lambda_2} - c_{m+1}\right| > |\lambda_2| \cdot r_m$$

and

$$\begin{aligned}\|x + \mu e_1\|_\nu &= \|\lambda_1 e_1 + \lambda_2 e_2 + \mu e_1\|_\nu = \lim_{n\to\infty} |\lambda_1 + \mu - \lambda_2 c_n| \\ &= \lim_{n\to\infty} |\lambda_1 + \lambda_2 c_{m+1} - \lambda_1 - \lambda_2 c_n| \\ &= |\lambda_2| \cdot \lim_{n\to\infty} |c_{m+1} - c_n| \leqslant |\lambda_2| \cdot r_{m+1} < |\lambda_2| \cdot r_m,\end{aligned}$$

a contradiction with (1.1). By Lemma 1.2.1, $\mathbb{K}^2_\nu$ has no two nonzero orthogonal elements (has no orthogonal base).

Let $I$ be an index set and for every $i \in I$ let $E_i$ be a normed linear space. Then, the product $\prod_{i\in I} E_i$ is in a natural way a linear space. By $\times_{i\in I} E_i$ we denote the *normed product* of $E_i$, i.e. the set of all elements of $\prod_{i\in I} E_i$ for which the set $\{\|x_i\| : i \in I\}$ is bounded, equipped with the norm $\|x\| := \sup\{\|x_i\| : i \in I\}$, $x \in \times_{i\in I} E_i$. The *normed direct sum* $\bigoplus_{i\in I} E_i$ of $E_i$ is the (normed) linear subspace of all $x \in \times_{i\in I} E_i$ such that for every $\varepsilon > 0$, the set $\{i \in I : \|x_i\| \geqslant \varepsilon\}$ is finite.

Let us note that the map $E \to \times_{n\in\mathbb{N}} E$, $x \mapsto (x, x, \dots)$ induces a linear isometry of $E$ into the spherically complete Banach space $\times_{n\in\mathbb{N}} E \Big/ \bigoplus_{n\in\mathbb{N}} E$ (see [57, 4.G and Theorem 4.1]). Hence, every $E$ can be linearly isometrically embedded into a spherically complete Banach space.

A non-Archimedean Banach space $\widehat{E}$ is called a *spherical completion* of $E$ if $\widehat{E}$ is spherically complete and there exists a linear isometry $j\colon E \to \widehat{E}$ such that $\widehat{E}$ has no spherically complete proper linear subspace containing $j(E)$.

**1.2.5. Theorem** ([57, Theorem 4.43]). *Every* $E$ *has a spherical completion* $\widehat{E}$ *and any two spherical completions of* $E$ *are isometrically isomorphic. The spherical completion* $\widehat{E}$ *of* $E$ *is a maximal immediate extension of* $E$*. Conversely, every spherically complete immediate extension of* $E$ *is a spherical completion of* $E$*.*

**1.2.6. Corollary.** *If* $E$ *is not spherically complete, then there exists an overspace, i.e. a normed space* $E_0$ *containing (an isometric image of)* $E$ *as a proper linear subspace such that* $E_0$ *is an immediate extension of* $E$*.*

**1.2.7. Corollary** ([57, Corollary 4.45]). *Let* $i\colon E \to F$ *be an isometric embedding of* $E$ *into a spherically complete Banach space* $F$*. Then,* $F$ *contains a spherical completion* $\widehat{E}$ *of* $i(E)$ *and every immediate extension of* $i(E)$ *is contained in* $\widehat{E}$*.*

**1.2.8. Corollary.** *Let* $D$ *be a linear subspace of* $E$*. If* $E$ *is spherically complete, then* $E$ *contains a spherical completion of* $D$*.*

**1.2.9. Proposition** ([33, Proposition 2.1]). *Let* $D_1, D_2$ *be closed linear subspaces of* $E$ *with* $D_1 \subset D_2$*.*

(1) *If* $E$ *is an immediate extension of* $D_2$ *and* $D_2$ *is an immediate extension of* $D_1$ *then* $E$ *is an immediate extension of* $D_1$

(2) *If* $E$ *is an immediate extension of* $D_1$ *then* $E$ *is an immediate extension of* $D_2$*.*

*Proof.* (1) Suppose for a contradiction that there is $x_0 \in E$ which is orthogonal to $D_1$. Since, by assumption, $D_2$ is an immediate extension of $D_1$, thus $x_0 \in E \setminus D_2$ and we can choose $y \in D_2 \setminus D_1$ satisfying $\|x_0\| = \|y\| > \|x_0 - y\|$. Similarly, we can select $z \in D_1$ for which $\|y\| = \|z\| > \|y - z\|$. But then

$$\|x_0 - z\| = \|x_0 - y + y - z\| \leqslant \max\{\|x_0 - y\|, \|y - z\|\} < \|x_0\| = \|z\|,$$

a contradiction with $x_0 \perp D_1$.

(2) Assume that there is $x_0 \in E$, orthogonal to $D_2$. But $D_1 \subset D_2$, thus $x_0 \perp D_1$, a contradiction. □

Proofs of two next propositions are straightforward.

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**1.2.10. Proposition** ([33, Proposition 2.2]). *Let* $(x_i)_{i\in I}$ *be an orthogonal set in* $E$. *If*

$$\operatorname{dist}(z, \overline{[(x_i)_{i\in I}]}) < \|z\| \quad \textit{for every } z \in E \setminus \overline{[(x_i)_{i\in I}]},$$

*then* $(x_i)_{i\in I}$ *is a maximal orthogonal set in* $E$. *If* $(x_i)_{i\in I}$ *is maximal in* $E$, *then* $E$ *is an immediate extension of* $\overline{[(x_i)_{i\in I}]}$.

**1.2.11. Proposition** ([33, Proposition 2.4]). *Let* $D$ *be a closed hyperplane (i.e. a linear subspace of* $E$ *with* $\dim(E/D) = 1$*) in* $E$. *The following conditions are equivalent:*

(1) *there exists* $x_0 \in E \setminus D$ *such that* $\operatorname{dist}(x_0, D)$ *is not attained;*
(2) $\operatorname{dist}(x, D)$ *is not attained for all* $x \in E \setminus D$;
(3) *there is no element* $x \in E \setminus D$ *orthogonal to* $D$.

**1.2.12. Proposition** ([33, Proposition 2.5]). *Let* $D$ *be a linear closed subspace of* $E$ *and* $(x_n)_n \subset D$ *be a sequence for which the sequence of closed balls* $(B_{E,\|x_n - x_{n+1}\|}(x_n))_n$ *is centered. Let* $V := \bigcap_n B_{E,\|x_n - x_{n+1}\|}(x_n)$. *If* $V \cap D = \emptyset$, *then the subspace* $D + [x]$ *is an immediate extension of* $D$ *for every* $x \in V$.

*Proof.* Assume the contrary and suppose that there is $x_0 \in V$ such that $D + [x_0]$ is not an immediate extension of $D$. Then, by Proposition 1.2.2, we can find $d_0 \in D$ such that $\operatorname{dist}(x_0, D) = \|x_0 - d_0\|$. But then $\|x_0 - x_n\| \geqslant \|x_0 - d_0\|$ for all $n \in \mathbb{N}$; hence $d_0 \in V$, a contradiction. □

In particular, $\mathbb{K}$ as a one-dimensional normed space has a spherical completion $\widehat{\mathbb{K}}$. $\widehat{\mathbb{K}}$ is an infinite-dimensional (even not of countable type) Banach space over $\mathbb{K}$. In $\widehat{\mathbb{K}}$ one can introduce a multiplication that extends the given multiplication of $\mathbb{K}$, such that $\widehat{\mathbb{K}}$ becomes a valued field ([57, Theorem 4.49]). $\mathbb{K}$ and $\widehat{\mathbb{K}}$ (as fields) have the same value group and the same residue class field.

**1.2.13. Remark.** $\operatorname{dist}(\lambda, \mathbb{K})$ is not attained for every $\lambda \in \widehat{\mathbb{K}} \setminus \mathbb{K}$ and every linear subspace of $\widehat{\mathbb{K}}$ has no orthogonal base.

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## 1.3 The spaces $c_0(I)$ and $l^\infty(I)$

The spaces $c_0(I)$ and $l^\infty(I)$ play fundamental role in the theory of non-Archimedean Banach spaces. All Banach spaces over discretely valued $\mathbb{K}$ are isomorphic with $c_0(I)$ for some I and all non-Archimedean Banach spaces of countable type are isomorphic with $c_0$. Every normpolar space $E$ can be linearly and isometrically embedded into some $l^\infty(I)$ (if $E$ is not normpolar, then $E$ can be linearly and isometrically embedded into some $l^\infty(I, \widehat{\mathbb{K}})$), see [56, Lemma 2.2] and [47, Theorem 4.4.9].

Let I be a nonempty set. Let $s\colon I \to (0, \infty)$ and $h\colon I \to \mathbb{K}$ be maps. Set $\|h\|_s := \sup\{|h(i)| \cdot s(i) : i \in I\}$. The maps $h\colon I \to \mathbb{K}$ for which $\|h\|_s$ is finite form a linear space $l^\infty(I : s, \mathbb{K})$, which is a non-Archimedean polar Banach space under the norm $\|\cdot\|_s$.

$c_0(I : s, \mathbb{K})$ will denote the closed linear subspace of $l^\infty(I : s, \mathbb{K})$, which consists of all $h \in l^\infty(I : s, \mathbb{K})$ such that for every $\varepsilon > 0$ the set $\{i \in I : |h(i)| \cdot s(i) \geqslant \varepsilon\}$ is finite. If $s(i) = 1$ for all $i \in I$, we will write $l^\infty(I, \mathbb{K})$ and $c_0(I, \mathbb{K})$, respectively.

In most places, when there is no risk of confusion, the ground field will be omitted; then we will write $l^\infty(I : s)$ and $c_0(I : s)$ instead of $l^\infty(I : s, \mathbb{K})$ and $c_0(I : s, \mathbb{K})$ (or $l^\infty(I)$ and $c_0(I)$ instead of $l^\infty(I, \mathbb{K})$ and $c_0(I, \mathbb{K})$). Note that $l^\infty(I) = \underset{i \in I}{\times} \mathbb{K}$ and $c_0(I) = \bigoplus_{i \in I} \mathbb{K}$.

In particular, we will write $l^\infty := l^\infty(\mathbb{N}, \mathbb{K})$ and $c_0 := c_0(\mathbb{N}, \mathbb{K})$.

According to this convention, $l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ denotes the linear space over $\mathbb{K}$ of all bounded maps $\mathbb{N} \to \widehat{\mathbb{K}}$ equipped with the supremum norm. Then, $l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ is a spherically complete Banach space (see [57, 4.A]); $c_0(\mathbb{N}, \widehat{\mathbb{K}})$ is a closed linear subspace of $l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ consisting of all sequences $(a_1, a_2, \ldots)$, such that $a_n \in \widehat{\mathbb{K}}$ for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = 0$. Clearly, $c_0 \subset c_0(\mathbb{N}, \widehat{\mathbb{K}}) \subset l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ and $c_0 \subset l^\infty \subset c_0(\mathbb{N}, \widehat{\mathbb{K}})$.

**1.3.1. Theorem.** *If* $\mathbb{K}$ *is discretely valued then for every infinite-dimensional non-Archimedean Banach space* $E$ *there exists an isomorphism* $T\colon E \to c_0(I)$ *such that*

$$|\rho| \cdot \|Tx\| < \|x\| \leqslant \|Tx\|$$

1

*for some infinite* $I$, *where* $\rho$ *is an uniformizing element of* $\mathbb{K}$. *Each maximal orthogonal system in* $E$ *is an orthogonal base of* $E$ *and every closed linear subspace of* $E$ *has an orthogonal complement. Additionally, if* $\|E\| = |\mathbb{K}|$, *then,* $T$ *can be defined as an isometric isomorphism.*

*Proof.* Follows from [47, Theorems 2.1.9 and 2.5.4] and [60, Theorem 20.5]. □

**1.3.2. Theorem** ([47, Theorems 2.3.7 and 2.3.11, Corollary 2.3.9] )**.** *labelisom-ct-c0 Every infinite-dimensional non-Archimedean Banach space of countable type is isomorphic with* $c_0$; *hence, it has a Schauder basis.*

**1.3.3. Theorem** ([47, Theorems 2.5.4 and 2.5.15])**.** *The space* $l^\infty$ *is not of countable type. For any set* $I$, *the space* $l^\infty(I)$ *has an orthogonal base if and only if* $\mathbb{K}$ *is discretely valued.*

The bilinear form $B\colon c_0(I) \times l^\infty(I) \to \mathbb{K}$ given by

$$B(x, y) := \sum_{i \in I} x^i y^i, \quad \text{for } x = (x^i)_{i \in I} \in c_0(I),\ y = (y^i)_{i \in I} \in l^\infty(I),$$

induces an isometric isomorphism $y \to B(\,\cdot\,, y)\colon l^\infty(I) \to [c_0(I)]^*$.

Recall that a set $I$ is *small* if it has non-measurable cardinality (the sets we meet in daily mathematical life are small), see also [57, p. 31–33].

**1.3.4. Proposition** (see [47, Theorem 7.4.3] and [57, Theorem 4.21])**.** *Let* $\mathbb{K}$ *be non-spherically complete and* $I$ *be a small set. Then,* $(l^\infty(I))^* = c_0(I)$ *and* $c_0(I)$ *and* $l^\infty(I)$ *are reflexive.*

From now we will assume that $I$ will always be a small set.

**1.3.5. Theorem** ([44, Theorem 3.6], [65, Theorem 2.3])**.** *Let* $D$ *be a closed linear subspace of* $l^\infty$. *The following assertions are equivalent*

(1) $D$ *is weakly closed in* $l^\infty$;
(2) $l^\infty/D \simeq l^\infty$ *or* $l^\infty/D \simeq \mathbb{K}^n$ *for some* $n \in \mathbb{N}$;
(3) $l^\infty/D$ *is reflexive*;
(4) *for every (for some) closed subspace* $L$ *of* $D$ *with* $\dim D/L = 1$, $L$ *is weakly closed in* $l^\infty$.

*Proof.* The implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are obvious.

$(1) \Rightarrow (2)$. Let $G$ be a closed linear subspace of $c_0$ and let $i : G \to c_0$ be the inclusion map. Then, $i^*\colon (c_0)^* \to G^*$, the adjoint of $i$, is a quotient map, thus $G^* \simeq (c_0)^*/G^0$. Applying this observation for $G = D^0$ and using $D = D^{00}$ we get

$$(D^0)^* \simeq (c_0)^*/D^{00} \simeq l^\infty/D.$$

Since $D^0$ is a closed linear subspace of $c_0$, we have $D^0 \simeq \mathbb{K}^n$ for some $n \in \mathbb{N}$ (and so $l^\infty/D \simeq \mathbb{K}^n$) or $D^0 \simeq c_0$ (and so $l^\infty/D \simeq l^\infty$).

$(1) \Rightarrow (4)$. If $L$ is a closed linear subspace of $D$ with $\dim D/L = 1$, then $L$ is weakly closed in $D$. By [65, Theorem 2.3, $(c) \implies (h)$] it follows that $L$ is also weakly closed in $l^\infty$.

$(4) \Rightarrow (1)$. Let $L$ be a closed linear subspace of $D$ as in $(4)$. Since, by assumption, $L$ is weakly closed in $l^\infty$, $l^\infty/L$ has a separating dual and we imply that $(l^\infty/L)/(D/L)$ has a separating dual, either. But $(l^\infty/L)/(D/L)$ is isometrically isomorphic to $l^\infty/D$, hence, $D$ is weakly closed in $l^\infty$. $\square$

By a standard application of the Open Mapping Theorem (see [57, Theorem 3.11]) we get the following result (note that $D$ is weakly closed if and only if $j_{E/D}$ is injective).

**1.3.6. Lemma** ([65, Lemmas 2.1 and 2.2])**.** *Let* $D$ *be a closed linear subspace of a Banach space* $E$*,* $i\colon D \to E$ *be the inclusion map and* $\pi\colon E \to E/D$ *be the quotient map. Assume that every* $f \in D^*$ *can be extended to a linear continuous functional defined on* $E$*. Then, in the commutative diagram*

$$\begin{tikzcd}
D \arrow[r, "i"] \arrow[d, "j_D"'] & E \arrow[r, "\pi"] \arrow[d, "j_E"] & E/D \arrow[d, "j_{E/D}"] \\
D^{**} \arrow[r, "i^{**}"'] & E^{**} \arrow[r, "\pi^{**}"'] & (E/D)^{**}
\end{tikzcd}$$

*we have* $\operatorname{Im} i^{**} = \ker \pi^{**}$ *and* $i^{**}$ *is injective. If, additionally,* $E$ *is polar then*:

(1) *if* $D$ *is reflexive then* $D$ *is weakly closed;*

(2) *if* $E$ *is reflexive and* $D$ *is weakly closed then* $D$ *is reflexive.*

If $\mathbb{K}$ is densely valued field, then every non-Archimedean Banach space of countable type can be isometrically embedded in $l^\infty$ as the next theorem shows.

**1.3.7. Theorem** (see [47, Theorem 2.5.13] )**.** *Let* $\mathbb{K}$ *be densely valued and* $E$ *be a non-Archimedean Banach space of countable type. Then,* $E$ *can be isometrically embedded into* $l^\infty$.

*Proof.* At the beginning, for each $n \in \mathbb{N}$ we construct a linear injection $T_n \colon E \to c_0 \hookrightarrow l^\infty$. By [47, Theorem 2.3.7], $E$ has an $(1 - 1/(n+1))$-orthogonal base $(x_m)_m$. Since $\mathbb{K}$ is densely valued, we can assume that

$$\frac{1}{(1-\frac{1}{n})} \geqslant \|x_m\| \geqslant \frac{1}{(1-\frac{1}{n+1})}$$

for all $m \in \mathbb{N}$ (for $n = 1$ the first inequality is skipped).

Next, for each $n \in \mathbb{N}$, define the map $T_n \colon E \to c_0$ setting

$$T_n\left(\sum_{m=1}^{\infty} a_m x_m\right) := \sum_{m=1}^{\infty} a_m e_m \in c_0,$$

where $(e_m)_m$ is the canonical base of $c_0$. For every $x \in E$, written as $x = \sum_{m=1}^{\infty} a_m x_m$ $a_m \in \mathbb{K}$ $(m \in \mathbb{N})$ we get

$$\begin{aligned}\left(1-\frac{1}{n}\right)\|x\| &\leqslant \left(1-\frac{1}{n}\right) \cdot \max_{m\in N} \|a_m x_m\| \\ &\leqslant \left(1-\frac{1}{n}\right) \cdot \max_{m\in N} |a_m| \cdot \max_{m\in N} \|x_m\| \\ &\leqslant \left(1-\frac{1}{n}\right) \cdot \|T_n x\| \cdot \frac{1}{(1-\frac{1}{n})} = \|T_n x\|\end{aligned}$$

and

$$\begin{aligned}\|x\| = \left\| \sum_{m=1}^{\infty} a_m x_m \right\| &\geqslant \left(1 - \frac{1}{n+1}\right) \cdot \max_{m \in \mathbb{N}} \|a_m x_m\| \\ &= \left(1 - \frac{1}{n+1}\right) \cdot \max_{m \in \mathbb{N}} (|a_m| \cdot \|x_m\|) \\ &\geqslant \left(1 - \frac{1}{n+1}\right) \cdot \max_{m \in \mathbb{N}} |a_m| \cdot \frac{1}{\left(1 - \frac{1}{n+1}\right)} = \|T_n x\|.\end{aligned}$$

Thus, for every $n \in \mathbb{N}$ we finally obtain

$$\left(1 - \frac{1}{n}\right)\|x\| \leqslant \|T_n x\| \leqslant \|x\|$$

Hence, the linear map $T\colon E \to \underset{n \in \mathbb{N}}{\times} l^\infty \simeq l^\infty$, defined as $T(x) := (T_1(x), T_2(x), \dots)$ is a required isometric embedding. $\square$

# Orthocomplemented subspaces in non-Archimedean Banach spaces 2

This Chapter contains results related to the properties of orthocomplemented linear subspaces in certain specific non-Archimedean Banach spaces. Section 2.1 is motivated by the question if every weakly closed, strict HB-subspace of a non-Archimedean Banach space over a non-spherically complete valued field $\mathbb{K}$ is orthocomplemented. We characterize in detail orthocomplemented linear subspaces of $c_0(I)$ and $l^\infty(I)$. Also, we construct a non-Archimedean space over $\mathbb{C}_p$ having a strict, weakly closed HB-subspace which is not orthocomplemented, solving negatively the problem stated above. Section 2.2 deals with the class of Hilbertian spaces, i.e. non-Archimedean spaces for which every finite-dimensional linear subspace is orthocomplemented. We prove, assuming that $\mathbb{K}$ is non-spherically complete, that all immediate extensions of $c_0$ which are contained in $l^\infty$ have such property and among them are those which have no orthogonal base. Section 2.3 refers to the problem if the finite-dimensional orthogonal decomposition of non-Archimedean Banach space is hereditary for closed linear subspaces. We determine the classes of non-Archimedean spaces having this property and show that the problem has a negative answer in general. We start with a simple observation.

**2.0.1. Lemma** ([47, Lemma 2.3.19]). *Assume that $E$ has an orthogonal base. Then, every one-dimensional linear subspace of $E$ is orthocomplemented in $E$.*

*Proof.* Let $\{x_i\}_{i\in I}$ be an orthogonal base of $E$ and let $z \in E \setminus \{0\}$; then, we can write $z = \sum\limits_{i\in I} \lambda_i x_i$ for some $\lambda_i \in \mathbb{K}$ $(i \in I)$. We show that $[z]$ is orthocomplemented in $E$. Since $\{x_i\}_{i\in I}$ is orthogonal, there is $i_0 \in I$ such that $\|z\| = \|\lambda_{i_0} x_{i_0}\|$. Define $D := \overline{\left[\{x_i\}_{i\in I\setminus\{i_0\}}\right]}$. Clearly, $[z] + D = E$. If $d \in D$ then

$$\begin{aligned}\|z - d\| &= \left\| \lambda_{i_0} x_{i_0} + \sum_{i\in I\setminus\{i_0\}} \lambda_i x_i - d \right\| \\ &= \max\left\{ \|\lambda_{i_0} x_{i_0}\|, \left\| \sum_{i\in I\setminus\{i_0\}} \lambda_i x_i - d \right\| \right\} \geqslant \|\lambda_{i_0} x_{i_0}\| = \|z\|,\end{aligned}$$

hence, $[z] \perp D$. □

From Proposition 1.1.7 and Lemma 2.0.1 follows immediately the following conclusion.

**2.0.2. Corollary.** *If $E$ is a non-Archimedean Banach space with an orthogonal base, then every finite-dimensional linear subspace of $E$ is orthocomplemented.*

As a direct consequence of Ingleton's theorem we imply that every spherically complete linear subspace $D$ of $E$ is orthocomplemented in $E$. Hence, if $\mathbb{K}$ is spherically complete, then every finite-dimensional linear subspace of a non-Archimedean Banach space over such $\mathbb{K}$ is orthocomplemented (see [57, Corollary 4.6]). However, the class of non-Archimedean Banach spaces for which every closed linear subspaces is orthocomplemented is much smaller. It is characterized by the following two results.

**2.0.3. Proposition** ([57, Theorem 5.15]). *Let $E$ be finite-dimensional. Then, every linear subspace of $E$ is orthocomplemented if and only if $E$ has an orthogonal base.*

**2.0.4. Proposition** (see [57, Theorems 5.13 and 5.16]). *Let $E$ be an infinite-dimensional non-Archimedean Banach space. Then, every closed linear subspace of $E$ is orthocomplemented if and only if one of the following equivalent conditions is satisfied*:

(1) *every closed linear subspace of countable type of* $E$ *is orthocomplemented;*
(2) *every closed linear subspace of* $E$ *is spherically complete;*
(3) $\mathbb{K}$ *is discretely valued and there is a nonempty set* $I$ *and a function* $s\colon I \to (|\rho|, 1]$, *where* $\rho \in \mathbb{K}$ *is an uniformizing element, such that* $E \simeq c_0(I : s)$ *while the set of values of* $s$ *is well-ordered.*



**2.0.5. Remark.** Note that the condition (3) cannot be restricted only to the assumption that $\mathbb{K}$ is discretely valued. Indeed, assume that $\mathbb{K}$ is discretely valued, $I = \mathbb{Q} \cap (|\rho|, 1]$ and $s\colon r \mapsto r$ for each $r \in I$. Select a strictly decreasing sequence $(p_n) \subset I$. Then, the linear subspace $D = \overline{[(x_n)_n]}$ of $c_0(I : s)$, where $x_n := e_{p_1} + \ldots + e_{p_n}$ $(n \in \mathbb{N})$, is non-spherically complete, since the sequence of balls $(B_{D,p_n}(x_n))_n$ has an empty intersection. Hence, the considered space $c_0(I : s)$ does not fulfill the conditions of Proposition 2.0.4.

## 2.1 Characterization of orthocomplemented subspaces in some concrete non-Archimedean Banach spaces

In non-Archimedean analysis some properties of non-Archimedean spaces strictly depend on the valued field $\mathbb{K}$, in particular, on whether it is spherically complete or not. The Ingleton's theorem (Theorem 1.1.11) is the one of the most important example. van Rooij's result (Theorem 1.1.13) implies that if $\mathbb{K}$ is non-spherically complete, every infinite-dimensional, non-Archimedean Banach space $E$ over such $\mathbb{K}$ has a closed linear subspace $D$ and $f \in D^*$ without preserving norm linear extension on $E$. Also, there exist numerous examples of non-Archimedean Banach spaces with closed, non-weakly closed linear subspaces (see for instance [10]). However, if in every dual separating non-Archimedean Banach space over $\mathbb{K}$ each closed linear subspace is weakly closed, then $\mathbb{K}$ is spherically complete (see [20]).

Let $D$ be a closed linear subspace of a non-Archimedean Banach space $E$. Consider the following properties of $D$:

(1) $D$ is *orthocomplemented* in $E$;

(2) D is a HB-*subspace* (has the HB-*property*), if each $f_0 \in D^*$ has a norm preserving extension $f \in E^*$;
(3) D is *strict*, if for every $x \in E/D$ there is $z \in E$ with $\pi(z) = x$ and $\|z\| = \|x\|$, where $\pi\colon E \to E/D$ is the quotient map (equivalently, see Lemma 2.1.5, if D is orthocomplemented in $[x] + D$ for every $x \in E \setminus D$);
(4) D is *weakly closed* if it is closed in the weak topology $\sigma(E, E^*)$.

The property (1) always implies (2) and (3) and if E has a separating dual also (4). If $\mathbb{K}$ is spherically complete, then every closed linear subspace of E is a weakly closed HB-subspace. However, in this case, we can construct an example (see Proposition 2.1.1) of a non-Archimedean Banach space having a strict, non-orthocomplemented, HB-subspace.

**2.1.1. Proposition** ([44]). *Let* $\mathbb{K}$ *be spherically complete such that* $|\mathbb{K}^\times| = (0, \infty)$. *Then there exists a strict, weakly closed* HB-*subspace of* $c_0(I)$ *for suitable* I *which is not orthocomplemented in* $c_0(I)$.

*Proof.* By [47, Theorem 2.5.6] there exists a strict quotient map $\pi\colon c_0(I) \to l^\infty$ for a suitable set I. Now, since $\mathbb{K}$ is spherically complete, we imply from Ingleton's theorem that $D := \ker \pi$ is a weakly closed and strict HB-subspace of $c_0(I)$. Assume that D is orthocomplemented in $c_0(I)$. Then $l^\infty$ is isometrically isomorphic to a closed subspace of $c_0(I)$ and by Theorem 1.1.4 it has an orthogonal base, a contradiction with Theorem 1.3.3. □

The situation differs substantially if $\mathbb{K}$ is non-spherically complete. Every infinite-dimensional non-Archimedean Banach space has a closed linear subspace without HB-property. The following problem, formulated in 1993 by Perez-Garcia and Schikhof (see [44] and [45]), is natural in this context.

**2.1.2. Problem.** Is every weakly closed, strict HB-subspace of a non-Archimedean Banach space over a non-spherically complete $\mathbb{K}$ orthocomplemented?

In the sequel we show that the answer for this question is affirmative for the spaces $c_0$ and $l^\infty$. On the other hand, we provide a counterexample, demonstrating that in general Problem 2.1.2 has a negative solution.



## The case of $c_0(I)$ and $l^\infty(I)$

Further consideration of this chapter, unless otherwise stated, we will assume that $\mathbb{K}$ is non-spherically complete.

This line of research was started by Perez-Garcia and Schikhof who proved that every one-dimensional, strict linear subspace of $l^\infty$ is orthocomplemented in $l^\infty$ and that every one-codimensional HB-subspace of $c_0$ is orthocomplemented in $c_0$ (see [45, Theorem 2.1] and [44, Theorems 3.4 and 4.3]). Theorems 2.1.13, 2.1.21 and Corollary 2.1.24 extend these results, showing among others that every HB-subspace of $c_0$ is orthocomplemented in $c_0$ and that every weakly closed, strict linear subspace of $l^\infty$ is orthocomplemented in $l^\infty$.

**2.1.3. Proposition** ([44, Remark 2.3 and Proposition 2.5])**.** *Let* $E$ *be a Banach space and* $f \in E^* \setminus \{0\}$. *Then* $\ker f$, *a closed hyperplane of* $E$, *is orthocomplemented in* $E$ *if and only if there exists a nonzero* $x \in E$ *with* $\|f\| = |f(x)|/\|x\|$. *If* $D$ *is an orthocomplemented linear subspace of* $E$, *then* $D^0$ *is orthocomplemented in* $E^*$.

*Proof.* Assume that $\ker f$ is orthocomplemented in $E$. Then $E = [x_1] \oplus \ker f$ for some $x_1 \in E \setminus \{0\}$. For every $x \in E$ we have $x = \lambda \cdot x_1 + x_0$, where $x_0 \in \ker f$ and $\lambda \in K$. Thus, if $x \neq 0$ we obtain

$$\frac{|f(x)|}{\|x\|} \leqslant \frac{|\lambda| \cdot |f(x_1)|}{\|\lambda x_1\|} = \frac{|f(x_1)|}{\|x_1\|}.$$

Now, suppose that there exists $x \in E$ with $\|f\| = |f(x)|/\|x\|$. Then $x \notin \ker f$. Taking $x_0 \in \ker f$, we get

$$\|f\| \cdot \|x + x_0\| \geqslant |f(x + x_0)| = |f(x)| = \|f\| \cdot \|x\|.$$

Hence, $\|x + x_0\| = \max\{\|x\|, \|x_0\|\}$ and we conclude that $[x] \perp \ker f$.

Let $D$ be an orthocomplemented linear subspace in $E$ and $G$ be an orthogonal complement of $D$ in $E$. Then, we can easily check that $G^0$ is an orthogonal complement of $D^0$ in $E^*$. $\square$

Next fact, extending the result of Perez-Garcia and Schikhof (see [44, Remark 2.3 and Theorem 3.3]) obtained for $l^\infty$, characterizes orthocomplemented, finite-dimensional linear subspaces of $l^\infty(I)$.



**2.1.4. Proposition** ([31, Proposition 3.1]). *Let* $D$ *be a finite-dimensional linear subspace of* $l^\infty(I)$. *Then, the following conditions are equivalent*:

(1) $D$ *is orthocomplemented in* $l^\infty(I)$.

(2) *Every one-dimensional subspace of* $D$ *is orthocomplemented in* $l^\infty(I)$.

(3) *For each* $x = (x_i)_{i\in I} \in D$, $\max\limits_{i\in I} |x_i|$ *exists.*

*Proof.* $(1) \Rightarrow (2)$. Assume that $D$ is orthocomplemented in $l^\infty(I)$. Then, by Proposition 2.1.3, there exists an orthogonal complement $H$ of $D^o$ in $[l^\infty(I)]^*$. But, by Proposition 1.3.4, $[l^\infty(I)]^* \simeq c_0(I)$; hence, we can write $c_0(I) = D^o + H$, where $H$ is a finite-dimensional linear subspace of $c_0(I)$. By Theorem 1.1.4, $H$ has an orthogonal base. Thus, $H^*$ has an orthogonal base, either. But $D$ is reflexive (see Proposition 1.1.8); hence, $H^* \simeq D^{**} \simeq D$. Thus, $D$ has an orthogonal base and every one-dimensional subspace $L$ of $D$ is orthocomplemented in $D$. But $D$ is orthocomplemented in $l^\infty(I)$, thus, $L$ is orthocomplemented in $l^\infty(I)$.

$(2) \Rightarrow (1)$. It suffices to prove that if $G$ is a linear subspace of $D$ of codimension 1 which is orthocomplemented in $l^\infty(I)$, then $D$ is orthocomplemented in $l^\infty(I)$. So, assume that there is an orthoprojection $P\colon l^\infty(I) \xrightarrow{\text{onto}} G$. Since $G \subset D$ and $G$ has a codimension 1 in $D$. there exists a nonzero $d \in D$ for which $P(d) = 0$. By assumption, there is an orthoprojection

$$Q\colon l^\infty(I) \xrightarrow{\text{onto}} [d]\,.$$

Since $(I - P)(d) = d$ and $\ker(I - P) = D$ we get that $Q\circ (I - P)$ is an orthoprojection of $l^\infty(I)$ onto $[d]$. Hence, $P \circ Q\circ (I - P) = 0$ and $Q\circ (I - P) \circ P = 0$. Thus, $P + Q\circ (I - P)$ is an orthoprojection of $l^\infty(I)$ onto $G + [d] = D$.

$(2) \Leftrightarrow (3)$. Let $f \in l^\infty(I) \simeq [c_0(I)]^*$. Then, if $[f]$ is orthocomplemented in $l^\infty(I)$, by Proposition 2.1.3,

$$[f]^o = \{x \in [l^\infty(I)]^* \simeq c_0(I) : f(x) = 0\} \simeq \ker f$$

is orthocomplemented in $[l^\infty(I)]^*$. Hence, using Proposition 2.1.3, we get the equivalence: $[f]$ is orthocomplemented in $l^\infty(I)$ if and only if $\ker f$ is orthocomplemented in $c_0(I)$. But, by Proposition 2.1.3, it is equivalent with

$$\|f\| = \max\{|f(x)| : \|x\| \leqslant 1\} = \max_{i\in I} |f(e_i)|,$$

where $(e_i)_{i\in I}$ is the canonical base of $c_0(I)$. □



The key tool for the proofs of the main results of this section is the characterizations of the strictness in terms of immediate extensions of linear subspaces provided in Theorem 2.1.6. First, a lemma.

**2.1.5. Lemma** (see [44, Proposition 1.2] and [32, Lemma 1]). *Let* $D$ *be a closed linear subspace of a non-Archimedean Banach space* $E$.

(1) $D$ *is strict in* $E$ *if and only if for each* $x \in E\backslash D$, $D$ *is orthocomplemented in* $D + [x]$.

(2) *Let* $x \in E \setminus D$. *Then* $D$ *is orthocomplemented in* $D + [x]$ *if and only if there exists* $d_0 \in D$ *for which* $\mathrm{dist}(x, D) = \|x - d_0\|$.

*Proof.* (1) ($\Rightarrow$) Take $x \in E \setminus D$ and assume that $D$ is strict in $E$. Let $\pi \colon E \to E/D$ be the quotient map. Then, there exists $u \in E$ with $\pi(u) = \pi(x)$ and $\|u\| = \|\pi(u)\|$. Thus $(u - x) \in \ker \pi$ and we can find $d_0 \in D$ such that $u = x + d_0$. We get

$$\|x + d_0\| = \|u\| = \|\pi(u)\| = \inf_{d\in D} \|u - d\| = \inf_{d\in D} \|(x + d_0) - d\|.$$

Hence, taking a nonzero $\lambda \in \mathbb{K}$ and $d \in D$ we obtain

$$\|\lambda(x + d_0) + d\| = |\lambda| \cdot \left\|(x + d_0) + \frac{d}{\lambda}\right\| \geqslant |\lambda| \cdot \|x + d_0\|$$

and conclude that $[x + d_0]$, the one-dimensional linear subspace generated by $x + d_0 \in E$, is an orthocomplement of $D$ in $D + [x]$.

($\Leftarrow$) If for each $x \in E$, $D$ is orthocomplemented in $D + [x]$ then for each $x \in E$, there exists $d_0 \in D$ such that $(x + d_0) \perp D$.Hence

$$\|\pi(x)\| = \mathrm{dist}(x, D) = \|x + d_0\|.$$

Since $\pi$ is surjective, we conclude that for every $y \in E/D$ there exists $x \in E$ such that $y = \pi(x)$ and $\|\pi(x)\| = \|x\|$.

(2) Let D be orthocomplemented in $D + [x]$. Then there exists $d_0 \in D$ such that $[(x - d_0)] \perp D$. For every $d \in D$ we have

$$\|x - d_0 + d\| = \max\{\|x - d_0\|, \|d\|\};$$

thus, $\|x - d_0 + d\| \geqslant \|x - d_0\|$ and finally $\operatorname{dist}(x, D) = \|x - d_0\|$. To prove the converse, assume that there exists $d_0 \in D$ for which $\operatorname{dist}(x, D) = \|x - d_0\|$. Suppose D is not orthocomplemented in $D + [x]$. Thus, for each $d \in D$ there exists $d_1 \in D$ and $\lambda \in \mathbb{K}$ with

$$\|\lambda(x - d) - d_1\| < \max\{\|\lambda(x - d)\|, \|d_1\|\}.$$

If $d = d_0$, then

$$\|\lambda(x - d_0) - d_1\| = |\lambda| \cdot \left\|x - d_0 - \frac{d_1}{\lambda}\right\| < |\lambda| \cdot \|x - d_0\|.$$

Thus, we conclude that $\|x - (d_0 + \frac{d_1}{\lambda})\| < \|x - d_0\|$ but $(d_0 + d_1/\lambda) \in D$, a contradiction. Since, for each $d \in D$,

$$\|\lambda(x - d_0) - d\| = \max\{\|\lambda(x - d_0)\|, \|d\|\},$$

we imply that $(x - d_0) \perp D$. □

**2.1.6. Theorem** ([32, Theorem 2.4]). *Let* E *be a non-Archimedean Banach space, and let* G *be a closed linear subspace of* E*. Then,* G *is strict in* E *if and only if for each linear subspace* L *of* G*, every immediate extension of* L *in* E *is contained in* G.

*Proof.* ($\Leftarrow$) Let $G \subset E$ be a closed linear subspace which is not strict. It means, applying Lemma 2.1.5, there exists $x \in E$ such that G is not orthocomplemented in $G + [x]$ and $\operatorname{dist}(x, G)$ is not attained. Hence, $\operatorname{dist}(x, G) < \|x\|$ and $G + [x]$ is an immediate extension of G, which is not contained in G.

($\Rightarrow$) Now, assume that there exists a linear subspace L of G and a linear subspace $L_0$ of E, which is an immediate extension of L but it is not contained in $G$. We show that $G + L_0$ is an immediate extension of $G$. Let $\widehat{E}$ be a spherical completion of $E$ and $i\colon E \to \widehat{E}$ be a suitable isometric embedding. Since $i(G) \subset \widehat{E}$, $\widehat{E}$ contains a spherical completion of $i(G)$ which we denote as $\widehat{G}$. Clearly, $i(L) \subset i(G)$ and $\widehat{G}$ contains a spherical completion of $i(L)$, denoted as $\widehat{L}$. Observe that $i(L_0)$ is an immediate extension of $i(L)$. Thus, $i(L_0) \subset \widehat{L}$; indeed, otherwise, assuming that there is $x_0 \in i(L_0) \setminus \widehat{L}$ we imply that $[x_0] + \widehat{L}$ is an immediate extension of $i(L)$, a contradiction with maximality of $\widehat{L}$ (see Corollary 1.2.7). Hence, we obtain

$$i(G) \subset i(G + L_0) = i(G) + i(L_0) \subset \widehat{G}$$

and conclude that $i(G + L_0)$ is an immediate extension of $i(G)$. Therefore, $G + L_0$ is an immediate extension of $G$. Take $z \in L_0 \setminus G$. Then, $\mathtt{dist}(z, G)$ is not attained; thus, applying Lemma 2.1.5, we conclude that $G$ is not strict in $E$. $\square$

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**2.1.7. Proposition** ([44, Proposition 2.1]). *Let $D$ be a closed linear subspace of $E$.*

(1) *If $D$ is strict in $E$ and $E/D \simeq c_0(I : s)$ for some set $I$ and $s\colon I \to (0, \infty)$, then $D$ is orthocomplemented in $E$;*

(2) *If $D$ is a HB-subspace of $E$ and $D \simeq l^\infty(I : s)$ for some set $I$ and $s\colon I \to (0, \infty)$, then $D$ is orthocomplemented in $E$.*

*Proof.* (1) Let $\pi_E\colon E \to E/D$ be the quotient map and $\{u_i\}_{i \in I}$ be an orthogonal base of $E/D$. Since, by assumption, $D$ is strict, there exists $\{z_i\}_{i \in I} \subset E$ such that $\pi_E(z_i) = u_i$ and $\|z_i\| = \|u_i\|$ for all $i \in I$. Then, the map $T\colon E/D \to E$ given by $\sum\limits_{i \in I} \lambda_i u_i \mapsto \sum\limits_{i \in I} \lambda_i z_i$ is a linear isometry for which $\pi_E \circ T$ is the identity on $E/D$. Hence, $D$ is orthocomplemented in $E$.

(2) For each $i \in I$ the coordinate functional $e_i^* \in D^*$ given by $e_i^*(x) = x_i$, where $x = (x_i)_i \in l^\infty(I : s)$ has the norm equal to $1/s(i)$. Since $D$ is a HB-subspace of $E$, there exists a preserving norm extension $f_i^* \in E^*$ of $e_i^*$. Then, the map $P\colon E \to D$ given by $x \mapsto (f_i^*(x))_{i \in I}$ is a required orthoprojection from $E$ onto $D$. $\square$

There is a duality between the HB-property and strictness which is shown by the following result.

**2.1.8. Proposition** ([44, Proposition 2.5]). *Let $D$ be a closed linear subspace of a non-Archimedean Banach space $E$. Then, the following assertions are satisfied*:

(1) *If $D$ is a HB-subspace of $E$, then $D^0$ is strict in $E^*$.*

(2) *If $D$ is strict in $E$ and $E/D$ is reflexive, then $D^0$ is a HB-subspace of $E^*$.*

(3) *If $D$ is orthocomplemented in $E$, then $D^0$ is orthocomplemented in $E^*$.*

*Proof.* (1) If $D$ is a HB-subspace of $E$ and $i\colon D \to E$ is the inclusion map, then its adjoint $i^*\colon E^* \to D^*$ is a strict map. But then, $\ker i^* = D^0$ is strict in $E$.

(2) Let $\pi_E\colon E \to E/D$ be the quotient map. Then its adjoint $\pi_E^*\colon (E/D)^* \to E^*$ is an isometric embedding for which $\pi_E^*((E/D)^*) = D^0$. Hence, to show that $D^0$ is a HB-subspace of $E^*$ it suffices to prove that for any $\phi \in (E/D)^{**}$ there exists $\phi_0 \in E^{**}$ such that $\|\phi\| = \|\phi_0\|$ and $\phi_0 \circ \pi_E^* = \phi$. Since, by assumption, $E/D$ is reflexive, there is $z \in E/D$ such that $\phi = j_{E/D}(z)$ ($j_{E/D}\colon E/D \to (E/D)^{**}$ is the natural map) and $\|z\| = \|\phi\|$. Also, by strictness of $D$, there is $x \in E$ with $\pi_E(x) = z$ and $\|x\| = \|z\|$. Then, $\phi_0 := j_E(x)$ satisfies the required conditions.

(3) Note that if $S$ is an orthogonal complement of $D$ in $E$, then $S^0$ is an orthogonal complement of $D^0$ in $E^*$. $\square$

Let $D$ be a closed linear subspace of $E$ and $S$ be a closed linear subspace of $D$. Consider the following commutative diagram, where $i_1, \pi_E, \pi_D$ are natural maps and $i_2$ makes the diagram commute

$$\begin{tikzcd} D \arrow[r, "i_1"] \arrow[d, "\pi_D"'] & E \arrow[d, "\pi_E"] \\ D/S \arrow[r, "i_2"'] & E/S \end{tikzcd}$$

**2.1.9. Proposition** ([44, Proposition 2.7]). *Let $D$ be a closed linear subspace of $E$ and let $S$ be a closed linear subspace of $D$. If $D$ is strict (resp. has* *the* HB*-property, is orthocomplemented*) *in* $E$*, then* $i_2(D/S)$ *is strict* (*resp. has the* HB*-property, is orthocomplemented*) *in* $E/S$*.*

*Proof.* (1) Assume that $D$ is strict. Let $x \in E$. Then we can find $d \in D$ such that $\|x - i_1(d)\| \leqslant \|x - i_1(d')\|$ for all $d' \in D$. Now, for all $s' \in S$ and $d' \in D$, we have

$$\begin{aligned}\|\pi_E(x) - i_2\pi_D(d)\| &= \|\pi_E(x) - \pi_E(i_1(d))\| \\ &\leqslant \|x - i_1(d)\| \leqslant \|x - i_1(d') - s'\|.\end{aligned}$$

Hence, $\|\pi_E(x) - i_2\pi_D(d)\| \leqslant \|\pi_E(x) - i_2\pi_D(d')\|$ for all $d' \in D$ and we see that $\mathrm{dist}(\pi_E(x), i_2(D/S))$ is attained. Thus, $i_2(D/S)$ is strict in $E/S$.

(2) Assume that $D$ is a HB-subspace. Let $f \in (D/S)^*$. Then, $f \circ \pi_D \in D^*$, so by assumption there is $g \in E^*$ such that $\|g\| = \|f \circ \pi_D\| = \|f\|$ and $g \circ i_1 = f \circ \pi_D$. Since $S \subset \ker g$, there is a unique $f' \in (E/S)^*$ such that $f' \circ \pi_E = g$. One verifies that then also $f' \circ i_2 = f$ and that $\|f'\| = \|f\|$.

(3) Suppose that $D$ is orthocomplemented and let $P\colon E \to D$ be an orthoprojection. Since $S \subset \ker(\pi_D \circ P)$, there is a unique continuous linear map $Q\colon E/S \to D/S$ such that $Q \circ \pi_E = \pi_D \circ P$ and $\|Q\| \leqslant 1$. Also, $Q \circ i_2\pi_D(x) = \pi_D(x)$ for all $x \in D$. So, since $\pi_D$ is surjective, we conclude that $D/S$, which implies that $i_2(D/S)$ is orthocomplemented in $E/S$. □

**2.1.10. Proposition** ([44, Proposition 2.8])**.** *Let* $D$ *be a closed subspace of* $E$*. If for each closed linear subspace* $S$ *of* $D$ *with* $\dim D/S = 1$ *we have that* $i_2(D/S)$ *has the* HB*-property in* $E/S$*, then* $D$ *has the* HB*-property in* $E$*.*

*Proof.* Let $f \in D^* \setminus \{0\}$ and let $S = \ker f$. Take $h_1 \in (D/S)^*$, then $f = h_1 \circ \pi_D$ and there exists $c > 0$ such that $|h_1(z)| = c \cdot \|z\|$ for all $z \in D/S$. By assumption and Proposition 2.1.7 there is an orthoprojection $h_2 : E/S \to D/S$ such that $h_2 \circ i_2$ is the identity on $D/S$. Now, set $f' := h_1 \circ h_2 \circ \pi_E$. Then, $\|f'\| = \|f\|$, $f' \circ i_1 = f$ and we are done. □

**2.1.11. Proposition** ([32, Proposition 3])**.** *Let* $x, y \in E$ *be non-zero elements for which* $\|x\| = \|y\|$ *and* $y \notin [x]$*. If the* $\mathrm{dist}(y, [x])$ *is not attained then there exists a centered sequence of closed balls* $(B_{\mathbb{K},r_m}(\lambda_m))_m$ *such that:*

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(1) $r_{m+1} < r_m$, $|\lambda_m| = 1$ *and* $\|y - \lambda_m x\| = |\lambda_m - \lambda_{m+1}| \cdot \|x\|$ *for all* $m \in \mathbb{N}$,

(2) $\lambda_m \notin B_{\mathbb{K},r_{m+1}}(\lambda_{m+1})$ *for all* $m \in \mathbb{N}$,

(3) $r = \inf_m r_m = \lim_{m\to\infty} r_m = \mathrm{dist}(y,[x])/\|y\| < 1$, $r > 0$,

(4) $\bigcap_{m=1}^{\infty} B_{\mathbb{K},r_m}(\lambda_m) = \emptyset$,

(5) $\|y - \lambda x\| = \lim_{m\to\infty} |\lambda_m - \lambda| \cdot \|x\|$ *for every* $\lambda \in \mathbb{K}$.

*Proof.* (1) Since $\mathrm{dist}(y,[x])$ is not attained, there exists a sequence $(\lambda_m)_m \subset \mathbb{K}$ such that $\lim_{m\to\infty} \|y - \lambda_m x\| = \mathrm{dist}(y,[x])$ and $|\lambda_m| = 1$ for all $m \in \mathbb{N}$. Without loss of generality, we may assume that $\|y-\lambda_m x\| > \|y - \lambda_{m+1}x\|$ for each $m \in \mathbb{N}$. Then, we obtain

$$\|y - \lambda_m x\| = \|y - \lambda_m x - (y - \lambda_{m+1}x)\| = |\lambda_m - \lambda_{m+1}| \cdot \|x\|$$

and similarly

$$\|y - \lambda_{m+1}x\| = |\lambda_{m+1} - \lambda_{m+2}| \cdot \|x\|.$$

Thus, we conclude that $|\lambda_m-\lambda_{m+1}| > |\lambda_{m+1}-\lambda_{m+2}|$. Now, we choose a sequence of real numbers $(r_m)_m$ for which

$$r_m > |\lambda_m - \lambda_{m+1}| > r_{m+1} > |\lambda_{m+1} - \lambda_{m+2}|$$

and form a sequence of closed balls $(B_{\mathbb{K},r_m}(\lambda_m))_m$.

Since $|\lambda_m-\lambda_{m+1}| < r_m$ and $r_{m+1} < r_m$, we get $\lambda_{m+1} \in B_{\mathbb{K},r_m}(\lambda_m)$ and $B_{\mathbb{K},r_{m+1}}(\lambda_{m+1}) \subset B_{\mathbb{K},r_m}(\lambda_m)$.

(2) From the inequality $|\lambda_m - \lambda_{m+1}| > r_{m+1}$ one gets

$$\lambda_m \notin B_{\mathbb{K},r_{m+1}}(\lambda_{m+1}) \quad \text{for all } m \in \mathbb{N}.$$

(3) We have $r = \inf_m r_m = \lim_{m\to\infty} r_m = \lim_{m\to\infty} |\lambda_m - \lambda_{m+1}|$. Since

$$\|y - \lambda_m x\| = |\lambda_m - \lambda_{m+1}| \cdot \|x\|,$$

by (1), we get $\|y - \lambda_m x\| \to \mathrm{dist}(y,[x])$ for $m \to \infty$ and

$$\frac{\mathrm{dist}(y,[x])}{\|x\|} = \frac{\mathrm{dist}(y,[x])}{\|y\|} = r.$$

Since $y \notin [x]$, $r > 0$.

(4) Assume that there exists $\lambda_0 \in \mathbb{K}$ such that $\lambda_0 \in \bigcap\limits_{m=1}^{\infty} B_{\mathbb{K},r_m}(\lambda_m)$. Then, for each $m \in \mathbb{N}$, we have $|\lambda_0 - \lambda_{m+1}| < r_{m+1}$. Since $|\lambda_m - \lambda_{m+1}| > r_{m+1}$, we obtain

$$|\lambda_m - \lambda_0| = |\lambda_m - \lambda_{m+1} + \lambda_{m+1} - \lambda_0| = |\lambda_m - \lambda_{m+1}|.$$

Thus, by (1)

$$\|(y-\lambda_m x)-(y-\lambda_0 x)\| = \|(\lambda_m-\lambda_0)x\| = \|(\lambda_m-\lambda_{m+1})x\| = \|y-\lambda_m x\|$$

and

$$\|y - \lambda_m x\| \geqslant \|y - \lambda_0 x\| \quad \text{for all } m \in \mathbb{N}.$$

Hence, we conclude that $\operatorname{dist}(y, [x]) = \|y - \lambda_0 x\|$, a contradiction.

(5) Fix $\lambda \in \mathbb{K}$. Since $\operatorname{dist}(y, [x]) = \lim\limits_{m\to\infty} \|y - \lambda_m x\|$, by (1), we can choose $m_\lambda \in \mathbb{N}$ such that

$$\|y - \lambda x\| > \|y - \lambda_{m_\lambda} x\| = |\lambda_{m_\lambda} - \lambda_{m_\lambda+1}| \cdot \|x\|.$$

Hence,

$$\|y - \lambda x\| = \|(y - \lambda x) - (y - \lambda_{m_\lambda} x)\| = |\lambda - \lambda_{m_\lambda}| \cdot \|x\|$$

and $|\lambda_{m_\lambda} - \lambda_{m_\lambda+1}| < |\lambda - \lambda_{m_\lambda}|$. Thus, we imply $\lambda \notin B_{\mathbb{K},r_{m_\lambda+1}}(\lambda_{m_\lambda+1})$ and

$$|\lambda - \lambda_m| = |\lambda - \lambda_{m_\lambda} + \lambda_{m_\lambda} - \lambda_m| = |\lambda - \lambda_{m_\lambda}|$$

for all $m > m_\lambda$. Finally we get $\|y - \lambda x\| = \lim\limits_{m\to\infty} |\lambda_m - \lambda| \cdot \|x\|$. □

**2.1.12. Proposition** ([31, Proposition 3.3]). *Let* $x = (x^i)_{i\in I}$, $y = (y^i)_{i\in I}$ *in* $l^\infty(I)$ *be such that* $\|y\| = \|x\|$ *and* $[x, y]$ *has no orthogonal bases. Denote* $N_0 := \{k \in I : |x^k| > \operatorname{dist}(y, [x])\}$. *Then*

1. $\max\limits_{i\in I} |x^i|$ *does not exist and* $|x^k| = |y^k|$ *for all* $k \in N_0$;
2. *set* $c_i := y^i/x^i$ *for* $i \in N_0$, *then* $|c_i| = 1$ *for every* $i \in N_0$. *If* $(x^{n_k})_k$ *is any sequence of elements of the set* $\{x^i : i \in N_0\}$ *such that* $|x^{n_k}| \to \|x\|$ *for* $k \to \infty$, *then* $\|y - c_{n_k} x\| \to \operatorname{dist}(y, [x])$ *if* $k \to \infty$ *and* $\|y - \lambda x\| = \lim\limits_{k\to\infty} |c_{n_k} - \lambda| \cdot \|x\|$ *for every* $\lambda \in \mathbb{K}$.

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*Proof.* (1) Since $[x, y]$ has no orthogonal base, $\mathtt{dist}(y, [x])$ is not attained by Lemma 1.2.1. Hence, $\mathtt{dist}(y, [x]) < \|x\|$ and the set $N_0$ is not empty. By Proposition 2.1.4, if $x \in l^\infty(I)$ and $\max_{i\in I}|x^i|$ exists, then the one-dimensional subspace $[x] \subset l^\infty(I)$ is orthocomplemented in $l^\infty(I)$. Then, we can write $y = \lambda_y x + y_0$, where $\lambda_y \in \mathbb{K}$, $y_0 \in l^\infty(I)$ and $y_0 \perp [x]$. We get

$$\mathtt{dist}(y, [x]) = \mathtt{dist}(\lambda_y x + y_0, [x]) = \mathtt{dist}(y_0, [x]) = \|y_0\|$$

and conclude that $\mathtt{dist}(y, [x])$ is attained, a contradiction.

Let $k \in N_0$. Hence, we can choose $\lambda \in \mathbb{K}$ for which $\|y - \lambda x\| < |x^k| < \|x\|$. Since $|y^k - \lambda x^k| \leqslant \|y - \lambda x\| < |x^k|$ we obtain $|x^k| = |\lambda x^k| = |x^k|$ and $|\lambda| = 1$.

(2) Let $(x^{n_k})_k \subset \{x_i : i \in N_0\}$ be a sequence of scalars, such that $|x^{n_k}| \to \|x\|$ if $k \to \infty$. Since $\max_{i\in I}|x^i|$ does not exist, $(x^{n_k})_k$ is infinite and $|x^{n_k}| < \|x\|$ for all indices $n_k$. By Proposition 2.1.11 we may choose a sequence $(\lambda_m)_m \subset \mathbb{K}$, $|\lambda_m| = 1$ for all $m \in \mathbb{N}$, such that $\|y - \lambda_m x\| \to \mathtt{dist}(y, [x])$ if $m \to \infty$, and the sequence of closed balls $(B_{\mathbb{K},r_m}(\lambda_m))_m$ which satisfies the conditions of Proposition 2.1.11. Fix $m_1 \in \mathbb{N}$. We shall prove that there exists $n_{k(m_1)} \in \{n_1, n_2, \ldots\}$ such that $c_{n_k} \in B_{\mathbb{K},r_{m_1}}(\lambda_{m_1})$ if $n_k > n_{k(m_1)}$. Then, by Proposition 2.1.11 (5), we obtain

$$\|y - c_{n_k}x\| = \lim_{m\to\infty} |\lambda_m - c_{n_k}|\cdot\|x\| < |\lambda_{m_1-1} - \lambda_{m_1}|\cdot\|x\| = \|y - \lambda_{m_1-1}x\|$$

and prove that $\|y - c_{n_k}x\| \to \|y - [x]\|$ for $k \to \infty$.

Since $|x^{n_k}| \to \|x\|$ we can choose an index $n_m$ with

$$\frac{\|x\|}{|x^{n_k}|} < \frac{r_{m_1}}{r_{m_1+1}}$$

for all $n_k > n_m$. From Proposition 2.1.11, we get

$$\|y - \lambda_{m_1+1}x\| = |\lambda_{m_1+2} - \lambda_{m_1+1}| \cdot \|x\| \leqslant r_{m_1+1} \cdot \|x\|.$$

Next, taking $c_{n_k}$ such that $n_k > n_m$, we obtain

$$\begin{aligned}|c_{n_k} - \lambda_{m_1+1}| \cdot |x^{n_k}| &= |y_{n_k} - \lambda_{m_1+1}x^{n_k}| \leqslant \|y - \lambda_{m_1+1}x\| \\ &\leqslant r_{m_1+1} \cdot \|x\| < r_{m_1+1} \cdot \frac{r_{m_1}}{r_{m_1+1}} \cdot |x^{n_k}| = r_{m_1} \cdot |x^{n_k}|.\end{aligned}$$

Hence, $|c_{n_k} - \lambda_{m_1+1}| < r_{m_1}$ and we finally conclude

$$c_{n_k} \in B_{\mathbb{K},r_{m_1}}(\lambda_{m_1+1}) = B_{\mathbb{K},r_{m_1}}(\lambda_{m_1})$$

for all $n_k > n_m$. By Proposition 2.1.11, the sequence of closed balls $(B_{\mathbb{K},r_m}(\lambda_m))_m$ has an empty intersection; thus, there is $p \in \mathbb{N}$ with

$$c_{n_m} \in B_{\mathbb{K},r_p}(\lambda_p) \setminus B_{\mathbb{K},r_{p+1}}(\lambda_{p+1}).$$

We see that $B_{\mathbb{K},r_p}(c_{n_m}) = B_{\mathbb{K},r_p}(\lambda_p)$. Taking in the next step $m_2 := p+1$ we can find $n_{k(m_2)}$ with $c_{n_{k(m_2)}} \in B(\lambda_{m_2}, r_{m_2})$ and $q \in \mathbb{N}$ such that

$$c_{n_{k(m_2)}} \in B_{\mathbb{K},r_q}(\lambda_q) \setminus B_{\mathbb{K},r_{q+1}}(\lambda_{q+1}).$$

This way we form inductively a subsequence $(B_{\mathbb{K},r_{m_k}}(c_{m_k}))_k$ of the sequence $(B_{\mathbb{K},r_m}(\lambda_m))_m$, which also satisfies the conditions of Proposition 2.1.11. Now, by Proposition 2.1.11 (5), we conclude

$$\|y - \lambda x\| = \lim_{k\to\infty} |c_{m_k} - \lambda| \cdot \|x\| = \lim_{k\to\infty} |c_{n_k} - \lambda| \cdot \|x\|. \qquad \square$$

**2.1.13. Theorem** ([32, Theorem 3.4]). *Let* $D \subset l^\infty(I)$ *be a finite dimensional linear subspace. Then,* $D$ *is strict in* $l^\infty(I)$ *if and only if* $D$ *is orthocomplemented in* $l^\infty(I)$.

*Proof.* Suppose that D is not orthocomplemented in $l^\infty(I)$. By Proposition 2.1.4, there is $x = (x^i)_{i\in I} \in D$ for which $\max_{i\in I} |x^i|$ does not exist. We shall prove that there is an infinite-dimensional subspace $F \subset l^\infty(I)$ which is an immediate extension of the one-dimensional subspace $[x]$. Then, applying Theorem 2.1.6, we conclude that D is not strict in $l^\infty(I)$.

Let $E_0 \subset \widehat{\mathbb{K}}$ be an arbitrary closed linear subspace of countable type; then, $E_0$ has no two nonzero mutually orthogonal elements as an immediate extension of one-dimensional linear space. By Theorem 1.3.7, there exists a linear isometry $T\colon E_0 \to T(E_0) \subset l^\infty(I)$. Thus, $T(E_0)$ has no two nonzero, orthogonal elements, either. In the next part of the proof we will construct an isomorphism $S\colon T(E_0) \to S(T(E_0)) \subset l^\infty(I)$ such that $x \in S(T(E_0))$. This way, we construct a required infinite-dimensional immediate extension of $[x]$.

Note that, since $\mathsf{T}(\mathsf{E}_0)$ has no two nonzero mutually orthogonal elements, we can choose a basis $(v_k)_k$ $(v_k = (v_k^i)_{i\in I})$ of $\mathsf{T}(\mathsf{E}_0)$ such that $\|v_k\| = \|v_{k+1}\|$ for all $k \in \mathbb{K}$ and for $j = 3, 4, \ldots$

$$\mathtt{dist}(v_j, [v_1, \ldots, v_{j-1}]) > \mathtt{dist}(v_{j-1}, [v_1, \ldots, v_{j-2}]).$$

Denote $v := (v^i)_{i\in I} = v_1$ and $r_j := \mathtt{dist}(v_j, [v]) \cdot \|v\|^{-1}$ for $j = 2, 3, \ldots$ Set $N_0 := \{i \in I : |v^i| > 0\}$. For each $i \in N_0$ we construct an infinite sequence of scalars $(\lambda_n^i)_n$ (possible $\lambda_n^i = 0$) such that $\lambda_1^i = 1$ and $v_k^i = \lambda_k^i \cdot v^i$ for $k = 2, 3, \ldots$

Now, define a map $h\colon I \to N_0$ which satisfies

$$|x^i| < |v^{h(i)}| \cdot \frac{\|x\|}{\|v\|} \quad \text{for every } i \in I$$

(recall that $\max_{i\in I} |v^i|$ does not exist by Proposition 2.1.4). Next, form an infinite sequence $(x_k)_k \subset l^\infty(I)$, setting $x_1 = x$, $x_k = (x_k^i)_{i\in I}$ where $x_k^i = \lambda_k^{h(i)} x^i$.

We shall prove that the linear map $S\colon \overline{[(v_k)_k]} \to \overline{[(x_k)_k]}$, defined by

$$S\Big(\sum_{n=1}^{\infty} a_n v_n\Big) := \sum_{n=1}^{\infty} a_n x_n,$$

where $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is a similarity, i.e. there exists $k(= \|x\|/\|v\|) > 0$ with $\|S(u)\| = k \cdot \|u\|$ for all $u \in \overline{[(v_k)_k]}$.

It is easy to see that $\|x_k\| = \|x\|$ for every $k \in \mathbb{N}$. We prove that

$$\frac{\|v\|}{\|x\|} \cdot \Big\| \sum_{i=1}^{m_0} a_i x_i \Big\| = \Big\| \sum_{i=1}^{m_0} a_i v_i \Big\|$$

for all $m_0 \in \mathbb{N}$ and $a_i \in \mathbb{K}$, $(i = 1, \ldots, m_0)$. First, suppose that there exists $i_0 \in N$ with

$$|a_{i_0}| > \max_{\substack{i=1,\ldots,m_0,\\ i\neq i_0}} |a_i|.$$

Then

$$\Big\| \sum_{i=1}^{m_0} a_i v_i \Big\| = |a_{i_0}| \cdot \|v_{i_0}\| = |a_{i_0}| \cdot \|v\|$$

and

$$\left\| \sum_{i=1}^{m_0} a_i x^i \right\| = |a_{i_0}| \cdot \|x_{i_0}\| = |a_{i_0}| \cdot \|x\|.$$

Thus, we are done.

Now, assume that there are indices $i_0 \neq i_1$ with $|a_{i_0}| = |a_{i_1}| = \max\limits_{i=1,\dots,m_0} |a_i|$. We can write

$$\left\| \sum_{i=1}^{m_0} a_i v_i \right\| = \left\| a_{i_0} v_{i_0} + \sum_{i=1,\, i \neq i_0}^{m_0} a_i v_i \right\|.$$

Set $w = \sum\limits_{i=1,\, i\neq i_0}^{m_0} a_i v_i \in l^\infty(I)$ and take a sequence $(x^{n_k})_k \subset \{x^i : i \in I\}$ with $|x^{n_k}| \to \|x\|$ if $k \to \infty$. Then, $|v^{h(n_k)}| \to \|v\|$ for $k \to \infty$. Without loss of generality, we may assume that $|v^{h(n_k)}| > \max\{r_2, \dots, r_{m_0}\} \cdot \|v\|$ for all $k \in \mathbb{N}$. Then, by Proposition 2.1.12, we have $|v_i^{h(n_k)}| = |v^{h(n_k)}|$ for all $k \in \mathbb{N}$ and $i \in \{1, \dots, m_0\}$. Since $[w, v_{i_0}]$ has no orthogonal base, by Proposition 2.1.12, we conclude that there is a subsequence $(m_k)_k$ such that

$$w^{h(m_k)} = g_{h(m_k)} v_{i_0}^{h(m_k)},$$

where $(g_{h(m_k)})_k$ is a sequence of scalars for which $|g_{h(m_k)}| = |a_{i_0}|$ for all $k \in \mathbb{N}$ and

$$\|a_{i_0} v_{i_0} + w\| = \lim_{k\to\infty} |a_{i_0} + g_{h(m_k)}| \cdot |v_{i_0}^{h(m_k)}|.$$

On the other hand,

$$\begin{aligned} w^{h(m_k)} &= \sum_{i=1,\, i\neq i_0}^{m_0} a_i v_i^{h(m_k)} = v^{h(m_k)} \cdot \sum_{i=1,\, i\neq i_0}^{m_0} a_i \lambda_i^{h(m_k)} \\ &= v_{i_0}^{h(m_k)} \cdot \frac{v^{h(m_k)}}{v_{i_0}^{h(m_k)}} \cdot \sum_{i=1, i\neq i_0}^{m_0} a_i \lambda_i^{h(m_k)}. \end{aligned}$$

Hence,

$$g_{h(m_k)} = \frac{v^{h(m_k)}}{v_{i_0}^{h(m_k)}} \cdot \sum_{i=1,\, i\neq i_0}^{m_0} a_i \lambda_i^{h(m_k)}.$$

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Thus, we obtain

$$\begin{aligned}
\Big\|\sum_{i=1}^{m_0} a_i v_i\Big\| &= \lim_{k\to\infty} |a_{i_0} + g_{h(m_k)}| \cdot |v_{i_0}^{h(m_k)}| \\
&= \lim_{k\to\infty} \Big| a_{i_0} v_{i_0}^{h(m_k)} + v^{h(m_k)} \cdot \sum_{i=1,\, i\neq i_0}^{m_0} a_i \lambda_i^{h(m_k)} \Big| \\
&= \lim_{k\to\infty} \Big| \sum_{i=1,\, i\neq i_0}^{m_0} a_i \lambda_i^{h(m_k)} | \cdot |v^{h(m_k)} \Big| \\
&= \frac{\|v\|}{\|x\|} \lim_{k\to\infty} \Big| \sum_{i=1}^{m_0} a_i \lambda_i^{h(m_k)} \Big| \cdot |x^{m_k}| \\
&= \frac{\|v\|}{\|x\|} \lim_{k\to\infty} \Big| \sum_{i=1}^{m_0} a_i \lambda_i^{h(m_k)} x^{m_k} \Big| \leqslant \frac{\|v\|}{\|x\|} \cdot \Big\| \sum_{i=1}^{m_0} a_i x_i \Big\|.
\end{aligned}$$

Assume that there exists $k_0 \in N_0$ such that

$$\Big| \sum_{i=1}^{m_0} a_i x_i^{k_0} \Big| \cdot \frac{\|v\|}{\|x\|} > \Big\| \sum_{i=1}^{m_0} a_i v_i \Big\|.$$

Then,

$$\begin{aligned}
\Big| \sum_{i=1}^{m_0} a_i x_i^{k_0} \Big| \cdot \frac{\|v\|}{\|x\|} &= \Big| \sum_{i=1}^{m_0} a_i \lambda_i^{h(k_0)} \Big| \cdot |x^{k_0}| \cdot \frac{\|v\|}{\|x\|} \\
&< \Big| \sum_{i=1}^{m_0} a_i \lambda_i^{h(k_0)} \Big| \cdot |v^{h(k_0)}| = \Big| \sum_{i=1}^{m_0} a_i v_i^{h(k_0)} \Big| \leqslant \Big\| \sum_{i=1}^{m_0} a_i v_i \Big\|,
\end{aligned}$$

a contradiction.

Now, setting $F := \overline{[(x_k)_k]}$ we provide a promised immediate extension of $[x]$. Clearly $F$, as an infinite dimensional subspace is not contained in D; thus, applying Theorem 2.1.6, we conclude that D is not strict in $l^\infty(I)$.

For the converse, observe that if $P\colon l^\infty(I) \to D$ is an orthoprojection, then the map $(I - P)\colon l^\infty(I) \to l^\infty(I)/D$ is the strict quotient. $\square$

The following conclusions are almost straightforward

**2.1.14. Corollary.** *Every strict, finite-dimensional subspace of* $l^\infty(I)$ *is a* HB*-subspace in* $l^\infty(I)$.

**2.1.15. Theorem.** *Every finite dimensional linear subspace of* $l^\infty(I)$ *which is strict has an orthonormal base.*

*Proof.* Let D be a finite dimensional linear subspace of $l^\infty(I)$ which is strict in $l^\infty(I)$. Then, by Theorem 2.1.13, D is orthocomplemented in $l^\infty(I)$. By Proposition 2.1.8, $D^o$ is orthocomplemented in $c_0(I) \simeq [l^\infty(I)]^*$. But $D^* \simeq c_0(I)/D^o$, hence $D^*$ is isometrically isomorphic to a closed subspace of $c_0(I)$. By Gruson's theorem (Theorem 1.1.4) it follows that $D^* \simeq \mathbb{K}^n$ for some $n \in \mathbb{N}$. Thus $D^{**} \simeq D \simeq \mathbb{K}^n$ as a reflexive Banach space by Proposition 1.1.8. □

Recall the following fact.

**2.1.16. Proposition** ([44, Corollary 3.7])**.** *If* D *is a weakly closed linear subspace of* $l^\infty$ *and* D *is strict in* $l^\infty$, *then* D *has the* HB*-property in* $l^\infty$.

*Proof.* Let S be a closed linear subspace of D with $\dim D/S = 1$. According to Proposition 2.1.10 it suffices to prove that $i_2(D/S)$ (where $i_2$ is the map in the diagram presented above Proposition 2.1.9) is a HB-subspace in $l^\infty/S$. Applying Proposition 2.1.9, since, by assumption, D is strict, $i_2(D/S)$ is a one-dimensional and strict subspace of $l^\infty/S$. But, by Theorem 1.3.5, $l^\infty/S \simeq \mathbb{K}^n$ for some $n \in \mathbb{N}$ or $l^\infty/S \simeq l^\infty$. In the first case the conclusion is obvious, in the second, follows form Theorem 2.1.13. □

Note that the converse is not true as the following example shows.

**2.1.17. Example** (see [44, Remark 2.3])**.** Let $E = \mathbb{K}_v^2$ (see Example 1.2.4). Then E has no orthogonal base. Since $\|\mathbb{E}\| = |\mathbb{K}|$, by [47, Theorem 2.5.6] there is a strict quotient map $\pi\colon c_0 \to E$ and $\ker\pi$ is a strict two-codimensional subspace of $c_0$. Thus, $\ker\pi$ cannot be orthocomplemented in $c_0$, since E has no orthogonal base. Let $D := (\ker\pi)^o$. Then D is a two-dimensional linear subspace of $l^\infty(\simeq c_0^*)$. Since $\ker\pi$ is strict in $c_0$, it follows from Theorem 2.1.8 that D is a HB-subspace of $l^\infty$. Assume that D is strict in $l^\infty$. Then, by Theorem 2.1.13, D is orthocomplemented in $l^\infty$ and by Theorem 2.1.15, it has an orthogonal base, a contradiction.



In the sequel, we need the following lemma, which was originally proved for $l^\infty$ by Perez-Garcia and Schikhof (see [44, Theorem 5.1 $i) \Leftrightarrow iv)$]). With cosmetic changes it works also in this context.

**2.1.18. Lemma.** *Let* $D$ *be a closed linear subspace of* $l^\infty(I)$ *such that* $D^*$ *is of countable type. Then,* $D$ *is orthocomplemented in* $l^\infty(I)$ *if and only if* $D$ *is weakly closed and for every closed subspace* $F$ *of* $D$ *with* $\dim D/F < \infty$, $D/F$ *is orthocomplemented in* $l^\infty(I)/F$.

*Proof.* $(\Rightarrow)$ Assume that $D$ is orthocomplemented in $l^\infty(I)$. Then, since $l^\infty(I)$ has a separating dual, $D$ is weakly closed in $l^\infty(I)$. The rest of this part of the proof follows from Proposition 2.1.9.

$(\Leftarrow)$ First we show that $D^*$ has an orthogonal base. By Proposition 2.1.10, $D$ is a HB-subspace of $l^\infty(I)$. Hence, the adjoint of the inclusion map $i^*\colon [l^\infty(I)]^* \to D^*$ is a strict quotient, $\ker i^* \simeq D^o$ and we imply $D^* \simeq c_0(I)/D^o$. Since, by assumption, $D^*$ is of countable type, it is enough to prove that every finite-dimensional subspace $G$ of $c_0(I)/D^o$ has an orthogonal base. So, assume that $G$ is a finite-dimensional linear subspace of $c_0(I)/D^o$ and $\pi_0\colon c_0(I) \to c_0(I)/D^o$ is the canonical surjection. Let $M$ be a subspace of $c_0(I)$ with $\pi_0(M) = G$. Note that

$$c_0(I)/(D^o + M) = (c_0(I)/D^o)/((D^o + M)/D^o).$$

Thus, the space $c_0(I)/(D^o + M)$ is of countable type as a quotient of a space of countable type. Therefore, $c_0(I)/(D^o + M)$ has a separating dual and we imply that $D^o + M$ is weakly closed in $c_0(I)$ as well as it is polar by [61, Corollary 4.8]. Let $S := (D^o + M)^o$ be a linear subspace of $l^\infty(I)$. Then,

$$S^o = (D^o + M)^{oo} = D^o + M.$$

Since $D/S$ is a finite-dimensional subspace of $l^\infty(I)/S$, thus, by assumption, $D/S$ is orthocomplemented in $l^\infty(I)/S$ and, by Proposition 2.1.8, $(D/S)^o$ is orthocomplemented in $(l^\infty(I)/S)^*$. Observe that

$(l^\infty(I)/S)^*$ is isometrically isomorphic to $S^o$; indeed, if $q\colon l^\infty(I) \to l^\infty(I)/S$ is the natural quotient map, then the required isometry $T\colon [l^\infty(I)/S]^* \to S^o$ is defined by $T(f) := f \circ q$. But then, $T((D/S)^o) = D^o$ and we imply that $D^o$ is orthocomplemented in $S^o$. Hence, there exists a closed linear subspace $M_1$ of $S^o$ which is an orthogonal complement of $D^o$ in $S^o$. Clearly, $D^o + M = D^o + M_1$. So, $\pi_0(M_1) = G$. But $M_1$, being a linear subspace of $c_0(I)$, has an orthogonal base by Theorem 1.1.4. Hence, so has $G$ and we conclude that $D^*$ has an orthogonal base.

As $D$ is weakly closed HB-subspace of $l^\infty(I)$, by 1.3.6 $D$ is reflexive. Thus, since $D^*$ has an orthogonal base, $D^{**} \simeq D \simeq l^\infty(J : s)$ for some set $J$ and a maps $s\colon J \to (0,\infty)$. Applying Proposition 2.1.7, we finally conclude that $D$ is orthocomplemented in $l^\infty(I)$. □

**2.1.19. Lemma** ([32, Lemma 3.6])**.** *Let* $D$ *be a closed linear subspace of a Banach space* $E$*. Let* $t \in (0,1)$*. If* $D$ *is* $t$*-orthocomplemented in* $E$ *then* $D^o$ *is* $t$*-orthocomplemented in* $E^*$*.*

*Proof.* Let $P\colon E \to D$ be a linear projection with $\|P\| \leqslant 1/t$. Define the map $q\colon E^* \to D^o$ by $q(f) := f - f_{/D} \circ P$. Then, $q$ is a projection. We get

$$\begin{aligned}
\|f - f_{/D} \circ P\| &= \sup_{x\neq 0} \frac{|(f - f_{/D} \circ P)(x)|}{\|x\|} \\
&\leqslant \sup_{x\neq 0} \max\left\{\frac{|f(x)|}{\|x\|}, \frac{|(f_{/D} \circ P)(x)|}{\|x\|}\right\} \\
&\leqslant \max\left\{\|f\|, \sup_{x\neq 0} \frac{|(f_{/D} \circ P)(x)|}{\|x\|}\right\} \\
&\leqslant \max\left\{\|f\|, \sup_{x\neq 0} \frac{\|f_{/D}\| \cdot \|P\| \cdot \|x\|}{\|x\|}\right\} \leqslant \frac{1}{t} \cdot \|f\|.
\end{aligned}$$

Thus, $D^o$ is t-orthocomplemented in $E^*$. □

**2.1.20. Lemma** ([31, Lemma 3.7])**.** *Let* $D$ *be a closed linear subspace of a Banach space* $E$*, such that the quotient space* $E/D$ *is of countable type. Then, for every* $t \in (0,1)$*, there exists a* $t$*-orthocomplement of* $D$*.*

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*Proof.* Let $t \in (0, 1)$. The quotient space $E/D$ is of countable type, so it has a $\sqrt{t}$-orthogonal base $\{e_1, e_2, \ldots\}$. Now let $q\colon E \to E/D$ be the quotient map, so we can choose $x_1, x_2, \ldots \in E$ such that $q(x_n) = e_n$ and $\|x_n\| \leqslant \|e_n\|/\sqrt{t}$ for each $n \in \mathbb{N}$. The formula

$$T\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) := \sum_{n=1}^{\infty} \lambda_n x_n, \quad \lambda_n \in \mathbb{K}$$

defines a linear map $T\colon E/D \to E$ for which $\|T\| \leqslant 1/t$ and $q \circ T$ is the identity on $E/D$. □

Using argumentation of Perez-Garcia and Schikhof (see [44, Problem 4]), we obtain.

**2.1.21. Theorem** ([32, Corollary 3.5]). *Let* $D$ *be a weakly closed linear subspace of* $l^\infty$ *such that* $D$ *is strict in* $l^\infty$. *Then* $D$ *is orthocomplemented in* $l^\infty$.

*Proof.* Let $D$ be a weakly closed subspace of $l^\infty$ such that $D$ is strict in $l^\infty$. By Proposition 2.1.16, $D$ is a HB-subspace. Let $F$ be a finite-codimensional closed subspace of $D$. We prove that $D/F$ is orthocomplemented in $l^\infty/F$. Using Lemma 2.1.20 we conclude that $F$ is weakly closed in $l^\infty$. From Proposition 2.1.9, since $D$ is strict in $l^\infty$, we imply that $D/F$ is a finite-dimensional and strict subspace of $l^\infty/F$. But $F$ is weakly closed in $l^\infty$ and by Theorem 1.3.5 and Theorem 1.3.7 either $l^\infty/F \simeq \mathbb{K}^n$ for some $n$ (then $D/F$ is orthocomplemented in $l^\infty/F$), or $l^\infty/F \simeq l^\infty$. If $l^\infty/F \simeq l^\infty$, it follows from Theorem 2.1.13 that $D/F$ is orthocomplemented in $l^\infty/F$. In this case, by Theorem 1.3.5, $D$ is isomorphic with $l^\infty$; hence, $D^*$ is of countable type. Applying Lemma 2.1.18 one gets that $D$ is orthocomplemented in $l^\infty$. □

However, it is unknown if the following question has an affirmative answer.

**2.1.22. Problem.** Let $D$ be a weakly closed, strict HB-subspace of $l^\infty(I)$. Is $D$ orthocomplemented in $l^\infty(I)$?

Applying the duality between strictness and HB-property, established in Proposition 2.1.8, we can characterize certain class of HB-subspaces of $c_0(I)$.



**2.1.23. Theorem** ([32, Theorem 7], [31, Theorem 3.8]). *Let* $H \subset c_0(I)$ *be a closed linear subspace such that* $c_0(I)/H$ *is of countable type. Then* $H$ *is a* HB*-subspace of* $c_0(I)$ *if and only if* $H$ *is orthocomplemented in* $c_0(I)$.

*Proof.* Assume that $H$ is a HB-subspace of $c_0(I)$ such that the quotient space $E_H := c_0(I)/H$ is of countable type. By [61, Theorem 4.4], $E_H$ is polar, thus it has a separating dual $(E_H)^*$. Hence, $H$ is weakly closed in $c_0(I)$. By [61, Corollary 4.8], $H$, as a weakly closed subspace of $c_0(I)$, is polar. Hence, $H^o = (H^{oo})^o = (H^o)^{oo}$ and the subspace $H^o \subset c_0(I)^*$ is polar, either. But $c_0(I)^* \simeq l^\infty(I)$ and $l^\infty(I)^* \simeq c_0(I)$. Thus, we can consider $H^o$ as a subspace of $l^\infty(I)$. From Proposition 2.1.8 we imply that $H^o$ is strict in $l^\infty(I)$.

Let $F \subset H^o$ be a finite-codimensional linear subspace of $H^o$. Then $H^o/F$ is a finite-dimensional subspace of $l^\infty(I)/F$ and by Proposition 2.1.9, (since $H^o$ is strict in $l^\infty(I)$) strict in $l^\infty(I)/F$. Let $t \in (0,1)$. Applying Lemma 2.1.20 we imply that $H$ is $\sqrt{t}$-orthocomplemented in $c_0(I)$. But then, by Lemma 2.1.19, $H^o$ is $\sqrt{t}$-orthocomplemented in $l^\infty(I)$. Using Lemma 2.1.20 again, we get that $F$ is $\sqrt{t}$-orthocomplemented in $H^o$ and finally conclude that $F$ is $t$-orthocomplemented in $l^\infty(I)$. By Lemma 2.1.19, $F^o$ is $t$-orthocomplemented in $(l^\infty(I))^*$. Let $j\colon F^o \to (l^\infty(I))^*$ be the inclusion map. Using [66, Proposition 6.1], we conclude that the adjoint

$$j^* : (l^\infty(I))^{**} \to (F^o)^* \cong (l^\infty(I))^{**}/F^{oo}$$

is a quotient map.

Since $F$, as a complemented linear subspace of $l^\infty(I)$, is weakly closed in $l^\infty(I)$, by [61, Corollary 4.8], it is polar. Thus, $F = F^{oo}$ and

$$l^\infty(I)/F \simeq (l^\infty(I))^{**}/F^{oo} \simeq (F^o)^*.$$

$F^o$ is a closed subspace of $c_0(I) \simeq (l^\infty(I))^*$, hence, it is isometrically isomorphic to $c_0(J)$ for some set $J$, or to $\mathbb{K}^n$. It follows that $(F^o)^* \simeq l^\infty(J)$ or $(F^o)^* \simeq \mathbb{K}^n$ and $l^\infty(I)/F \simeq l^\infty(J)$ or $l^\infty(I)/F \simeq \mathbb{K}^n$.

By Proposition 2.0.3, every linear subspace of $\mathbb{K}^n$ is orthocomplemented in $\mathbb{K}^n$. If $l^\infty(I)/F \simeq l^\infty(J)$, we can apply Theorem 2.1.13, and conclude that $H^o/F$ is orthocomplemented in $l^\infty(I)/F$. By Lemma 2.1.18, $H^o$ is orthocomplemented in $l^\infty(I)$. Finally, using Proposition 2.1.8, we deduce that $(H^o)^o = H$ is orthocomplemented in $c_0(I)$. □

**2.1.24. Corollary.** *Let* $H \subset c_0$ *be a closed, linear subspace. Then* $H$ *is a* HB*-subspace of* $c_0$ *if and only if* $H$ *is orthocomplemented in* $c_0$.

**2.1.25. Corollary.** *Let* $H \subset c_0(I)$ *be a closed, linear subspace of countable type. Then* $H$ *is a* HB*-subspace of* $c_0(I)$ *if and only if* $H$ *is orthocomplemented in* $c_0(I)$.

*Proof.* Clearly, $H$ is a HB-subspace if $H$ is orthocomplemented in $c_0(I)$ (then every $f \in H^*$ has a linear, preserving norm extension on $c_0(I)$ defined by $f \circ P$, where $P\colon c_0(I) \to H$ is an orthoprojection). So, assume that $H$ is a HB-subspace of $c_0(I)$. Let $(e_i)_{i\in I}$ be standard base of $c_0(I)$. By Gruson's theorem (Theorem 1.1.4) $H$ has an orthonormal base, say $(x_n)_n$. Then $x_n = \sum_{i\in I} a_i^n e_i$ $(n \in \mathbb{N})$, where $a_i^n \in \mathbb{K}$ and for every $n \in \mathbb{N}$ the set $\{i \in I : a_i^n \neq 0\}$ is countable. Hence, the set $I_0 = \{i \in I : a_i^n \neq 0, n \in \mathbb{N}\}$ is also countable. Let $D := \overline{[(e_i)_{i\in I_0}]}$. Then $H \subset D$ and $D$ is a linear subspace of countable type which is orthocomplemented in $c_0(I)$. Obviously, $H$ is a HB-subspace of $D$. From Corollary 2.1.24, we conclude that $H$ is orthocomplemented in $D$, hence in $c_0(I)$. □

We left as open the following question.

**2.1.26. Problem.** Let $H$ be any closed linear HB-subspace of $c_0(I)$ which is not of countable type. Is $H$ orthocomplemented in $c_0(I)$?

## The solution of Problem 2.1.2

Theorem 2.1.21 and Corollary 2.1.24 show that the question formulated in Problem 2.1.2 has an affirmative answer for the spaces $c_0$ and $l^\infty$. However, in general the answer is negative. Theorem 2.1.30 presents an example of the 4-dimensional normed space $E_4$ over $\mathbb{C}_p$, and its strict, weakly closed HB-subspace which is not orthocomplemented. The construction of such space requires to select a sequence of elements of $\mathbb{C}_p^3$, with very special properties. To prove the main result we need to prepare.

If $\mathbb{K}$ is non-spherically complete, we can select a centered sequence of closed balls $(B_{\mathbb{K},r_n}(c_n))_n$ with an empty intersection. Then, we can define the non-Archimedean norm on the linear space $\mathbb{K}^2$ (see Example 1.2.4), setting

$$\|(x_1, x_2)\|_v := \lim_{n\to\infty} |x_1 - x_2 c_n|, \quad (x_1, x_2) \in \mathbb{K}^2.$$

The normed space $(\mathbb{K}^2, \|\cdot\|_v)$ has no orthogonal base. It is quite natural to ask whether we can find a centered sequence of closed balls in the finite-dimensional space $(\mathbb{K}^n, \|.\|)$ for $n > 2$ (with the norm $\|(x_1, \ldots, x_n)\| = \max_i |x_i|$) which has not only an empty intersection, but it has some other special properties, crucial for defining specific norms on $\mathbb{K}^{n+1}$. The answer to this question, which was given for $n = 3$ by van Rooij (see [56, Theorem 1.14]), is contained in Theorem 2.1.28.

Recall that a subset $L \subset E$ is a *linear submanifold* in $E$ if there exist a linear subspace $D \subset E$ and $x \in E$ such that $L = x + D$.

The following lemma results almost directly from Proposition 2.0.3.

**2.1.27. Lemma.** *Let $E$ be a finite-dimensional normed space with an orthogonal base. Then for any $x \in E$ and for any linear submanifold $L$ in $E$ there exists $y \in L$ such that $\operatorname{dist}(x, L) = \|x - y\|$.*

*Proof.* Let $L = z + F_a$ for some $z \in E$ and linear subspace $F_a \subset E$. By Proposition 2.0.3, there is an orthocomplement $F_b$ of $F_a$ in $E$. Then, $x = x_a + x_b$, $z = z_a + z_b$, $x_a, z_a \in F_a$, $x_b, z_b \in F_b$ and

$$\begin{aligned}\operatorname{dist}(x, L) &= \inf_{u\in F_a} \|x - (z + u)\| = \inf_{u\in F_a} \|x_a + x_b - (z_a + z_b + u)\| \\ &= \inf_{u\in F_a} \max\{\|x_a - z_a - u\|, \|x_b - z_b\|\} = \|x_b - z_b\|.\end{aligned}$$

Setting $y := z + (x_a - z_a)$ we are done. □

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**2.1.28. Theorem** ([31, Theorem 2.10]). *Let $\mathbb{K}$ be separable and densely valued. For every $n \in \mathbb{N}$ there exists a centered sequence of closed balls $(B_{\mathbb{K}^n, r_k}(c_k))_k$ such that for every submanifold $L$ in $\mathbb{K}^n$ (where $\mathbb{K}^n$ is equipped with the standard maximum norm) there exists $k_0 \in \mathbb{N}$ for which*

$$L \cap B_{\mathbb{K}^n, r_{k_0}}(c_{k_0}) = \emptyset.$$

*Proof.* Denote by

$$S := \{(a, b^1, \ldots, b^{n-1}) \in \underbrace{\mathbb{K}^n \times \ldots \times \mathbb{K}^n}_{n} : \\ \|a\| \leqslant 1,\ \|b^j\| = 1 \text{ for } j = 1, \ldots, n-1\}.$$

Since $\mathbb{K}$ is separable, thus $\mathbb{K}^{n^2}$ (equipped with the standard maximum norm) and $S$ are separable. Let $(a_k, b_k^1, \ldots, b_k^{n-1})_k$ be a dense sequence in $S$. Denote by $L_k := a_k + [b_k^1, \ldots, b_k^{n-1}]$ ($k \in \mathbb{N}$), the linear submanifold in $\mathbb{K}^n$. Let $(r_k)_k$ be a decreasing sequence of elements of $|\mathbb{K}^\times|$ such that $1 > r_1 > r_2 \ldots > 1/2$. First, we select inductively a sequence of balls $B_{\mathbb{K}^n,1}(0) \supset B_{\mathbb{K}^n, r_1}(c_1) \supset B_{\mathbb{K}^n, r_2}(c_2) \ldots$ such that $L_k \cap B_{\mathbb{K}^n, r_k}(c_k) = \emptyset$ for all $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. Assume that $L_k \cap B_{\mathbb{K}^n, r_{k-1}}(c_{k-1}) \neq \emptyset$ (taking $r_0 := 1$, $c_0 := (\underbrace{1, \ldots, 1}_{n})$). We proceed to choose such $c_k \in \mathbb{K}^n$ that $L_k \cap B_{\mathbb{K}^n, r_k}(c_k) = \emptyset$. If $\operatorname{dist}(c_{k-1}, L_k) > r_k$, then it is nothing to prove, we take $c_k := c_{k-1}$.

Suppose that $\operatorname{dist}(c_{k-1}, L_k) \leqslant r_k$ and consider two cases:

(a) Assume $\|c_{k-1} - a_k\| \leqslant r_k$. Using [57, Lemma 3.14], we choose $x \in \mathbb{K}^n$ such that $r_{k-1} > \operatorname{dist}(x, [b_k^1, \ldots, b_k^{n-1}]) > r_k$ and $r_{k-1} > \|x\| > r_k$. Taking $c_k := c_{k-1} + x$ we obtain

$$\|c_k - c_{k-1}\| = \|x\| < r_{k-1};$$

hence, $c_k \in B_{r_{k-1}}(c_{k-1})$ and

$$\begin{aligned} &\|c_k - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\ &\quad = \|c_{k-1} - a_k + x - (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\ &\quad = \|x - (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| > r_k \end{aligned}$$

for all $\lambda_1, \ldots, \lambda_{n-1} \in K$, since

$$\|c_{k-1} - a_k\| < \|x - (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\|.$$

(b) Now, assume $\|c_{k-1} - a_k\| > r_k$. First, we select $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{K}$ such that

$$\|c_{k-1} - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \leqslant r_k. \tag{2.1}$$

Recall that by Lemma 2.1.27, there exist $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{K}$ for which

$$\|c_{k-1} - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\ = \mathtt{dist}(c_{k-1} - a_k, [b_k^1, \ldots, b_k^{n-1}]).$$

Applying [57, Lemma 3.14], we choose $w \in \mathbb{K}^n$ satisfying $r_{k-1} > \|w\| > r_k$ and $r_{k-1} > \mathtt{dist}(w, [b_k^1, \ldots, b_k^{n-1}]) > r_k$. Using Lemma 2.1.27 again, we can find $\mu_1, \ldots, \mu_{n-1} \in \mathbb{K}$ such that

$$r_{k-1} > \|w + (\mu_1 b_k^1 + \ldots + \mu_{n-1} b_k^{n-1})\| > r_k. \tag{2.2}$$

Taking

$$c_k := a_k - w - (\mu_1 b_k^1 + \ldots + \mu_{n-1} b_k^{n-1}) + (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1}),$$

we verify

$$\begin{aligned} \|c_k - c_{k-1}\| &= \|a_k - c_{k-1} - (\mu_1 b_k^1 + \ldots + \mu_{n-1} b_k^{n-1}) \\ &\qquad - w + (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\ &\leqslant \max\{\|c_{k-1} - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\|, \\ &\qquad \|w + (\mu_1 b_k^1 + \ldots + \mu_{n-1} b_k^{n-1})\|\} < r_{k-1} \end{aligned}$$

by (2.1) and (2.2). Consequently, for all $\nu_1, \ldots, \nu_{n-1} \in \mathbb{K}$,

$$\begin{aligned} &\|c_k - (a_k + \nu_1 b_k^1 + \ldots + \nu_{n-1} b_k^{n-1})\| \\ &= \|a_k - w - (\mu_1 b_k^1 + \ldots + \mu_{n-1} b_k^{n-1}) \\ &\quad + (\lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1}) - a_k - (\nu_1 b_k^1 + \ldots + \nu_{n-1} b_k^{n-1})\| \\ &= \left\| w - \sum_{i=1}^{n-1} (\lambda_i - \mu_i - \nu_i) b_k^i \right\| \geqslant \mathtt{dist}(w, [b_k^1, \ldots, b_k^{n-1}]) > r_k. \end{aligned}$$

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Thus, in both considered cases, $c_k \in B_{r_k}(c_{k-1})$ and $\mathrm{dist}(c_k, L_k) > r_k$.

Now, let L be an arbitrary linear submanifold in $\mathbb{K}^n$. Then $L = x_0 + F$, where F is a proper linear subspace of $\mathbb{K}^n$. Without loss of generality, we can suppose that $\dim F = n-1$. We prove that there exists $k \in \mathbb{N}$ such that $L \cap B_{\mathbb{K}^n, r_k}(c_k) = \emptyset$. We may assume that $L \cap B_{\mathbb{K}^n}(0) \neq \emptyset$. Thus, $L = a + \left[b^1, \dots, b^{n-1}\right]$ for some $(a, b^1, \dots, b^{n-1}) \in S$, where $b^1, \dots, b^{n-1}$ can be selected as an orthogonal sequence, thanks to Proposition 2.0.3. Since $(a_k, b_k^1, \dots, b_k^{n-1})_k$ is a dense sequence in S, we can choose such $k \in \mathbb{N}$ that $\|a - a_k\| < 1/2$, $\|b^i - b_k^i\| < 1/2$ for all $i = 1, \dots, n-1$. Suppose that there exists $x \in L \cap B_{\mathbb{K}^n, r_k}(c_k)$. Then $x = a + \lambda_1 b^1 + \ldots + \lambda_{n-1} b^{n-1}$ for some $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{K}$. Since $\|x\| \leqslant 1$, $\|a\| \leqslant 1$, we obtain $\|x - a\| = \|\lambda_1 b^1 + \ldots + \lambda_{n-1} b^{n-1}\| \leqslant 1$ and conclude that $|\lambda_i| \leqslant 1$ for $i = 1, \dots, n-1$ as $b^1, \dots, b^{n-1}$ is an orthogonal sequence. Then

$$
\begin{aligned}
&\|c_k - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\
&\quad = \|c_k - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1}) - x + x\| \\
&\quad = \|c_k - x + (a + \lambda_1 b^1 + \ldots + \lambda_{n-1} b^{n-1}) \\
&\qquad - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\| \\
&\quad \leqslant \max\big\{\|c_k - x\|, \|(a + \lambda_1 b^1 + \ldots + \lambda_{n-1} b^{n-1}) \\
&\qquad\quad - (a_k + \lambda_1 b_k^1 + \ldots + \lambda_{n-1} b_k^{n-1})\|\big\} \\
&\quad \leqslant \max\Big\{\|c_k - x\|, \|a - a_k\|, \max_{i=1,\dots,n-1} \|\lambda_i (b^i - b_k^i)\|\Big\} \leqslant r_k,
\end{aligned}
$$

a contradiction with $L_k \cap B_{\mathbb{K}^n, r_k}(c_k) = \emptyset$. □

Next result, applying Theorem 2.1.28, allows to select sequences of elements of separable non-spherically complete $\mathbb{K}$ and its spherical completion $\widehat{\mathbb{K}}$ with very special properties. Let $j_{\mathbb{K}} \colon \mathbb{K} \to \widehat{\mathbb{K}}$ denote the natural isometric embedding. Recall that every separable and densely valued field is non-spherically complete ([60, Theorem 20.5]).

**2.1.29. Lemma.** *Let $\mathbb{K}$ be separable and densely valued and let $n \in \mathbb{N}$. Then, there exists a sequence $(c_k)_k \subset \mathbb{K}^n$ ($\mathbb{K}^n$ is equipped with the standard maximum norm), where $c_k = (c_k^1, \dots, c_k^n)$, $|c_k^1| = \ldots = |c_k^n| = 1$ for all $k \in \mathbb{N}$, such that*

(1) *the sequence of closed balls* $(B_{\mathbb{K}^n, r_k}(c_k))_k$, *where* $r_k := \|c_k - c_{k+1}\|$ $(k \in \mathbb{N})$, *is centered,* $r := \lim_k r_k > 0$ *and for every linear submanifold* $L$ *in* $\mathbb{K}^n$ *there exists* $k_0 \in \mathbb{N}$ *for which* $L \cap B_{\mathbb{K}^n, r_{k_0}}(c_{k_0}) = \emptyset$;

(2) *for each* $i \in \{1, \dots, n\}$ *the sequence of closed balls* $(B_{\mathbb{K}, r_k}(c_k^i))_k$ *is centered and has an empty intersection;*

(3) *for each* $i \in \{1, \dots, n\}$, *for every* $\lambda, \lambda_j \in \mathbb{K}$ $(j = 1, \dots, n,\ j \neq i)$ *there is* $k_0 \in \mathbb{N}$ *such that*

$$\left| c_k^i - \sum_{j=1, j \neq i}^{n} \lambda_j c_k^j - \lambda \right| > r_{k_0} \quad \textit{for all } k > k_0;$$

(4) *if* $x_1, \dots, x_n \in \widehat{\mathbb{K}} \setminus \mathbb{K}$ *and* $x_j \in \bigcap_k B_{\widehat{\mathbb{K}}, r_k}(j_{\mathbb{K}}(c_k^i))$ *for each* $j = 1, \dots, n$, *then, for every* $j \in \{1, \dots, n\}$,

$$\operatorname{dist}(x_j, [x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, 1]) = r,$$

*for every* $j \in \{1, \dots, n\}$.

*Proof.* (1) Observe, that the sequence $(c_k)_k$ $(c_k = (c_k^1, \dots, c_k^n) \in \mathbb{K}^n$, $k \in \mathbb{N})$ constructed in the proof of Theorem 2.1.28 can be selected so that it is $|c_k^i| = |c_l^j| = 1$ for all $k, l \in \mathbb{N}$ and $i, j = 1, \dots, n$. Indeed, taking $c_0 = (\underbrace{1, \dots, 1}_{n})$, it is clear that if $r_{k-1} < 1$ then $|c_k^i| = 1$ for all $i = 1, \dots, n$ and $k \in \mathbb{N}$. Hence, by Theorem 2.1.28, there exists a required centered sequence of closed balls $(B_{\mathbb{K}^n, r_k}(c_k))_k$. If $\lim_k r_k = 0$ then there exists $c' = \lim_k c_k$ and $[c'] \cap B_{\mathbb{K}^n, r_k}(c_k) \neq \emptyset$ for each $k \in \mathbb{N}$, a contradiction.

(2) Fix $i \in \{1, \dots, n\}$. Then $|c_{k+1}^i - c_k^i| \leqslant \|c_{k+1} - c_k\| = r_k$; hence, the sequence $(B_{\mathbb{K}, r_k}(c_k^i))_k$ is centered. Assume that for some $i_0 \in \{1, \dots, n\}$ there exists $\gamma \in \mathbb{K}$ such that $\gamma \in \bigcap_k B_{\mathbb{K}, r_k}(c_k^{i_0})$. Then, taking a linear submanifold $L := \gamma e_{i_0} + [e_1, \dots, e_{i_0-1}, e_{i_0+1}, \dots, e_n]$ in $\mathbb{K}^n$, where $e_1, \dots, e_n$ is the standard base of $\mathbb{K}^n$, we obtain

$$\begin{aligned} \operatorname{dist}(c_k, L) &\leqslant \lim_{m \to \infty} \|(c_m^1, \dots, c_m^{i_0-1}, \gamma, c_m^{i_0+1}, \dots, c_m^n) - (c_k^1, \dots, c_k^n)\| \\ &= \lim_{m \to \infty} \max \left\{ |\gamma - c_k^{i_0}|, \max_{j=1,\dots,i_0-1,i_0+1,\dots,n} |c_m^j - c_k^j| \right\} \leqslant r_k \end{aligned}$$

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for all $k \in \mathbb{N}$, a contradiction with (1).

(3) Take $i \in \{1, \ldots, n\}$. Assume the contrary and suppose that there exist $\lambda, \lambda_j \in \mathbb{K}$ $(j = 1, \ldots, n,\ j \neq i)$ such that for every $k \in \mathbb{N}$ we can select $n_k \in \mathbb{N}$, $n_k > k$ for which

$$\left| c^i_{n_k} - \sum_{j=1,\, j\neq i}^{n} \lambda_j c^j_{n_k} - \lambda \right| \leqslant r_k.$$

Let $L := \lambda \cdot e_i + [e_1 + \lambda_1 e_i, \ldots, e_{i-1} + \lambda_{i-1} e_i, e_{i+1} + \lambda_{i+1} e_i, \ldots, e_n + \lambda_n e_i]$ be a linear submanifold in $\mathbb{K}^n$. Then we get

$$\begin{aligned}
\operatorname{dist}(c_{n_k}, L) &= \inf_{x \in L} \|x - c_{n_k}\| \\
&= \inf_{\mu_1, \ldots, \mu_n \in K} \|\lambda e_i + \mu_1(e_1 + \lambda_1 e_i) + \ldots + \mu_{i-1}(e_{i-1} + \lambda_{i-1} e_i) \\
&\quad + \mu_{i+1}(e_{i+1} + \lambda_{i+1} e_i) + \ldots + \mu_n(e_n + \lambda_n e_i) - c_{n_k}\| \\
&= \inf_{\mu_1, \ldots, \mu_n \in \mathbb{K}} \max \left\{ \max_{j=1, \ldots, n,\, j\neq i} |\mu_j - c^j_{n_k}|, \left| c^i_{n_k} - \sum_{j=1, j\neq i}^{n} \lambda_j \mu_j - \lambda \right| \right\} \\
&\leqslant \max \left\{ \max_{j=1, \ldots, n,\ j\neq i} |c^j_{n_k} - c^j_{n_k}|, \left| c^i_{n_k} - \sum_{j=1,\, j\neq i}^{n} \lambda_j c^j_{n_k} - \lambda \right| \right\} \leqslant r_k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{dist}(c_k, L) &= \inf_{x \in L} \|x - c_k\| = \inf_{x \in L} \|x - c_{n_k} + c_{n_k} - c_k\| \\
&\leqslant \inf_{x \in L} \max\{\|x - c_{n_k}\|, \|c_{n_k} - c_k\|\} \leqslant r_k
\end{aligned}$$

for all $k \in \mathbb{N}$, a contradiction with (1).

(4) Assume the contrary and suppose that there exist $\lambda_0, \lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$, $\lambda_i = 1$ for some $i \in \{1, \ldots, n\}$, such that $\left| \sum_{j=1}^{n} \lambda_j x_j + \lambda_0 \right| < r$. Then we get

$$\left| \sum_{j=1}^{n} \lambda_j x_j + \lambda_0 \right| = \left| \sum_{j=1}^{n} \lambda_j (x_j - j_{\mathbb{K}}(c^j_k)) + \sum_{j=1}^{n} \lambda_j j_{\mathbb{K}}(c^j_k) + \lambda_0 \right| < r. \quad (2.3)$$

But, applying (3), we can select $k_0 \in \mathbb{N}$ such that

$$\left| \sum_{j=1}^{n} \lambda_j j_{\mathbb{K}}(c^j_k) + \lambda_0 \right| = \left| \sum_{j=1}^{n} \lambda_j c^j_k + \lambda_0 \right| > r_{k_0} > r.$$

Hence, for validity of (2.3),

$$\left|\sum_{j=1}^{n} \lambda_j (x_j - j_{\mathbb{K}}(c_k^j))\right| > r_{k_0} > r.$$

But $|x_j - j_{\mathbb{K}}(c_k^j)| \to r$ for each $j \in \{1, \dots, n\}$ if $k \to \infty$, a contradiction. $\square$

Now, we are ready to prove

**2.1.30. Theorem.** *There exists a four-dimensional normed space* $E_4$ *over* $\mathbb{C}_p$ *having a two-dimensional strict* HB*-subspace* $D$ *such that* $D$ *is non-orthocomplemented in* $E_4$*.*

*Proof.* Let $\mathbb{K} = \mathbb{C}_p$ and let $(B_{\mathbb{K}^3, r_n}(c_n))_n$ ($c_n := \left(c_n^1, c_n^2, c_3^3\right)$, $|c_n^1| = |c_n^2| = |c_n^3| = 1$, $n \in \mathbb{N}$) be a centered sequence of closed balls which satisfies the conditions of Lemma 2.1.29 (i.e. the sequence of closed balls $(B_{\mathbb{K}^3, r_k}(c_k))_k$, where $r_k := \|c_k - c_{k+1}\|$ ($k \in \mathbb{N}$), which is centered, $r := \lim_k r_k > 0$ and for every linear submanifold $L$ in $\mathbb{K}^3$ there exists $k_0 \in \mathbb{N}$ for which $L \cap B_{\mathbb{K}^3, r_{k_0}}(c_{k_0}) = \emptyset$).

Denote $\lambda_n := c_n^1$, $\mu_n := c_n^2$, $\nu_n := c_n^3$ ($n \in \mathbb{N}$) and define $u_1, u_2, u_3, u_4 \in l^\infty$ by

$$\begin{aligned} u_1 &:= (1, 0, 1, 0, \dots), \\ u_2 &:= (0, 1, 0, 1, 0, \dots), \\ u_3 &:= (\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3, \dots), \\ u_4 &:= (\nu_1, 0, \nu_2, 0, \nu_3, 0, \dots). \end{aligned}$$

Let $\pi\colon l^\infty \to l^\infty/c_0$ be the natural quotient map and let $x_i := \pi(u_i)$ for $i = 1, \dots, 4$. Let $E_4 := [x_1, x_2, x_3, x_4]$ and $D := [x_1, x_4]$. We prove that $D$ is a strict, non-orthocomplemented HB-subspace of $E_4$.

Clearly $\{x_1, \dots, x_4\}$ is a base of $E_4$, thus any $x \in E_4$ can be written as $x = \sum_{i=1}^{4} a_i x_i$ for some $a_i \in \mathbb{K}$. The restricted quotient norm of such

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$x$ is given by

$$\begin{aligned}\|x\| &= \left\|\sum_{i=1}^{4} a_i x_i\right\| = \inf_{z\in c_0}\left\|\sum_{i=1}^{4} a_i u_i - z\right\| \\ &= \inf_{z=(z_1,z_2,\dots)\in c_0}\max_{n\in\mathbb{N}}\{|a_1 + a_3\lambda_n + a_4\nu_n - z_{2n-1}|, |a_2 + a_3\mu_n - z_{2n}|\} \\ &= \lim_{n\to\infty}\max\{|a_1 + a_3\lambda_n + a_4\nu_n|, |a_2 + a_3\mu_n|\}.\end{aligned}$$

Part A. First, we prove that every maximal orthogonal set in $E_4$ consists of two elements. It is easy to see that $x_1 \perp x_2$. Assume that there exists $x \in E_4$, where $x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, such that $x \perp [x_1, x_2]$. We derive a contradiction. Observe that

$$\begin{aligned}\operatorname{dist}(x,[x_1,x_2]) &= \operatorname{dist}(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, [x_1,x_2]) \\ &= \operatorname{dist}(a_3x_3 + a_4x_4, [x_1,x_2]).\end{aligned}$$

Denoting $u := a_3x_3 + a_4x_4$, we get

$$\begin{aligned}\operatorname{dist}(x,[x_1,x_2]) &= \operatorname{dist}(u,[x_1,x_2]) \\ &= \inf_{h_1,h_2\in\mathbb{K}}\|a_3x_3 + a_4x_4 - (h_1x_1 + h_2x_2)\| \\ &= \inf_{h_1,h_2\in\mathbb{K}}\lim_{n\to\infty}\max\{|a_3\lambda_n + a_4\nu_n - h_1|, |a_3\mu_n - h_2|\} \\ &= r\cdot\max\{|a_3|,|a_4|\}.\end{aligned}$$

Indeed, let $h_1^m := a_3\lambda_m + a_4\nu_m$ and $h_2^m := a_3\mu_m$, $m \in \mathbb{N}$. Then

$$\begin{aligned}&\operatorname{dist}(u,[x_1,x_2]) \leqslant \inf_{m\in\mathbb{N}}\|u - (h_1^m x_1 + h_2^m x_2)\| \\ &= \inf_{m\in\mathbb{N}}\lim_{n\to\infty}\max\{|a_3\lambda_n + a_4\nu_n - h_1^m|, |a_3\mu_n - h_2^m|\} \\ &= \inf_{m\in\mathbb{N}}\lim_{n\to\infty}\max\{|a_3\lambda_n + a_4\nu_n - (a_3\lambda_m + a_4\nu_m)|, |a_3\mu_n - a_3\mu_m|\}.\end{aligned}$$

But

$$\begin{aligned}&\lim_{n\to\infty}|a_3\lambda_n + a_4\nu_n - (a_3\lambda_m + a_4\nu_m)| \\ &\qquad\qquad \leqslant \lim_{n\to\infty}\max\{|a_3(\lambda_n - \lambda_m)|, |a_4(\nu_n - \nu_m)|\},\end{aligned}$$

thus

$$\begin{aligned}
&\inf_{m\in\mathbb{N}} \lim_{n\to\infty} \max\{|a_3\lambda_n + a_4\nu_n - (a_3\lambda_m + a_4\nu_m)|, |a_3\mu_n - a_3\mu_m|\}\\
&\leqslant \lim_{m\to\infty} \lim_{n\to\infty} \max\{|a_3(\lambda_n - \lambda_m)|, |a_4(\nu_n - \nu_m)|, |a_3\mu_n - a_3\mu_m|\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad = r \cdot \max\{|a_3|, |a_4|\}.
\end{aligned}$$

Hence $\mathrm{dist}(u, [x_1, x_2]) \leqslant r \cdot \max\{|a_3|, |a_4|\}$. But by Lemma 2.1.29 (3)

$$\begin{aligned}
\lim_{n\to\infty} |a_3\lambda_n + a_4\nu_n - h_1| &> r \cdot \max\{|a_3|, |a_4|\},\\
\lim_{n\to\infty} |a_3\mu_n - h_2| &> r|a_3|,
\end{aligned} \tag{2.4}$$

for every $h_1, h_2 \in \mathbb{K}$, hence, $\mathrm{dist}(x, [x_1, x_2]) = r \cdot \max\{|a_3|, |a_4|\}$.

It follows also from (2.4) that $\mathrm{dist}(x, [x_1, x_2])$ is not attained, conflicting with the assumption $x \perp [x_1, x_2]$. By [57, Theorem 5.4] all maximal orthogonal sets in $E_4$ have the same cardinality, thus, every maximal orthogonal sequence in $E_4$ consists of two elements.

Part B. Let $E_3 := [x_1, x_2, x_3]$. We show that every two-dimensional linear subspace of $E_3$ has an orthogonal base. Clearly, $\{x_1, x_2\}$ is an orthogonal base of $[x_1, x_2]$. Thus, to finish this part of the proof it is enough to show that taking any nonzero $w_1 := a_1x_1 + a_2x_2$ and $w_2 := b_1x_1 + b_2x_2 + x_3$ ($a_1, a_2, b_1, b_2 \in \mathbb{K}$), $\mathrm{dist}(w_2, [w_1])$ is attained. Note that

$$\begin{aligned}
\mathrm{dist}(w_2, [w_1]) &= \inf_{h\in\mathbb{K}} \|b_1x_1 + b_2x_2 + x_3 - h \cdot (a_1x_1 + a_2x_2)\|\\
&= \inf_{h\in\mathbb{K}} \lim_{n\to\infty} \max\{|b_1 - h \cdot a_1 + \lambda_n|, |b_2 - h \cdot a_2 + \mu_n|\}.
\end{aligned}$$

Clearly, $a_1 \neq 0$ or $a_2 \neq 0$. So, suppose that $a_1 \neq 0$ (assuming $a_2 \neq 0$ we work almost identically). Set

$$h_m := \frac{\lambda_m}{a_1} + \frac{b_1}{a_1}, \quad m \in \mathbb{N}.$$

Then, for fixed $m \in \mathbb{N}$,

$$|b_1 - h_m \cdot a_1 + \lambda_n| = \left|b_1 - \left(\frac{\lambda_m}{a_1} + \frac{b_1}{a_1}\right) \cdot a_1 + \lambda_n\right| = |\lambda_m - \lambda_n| \tag{2.5}$$

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and

$$|b_2 - h_m \cdot a_2 + \mu_n| = \left|b_2 - \left(\frac{\lambda_m}{a_1} + \frac{b_1}{a_1}\right) \cdot a_2 + \mu_n\right|$$
$$= \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_n\right|. \tag{2.6}$$

Assume now that $|a_2/a_1 \leqslant 1$. Then, applying Lemma 2.1.29 (3), we imply that there exists $m_0 \in \mathbb{N}$ such that for any $m > m_0$

$$\left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_m\right| > r_{m_0}. \tag{2.7}$$

We can find $M_0 \in \mathbb{N}$, $M_0 > m_0$ such that $|\mu_m - \mu_n| < r_{m_0}$ and $|\lambda_m - \lambda_n| < r_{m_0}$ for every $m, n > M_0$. Thus, for $m, n > M_0$ we get

$$\left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_n\right| = \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_m - \mu_m + \mu_n\right|$$
$$= \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_m\right| > r_{m_0}$$

and, for $m > M_0$, using (2.5) and (2.6) we obtain

$$\|w_2 - h_m w_1\| = \lim_{n\to\infty} \max\{|b_1 - h_m \cdot a_1 + \lambda_n|, |b_2 - h_m \cdot a_2 + \mu_n|\}$$
$$= \lim_{n\to\infty} \max\left\{|\lambda_m - \lambda_n|, \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_n\right|\right\}$$
$$= \lim_{n\to\infty} \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_n\right| > r_{m_0}.$$

Fix $m > M_0$. Then, for $k > m$, we obtain

$$\|w_2 - h_k w_1\| = \|(w_2 - h_m w_1) + (h_m w_1 - h_k w_1)\|$$
$$= \lim_{n\to\infty} \left|b_2 - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_m + \mu_n\right|$$

since

$$\|h_m w_1 - h_k w_1\| = \|(h_m - h_k) \cdot (a_1 x_1 + a_2 x_2)\|$$
$$= \max\{|(h_m - h_k) \cdot a_1|, |(h_m - h_k) \cdot a_2|\}$$
$$= \max\left\{|\lambda_m - \lambda_k|, \left|\frac{a_2}{a_1}(\lambda_m - \lambda_k)\right|\right\} \leqslant r_{m_0}.$$

Hence, for any $k > m > M_0$,

$$\mathtt{dist}(w_2, [w_1]) \leqslant \lim_{n\to\infty} \|w_2 - h_n w_1\| = \|w_2 - h_k w_1\|.$$

Now, suppose that there is $h \in \mathbb{K}$ for which

$$\|w_2 - h w_1\| < \lim_{n\to\infty} \|w_2 - h_n w_1\|.$$

Then

$$\lim_{n\to\infty} \max\{|b_1 - h\cdot a_1 + \lambda_n|, |b_2 - h\cdot a_2 + \mu_n|\} \leqslant r_{m_0}. \tag{2.8}$$

On the other hand, by (2.7), for large $n$, we have

$$\begin{aligned}\left|(b_2 - h\cdot a_2 + \mu_n) - \frac{a_2}{a_1}(b_1 - h\cdot a_1 + \lambda_n)\right|& \\ = \left|b_2 + \mu_n - \frac{a_2 b_1}{a_1} - \frac{a_2}{a_1}\lambda_n\right| &\geqslant r_{m_0},\end{aligned}$$

a contradiction with (2.8). Now, suppose that $|a_2/a_1| > 1$. Then, obviously $a_2 \neq 0$. Set

$$h_m := \frac{\mu_m}{a_2} + \frac{b_2}{a_2}, \quad m \in \mathbb{N}.$$

Following similarly like in the previous part, for fixed $m \in \mathbb{N}$ we get

$$\begin{aligned}|b_2 - h_m\cdot a_2 + \mu_n| &= \left|b_2 - \left(\frac{\mu_m}{a_2} + \frac{b_2}{a_2}\right)\cdot a_2 + \mu_n\right| \\ &= |\mu_m - \mu_n|, \qquad (2.9)\\ |b_1 - h_m\cdot a_1 + \lambda_n| &= \left|b_1 - \left(\frac{\mu_m}{a_2} + \frac{b_2}{a_2}\right)\cdot a_1 + \lambda_n\right| \\ &= \left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right|. \qquad (2.10)\end{aligned}$$

Applying Lemma 2.1.29 (3) again, we imply that there exists $m_0 \in \mathbb{N}$ such that, for any $m > m_0$,

$$\left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right| > r_{m_0}. \tag{2.11}$$

We can find $M_0 \in \mathbb{N}$, $M_0 > m_0$ such that $|\mu_m - \mu_n| < r_{m_0}$ and $|\lambda_m - \lambda_n| < r_{m_0}$ for every $m, n > M_0$. Thus, for $m, n > M_0$, we get

$$\begin{aligned}\left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\lambda_m + \lambda_n\right| &= \left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\lambda_m + \lambda_m - \lambda_m + \lambda_n\right| \\ &= \left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_m\right| > r_{m_0}.\end{aligned}$$

and, for $m > M_0$, using (2.9) and (2.10), we obtain

$$\begin{aligned}\|w_2 - h_m w_1\| &= \lim_{n\to\infty} \max\{|b_1 - h_m \cdot a_1 + \lambda_n|, |b_2 - h_m \cdot a_2 + \mu_n|\} \\ &= \lim_{n\to\infty} \max\left\{\left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right|, |\mu_m - \mu_n|\right\} \\ &= \lim_{n\to\infty}\left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right| > r_{m_0}.\end{aligned}$$

Fix $m > M_0$. Then, for $k > m$, we obtain

$$\begin{aligned}\|w_2 - h_k w_1\| &= \|(w_2 - h_m w_1) + (h_m w_1 - h_k w_1)\| \\ &= \lim_{n\to\infty}\left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right|\end{aligned}$$

since

$$\begin{aligned}\|h_m w_1 - h_k w_1\| &= \|(h_m - h_k)\cdot(a_1 x_1 + a_2 x_2)\| \\ &= \max\{|(h_m - h_k)\cdot a_1|, |(h_m - h_k)\cdot a_2|\} \\ &= \max\left\{\left|\frac{a_1}{a_2}(|\mu_m - \mu_k|)\right|, |\mu_m - \mu_k|\right\} \leqslant r_{m_0}.\end{aligned}$$

Hence, $\mathrm{dist}(w_2, [w_1]) \leqslant \lim_{n\to\infty} \|w_2 - h_n w_1\| = \|w_2 - h_k w_1\|$ for any $k > m > M_0$.

Now, suppose that there is $h \in \mathbb{K}$ for which

$$\|w_2 - h w_1\| < \lim_{n\to\infty} \|w_2 - h_n w_1\|.$$

It means that

$$\lim_{n\to\infty} \max\{|b_1 - h\cdot a_1 + \lambda_n|, |b_2 - h\cdot a_2 + \mu_n|\} \leqslant r_{m_0}. \tag{2.12}$$

But, by (2.11), we get for large $n$

$$\left|\frac{a_1}{a_2}(b_2 - h\cdot a_2 + \mu_n) - (b_1 - h\cdot a_1 + \lambda_n)\right| \\ = \left|b_1 - \frac{a_1 b_2}{a_2} - \frac{a_1}{a_2}\mu_m + \lambda_n\right| \geqslant r_{m_0},$$

a contradiction with (2.12).

Hence, $\operatorname{dist}(w_2, [w_1]) = \lim_{n\to\infty} \|w_2 - h_n w_1\|$ and $\operatorname{dist}(w_2, [w_1])$ is attained.

Part C. The most laborious part of this proof is showing that $D$ is not orthocomplemented in $E_4$. Assume the contrary and suppose that there exists a linear subspace $D_0 \subset E_4$ which is an orthocomplement of $D$. Note that $D_0$, two-dimensional linear subspace of $E_4$, cannot have an orthogonal base, otherwise we can select an orthogonal sequence in $E_4$ consisting of three elements, contradicting the conclusion of Part A. By Part B, we can deduce $D_0 \subsetneq E_3$.

We can write $D_0 = [w_1, w_2]$ for some $w_1, w_2 \in E_4$. In fact, it is enough to consider the following two cases:

(a) $w_1 := a_1x_1 + a_2x_2 + x_4, w_2 := b_1x_1 + b_2x_2 + x_3$ ($a_1, a_2, b_1, b_2 \in \mathbb{K}$; note that $a_2 \neq 0$ since $w_1 \notin D$),

(b) $w_1 := a_1x_1 + x_3 + a_4x_4, w_2 := b_1x_1 + b_2x_2$ ($a_1, a_4, b_1, b_2 \in \mathbb{K}$).

In order to finish this part of the proof, we demonstrate that in both considered cases $D_0$ has an orthogonal base (note that using Lemma 1.2.1 it is equivalent to finding such $k_0 \in \mathbb{K}$ for which $\|w_2 - k_0 w_1\| = \operatorname{dist}(w_2, [w_1])$), deriving a contradiction.

Consider the case (a). By assumption that $w_1 \perp D$ and $w_2 \perp D$, we imply that $\lim_{n\to\infty} |a_1 + \nu_n| \leqslant |a_2|$ and $\lim_{n\to\infty} |b_1 + \lambda_n| \leqslant \lim_{n\to\infty} |b_2 + \mu_n|$. Let $k \in \mathbb{K}$. Then

$$\|w_2 - kw_1\| = \lim_{n\to\infty} \max\{|b_1 + \lambda_n + k(a_1 + \nu_n)|, |b_2 + \mu_n + ka_2|\}.$$

If $\lim_{n\to\infty}(|b_1 + \lambda_n| \cdot |a_2|) \neq \lim_{n\to\infty}(|b_2 + \mu_n| \cdot |a_1 + \nu_n|)$ then, taking $k := -(b_2 + \mu_m)/a_2$ where $m \in \mathbb{N}$ is chosen in such a way that

$$\lim_{n\to\infty} |b_1 + \lambda_n| > \lim_{n\to\infty} |\mu_n - \mu_m|,$$

we get

$$\|w_2 - kw_1\| = \lim_{n\to\infty} \max\left\{\left|b_1 + \lambda_n - \frac{b_2+\mu_m}{a_2}(a_1+\nu_n)\right|, |\mu_n - \mu_m|\right\}$$

and observe that

$$\inf_{k\in\mathbb{K}} \|w_2 - kw_1\| = \lim_{n\to\infty} \max\left\{|b_1+\lambda_n|, \left|\frac{b_2+\mu_n}{a_2}\right| \cdot |a_1+\nu_n|\right\}.$$

Hence, $\mathtt{dist}(w_2, [w_1])$ is attained and we conclude that $D_0$ has an orthogonal base, a contradiction.

Now, we assume that

$$\lim_{n\to\infty}(|b_1+\lambda_n|\cdot|a_2|) = \lim_{n\to\infty}(|b_2+\mu_n|\cdot|a_1+\nu_n|). \tag{2.13}$$

Let $j_{\mathbb{K}}\colon \mathbb{K} \hookrightarrow \widehat{\mathbb{K}}$ be a natural embedding of $\mathbb{K}$ into its spherical completion $\widehat{\mathbb{K}}$. Fix $\lambda_0, \mu_0, \nu_0 \in \widehat{\mathbb{K}}$ such that $\lambda_0 \in \bigcap_n B_{\widehat{\mathbb{K}},r_n}(j_{\mathbb{K}}(\lambda_n)), \mu_0 \in \bigcap_n B_{\widehat{\mathbb{K}},r_n}(j_{\mathbb{K}}(\mu_n))$, $\nu_0 \in \bigcap_n B_{\widehat{\mathbb{K}},r_n}(j_{\mathbb{K}}(\nu_n))$. Then, applying simplifications suggested at the beginning of this section, we get

$$\lim_{n\to\infty}|b_1+\lambda_n| = \lim_{n\to\infty}|b_1+\lambda_0-\lambda_0+\lambda_n| = |b_1+\lambda_0|,$$

since we can choose such $n_0 \in \mathbb{N}$ that $|b_1+\lambda_0| > |\lambda_n - \lambda_0|$ for all $n > n_0$. Using the same argumentation, we obtain

$$\lim_{n\to\infty}(|b_2+\mu_n|\cdot|a_1+\nu_n|) = |b_2+\mu_0|\cdot|a_1+\nu_0|$$

and conclude that (2.13) is equivalent to

$$|b_1+\lambda_0|\cdot|a_2| = |b_2+\mu_0|\cdot|a_1+\nu_0|. \tag{2.14}$$

Observe that

$$|b_1+\lambda_0+k(a_1+\nu_0)| = \lim_{n\to\infty}|b_1+\lambda_n+k(a_1+\nu_n)| \tag{2.15}$$

$$|b_2+\mu_0+ka_2| = \lim_{n\to\infty}|b_2+\mu_n+ka_2|. \tag{2.16}$$

Indeed, using Lemma 2.1.29 (3), we can find such $n_0 \in \mathbb{N}$ that

$$\max\{|\lambda_0-\lambda_n|, |k|\cdot|\nu_0-\nu_n|\} < |\lambda_n + b_1 + ka_1 + k\nu_n|$$

for all $n > n_0$. Then

$$\begin{aligned}
&|b_1 + \lambda_0 + k(a_1 + \nu_0)| \\
&\qquad = |(\lambda_0 - \lambda_n) + \lambda_n + b_1 + k(a_1 + \nu_n) + k(\nu_0 - \nu_n)| \\
&\qquad\qquad = |b_1 + \lambda_n + k(a_1 + \nu_n)|
\end{aligned}$$

for all $n > n_0$; hence, the condition (2.15) is valid. Using the same argumentation we can prove (2.16).

By simple calculations we obtain

$$\begin{aligned}
&|b_2 + \mu_0 + ka_2| \cdot \left|\frac{a_1 + \nu_0}{a_2}\right| = \left|\frac{(b_2 + \mu_0)(a_1 + \nu_0)}{a_2} + k(a_1 + \nu_0)\right| \\
&= \left|\frac{(b_2 + \mu_0)(a_1 + \nu_0)}{a_2} - (b_1 + \lambda_0) + (b_1 + \lambda_0) + k(a_1 + \nu_0)\right| \\
&= \left|b_1 + \lambda_0 - \frac{(b_2 + \mu_0)(a_1 + \nu_0)}{a_2} - (b_1 + \lambda_0 + k(a_1 + \nu_0))\right|. \qquad (2.17)
\end{aligned}$$

By assumption $w_1 \perp D$, thus $\lim\limits_{n\to\infty} |a_1 + \nu_n| \leqslant |a_2|$ and $|a_2| \geqslant |a_1 + \nu_0| > r$, since $\lim\limits_{n\to\infty} |a_1 + \nu_n| = |a_1 + \nu_0|$. Observe, that

$$\begin{aligned}
r^2 &= \lim_{k\to\infty} (|\mu_k - \mu_0| \cdot |\nu_k - \nu_0|) \\
&= \lim_{k\to\infty} |\mu_0\nu_0 - \nu_k\mu_0 - \mu_k\nu_0 + \mu_k\nu_k| \geqslant \mathrm{dist}(\mu_0\nu_0, [\mu_0, \nu_0, 1]).
\end{aligned}$$

Hence, we can choose $z \in [\{\mu_0, \nu_0, 1\}]$ (where $[\{\mu_0, \nu_0, 1\}]$ is the $\mathbb{K}$-vector linear subspace of $\widehat{\mathbb{K}}$ spanned by $\{\mu_0, \nu_0, 1\}$) such that $|\mu_0\nu_0 - z| < r \cdot |a_1 + \nu_0| \leqslant r \cdot |a_2|$. Then, $|1/a_2| \cdot |\mu_0\nu_0 - z| = |\mu_0\nu_0/a_2 - z/a_2| < r$. By Lemma 2.1.29 (4), we get

$$d_0 := \left|\lambda_0 - \left(\frac{a_1 b_1}{a_2} - b_1 + \frac{b_2}{a_2}\nu_0 + \frac{a_1}{a_2}\mu_0 - \frac{1}{a_2}z\right)\right| > r$$

and

$$\begin{aligned}
&\left|b_1 + \lambda_0 - \frac{(b_2 + \mu_0)(a_1 + \nu_0)}{a_2}\right| \\
&= \left|\lambda_0 - \left(\frac{a_1 b_1}{a_2} - b_1 + \frac{b_2}{a_2}\nu_0 + \frac{a_1}{a_2}\mu_0 + \frac{1}{a_2}\nu_0\mu_0\right)\right| \\
&= \left|\lambda_0 - \left(\frac{a_1 b_1}{a_2} - b_1 + \frac{b_2}{a_2}\nu_0 + \frac{a_1}{a_2}\mu_0 - \frac{1}{a_2}z\right) + \frac{1}{a_2}(\nu_0\mu_0 - z)\right| \\
&= d_0 > r. \qquad (2.18)
\end{aligned}$$

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Now, assume that $k \in \mathbb{K}$ is chosen in such a way that

$$\lim_{n\to\infty} |b_2 + \mu_n + ka_2| < d_0.$$

Thus, $|b_2 + \mu_0 + ka_2| < d_0$ by (2.15) and we get

$$|b_2 + \mu_0 + ka_2| \cdot \left|\frac{a_1 + \nu_0}{a_2}\right| < d_0, \tag{2.19}$$

since $|(a_1 + \nu_0)/a_2| \leqslant 1$ follows from the assumption $w_1 \perp D$. Now, from (2.19), applying (2.17) and (2.18), we obtain $|b_1+\lambda_0+k(a_1+\nu_0)| = d_0$ and conclude that

$$\begin{aligned}
&\mathtt{dist}(w_2 - [w_1]) \\
&\qquad = \inf_{k\in\mathbb{K}} \lim_{n\to\infty} \max\{|b_1 + \lambda_n + k(a_1 + \nu_n)|, |b_2 + \mu_n + ka_2|\} \\
&\qquad = \inf_{k\in\mathbb{K}} \max\{|b_1 + \lambda_0 + k(a_1 + \nu_0)|, |b_2 + \mu_0 + ka_2|\} = d_0.
\end{aligned}$$

Since $|b_2 + \mu_0 + ka_2| < d_0$, we obtain $|b_1 + \lambda_0 + k(a_1 + \nu_0)| = d_0$ if a scalar $k \in \mathbb{K}$ satisfies $|b_2 + \mu_0 + ka_2| = \lim_{n\to\infty} |b_2 + \mu_n + ka_2| < d_0$. Hence, $\mathtt{dist}(w_2, [w_1])$ is attained and $D_0$ has an orthogonal base, a contradiction.

Now, consider the case (b). Assuming $w_1 \perp D$ and $w_2 \perp D$ we get $\lim_{n\to\infty} |a_1 + \lambda_n + a_4\nu_n| \leqslant |\mu_n|$ for all $n \in \mathbb{N}$ and $|b_1| \leqslant |b_2|$. Let $k \in \mathbb{K}$. Then

$$\|w_1 - kw_2\| = \lim_{n\to\infty} \max\{|a_1 + \lambda_n + a_4\nu_n + kb_1|, |\mu_n + kb_2|\}.$$

If $\lim_{n\to\infty} (|a_1 + \lambda_n + a_4\nu_n| \cdot |b_2|) \neq \lim_{n\to\infty} (|\mu_n| \cdot |b_1|)$, then

$$\inf_{k\in\mathbb{K}} \|w_1 - kw_2\| = \lim_{n\to\infty} \max\left\{|a_1 + \lambda_n + a_4\nu_n|, \left|\frac{b_1}{b_2}\mu_n\right|\right\}.$$

Taking $k := \mu_m/b_2$ where $m \in \mathbb{N}$ is chosen, thanks to Lemma 2.1.29 (3), in such a way that $\lim_{n\to\infty} |a_1+\lambda_n+a_4\nu_n| > \lim_{n\to\infty} |\mu_n-\mu_m|$, we conclude that $\mathtt{dist}(w_1, [w_2])$ is attained, a contradiction. Assume now that

$$\lim_{n\to\infty} (|a_1 + \lambda_n + a_4\nu_n| \cdot |b_2|) = \lim_{n\to\infty} (|\mu_n| \cdot |b_1|).$$

Then, since $\lim_{n\to\infty} |a_1 + \lambda_n + a_4 \nu_n| > r$, we get $|b_1|/|b_2| > r$. Recall that $|b_1|/|b_2| \leqslant 1$ by the assumption $w_2 \perp D$. Observe that

$$\|w_1 - kw_2\| = \lim_{n\to\infty} \max\left\{\left|\frac{b_1}{b_2}\right| \cdot \left|\frac{b_2}{b_1}(a_1 + \lambda_n + a_4\nu_n) + kb_2\right|, \right.$$
$$\left. |\mu_n + kb_2|\right\}. \quad (2.20)$$

Let $R := \lim_{n\to\infty} |b_2(a_1 + \lambda_n + a_4\nu_n)/b_1 - \mu_n|$. By Lemma 2.1.29 (3), $R > |b_2|r/|b_1|$. Suppose that $\lim_{n\to\infty} |\mu_n + kb_2| < R$. Then, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{b_1}{b_2}\right|\left|\frac{b_2}{b_1}(a_1 + \lambda_n + a_4\nu_n) + kb_2\right| \\
&\quad = \lim_{n\to\infty}\left|\frac{b_1}{b_2}\right|\left|\frac{b_2}{b_1}(a_1 + \lambda_n + a_4\nu_n) - \mu_n + \mu_n + kb_2\right| \\
&\quad = \left|\frac{b_1}{b_2}\right| \cdot R > r. \qquad (2.21)
\end{aligned}$$

Hence, from (2.20), applying (2.21), we get that $\inf_{k\in\mathbb{K}} \|w_1 - kw_2\| = |b_1/b_2| \cdot R$. But, it follows that $\mathrm{dist}(w_1, [w_2])$ is attained for $k_0 \in \mathbb{K}$, satisfying $\lim_{n\to\infty} |\mu_n + k_0 b_2| < R$; thus, $D_0$ has an orthogonal base, a contradiction.

Part D. We demonstrate that D is a HB-subspace. Let $f\colon D \to \mathbb{K}$ be a linear functional, given by $f(a_1x_1 + a_4x_4) := a_1\lambda_1 + a_4\lambda_4$ ($a_1, a_4, \lambda_1, \lambda_4 \in K$). First, suppose that $\lambda_1 = 0$. Then, we obtain $\|f\| = |\lambda_4|/r$ and

$$f_0(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) := a_4\lambda_4$$

is the linear extension on the whole of $E_4$ with the same norm.

Assume now $\lambda_1 \neq 0$. Then

$$\begin{aligned}
\|f\| &:= \sup_{x\in D} \frac{|f(x)|}{\|x\|} = \sup_{a_1,a_4\in\mathbb{K}} \frac{|a_1\lambda_1 + a_4\lambda_4|}{\|a_1x_1 + a_4x_4\|} = \sup_{k\in\mathbb{K}} \lim_{n\to\infty} \frac{|\lambda_1||k + \lambda_4/\lambda_1|}{|k + \nu_n|} \\
&= \sup_{k\in\mathbb{K}} \lim_{n\to\infty} \frac{|\lambda_1||k + \nu_n - \nu_n + \lambda_4/\lambda_1|}{|k + \nu_n|} = \lim_{n\to\infty} \frac{|\lambda_4 - \lambda_1\nu_n|}{r}.
\end{aligned}$$

Choose $\lambda_3 \in \mathbb{K}$ such that $\lim_{n\to\infty} |\lambda_3/\lambda_1 - \lambda_n| < \lim_{n\to\infty} |\lambda_4/\lambda_1 - \nu_n|$. Let $f_0\colon E_4 \to \mathbb{K}$ be a linear extension of f, defined by

$$f_0(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) := a_1\lambda_1 + a_3\lambda_3 + a_4\lambda_4.$$

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Then

$$\begin{aligned}
\frac{|f_0(x)|}{\|x\|} &= \lim_{n\to\infty} \frac{|a_1\lambda_1 + a_3\lambda_3 + a_4\lambda_4|}{\max\{|a_1 + a_3\lambda_n + a_4\nu_n|, |a_2 + a_3\mu_n|\}} \\
&\leqslant \lim_{n\to\infty} \frac{|a_1\lambda_1 + a_3\lambda_3 + a_4\lambda_4|}{|a_1 + a_3\lambda_n + a_4\nu_n|} = \lim_{n\to\infty} \frac{|\lambda_1| \cdot |a_1 + a_3\lambda_3/\lambda_1 + a_4\lambda_4/\lambda_1|}{|a_1 + a_3\lambda_n + a_4\nu_n|} \\
&= \lim_{n\to\infty} \frac{|\lambda_1| \cdot |a_1 + a_3\lambda_n + a_4\nu_n + a_3(\lambda_3/\lambda_1 - \lambda_n) + a_4(\lambda_4/\lambda_1 - \nu_n)|}{|a_1 + a_3\lambda_n + a_4\nu_n|} \\
&\leqslant \lim_{n\to\infty} \max\left\{ \frac{|\lambda_1| \cdot |a_1 + a_3\lambda_n + a_4\nu_n|}{|a_1 + a_3\lambda_n + a_4\nu_n|}, \right. \\
&\qquad \left. \frac{|\lambda_1| \cdot |a_3(\lambda_3/\lambda_1 - \lambda_n)|}{|a_1 + a_3\lambda_n + a_4\nu_n|}, \frac{|\lambda_1| \cdot |a_4(\lambda_4/\lambda_1 - \nu_n)|}{|a_1 + a_3\lambda_n + a_4\nu_n|} \right\} \\
&\leqslant \max\left\{ |\lambda_1|, \lim_{n\to\infty} \frac{|\lambda_3 - \lambda_1\lambda_n|}{r}, \lim_{n\to\infty} \frac{|\lambda_4 - \lambda_1\nu_n|}{r} \right\} = \lim_{n\to\infty} \frac{|\lambda_4 - \lambda_1\nu_n|}{r}.
\end{aligned}$$

Hence $\|f_0\| \leqslant \|f\|$.

Part E. We prove that $D$ is strict in $E_4$. Let $x \in E_4 \setminus D$. We can write $x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ for some $a_1, a_2, a_3, a_4 \in \mathbb{K}$. Let $u := a_2x_2 + a_3x_3$. Since, applying Part B, $[x_1, u]$ has an orthogonal base as a two-dimensional subspace of $E_3$, we can choose $\lambda \in \mathbb{K}$ such that $(u + \lambda x_1) \perp [x_1]$.

Now, we show that the element $x - (a_1 - \lambda)x_1 - a_4x_4$ is orthogonal to $D$. Denoting $d := \lambda_1x_1 + \lambda_4x_4 \in D$, we get

$$\begin{aligned}
\|(x - (a_1 - \lambda)x_1 - a_4x_4) + d\| &= \|(x - (a_1 - \lambda)x_1 - a_4x_4) + (\lambda_1x_1 + \lambda_4x_4)\| \\
&= \|(u + \lambda x_1) + (\lambda_1x_1 + \lambda_4x_4)\|.
\end{aligned}$$

But, it is easy to observe, $[x_1, x_4]$ has no orthogonal base; hence, we can find $\mu \in \mathbb{K}$ such that

$$\|\mu x_1\| = \|\lambda_1x_1 + \lambda_4x_4\| \quad \text{and} \quad \|\mu x_1 + (\lambda_1x_1 + \lambda_4x_4)\|\| < \|\lambda_1x_1 + \lambda_4x_4\|.$$

Applying $(u + \lambda x_1) \perp [x_1]$, we obtain

$$\begin{aligned}
&\|(u + \lambda x_1) + (\lambda_1x_1 + \lambda_4x_4)\| \\
&\qquad = \|(u + \lambda x_1) - \mu x_1 + \mu x_1 + (\lambda_1x_1 + \lambda_4x_4)\| \\
&\qquad = \|(u + \lambda x_1) - \mu x_1\| = \max\{\|(u + \lambda x_1)\|, \|\mu x_1\|\} \\
&\qquad \geqslant \|\mu x_1\| = \|(\lambda_1x_1 + \lambda_4x_4)\| = \|d\|
\end{aligned}$$

and we conclude that $D$ is orthocomplemented in $D + [x]$; hence, $D$ is strict in $E_4$. □

**2.1.31. Remark.** It is worthwhile to note that the dimension of the constructed normed space $E_4$ is the lowest possible. In fact, taking an arbitrary normed space $E$ with $\dim E = 3$ and its strict HB-subspace $D$, we observe that if $\dim D = 1$, the orthocomplementation of $D$ follows from the HB-property and if $\dim D = 2$, it follows from the strictness.



## 2.2 Hilbertian spaces

A non-Archimedean normed space $E$ is called *Hilbertian*, if every finite-dimensional linear subspace of $E$ has an orthogonal complement. We say that $E$ is *Cartesian* if its every finite-dimensional linear subspace has an orthogonal base.

In the classical functional analysis (i.e. where the scalar field is $\mathbb{R}$ or $\mathbb{C}$) Hilbert spaces play an especially important role. Unfortunately, their non-Archimedean infinite-dimensional counterparts do not exist, i.e. there is no infinite-dimensional Banach space with an inner product for which every closed linear subspace has an orthogonal complement. Quite naturally, one looks for classes of Banach spaces with similar, although weaker properties. Cartesian and Hilbertian spaces are examples of such classes. Note that if $E$ is Hilbertian and $\|E\| \subset |\mathbb{K}|^{1/2}$ then $E$ admits an inner product that induces the norm on $E$ ([40, Theorem 4.1]). Hilbertian spaces were developed by several authors, see for instance [41], [43], [46] and [57, Chapters 4 and 5]. Cartesian spaces are studied in detail in [7, Chapter 2].

The contents of this section concentrates around the following three properties:

(1) $E$ has an orthogonal base;

(2) $E$ is Hilbertian;

(3) $E$ is Cartesian.

In general, $(1) \Rightarrow (2)$ (see Corollary 2.0.2) and $(2) \Rightarrow (3)$ (see Proposition 2.2.2). If $\mathbb{K}$ is spherically complete, all non-Archimedean normed spaces over $\mathbb{K}$ are Hilbertian, thus Cartesian. If $\mathbb{K}$ is densely valued, there are many examples of normed spaces without orthogonal bases, for instance $l^\infty$; van Rooij and Schikhof proved ([58, Problem 4]) that in this case the implication (3) $\Rightarrow$ (1) does not work in general (see Proposition 2.2.19). They also formulated the problem if the implication (3) $\Rightarrow$ (2) is true when $\mathbb{K}$ is non-spherically complete. The question if (2) $\Rightarrow$ (1) works for all non-Archimedean Banach spaces over non-spherically complete $\mathbb{K}$, was formulated several times (among others in [43, Problem preceded by Proposition 3.5] and [41, Remark after Proposition 2.3.2]).

We show, presenting counterexamples, that both implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are not true in general. We demonstrate that all immediate extensions of $c_0$ which are contained in $l^\infty$ are Hilbertian and among them are those which do not have orthogonal bases (Theorem 2.2.10 and Corollary 2.2.11). We prove also that there exists an immediate extensions of $c_0$ which is Hilbertian but it is not Cartesian (Theorem 2.2.27).

## General properties of Hilbertian spaces

At the beginning of this section we recall some known and new properties of Hilbertian spaces.

**2.2.1. Proposition** ([43, Theorem 3.1]). *Every linear subspace of a Hilbertian space $E$ is Hilbertian. If $D$ is a finite-dimensional linear subspace of $E$ then $E/D$ is Hilbertian. Normed direct sums and finite normed products of Hilbertian spaces are Hilbertian.*

*Proof.* Let $E$ be a Hilbertian space and $D$ be its linear subspace. For each $x \in D \setminus \{0\}$, $[x]$ is orthocomplemented in $E$, thus, in $D$. Hence, by Proposition 1.1.7, $D$ is Hilbertian. Now, assume that $D$ is finite-dimensional. Let $x \in E/D$, $x \neq 0$; then, there is $x_E \in E$ such that $\pi(x_E) = x$, where $\pi\colon E \to E/D$ is the canonical map. As $E$ is Hilbertian, there exists a closed linear subspace $D_0 \subset E$ which is an orthocomplement of $[x_E] + D$. Then, for each $z \in D_0$, we get

$$\|\pi(z) - x\| = \inf_{y \in D} \|z - x_E - y\| \geqslant \inf_{y \in D} \|x_E - y\| = \|\pi(x_E)\|.$$

Thus, $\pi(D_0)$ is an orthocomplement of $[x]$.

Let $\{E_i\}_{i\in I}$ be a family of Hilbertian spaces and let $x = (x_i)_{i\in I} \in E = \bigoplus_{i\in I} E_i$, $x \neq 0$. There is $i_0 \in I$ for which $\|x\| = \|x_{i_0}\|$. As $E_{i_0}$ is Hilbertian, $[x_{i_0}]$ has an orthocomplement $D_{i_0}$ in $E_{i_0}$. Let $D := \bigoplus_{i\in I} D_i$, where $D_i = E_i$ if $i \neq i_0$. Since for each $z = (z_i)_{i\in I} \in D$

$$\|x + z\| = \max_{i\in I} \|x_i + z_i\| \geqslant \|x_{i_0} + z_{i_0}\| \geqslant \|x_{i_0}\| = \|x\|,$$

we imply that D is an orthocomplement of $[x]$ in E. Now, apply Proposition 1.1.7 and conclude that E is Hilbertian. If $E = \times_{i\in I} E_i$ and I is finite, then $\times_{i\in I} E_i = \bigoplus_{i\in I} E_i$ and the conclusion follows from the above. □

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Note that (see [43, Remarks 3.2]), there exist products of infinitely many Hilbertian spaces and quotients of Hilbertian spaces which are not Hilbertian.

**2.2.2. Proposition.** *If* $E$ *is of countable type, then* $E$ *is Hilbertian if and only if* $E$ *has an orthogonal base.*

*Proof.* If E has an orthogonal base, the conclusion follows from Corollary 2.0.2. Assume that E is a Hilbertian space which is of countable type. Then, there are finite-dimensional linear subspaces $D_n$, $n \in \mathbb{N}$, of E such that $D_1 \subset D_2 \subset \ldots$, $\dim(D_n) = n$ $(n \in \mathbb{N})$ and $E = \overline{\bigcup_n D_n}$. Take $x_1 \in D_1$, $x_1 \neq 0$. By Proposition 2.2.1 for each $n \in \mathbb{N}$ the finite-dimensional $D_n$ is orthocomplemented in $D_{n+1}$; hence, there is $x_{n+1} \in D_{n+1}$, $x_{n+1} \neq 0$, such that $x_{n+1} \perp D_n$ . Clearly, $D_{n+1} = D_n + [x_{n+1}]$. Thus, by [47, Theorem 2.2.7], $\{x_1, x_2, \ldots\}$ is orthogonal and by [47, Theorem 2.3.6] it is an orthogonal base. □

The following theorem gives us the necessary and sufficient conditions for a non-Archimedean space of being Hilbertian.

**2.2.3. Theorem** ([33, Theorem 3.5]). $E$ *is Hilbertian if and only if for every nonzero* $x \in E$ *there exists a set* $\{w_i\}_{i\in I} \subset E$ *such that* $\{x\} \cup \{w_i\}_{i\in I}$ *is a maximal orthogonal set in* $E$ *and* $E = [x] + D$, *where* $D$ *is an immediate extension of* $\overline{[\{w_i\}_{i\in I}]}$.

*Proof.* ($\Rightarrow$) Suppose that $E$ is Hilbertian. Let $x \in E$ ($x \neq 0$) and let $D$ be an orthogonal complement of $[x]$ in $E$. Take $\{w_i\}_{i\in I}$, a maximal orthogonal set in $D$. Obviously, $\{x\} \cup \{w_i\}_{i\in I}$ is orthogonal. We prove that $\{x\}\cup\{w_i\}_{i\in I}$ is a maximal orthogonal set in $E$. Let $z \in E\setminus\overline{[\{x\} \cup \{w_i\}_{i\in I}]}$; then, $z = \lambda x+d$ for some $\lambda \in \mathbb{K}$ and $d \in D$. Since, by Proposition 1.2.10, $D$ is an immediate extension of $\overline{[\{w_i\}_{i\in I}]}$, we can select $w \in \overline{[\{w_i\}_{i\in I}]}$ which satisfies $\|d - w\| < \|d\|$. Next, we obtain

$$\|z - (\lambda x + w)\| = \|d - w\| < \|d\| \leqslant \|\lambda x + d\| = \|z\|;$$

hence, $\mathrm{dist}(z, \overline{[\{x\} \cup \{w_i\}_{i\in I}]}) < \|z\|$. By Proposition 1.2.10, $\{x\} \cup \{w_i\}_{i\in I}$ is a maximal orthogonal set in $E$.

($\Leftarrow$) Assume the contrary and suppose that $E$ is not Hilbertian. Then, there exists $x \in E$ ($x \neq 0$) such that $[x]$ has no orthogonal complement in $E$. By assumption, there exists $\{w_i\}_{i\in I}$, an orthogonal set in $E$ such that $\{x\} \cup \{w_i\}_{i\in I}$ is a maximal orthogonal set in $E$ and $E = [x] + D$, where $D$ is an immediate extension of $\overline{[\{w_i\}_{i\in I}]}$. Since, by assumption, $D$ is not an orthogonal complement of $[x]$ in $E$, we can find $d \in D$ with $\|x\| = \|d\|$ and $\|x + d\| < \|x\|$. Since $x \perp \overline{[\{w_i\}_{i\in I}]}$, we have $d \in D \setminus \overline{[\{w_i\}_{i\in I}]}$. But then there is $w \in \overline{[\{w_i\}_{i\in I}]}$ satisfying $\|w - d\| < \|d\|$; thus, we get

$$\|x + w\| = \|x + d - d + w\| \leqslant \max\{\|x + d\|, \|w - d\|\} < \|x\| = \|d\|,$$

a contradiction with $x \perp \overline{[\{w_i\}_{i\in I}]}$. □

## Hilbertian subspaces of $l^\infty$

The main result of this section, Theorem 2.2.10, characterizes the specific class of Hilbertian spaces over non-spherically complete $\mathbb{K}$, linear subspaces of $l^\infty$ among which are those which have no orthogonal base. Thus, Theorem 2.2.10 enables to construct a counterexample with respect to the implication (2) $\Rightarrow$ (1).

**2.2.4. Example** ([33, Example 2.6]). Choose a sequence $(a_n)_n \subset \mathbb{K}$ such that $|a_1| > \ldots > |a_n| > |a_{n+1}| > \ldots > 1$ for $n \in \mathbb{N}$. Let $a := (a_1, a_2, \ldots)$, $x_n := (a_1, \ldots, a_n, 0, \ldots)$ ($n \in \mathbb{N}$) be elements of $l^\infty$. We can easily observe that $(B_{l^\infty,|a_{n+1}|}(x_n))_n$ is a centered sequence of closed balls and

$$a \in \bigcap_n B_{l^\infty,|a_{n+1}|}(x_n, |a_{n+1}|).$$

Applying Proposition 1.2.12, we deduce that $c_0 + [a]$, a closed linear subspace of $l^\infty$, is an immediate extension of $c_0$. We can easily check that $\mathrm{dist}(a, c_0) = \lim_{n\to\infty} \|a - x_n\|$ and prove that $(y_n)_n$, where $y_n = (0, \dots, 0, a_n, a_{n+1}, \dots)$, $n \in \mathbb{N}$, is an orthogonal base of $c_0 + [a]$.

The Example 2.2.4 shows us that $l^\infty$ contains closed linear subspaces, which are immediate extensions of $c_0$. Note that by Zorn's lemma, among all immediate extensions of $c_0$ contained in $l^\infty$ there exists a maximal one, clearly not unique. Next results characterize immediate extensions of $c_0$ contained in $l^\infty$ more precisely.

**2.2.5. Proposition** ([33, Proposition 2.8]). *Let* $E_0$ *be an immediate extension of* $c_0$ *contained in* $l^\infty$ *and let* $x = (x_1, x_2, \dots) \in l^\infty$. *If* $x \in E_0$ *then for every* $m \in \mathbb{N}$ *the set*

$$M_m(x) := \Big\{ n \in \mathbb{N} : n > m \textit{ and } |x_n| = \sup_{k>m} |x_k| \Big\}.$$

*is nonempty and finite. If* $x \in E_0 \setminus c_0$, *then* $\mathrm{dist}(x, c_0) = \lim\limits_{n\to\infty} \|x - y_n\|$, *where* $y_n = \sum\limits_{i=1}^{n} x_i e_i$ ($e_i, i \in \mathbb{N}$, *are unit vectors*).

*Proof.* First, assume that for some $m_0 \in \mathbb{N}$ the set $M_{m_0}(x)$ is empty. Define $z := x - \sum\limits_{i=1}^{m_0} x_i e_i$. Clearly, $z \in E_0$ and $\|z\| > |x_n|$ for all $n > m_0$. We can choose a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $|x_{n_{k+1}}| > |x_{n_k}|$ for every $k \in \mathbb{N}$ and $\|z\| = \lim\limits_{k\to\infty} |x_{n_k}|$. Hence,

$$\mathrm{dist}(z, c_0) = \lim_{k\to\infty} |x_{n_k}|.$$

Thus, we conclude that $z \perp c_0$, a contradiction.

Next, suppose that there exists $m_0$ such that $M_{m_0}(x)$ is infinite. Setting again $z := x - \sum\limits_{i=1}^{m_0} x_i e_i$, we see that $\|z\| = |x_j|$ if $j \in M_{m_0}$. Hence, $\mathrm{dist}(z, c_0) = \|z\|$, a contradiction.

Now, let $x \in E_0 \setminus c_0$. Then, there exists $r > 0$ and an infinite subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $|x_{n_k}| > r$ for all $k \in \mathbb{N}$. Defining $r_0 := \sup\{r > 0 :$ there exists an infinite subsequence $(x_{n_k})_k$ with $|x_{n_k}| > r\}$, we see that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n| > r_0 + \varepsilon\}$ is finite and $\mathrm{dist}(x, c_0) \leqslant r_0 + \varepsilon$. On the other hand, taking $y_n = \sum_{i=1}^{n} x_i e_i$, we get $\lim_{n\to\infty} \|x - y_n\| = r_0$ and finish the proof. □

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**2.2.6. Remark.** Observe that if $E_0$ is a maximal immediate extension of $c_0$ contained in $l^\infty$, then there exists $x \in l^\infty$ such that $M_m(x)$ is nonempty and finite for all $m \in \mathbb{N}$, but $x \notin E_0$. Indeed, take $y = (y_1, y_2, \dots) \in E_0 \setminus c_0$ and nonzero $\lambda \in \mathbb{K}$ with $|\lambda| < \mathrm{dist}(y, c_0)$. Setting $x = (x_1, x_2, \dots)$, where $x_n := y_n + \lambda$ $(n \in \mathbb{N})$ we see that $M_m(x) = M_m(y)$ for all $m \in \mathbb{N}$. On the other hand, $y - x = (\lambda, \lambda, \dots)$ and, by Proposition 2.2.5, $y - x \notin E_0$; thus, $x \notin E_0$. We can easily verify that $c_0 + [x]$ is an immediate extension of $c_0$ and conclude that a maximal immediate extension of $c_0$ which contains $c_0 + [x]$ is not equal to $E_0$.

**2.2.7. Proposition** ([34, Proposition 3])**.** *Let $(p_n)_n$ be a sequence of non-negative reals such that*

$$M_m := \Big\{ n \in \mathbb{N} : n > m \text{ and } p_n = \sup_{k>m} p_k \Big\}$$

*is nonempty and finite for each $m \in \mathbb{N}$. Let $x = (x_1, x_2, \dots) \in l^\infty$ and $M_0 = \{n \in \mathbb{N} : |x_n| > \mathrm{dist}(x, c_0)\}$. If $|x_n| = p_n$ for every $n \in \mathbb{N}$, then $c_0 + [x]$ is an immediate extension of $c_0$. If $E$ is a maximal immediate extension of $c_0$ contained in $l^\infty$, then, there exists $z = (z_1, z_2, \dots) \in E$ such that $|z_n| = |x_n| = p_n$ for all $n \in M_0$ and $|x_n - z_n| \leqslant \mathrm{dist}(x, c_0)$.*

*Proof.* If $x \in c_0$, the conclusion is trivial. So, assume that $x \in l^\infty \setminus c_0$. First, we prove that $c_0 + [x]$ is an immediate extension of $c_0$. Assume for a contradiction that there exists $y = (y_1, y_2, \dots) \in c_0$ such that $\|x - y\| = \mathrm{dist}(x, c_0)$. Clearly, $N_0 = \{n \in \mathbb{N} : |y_n| \geqslant \|x - y\|\}$ is finite. Let $n_0 = \max\{n \in N_0\}$. Without loss of generality we can assume that $y \in [e_1, \dots, e_{n_0}]$. Note that $N_0$ and $M_{n_0}$ are disjoint, $M_{n_0}$ is finite by assumption. Define $z = (z_1, z_2, \dots) \in c_0$, where $z_i = x_i - y_i$ if

$i \in [1, \dots, n_0]$, $z_i = x_i$ if $i \in M_{n_0}$ and $z_i = 0$ otherwise. We obtain

$$\begin{aligned}\|x - y - z\| &= \sup_{i \in \mathbb{N}} |x_i - y_i - z_i| \\ &= \max\Big\{ \max_{i \in [1, \dots, n_0]} |x_i - y_i - z_i|, \max_{i \in M_{n_0}} |x_i - y_i - z_i|, \\ &\quad \sup\{|x_i - y_i - z_i| : i \in \mathbb{N} \setminus ([1, \dots, n_0] \cup M_{n_0})\Big\} \\ &= \sup\{|x_i| : i \in \mathbb{N} \setminus ([1, \dots, n_0] \cup M_{n_0})\} < \|x - y\|,\end{aligned}$$

but this contradicts with $\|x - y\| = \mathrm{dist}(x, c_0)$.

Assume now that $x \notin E$. By maximality of $E$, $E + [x]$ is not an immediate extension of $c_0$ and by Proposition 1.2.9, $E + [x]$ is not an immediate extension of $E$; thus, there exists $z = (z_1, z_2, \dots) \in E$ such that $\mathrm{dist}(x, E) = \|x - z\|$. Clearly, $\mathrm{dist}(x, E) \leqslant \mathrm{dist}(x, c_0)$. Thus, we obtain

$$\|x - z\| = \sup_{n \in \mathbb{N}} |x_n - z_n| \leqslant \mathrm{dist}(x, c_0).$$

Hence, $|z_n| = |x_n| = p_n$ for all $n \in M_0$. □

**2.2.8. Proposition** ([33, Proposition 2.10]). *Let* $E_0 \subset l^\infty$ *be a maximal immediate extension of* $c_0$ *and let* $(p_n)_n$ *be a strictly decreasing sequence of reals for which* $\inf_{n \in \mathbb{N}} p_n > 0$ *and* $\{p_n : n \in \mathbb{N}\} \subset |\mathbb{K}^\times|$. *Then, there exists* $y = (y_1, y_2, \dots) \in E_0$ *such that* $|y_n| = p_n$ *for all* $n \in \mathbb{N}$.

*Proof.* Let $x = (x_1, x_2, \dots) \in l^\infty \setminus E_0$ be such that $|x_n| = p_n$ for all $n \in \mathbb{N}$. By maximality of $E_0$, $E_0 + [x]$ is not an immediate extension of $c_0$. Hence, by Proposition 1.2.9, $E_0 + [x]$ is not an immediate extension of $E_0$. Applying Lemma 1.2.2, we imply that there is $y = (y_1, y_2, \dots) \in E_0$ for which $\mathrm{dist}(x, E_0) = \|x - y\|$. Clearly, $\mathrm{dist}(x, E_0) \leqslant \mathrm{dist}(x, c_0)$; thus,

$$\|x - y\| \leqslant \inf_{n \in \mathbb{N}} p_n.$$

But $\|x - y\| = \sup_{n \in \mathbb{N}} |x_n - y_n|$; hence, $|y_n| = |x_n| = p_n$ for all $n \in \mathbb{N}$. This shows that $y$, an element of $E_0$, satisfies the required conditions. □

**2.2.9. Proposition** ([35, Proposition 3.2] ). *Let* $a = (a_1, a_2, \dots) \in l^\infty$. *There exists* $b = (b_1, b_2, \dots) \in l^\infty$ *such that* $[a, b]$ *is a two-dimensional*

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*linear subspace without an orthogonal base if and only if* $|a_n| < \|a\|$ *for all* $n \in \mathbb{N}$.

*Proof.* Assume that there exists $n_0 \in \mathbb{N}$ such that $\|a\| = |a_{n_0}|$. Then, by Proposition 2.1.4, $[a]$ is orthocomplemented in $l^\infty$. Thus, for every $b \in l^\infty$, there exists $\lambda \in \mathbb{K}$ for which $b = \lambda a + (b - \lambda a)$ and $(b - \lambda a) \perp [a]$. It means that $\{a, b - \lambda a\}$ is an orthogonal base of $[a, b]$.

Suppose that $|a_n| < \|a\|$ for all $n \in \mathbb{N}$. Applying Proposition 2.1.4 and Theorem 2.1.13, we imply that $[a]$ is not strict in $l^\infty$. Hence, we can select $b \in l^\infty$ such that $[a]$ is not orthocomplemented in $[a, b]$. Therefore, by Corollary 2.0.2, $[a, b]$ does not have an orthogonal base and we are done. □

Now, we are ready to obtain a characterization of a maximal immediate extension of $c_0$ contained in $l^\infty$.

**2.2.10. Theorem** ([33, Theorem 3.6]). *Let* $E_0$ *be a maximal immediate extension of* $c_0$ *contained in* $l^\infty$. *Then*

(1) $E_0$ *is Hilbertian;*

(2) $E_0$ *is not of countable type;*

(3) $E_0$ *has no orthogonal base.*

*Proof.* First, we prove that $E_0$ is Hilbertian. Take a nonzero $a = (a_1, a_2, \dots) \in E_0$. By Proposition 2.2.5, there exists a nonempty and finite $M_a \subset \mathbb{N}$ with $\|a\| = |a_i|$ if $i \in M_a$ and $\|a\| > |a_j|$ if $j \in \mathbb{N} \setminus M_a$.

Take $i_0 \in M_a$. Let $X_0 = \{e_1, \dots, e_{i_0-1}, e_{i_0+1}, \dots\}$ and let $D_0$ be a maximal immediate extension of $\overline{[X_0]}$ in $E_0$. We see that $\{a\} \cup X_0$ is an orthogonal set. We prove that it is a maximal orthogonal set in $E_0$, i.e. there is no element in $E_0$ orthogonal to $[\{a\} \cup X_0]$.

Indeed, taking $b = (b_1, b_2, \dots) \in E_0 \setminus [\{a\} \cup X_0]$ and applying Proposition 2.2.5 again, we can select a finite subset $M_b \subset \mathbb{N}$ such that $\|b\| = |b_i|$ for every $i \in M_b$ and $\|b\| > |b_i|$ for all $i \in \mathbb{N} \setminus M_b$. Assume that $i_0 \notin M_b$ and define $z := \sum_{i \in M_b} b_i e_i$; then $z \in [X_0]$. Next, we obtain

$$\|b - z\| = \left\| b - \sum_{i \in M_b} b_i e_i \right\| = \max_{i \in \mathbb{N} \setminus M_b} |b_i| < \|b\|.$$

If $i_0 \in M_b$, defining $a' \in [\{a\} \cup X_0]$ by

$$a' := a - \sum_{i \in M_a \setminus \{i_0\}} a_i e_i$$

and $z \in [\{a\} \cup X_0]$ by

$$z := \sum_{i \in M_b \setminus \{i_0\}} b_i e_i + \frac{b_{i_0}}{a_{i_0}} a',$$

we get

$$\begin{aligned}
\|b - z\| &= \left\| b - \sum_{i \in M_b \setminus \{i_0\}} b_i e_i - \frac{b_{i_0}}{a_{i_0}} a + \frac{b_{i_0}}{a_{i_0}} \sum_{i \in M_a \setminus \{i_0\}} a_i e_i \right\| \\
&= \Bigg\| \Bigg( b - \sum_{i \in M_b \setminus \{i_0\}} b_i e_i - b_{i_0} e_{i_0} \Bigg) \\
&\quad + \frac{b_{i_0}}{a_{i_0}} \Bigg( a_{i_0} e_{i_0} - a + \sum_{i \in M_a \setminus \{i_0\}} a_i e_i \Bigg) \Bigg\| \\
&= \left\| \left( b - \sum_{i \in M_b} b_i e_i \right) - \frac{b_{i_0}}{a_{i_0}} \left( a - \sum_{i \in M_a} a_i e_i \right) \right\| \\
&\leqslant \max \left\{ \max_{i \in N \setminus M_b} |b_i|, \max_{i \in N \setminus M_a} \left\{ |a_i| \cdot \left| \frac{b_{i_0}}{a_{i_0}} \right| \right\} \right\} < \|b\|,
\end{aligned}$$

since

$$|b_{i_0}| = \|b\| > \max_{i \in \mathbb{N} \setminus M_b} |b_i| \text{ and } |a_{i_0}| > \max_{i \in \mathbb{N} \setminus M_a} |a_i|.$$

Hence, $b$ is not orthogonal to $\overline{[\{a\} \cup X_0]}$.

Now, we prove that $D_0$ is an orthogonal complement of $[a]$ in $E_0$. Let $b = (b_1, b_2, \dots) \in E_0$. If $b_{i_0} = 0$, then, applying Proposition 2.2.5, we deduce that $b \in D_0$. Assuming that $b_{i_0} \neq 0$, we can write $b = b_{i_0} a / a_{i_0} + d$, where $d = b - b_{i_0} a / a_{i_0} a$. Since $d_{i_0} = b_{i_0} - b_{i_0} a_{i_0} / a_{i_0} = 0$, we conclude that $d \in D_0$ and finally $E_0 = [a] + D_0$; hence, by Theorem 2.2.3, $D_0$ is an orthogonal complement of $[a]$ and $E_0$ is Hilbertian.

Next, we prove that $E_0$ is not of countable type. Assuming that $|\mathbb{K}^\times|$ is countable, we can choose an uncountable $S \subset (1, 2)$, such that $\pi(r) \neq \pi(s)$ for $r \neq s$ ($r, s \in S$), where $\pi \colon \mathbb{R}_+ \to \mathbb{R}_+ / |\mathbb{K}^\times|$ is the natural map (then, elements of $S$ are in different cosets of $|\mathbb{K}^\times|$). Using Proposition 2.2.8, for every $r \in S$ we construct $x^r = (x_1^r, x_2^r, \dots) \in E_0$ such that $|x_1^r| \leqslant 2$, $(|x_n^r|)_n$ is a strictly decreasing sequence of reals and $\lim_{n\to\infty} |x_n^r| = r$. We verify that $\{x^r : r \in S\}$ is an $1/2$-orthogonal set. Take a finite subset $P \subset S$ and nonzero $\lambda_r \in \mathbb{K}$ $(r \in P)$. Then, by assumption, we can find $r_0 \in P$ such that

$$|\lambda_{r_0}| \cdot r_0 > \max_{r\in P,\, r\neq r_0} \{|\lambda_r| \cdot r\}.$$

But then, there exists $n_0 \in \mathbb{N}$ for which $|\lambda_{r_0} x_n^{r_0}| > |\lambda_r x_n^r|$ for each $r \in P$, $r \neq r_0$ and all $n > n_0$. Taking $m > n_0$, we have

$$\Big\| \sum_{r\in P} \lambda_r x^r \Big\| \geqslant \Big| \sum_{r\in P} \lambda_r x_m^r \Big| = |\lambda_{r_0} x_m^{r_0}| > |\lambda_{r_0}| \cdot r_0 > \frac{1}{2} \|\lambda_{r_0} x^{r_0}\|.$$

If $|\mathbb{K}^\times|$ is not countable, for every $r \in (1,2) \cap |\mathbb{K}^\times|$ we select $x^r = (x_1^r, x_2^r, \dots) \in E_0$, assuming that

$$|x_1^r| = r, \quad |x_{n-1}^r| > |x_n^r|, \quad \sqrt[n-1]{r} > |x_n^r| \geqslant \sqrt[n]{r} \quad \text{for } n = 2, 3, \dots$$

Take a finite subset $P \subset (1,2) \cap |\mathbb{K}^\times|$. Then, if

$$\Big\| \sum_{r\in P} \lambda_r x^r \Big\| < \max_{r\in P} \|\lambda_r x^r\|$$

for some $\lambda_r \in \mathbb{K}$ $(r \in P)$, we can choose $P_0 \subset P$ such that $\|\lambda_q x^q\| = \max_{r\in P} \|\lambda_r x^r\|$ for all $q \in P_0$. Hence,

$$|\lambda_q x_1^q| = |\lambda_q| \cdot q = \max_{r\in P} \|\lambda_r x^r\|$$

for all $q \in P_0$. But we can find $n \in \mathbb{N}$ for which $|\lambda_q x_n^q| \neq |\lambda_r x_n^r|$ if $q \neq r$ $(q, r \in P_0)$. Thus,

$$\Big\| \sum_{r\in P} \lambda_r x^r \Big\| \geqslant \max_{r\in P_0} |\lambda_r x_n^r|$$

and we finally conclude that $\{x^r : r \in (1,2) \cap |\mathbb{K}^\times|\}$ is an uncountable $1/2$-orthogonal set in $E_0$; hence, $E_0$ is not of countable type.

Since $E_0$ is an immediate extension of $c_0$, by [57, Theorem 5.4], every maximal orthogonal set in $E_0$ is countable. But $E_0$ is not of countable type, thus, by Proposition 2.2.2, $E_0$ has no orthogonal base. $\square$

**2.2.11. Corollary.** *Every immediate extension of* $c_0$ *contained in* $l^\infty$ *is Hilbertian.*

*Proof.* Let $E$ be an immediate extension of $c_0$ contained in $l^\infty$. Then, there exists $E_0$, a maximal immediate extension of $c_0$, which is Hilbertian by Theorem 2.2.10, and such that $E \subset E_0$. From Proposition 2.2.1 we conclude that $E$ is Hilbertian. □



**2.2.12. Corollary.** *Every immediate extension of* $c_0$ *contained in* $l^\infty$ *which is of countable type has an orthogonal base.*

*Proof.* Follows immediately from Corollary 2.2.11 and Proposition 2.2.2. □

Theorem 2.2.10 shows that all immediate extensions of $[(e_n)_n]$, where $(e_n)_n$ is the standard base of $c_0$, contained in $l^\infty$ are Hilbertian. Now, we extend this result, characterizing linear subspaces of $l^\infty$ which are maximal immediate extensions of linear spans of their maximal orthogonal sets, giving equivalent conditions for being Cartesian and Hilbertian. Note (see Remark 2.2.14) that this result cannot be generalized for all linear subspaces of $l^\infty$.

**2.2.13. Theorem** ([35, Theorem 3.3]). *Let* $E_0$ *be a linear subspace of* $l^\infty$ *and let* $(x_i)_{i\in I}$ *be a maximal orthogonal set in* $E_0$. *If* $E_0$ *is a maximal immediate extension of* $\overline{[(x_i)_{i\in I}]}$ *contained in* $l^\infty$, *then the following are equivalent*

(1) $E_0$ *is Hilbertian;*
(2) $E_0$ *is Cartesian;*
(3) *for every* $u = (u_1, u_2, \dots) \in E_0$, $\max\limits_{n\in\mathbb{N}} |u_n|$ *exists.*

*Proof.* (1) $\Rightarrow$ (2). Follows from [43, Theorem 3.1 and Proposition 3.5].

(2) $\Rightarrow$ (3). Assume the contrary and suppose that there exists $u = (u_1, u_2, \dots) \in E_0$ such that $\max\limits_{n\in\mathbb{N}} |u_n|$ does not exist. Using Proposition 2.2.9, we choose $b = (b_1, b_2, \dots) \in l^\infty$ for which $[u, b]$ has no orthogonal base. If $b \in E_0$ then $E_0$ is not Cartesian; thus, we are done. Assume that $b \notin E_0$. Then, since $E_0$ is a maximal immediate extension of $\overline{[(x_i)_{i\in I}]}$ and $E_0 + [b]$ is not an immediate extension of $\overline{[(x_i)_{i\in I}]}$, by Proposition 1.2.9, $E_0 + [b]$ is not an immediate extension of $E_0$.

Hence, we can find $d \in E_0$ with $\|b - d\| = \mathtt{dist}(b, E_0)$. By Lemma 1.2.1, $\|b - d\| < \|b - \lambda u\|$ for every $\lambda \in \mathbb{K}$. Taking any nonzero $\mu \in \mathbb{K}$, we get

$$\begin{aligned}\|u - \mu d\| = |\mu| \cdot \left\| \frac{1}{\mu} u - d \right\| &= |\mu| \cdot \left\| \frac{1}{\mu} u - b + b - d \right\| \\ &= |\mu| \cdot \left\| \frac{1}{\mu} u - b \right\| = \|u - \mu b\|.\end{aligned}$$

Thus, we conclude that $\mathtt{dist}(u, [d])$ is not attained. Using Lemma 1.2.1 again, we imply that $[u, d]$ has no orthogonal base; hence, $E_0$ is not Cartesian.

(3) $\Rightarrow$ (1). If $\max_{n \in \mathbb{N}} |u_n|$ exists for every $u = (u_1, u_2, \dots) \in E_0$, then, by Proposition 2.1.4, $[u]$ is orthocomplemented in $l^\infty$; thus $[u]$ is orthocomplemented in $E_0$ and $E_0$ is Hilbertian. □

**2.2.14. Remark.** The conclusion (1) $\Rightarrow$ (3) of Theorem 2.2.13 does not work if $E_0$ is an immediate extension of $\overline{[(x_i)_{i \in I}]}$, but not maximal. See Example 2.2.17.

**2.2.15. Remark.** Theorem 2.2.13 is not valid if $\mathbb{K}$ is spherically complete. In this case all normed spaces over $\mathbb{K}$ are Hilbertian and Cartesian (see [57, Lemma 4.35] and [43, Theorem 3.1 and Proposition 3.5]). Let $(x_i)_{i \in I}$ be a maximal orthogonal set in $l^\infty$, then $l^\infty$ is a maximal immediate extension of its linear span. However, in this case the implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are false.

Next result, which seems to be interesting on its own right, provides Example 2.2.17 announced in Remark 2.2.14.

**2.2.16. Proposition** ([35, Proposition 3.7]). *Let* $E_0$*be a closed Hilbertian linear subspace of* $E$. *If* $x \in E \setminus E_0$ *and* $\mathtt{dist}(x, E_0)$ *is attained then* $[x] + E_0$ *is Hilbertian, either.*

*Proof.* Since $\mathtt{dist}(x, E_0)$ is attained there exists $z_0 \in E_0$ for which

$$\|x - z_0\| = \mathtt{dist}(x, E_0) = \mathtt{dist}(x - z_0, E_0),$$

i.e. $[x - z_0] \perp E_0$. Also, it is clear that $[x] + E_0 = [x - z_0] + E_0$, so the conclusion follows as soon we prove that $[x - z_0] + E_0$ is Hilbertian. For that, note that from orthogonality of $[x - z_0]$ and $E_0$, we imply that $[x - z_0] + E_0$ is isometrically isomorphic to the Banach space $[x - z_0] \oplus E_0$. Obviously, $[x - z_0]$ is Hilbertian, $E_0$ is Hilbertian by assumption. Applying Proposition 2.2.1, we conclude that $[x - z_0] \oplus E_0$, thus $[x] + E_0$, is Hilbertian. $\square$

**2.2.17. Example** ([35, Example 3.8])**.** Let $E_0$ be a maximal immediate extension of $c_0$ contained in $l^\infty$. Choose a bounded sequence $(u_n)_n \subset \mathbb{K}$ such that $|u_n| < |u_{n+1}|$ for every $n \in \mathbb{N}$ and define $u = (u_1, u_2, \dots) \in l^\infty$. By Proposition 2.2.5, $u \notin E_0$. Let $E = [u] + E_0$. Then $E$ is Hilbertian. Indeed, first observe that $u$ is orthogonal to $E_0$. By Proposition 2.2.5, for any $x = (x_1, x_2, \dots) \in E_0$ there exists $N_0$ such that $|x_n| < |x_{N_0}|$ if $n > N_0$. Thus

$$\begin{aligned}\|x - u\| &= \sup_{n \in \mathbb{N}} |x_n - u_n| \\ &= \max\left\{ \max_{n \leqslant N_0} |x_n - u_n|, \sup_{n > N_0} |x_n - u_n| \right\} = \max\{\|x\|, \|u\|\}.\end{aligned}$$

Now, by Theorem 2.2.10, $E_0$ is Hilbertian; thus, applying Proposition 2.2.16, we conclude that $E$ is Hilbertian.

However, Theorem 2.2.10 implies that $E$ is not a maximal immediate extension of the linear span of any maximal orthogonal set.

At the end of this section, let us to get know another interesting property of a maximal immediate extension of $c_0$ contained in $l^\infty$. Recall, that by [57, Theorem 4.1], $l^\infty / c_0$ is spherically complete for any (spherically complete and non-spherically complete) $\mathbb{K}$.

**2.2.18. Theorem** ([35, Theorem 3.9])**.** *Let $E$ be a maximal immediate extension of $c_0$ contained in $l^\infty$. Then $E/c_0$ is spherically complete for any $\mathbb{K}$.*

*Proof.* If $\mathbb{K}$ is spherically complete then $E$ as a maximal immediate extension of $c_0$ contained in $l^\infty$ is spherically complete. Thus, the conclusion follows from [57, Theorem 4.2].

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Let $\mathbb{K}$ be non-spherically complete, $\pi\colon l^\infty \to l^\infty/c_0$ be the quotient map and $(B_{l^\infty/c_0,r_n}(x_n))_n$ be a centered sequence of closed balls such that $(x_n)_n \subset \pi(E)$. Suppose that $r_1 > r_2 > \ldots$ and $r_0 := \lim_n r_n > 0$. We prove that $\bigcap_n B_{l^\infty/c_0,r_n}(x_n) \cap \pi(E)$ is nonempty. Since, by [57, Theorem 4.1], $l^\infty/c_0$ is spherically complete, we can choose $x_0 \in l^\infty/c_0$ with $x_0 \in \bigcap_n B_{l^\infty/c_0,r_n}(x_n)$. Suppose that $x_0 \notin \pi(E)$. Select a sequence $(a_n)_n \subset E$ for which $\pi(a_n) = x_n$ $(n \in \mathbb{N})$. Choose $a_0 \in l^\infty$ such that $\pi(a_0) = x_0$. Then, $a_0 \notin E$. Next, for every $n > 1$ take $g_n \in c_0$ for which $\|a_0 - (a_n + g_n)\| < r_{n-1}$. Since $(a_n + g_n) \in E$, $\mathrm{dist}(a_0, E) \leqslant r_0$. By assumption and Proposition 1.2.9, $[a_0] + E$ is not an immediate extension of $E$; thus, there exists $a \in E$ such that $\|a_0 - a\| \leqslant r_0$. Hence, $\|x_0 - \pi(a)\| \leqslant r_0$ and $\pi(a) \in \bigcap_n B_{l^\infty/c_0,r_n}(x_n) \cap \pi(E)$. $\square$

## An example of Cartesian space which is not Hilbertian

This section complements the previous one providing an example of a Cartesian space which is not Hilbertian. Let us start by giving an example of the Cartesian space without an orthogonal base obtained by van Rooij and Schikhof (see [58, Problem 4]).

**2.2.19. Proposition.** *Let $\mathbb{K}$ be densely valued. Then, the spherical completion $\widehat{c_0}$ of $c_0$ contains a linear subspace which is Cartesian but it has no orthogonal base.*

*Proof.* By Zorn's Lemma, there is a maximal Cartesian subspace $D$ of $\widehat{c_0}$ containing $c_0$. We show that $D$ is a required example of a Cartesian space without an orthogonal base. Assume the contrary and suppose that $D$ has an orthogonal base. Since, by [57, Theorem 5.2], every maximal orthogonal set in $\widehat{c_0}$ is countable, $D$ has a countable orthonormal base $(x_n)_n$.

Select $\lambda_1, \lambda_2, \ldots \in \mathbb{K}$ such that $|\lambda_1| > |\lambda_2| > \ldots \to 1$ and take $z_0 \in \widehat{c_0}$ such that

$$\left\| z_0 - \sum_{n=1}^{m} \lambda_n x_n \right\| \leqslant |\lambda_{m+1}| \quad (m = 1, 2, \ldots).$$

Then, $z_0 \notin D$. Set $D_0 = D + [z_0]$ and define

$$z_m := z_0 - \sum_{n=1}^{m} \lambda_n x_n \quad (m = 1, 2, \dots). \tag{2.22}$$

We will show that $\{z_0, z_1, \dots\}$ is an orthogonal base of $D_0$. First, observe that $\|z_n\| = |\lambda_{n+1}|$ for $n = 0, 1, \dots$ We demonstrate that for each $m \in \mathbb{N}$ if $\{z_0, \dots, z_{m-1}\}$ is orthogonal, then $z_m \perp [z_0, \dots, z_{m-1}]$. Assume the contrary and suppose that there is $k \in \mathbb{N}$ and $\mu_0, \dots, \mu_{k-1} \in \mathbb{K}$ such that

$$\|z_k + \mu_0 z_0 + \dots + \mu_{k-1} z_{k-1}\| < \|z_k\| = |\lambda_{k+1}|. \tag{2.23}$$

Then, $\|\mu_0 z_0 + \dots + \mu_{k-1} z_{k-1}\| = |\lambda_{k+1}|$. Since, by assumption, $\{z_0, \dots, z_{m-1}\}$ is orthogonal,

$$\max_{i=0,\dots,k-1} \|\mu_i z_i\| = |\lambda_{k+1}|.$$

Hence, $|\mu_i| < 1$ for each $i \in \{0, \dots, k-1\}$ and

$$\|(1 + \mu_0 + \dots + \mu_{k-1}) z_{k+1}\| = |\lambda_{k+2}| < |\lambda_{k+1}|. \tag{2.24}$$

From (2.22) and (2.23) we get

$$\begin{aligned}
&\|z_k + \mu_0 z_0 + \dots + \mu_{k-1} z_{k-1} - (1 + \mu_0 + \dots + \mu_{k-1}) z_{k+1}\| \\
&\quad = \Bigg\| (1 + \mu_0 + \dots + \mu_{k-1}) z_0 + \sum_{n=1}^{k-1} (1 + \mu_n + \dots + \mu_{k-1}) \lambda_n x_n \\
&\qquad + \mu_k \lambda_k x_k - (1 + \mu_0 + \dots + \mu_{k-1}) z_{k+1} \Bigg\| \\
&\quad = \Bigg\| \sum_{n=1}^{k} (\mu_0 + \dots + \mu_{n-1}) \lambda_n x_n + (1 + \mu_0 + \dots + \mu_{k-1}) \lambda_{k+1} x_{k+1} \Bigg\| \\
&\quad \geqslant \|(1 + \mu_0 + \dots + \mu_{k-1}) \lambda_{k+1} x_{k+1}\| = |\lambda_{k+1}|
\end{aligned}$$

since $\{x_1, x_2, \dots\}$ is orthogonal. But, it contradicts with (2.23) and (2.24). Hence, $\{z_0, z_1, \dots\}$ is an orthogonal base of $D_0$. Thus, $D_0$ is Cartesian. However, we assumed that $D$ is a maximal Cartesian subspace of $\widehat{c_0}$, a contradiction. □

Recall that the space $l^\infty(\mathbb{N},\widehat{\mathbb{K}})$, the linear space over $\mathbb{K}$ of all bounded maps $\mathbb{N}\to\widehat{\mathbb{K}}$ equipped with the supremum norm, is spherically complete (see [57, 4.A]), thus, it contains a spherical completion of $c_0(\mathbb{N},\widehat{\mathbb{K}})$ (note that by [57, 4.B], $c_0(\mathbb{N},\widehat{\mathbb{K}})$ is not spherically complete).



**2.2.20. Remark.** Note that $l^\infty(\mathbb{N},\widehat{\mathbb{K}})$ contains elements which are orthogonal to $l^\infty$ (considered as a linear subspace of $l^\infty(\mathbb{N},\widehat{\mathbb{K}})$). Hence, by Lemma 1.2.2, $l^\infty(\mathbb{N},\widehat{\mathbb{K}})$ contains a proper linear subspace which is a spherical completion of $l^\infty$. Indeed, let $\lambda\in\widehat{\mathbb{K}}\backslash\mathbb{K}$ and let $r:=\mathrm{dist}(\lambda,\mathbb{K})$. Then, there exists a sequence $(c_n)_n\subset\mathbb{K}$ such that $|c_n-\lambda|\to r$ if $n\to\infty$. We can assume that $|c_n-\lambda|>|c_{n+1}-\lambda|$ for each $n\in\mathbb{N}$. Set

$$\mu_n:=\frac{c_n-\lambda}{c_n-c_{n+1}}\ (\mu_n\in\widehat{\mathbb{K}}),\quad n\in\mathbb{N}.$$

Then,

$$|\mu_n|=\left|\frac{c_n-\lambda}{c_n-c_{n+1}}\right|=\left|\frac{c_n-\lambda}{c_n-\lambda+\lambda-c_{n+1}}\right|=\left|\frac{c_n-\lambda}{c_n-\lambda}\right|=1$$

and

$$\mathrm{dist}(\mu_n,\mathbb{K})=\mathrm{dist}\left(\frac{\lambda}{c_n-c_{n+1}},\mathbb{K}\right)=\frac{r}{|c_n-c_{n+1}|}.$$

Since $|c_n-c_{n+1}|=|c_n-\lambda+\lambda-c_{n+1}|=|c_n-\lambda|$, $\mathrm{dist}(\mu_n,\mathbb{K})\to 1$ if $n\to\infty$.

Set $x=(\mu_1,\mu_2,\dots)\in l^\infty(\mathbb{N},\widehat{\mathbb{K}})$. Then, $\|x\|=1$ and $\mathrm{dist}(x,l^\infty)=\sup_n \mathrm{dist}(\mu_n,\mathbb{K})=1$; thus, $x$ is orthogonal to $l^\infty$.

**2.2.21. Proposition** ([34, Proposition 6]). *The space $c_0(\mathbb{N},\widehat{\mathbb{K}})$ is an immediate extension of $c_0$. A linear subspace $G$ of $l^\infty(\mathbb{N},\widehat{\mathbb{K}})$ is an immediate extension of $c_0(\mathbb{N},\widehat{\mathbb{K}})$ if and only if $G$ is an immediate extension of $c_0$ and $c_0(\mathbb{N},\widehat{\mathbb{K}})\subset G$.*

*Proof.* Take $x=(x_1,x_2,\dots)\in c_0(\mathbb{N},\widehat{\mathbb{K}})\setminus c_0$, then

$$\mathrm{dist}(x,c_0)=\max_{n\in\mathbb{N}}\mathrm{dist}(x_n,\mathbb{K})>0,$$

where $\mathbb{K}$ in this case denotes a one-dimensional linear subspace of $\widehat{\mathbb{K}}$ generated by element 1. Let $M_0=\{n\in\mathbb{N}:\mathrm{dist}(x_n,\mathbb{K})=\mathrm{dist}(x,c_0)\}$.

Clearly, $M_0$ is nonempty and finite. Take $n \in M_0$. Since $x_n \in \widehat{\mathbb{K}} \setminus \mathbb{K}$, applying Remark 1.2.13, $\mathrm{dist}(x_n, \mathbb{K})$ is not attained; hence, $c_0(\mathbb{N}, \widehat{\mathbb{K}})$ is an immediate extension of $c_0$. Using Proposition 1.2.9, since $c_0 \subset c_0(\mathbb{N}, \widehat{\mathbb{K}}) \subset G$, we finish the proof. □



**2.2.22. Corollary** ([34, Corollary 7]). *Let $x = (x_1, x_2, \dots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$. If $[x] + c_0(\mathbb{N}, \widehat{\mathbb{K}})$ is an immediate extension of $c_0(\mathbb{N}, \widehat{\mathbb{K}})$ then $[x] + c_0$ is an immediate extension of $c_0$.*

*Proof.* Since $c_0 \subset c_0(\mathbb{N}, \widehat{\mathbb{K}}) \subset [x] + c_0(\mathbb{N}, \widehat{\mathbb{K}})$, it follows readily from Propositions 1.2.9 and 2.2.21 that $[x] + c_0(\mathbb{N}, \widehat{\mathbb{K}})$ is an immediate extension of $c_0$; thus, since $[x] + c_0 \subset [x] + c_0(\mathbb{N}, \widehat{\mathbb{K}})$, $[x] + c_0$ is an immediate extension of $c_0$. □

Next, we want to show that the converse of Corollary 2.2.22 is not true.

**2.2.23. Example.** Take $a \in \mathbb{K} \setminus \{0\}$ and $a_0 \in \widehat{\mathbb{K}} \setminus \mathbb{K}$ such that $\mathrm{dist}(a_0, \mathbb{K}) > |a|$. Define $\widehat{a} = (a_0, a, a, \dots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$. Then,

$$\mathrm{dist}(\widehat{a}, c_0)(= \mathrm{dist}(a_0, \mathbb{K}))$$

is not attained; hence, $[\widehat{a}] + c_0$ is an immediate extension of $c_0$. But $[\widehat{a}] + c_0(\mathbb{N}, \widehat{\mathbb{K}})$ is not an immediate extension of $c_0(\mathbb{N}, \widehat{\mathbb{K}})$ since

$$\mathrm{dist}(\widehat{a}, c_0(\mathbb{N}, \widehat{\mathbb{K}})) = \|\widehat{a} - (a_0, 0, 0, \dots)\| = |a|.$$

**2.2.24. Proposition** ([34, Proposition 9]). *Let*

$$x = (x_1, x_2, \dots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}}) \setminus l^\infty$$

*be such that $[x] + c_0$ is an immediate extension of $c_0$. Assume that*

$$\sup_{n \in \mathbb{N}} \mathrm{dist}(x_n, \mathbb{K}) \geqslant \mathrm{dist}(x, c_0) \tag{2.25}$$

*and there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{n \in \mathbb{N}} \mathrm{dist}(x_n, \mathbb{K}) = \mathrm{dist}(x_{n_0}, \mathbb{K}). \tag{2.26}$$

*If $E$ is a maximal immediate extension of $c_0$ contained in $l^\infty$, then*

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(1) $[x] + E$ *is an immediate extension of* $E$;
(2) $[x] + E$ *is an immediate extension of* $c_0$.

*Proof.* Assume the contrary and suppose that there is $u = (u_1, u_2, \dots)$ in $E$ for which $\mathrm{dist}(x, E) = \|x - u\|$. Using Remark 1.2.13, (2.25) and (2.26) we obtain

$$\begin{aligned}\mathrm{dist}(x, E) = \|x - u\| &\geqslant |x_{n_0} - u_{n_0}| \\ &> \mathrm{dist}(x_{n_0}, \mathbb{K}) \geqslant \mathrm{dist}(x, c_0) \geqslant \mathrm{dist}(x, E),\end{aligned}$$

a contradiction. Hence, $[x] + E$ is an immediate extension of $E$. Applying Proposition 1.2.9, we conclude that $[x] + E$ is an immediate extension of $c_0$. □

Note that the condition (2.26) is crucial for the proof of Proposition 2.2.24, as the following example shows.

**2.2.25. Example** ([34, Example 10])**.** Let $b = (b_1, b_2, \dots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}}) \setminus l^\infty$ be such that for every $n \in \mathbb{N}$

$$|b_n| > |b_{n+1}|, \ \mathrm{dist}(b_n, \mathbb{K}) < \mathrm{dist}(b_{n+1}, \mathbb{K})$$

and

$$\lim_{n\to\infty} \mathrm{dist}(b_n, \mathbb{K}) = r_1 > 0, \quad \lim_{n\to\infty} |b_n| = r_0 > r_1.$$

For every $n \in \mathbb{N}$ choose $c_n \in \mathbb{K}$ for which

$$\mathrm{dist}(b_n, \mathbb{K}) < |b_n - c_n| < \mathrm{dist}(b_{n+1}, \mathbb{K}).$$

Then, $|c_n| = |b_n|$ and $|c_n - b_n| < |c_{n+1} - b_{n+1}|$ for all $n \in \mathbb{N}$. Define $c = (c_1, c_2, \dots) \in l^\infty$. Then, by Proposition 2.2.8, $[c] + c_0$ is an immediate extension of $c_0$. Let $E$ be a maximal immediate extension of $[c] + c_0$, contained in $l^\infty$. By Proposition 1.2.9, $E$ is a maximal immediate extension of $c_0$. Let $x := b - c$. Then, $\sup_{n\in\mathbb{N}} \mathrm{dist}(x_n, \mathbb{K}) = \mathrm{dist}(x, c_0) = r_1$ but $\mathrm{dist}(x_n, \mathbb{K}) < r$ for every $n \in \mathbb{N}$; i.e. the condition (2.26) is not satisfied. We see that $x \perp E$; thus, by Proposition 1.2.9, $[x] + E$ is not an immediate extension of $c_0$.

All immediate extensions of $c_0$ contained in $l^\infty$ are Hilbertian (see Corollary 2.2.11). However, among linear subspaces of $l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ we can easily find an immediate extension of $c_0$ which is not Hilbertian, $\widehat{c_0}$ for instance. Even more, taking $E$, a maximal immediate extension of $c_0$ contained in $l^\infty$, we can find $x \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ such that $[x] + E$ is not Hilbertian. Theorem 2.2.27 shows, assuming that $\mathbb{K}$ is separable and non-spherically complete, that for some $x \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ the space $[x] + E$ is Cartesian but not Hilbertian. First, a lemma.

**2.2.26. Lemma.** ([34, Lemma 11]) *Let* $x = (x_1, x_2, \dots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$. *Assume that* $x_k \in \widehat{\mathbb{K}} \setminus \mathbb{K}$, $|x_k| > |x_{k+1}|$, $\mathtt{dist}(x_k, \mathbb{K}) = r$ *for every* $k \in \mathbb{N}$ *and* $\lim\limits_{k\to\infty} |x_k| = r$. *Let* $D$ *be a linear subspace of* $c_0 + [x]$ *such that* $[e_1]$ *is an orthocomplement of* $D$. *Then*

(1) *there exist* $\lambda_x, \lambda_k \in \mathbb{K}$, $|\lambda_k| \leqslant 1$ $(k = 2, 3, \dots)$ *such that* $x - \lambda_x e_1$, $e_k - \lambda_k e_1 \in D$;

(2) *for every* $k = 2, 3, \dots$,

$$\left| x_1 - \lambda_x + \sum_{i=2}^{k} \lambda_i x_i \right| \leqslant |x_{k+1}|.$$

*Proof.* (1) Let $D$ be an orthocomplement of $[e_1]$ in $c_0 + [x]$. Then, for every $z \in c_0 + [x]$ there exists unequivocally selected $\lambda_z \in \mathbb{K}$ with $z - \lambda_z e_1 \in D$. In particular, there exist $\lambda_x, \lambda_k \in \mathbb{K}$ $(k = 2, 3, \dots)$ for which $x - \lambda_x e_1, e_k - \lambda_k e_1 \in D$ $(k = 2, 3, \dots)$. We see that $|\lambda_k| \leqslant 1$ for every $k = 2, 3, \dots$; otherwise, $e_k - \lambda_k e_1$ is not orthogonal to $[e_1]$.

(2) Assume the contrary and suppose that there exists $k_0 \in \mathbb{N}$ for which we get

$$\left| x_1 - \lambda_x + \sum_{i=2}^{k_0} \lambda_i x_i \right| > |x_{k_0+1}|. \tag{2.27}$$

By assumption, we can select $a_2, \dots, a_{k_0} \in \mathbb{K}$ such that

$$|x_i + a_i| < |x_{k_0+1}|; \tag{2.28}$$

thus,

$$|\lambda_i| \cdot |x_i + a_i| < |x_{k_0+1}|. \tag{2.29}$$

for $i = 2, \ldots, k_0$. Using (2.27) and (2.29) we get

$$\begin{aligned}\left|x_1 - \lambda_x - \sum_{i=2}^{k_0} a_i\lambda_i\right| &= \left|x_1 - \lambda_x + \sum_{i=2}^{k_0} \lambda_i x_i - \sum_{i=2}^{k_0} \lambda_i(x_i + a_i)\right| \\ &= \left|x_1 - \lambda_x + \sum_{i=2}^{k_0} \lambda_i x_i\right| > |x_{k_0+1}|.\end{aligned}$$

Hence, applying (2.28), we obtain

$$\begin{aligned}\left\|x - \lambda_x e_1 + \sum_{i=2}^{k_0} a_i(e_i - \lambda_i e_1)\right\| &= \max\left\{\left|x_1 - \lambda_x - \sum_{i=2}^{k_0} a_i\lambda_i\right|,\right. \\ &\qquad \left.|x_2 + a_2|, \ldots, |x_{k_0} + a_{k_0}|, |x_{k_0+1}|, |x_{k_0+2}|, \ldots\right\} \\ &= \left|x_1 - \lambda_x + \sum_{i=2}^{k_0} \lambda_i x_i\right| > |x_{k_0+1}|.\end{aligned}$$

But then, choosing $\lambda_0 \in \mathbb{K}$ such that $|x_1 - \lambda_0| < |x_{k_0+1}|$, from (2.28) we obtain

$$\begin{aligned}&\left\|(x - \lambda_x e_1 + \sum_{i=2}^{k_0} a_i(e_i - \lambda_i e_1)) + (\lambda_x + \sum_{i=2}^{k_0} a_i\lambda_i - \lambda_0)e_1\right\| \\ &= \left\|x + \sum_{i=2}^{k_0} a_i e_i - \lambda_0 e_1\right\| \\ &= \max\{|x_1 - \lambda_0|, |x_2 + a_2|, \ldots, |x_{k_0} + a_{k_0}|, |x_{k_0+1}|, |x_{k_0+2}|, \ldots\} \\ &= |x_{k_0+1}| < \left\|x - \lambda_x e_1 + \sum_{i=2}^{k_0} a_i(e_i - \lambda_i e_1)\right\|.\end{aligned}$$

Since $x - \lambda_x e_1 + \sum_{i=2}^{k_0} a_i(e_i - \lambda_i e_1) \in D$, we contradict to $[e_1] \perp D$. $\square$

**2.2.27. Theorem** ([34, Theorem 12]). *Let* $x = (x_1, x_2, \ldots) \in l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$. *Assume that for every* $k \in \mathbb{N}$ $x_k \in \widehat{\mathbb{K}} \setminus \mathbb{K}$, $|x_k| > |x_{k+1}|$, $\operatorname{dist}(x_k, \mathbb{K}) = r > 0$, *for every finite subset* $\{k_1, \ldots, k_n\} \subset \mathbb{N} \cup \{0\}$

$$\operatorname{dist}(x_{k_i}, [x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, x_{k_n}]) \geqslant r \quad (i = 1, \ldots, n), \tag{2.30}$$

*where* $x_0 = 1$ *and* $\lim\limits_{k\to\infty} |x_k| = r$. *If* $E$ *is a maximal immediate extension of* $c_0$ *contained in* $l^\infty$*, then,*

(1) $[x] + E$ *is not Hilbertian;*

(2) $[x] + E$ *is Cartesian.*



*Proof.* (1) Assume for a contradiction that $[x] + E$ is Hilbertian. Then, there exists $D$, an orthogonal complement of $[e_1]$ in $[x] + E$. Since $[x] + c_0 \subset [x] + E$, $D_0 := D \cap ([x] + c_0)$ is an orthogonal complement of $[e_1]$ in $[x] + c_0$. By Lemma 2.2.26, there exist $\lambda_x, \lambda_k \in \mathbb{K}$, $|\lambda_x|, |\lambda_k| \leqslant 1$ $(k = 2, 3, \dots)$ such that $x - \lambda_x e_1, e_k - \lambda_k e_1 \in D_0$ $(k = 2, 3, \dots)$ and

$$\left| x_1 - \lambda_x + \sum_{i=2}^{k} \lambda_i x_i \right| \leqslant |x_{k+1}| \tag{2.31}$$

for every $k = 2, 3, \dots$

Now, we find a subsequence $(n_k)_k \subset \mathbb{N}$ for which $|\lambda_{n_k}| > (k-1)/k$ $(k \in \mathbb{N})$. Fix $k \in \mathbb{N}$ $(k > 1)$. Then, we choose $k_1 \in \mathbb{N}$ $(k_1 > k)$ such that

$$|x_{k_1}| < \frac{k}{k-1} \cdot r. \tag{2.32}$$

Consider two cases:

(i) $\left| x_1 - \lambda_x + \sum\limits_{i=2}^{k_1-1} \lambda_i x_i \right| = |x_{k_1}|$. By assumption and (2.31),

$$|x_1 - \lambda_x + \sum_{i=2}^{k_1} \lambda_i x_i| \leqslant |x_{k_1+1}| < |x_{k_1}|;$$

thus, we imply that $|\lambda_{k_1}| = 1$. We take $n_k := k_1$.

(ii) $\left| x_1 - \lambda_x + \sum\limits_{i=2}^{k_1-1} \lambda_i x_i \right| < |x_{k_1}|$. Then, applying (2.30), we choose $k_2 > k_1$ with $\left| x_1 - \lambda_x + \sum\limits_{i=2}^{k_1-1} \lambda_i x_i \right| > |x_{k_2}|$. Since, by (2.31)

$$\left| x_1 - \lambda_x + \sum_{i=2}^{k_2-1} \lambda_i x_i \right| \leqslant |x_{k_2}|,$$

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there exists $k_3 \in \mathbb{N}$ $(k_1 < k_3 < k_2)$ such that

$$\left|x_1 - \lambda_x + \sum_{i=2}^{k_1-1} \lambda_i x_i\right| = |\lambda_{k_3} x_{k_3}|.$$

Then $|\lambda_{k_3} x_{k_3}| > |x_{k_2}|$; hence, using (2.32) we get,

$$|\lambda_{k_3}| > \frac{|x_{k_2}|}{|x_{k_3}|} > \frac{|x_{k_2}|}{r}\frac{k-1}{k} > \frac{k-1}{k},$$

since $|x_{k_3}| < |x_{k_1}|$; we take $n_k := k_3$.

There exists a sequence $(c_k)_k \subset \mathbb{K}$ such that $\lim\limits_{k\to\infty} |c_k - x_1| = r$ and $(B_{\mathbb{K},|c_k - c_{k+1}|}(c_k))_k$ is a centered sequence of closed balls with an empty intersection. Without loss of generality, we can assume that $|c_k - c_{k+1}| > |c_{k+1} - c_{k+2}|$ $(k \in \mathbb{N})$ and for some $k_0 \in \mathbb{N}$

$$|c_k - c_{k+1}| < \frac{k-1}{k+1}|c_{k-1} - c_k|$$

if $k > k_0$. Then, for $k > k_0$,

$$\begin{aligned} |\lambda_{n_k - 1}| \cdot |c_k - c_{k+1}| \leqslant |c_k - c_{k+1}| &< \frac{k-1}{k+1}|c_{k-1} - c_k| \\ &< \frac{k-1}{k}|c_{k-1} - c_k| < |\lambda_{n_k}| \cdot |c_{k-1} - c_k|; \end{aligned}$$

thus

$$\frac{|c_k - c_{k+1}|}{|\lambda_{n_k}|} < \frac{|c_{k-1} - c_k|}{|\lambda_{n_k - 1}|}. \tag{2.33}$$

Let $N_0 := \{n_k : k \in \mathbb{N}\}$. Define $b' = (b'_1, b'_2, \dots) \in l^\infty$ by setting

$$b'_{n_1} := \frac{-c_1}{\lambda_{n_1}}, \quad b'_{n_k} := \frac{c_k - c_{k+1}}{\lambda_{n_k}}, \quad k = 2, 3, \dots$$

and $b'_i = 0$ if $i \notin N_0$. It follows from (2.33) and Proposition 2.2.5, that $[b'] + c_0$ is an immediate extension of $c_0$. If $b' \notin E$, by Proposition 2.2.8 there exists $g_1 \in l^\infty$ such that $b' + g_1 \in E$ and $\|g_1\| \leqslant \mathtt{dist}(b', c_0) = r$. Define $b = b' + g$, taking $g = g_1$ if $b' \notin E$ and $g = 0$, otherwise. By assumption there exist $\lambda_0 \in \mathbb{K}$ and $\overline{b} \in D$ such that $b = \lambda_0 e_1 + \overline{b}$.

Since $(B_{\mathbb{K},|c_k-c_{k+1}|}(c_k))_k$ has an empty intersection, we can find $m_1 \in \mathbb{N}$ such that $(b'_1 - \lambda_0) \notin B_{\mathbb{K},|c_{m_1}-c_{m_1+1}|}(c_{m_1})$. Then, we can easily verify that

$$|b'_1 - \lambda_0 + c_{m_1}| = |b'_1 - \lambda_0 + c_n| > |c_{m_1} - c_{m_1+1}| \tag{2.34}$$



for all $n > m_1$ $(n \in \mathbb{N})$. Next, we find $m_2 \in \mathbb{N}$ for which

$$|c_{m_2} - c_{m_2+1}| \cdot \frac{m_2}{m_2 - 1} < |c_{m_1} - c_{m_1+1}|.$$

Hence,

$$\begin{aligned}|b'_{n_p}| < |b'_{n_{m_2}}| &= \frac{|c_{m_2} - c_{m_2+1}|}{|\lambda_{n_{m_2}}|}\\ &< |c_{m_2} - c_{m_2+1}| \cdot \frac{m_2}{m_2 - 1} < |c_{m_1} - c_{m_1+1}| \end{aligned}\tag{2.35}$$

for every $p > m_2$. Since

$$\begin{aligned}&\overline{b} - \sum_{k=1}^{m} b'_{n_k}(\lambda_{n_k} e_1 - e_{n_k}) = b' + g - \lambda_0 e_1 - \sum_{k=1}^{m} b'_{n_k}(\lambda_{n_k} e_1 - e_{n_k})\\ &= g + (b'_1 - \lambda_0 - \sum_{k=1}^{m} b'_{n_k}\lambda_{n_k})e_1 + b'_{n_{m+1}} e_{n_{m+1}} + b'_{n_{m+2}} e_{n_{m+2}} + \ldots\\ &= g + (b'_1 - \lambda_0 + c_1 - (c_1 - c_2) - \ldots\\ &\quad - (c_m - c_{m+1}))e_1 + b'_{n_{m+1}} e_{n_{m+1}} + b'_{n_{m+2}} e_{n_{m+2}} + \ldots\\ &= g + (b'_1 - \lambda_0 + c_{m+1})e_1 + b'_{n_{m+1}} e_{n_{m+1}} + b'_{n_{m+2}} e_{n_{m+2}} + \ldots,\end{aligned}$$

taking $m_0 > \max\{m_1, m_2\}$ from (2.35) and (2.34) we get

$$\begin{aligned}&\left\| \left( \overline{b} - \sum_{k=1}^{m_0} b'_{n_k}(\lambda_{n_k} e_1 - e_{n_k}) \right) - (b'_1 - \lambda_0 + c_{m_0+1})e_1 \right\|\\ &\quad = \|g + b'_{n_{m_0+1}} e_{n_{m_0+1}} + b'_{n_{m_0+2}} e_{n_{m_0+2}} + \ldots\| = \max_{m > m_0} |b'_{n_m}|\\ &\quad < |c_{m_1} - c_{m_1+1}| < |b'_1 - \lambda_0 + c_{m_0+1}| \leqslant \|(b'_1 - \lambda_0 + c_{m_0+1})e_1\|,\end{aligned}$$

a contradiction with orthogonality of $D$ and $[e_1]$.

(2) Since, by Theorem 2.2.10 and Propositions 2.2.1 and 2.2.2, $E$ is Cartesian, it is enough to prove that for every finite-dimensional linear subspace $F \subset E$ there exists $x_F \in F$ such that $\|x - x_F\| = \mathrm{dist}(x, F)$.

Let $n = \dim(F)$. Choose an orthonormal base $(v_k)_k$, where $v_k = (v_k^1, v_k^2, \dots)$ $(k \in \mathbb{N})$, of $F$. By Proposition 2.2.5 and assumption of orthogonality, for every $i \in \{1, \dots, n\}$ there exists $k_i \in \mathbb{N}$ such that $\|v_i\| = |v_i^{k_i}|$ and $k_i \neq k_j$ if $i \neq j$. Even more, we can choose $(v_k)_k$ that for each $i = 1, \dots, n$ we have

$$v_j^{k_i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad (j = 1, \dots, n).$$

Taking $a_1, \dots, a_n \in \mathbb{K}$ and denoting $M_n := \mathbb{N} \setminus \{k_1, \dots, k_n\}$, we get

$$\begin{aligned}\left\| x - \sum_{i=1}^{n} a_i v_i \right\| \\ &= \max\left\{ \max_{i \in \{1,\dots,n\}} |x_{k_i} - a_i|, \sup_{m \in M_n} \left| x_m - \sum_{i=1}^{n} a_i v_i^m \right| \right\}. \end{aligned} \tag{2.36}$$

By (2.30), for every $m \in M_n$,

$$\left| x_m - \sum_{i=1}^{n} v_i^m x_{k_i} \right| > r.$$

Let

$$d := \sup_{m \in M_n} \left| x_m - \sum_{i=1}^{n} v_i^m x_{k_i} \right| \tag{2.37}$$

and assume that, for every $i \in \{1, \dots, n\}$, $|x_{k_i} - a_i| < d$. Thus, there exists $\varepsilon > 0$ such that

$$\max_{i \in \{1,\dots,n\}} |x_{k_i} - a_i| = (1 - \varepsilon) \cdot d \quad \text{and} \quad |(x_{k_i} - a_i) v_i^m| \leqslant (1 - \varepsilon) \cdot d$$

for every $i \in \{1, \dots, n\}$ and $m \in M_n$. Hence, for every $m \in M_n$, we get

$$\sum_{i=1}^{n} (x_{k_i} - a_i) v_i^m \leqslant (1 - \varepsilon) \cdot d. \tag{2.38}$$

Note that by $(2.37)$, there exists $m_0 \in M_n$ with

$$\left| x_{m_0} - \sum_{i=1}^{n} v_i^{m_0} x_{k_i} \right| > (1-\varepsilon) \cdot d \tag{2.39}$$

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and observe that taking $m \in M_n$ we get

$$\left| x_m - \sum_{i=1}^{n} a_i v_i^m \right| = \left| x_m - \sum_{i=1}^{n} v_i^m x_{k_i} + \sum_{i=1}^{n} (x_{k_i} - a_i) v_i^m \right|;$$

hence, by $(2.38)$ and $(2.39)$

$$\sup_{m \in M_n} \left| x_m - \sum_{i=1}^{n} a_i v_i^m \right| = \sup_{m \in M_n} \left| x_m - \sum_{i=1}^{n} v_i^m x_{k_i} \right| = d.$$

Now, applying $(2.36)$ we conclude that

$$\left\| x - \sum_{i=1}^{n} a_i v_i \right\| = \sup_{m \in M_n} \left| x_m - \sum_{i=1}^{n} a_i v_i^m \right| = d.$$

Consequently, there exist $a_1, \dots, a_n \in \mathbb{K}$ such that

$$\operatorname{dist}(x, F) = \left\| x - \sum_{i=1}^{n} a_i v_i \right\|.$$

This shows that $E + [x]$ is Cartesian. □

**2.2.28. Remark.** Note that the valued field $\mathbb{C}_p$, a completion of an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$ (see Proposition 2.12 of [30]) is an example of non-Archimedean field for which the condition $(2.30)$ satisfies.

The following observation is worth mentioning.

**2.2.29. Proposition** ([34, Proposition 14]). *Let* $E_0$ *be a maximal immediate extension of* $c_0$*, contained in* $l^\infty$ *and let* $x = (x_1, x_2, \dots) \in \widehat{c_0} \setminus E_0$*. Assume that* $\operatorname{dist}(x, E_0) = \operatorname{dist}(x, c_0) = r$*. Denote* $N_0 := \{k : \operatorname{dist}(x_k, \mathbb{K}) = r\}$*. If* $N_0$ *is nonempty and finite, then* $E_0 + [x]$ *is not Cartesian.*

*Proof.* Take $k \in \mathbb{N} \setminus N_0$. Then, $\mathtt{dist}(x_k, \mathbb{K}) < r$ and we can find $a_k \in \mathbb{K}$, $|a_k| = |x_k|$ which satisfies $|x_k - a_k| \leqslant r$. Define $y = (y_1, y_2, \dots) \in l^\infty$, taking $y_k = 0$ if $k \in N_0$ and $y_k = a_k$, otherwise. By Proposition 2.2.8, there exists $z = (z_1, z_2, \dots) \in E_0$ such that $|z_n - y_n| \leqslant r$ for all $n \in \mathbb{N}$. Let $F = [(e_i)_{i \in N_0}]$. Then, for any $\lambda_k \in \mathbb{K}$ $(k \in N_0)$ we obtain

$$\left\| (x - z) - \sum_{k \in N_0} \lambda_k e_k \right\| = \max_{k \in N_0} |x_k - \lambda_k| > r = \mathtt{dist}(y - x, F)$$

and conclude that $[x - z] + F$ has no orthogonal base. □

## 2.3 The finite-dimensional decompositions in non-Archimedean Banach spaces

Recall that a real or complex separable Banach space $X$ has the *finite-dimensional decomposition* if there exists a sequence $(D_n)_n$ of finite-dimensional subspaces of $X$ such that every $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in D_n$ for all $n \in \mathbb{N}$. Clearly, every Banach space with a Schauder basis has the finite-dimensional decomposition, but the converse is false. There exist separable Banach spaces without finite-dimensional decomposition. Also, a closed linear subspace of a real or complex Banach space with the finite-dimensional decomposition needs not have the finite-dimensional decomposition (see [8]).

In the non-Archimedean context the situation differs substantially, every non-Archimedean Banach space of countable type has a Schauder base; thus, all such spaces and their closed subspaces have the finite-dimensional decomposition (although, as proved Śliwa in [69], there exist non-Archimedean Fréchet spaces of countable type without the finite-dimensional decomposition).

A natural modification of the above classical concept reads as follows:

Let $E$ be a non-Archimedean Banach space of countable type. We say that $E$ has the *orthogonal finite-dimensional decomposition* (OFDD) or $E$ has the *orthogonal finite-dimensional decomposition property* (OFDDP) if $E$ is the orthogonal direct sum of a sequence of finite-dimensional subspaces $D_1, D_2, \ldots$, i.e. every $x \in E$ can be unequivocally written as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in D_n$ $(n \in \mathbb{N})$, and we have $\|x\| = \max_n \{\|x_n\|\}$.

If $\mathbb{K}$ is spherically complete, every non-Archimedean Banach space of countable type has an orthogonal base ([57, Lemma 5.5]), thus, it has the orthogonal finite-dimensional decomposition. If $\mathbb{K}$ is not spherically complete, there exist various kinds, even of finite-dimensional non-Archimedean spaces, without an orthogonal base as well as examples of non-Archimedean Banach spaces without the orthogonal finite-dimensional decomposition property (Proposition 2.3.1 shows simple examples). Thus, for these $\mathbb{K}$, the class of Banach spaces with the orthogonal finite-dimensional decomposition can be considered as a proper generalization of the class of non-Archimedean Banach spaces of countable type with an orthogonal base.

**2.3.1. Proposition.** *Let $\mathbb{K}$ be non-spherically complete and let $E = \widehat{\mathbb{K}}$ (the spherical completion of $\mathbb{K}$). Let $D$ be a closed subspace of countable type of $E$ and $F$ be a finite-dimensional linear subspaces of $E$, respectively. Then,*

(1) *$D$ does not have the orthogonal finite-dimensional decomposition property;*
(2) *the space $F \oplus c_0$ has the orthogonal finite-dimensional decomposition property, but it has no orthogonal base.*

*Proof.* Recall that $E$, thus $D$ and $F$, are immediate extensions of any one-dimensional linear subspaces; hence, assuming that $D$ has OFDD, we imply that $D$ contains a countable orthogonal set, a contradiction. Clearly, $F \oplus c_0$ has the orthogonal finite-dimensional decomposition. As, $F$ has no orthogonal base, it follows from Theorem 1.1.4 that $F \oplus c_0$ has no orthogonal base, either. □

By Gruson's theorem (Theorem 1.1.4 ) every closed linear subspace of a non-Archimedean Banach space with an orthogonal base has an orthogonal base, either. The following question, formulated by Perez-Garcia and Schikhof in [49, Remark 4.10], is quite natural:

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**2.3.2. Problem.** Does any closed linear subspace of a non-Archimedean Banach space $E$ with the orthogonal finite-dimensional decomposition property have the orthogonal finite-dimensional decomposition, either?

Perez-Garcia and Schikhof proved that the answer for this question is affirmative if $D$ is orthocomplemented in $E$ (see [49, Theorems 4.1, 4.3 and Remark 10] and [48]). In Theorem 2.3.6 we present a counterexample, a non-Archimedean Banach space $E$ with the OFDDP and its closed linear subspace without this property.

To prove Theorem 2.3.6 we need the following lemmas.

**2.3.3. Lemma** ([36, Lemma 2.2]). *Let* $\lambda, \nu \in \widehat{\mathbb{K}}$ *and let* $r > 0$*. Suppose there exists* $n_0 \in \mathbb{N}$ *such that*

$$|\lambda - c_n| < \left(1 + \frac{1}{n}\right) r \tag{2.40}$$

*and*

$$|\nu + b - a c_n| < \left(1 + \frac{1}{n}\right) r \tag{2.41}$$

*hold for some* $a, b, c_1, c_2, \ldots \in \mathbb{K}$ *and all* $n \geqslant n_0$*. Then,* $a\lambda - b \in B_{\widehat{\mathbb{K}},r}(\nu)$ *if* $|a| \leqslant 1$ *and* $\nu/a + b/a \in B_{\widehat{\mathbb{K}},r}(\lambda)$*, otherwise.*

*Proof.* Suppose $|a| \leqslant 1$. In this case, by (2.40), we have $|a\lambda - a c_n| < (1 + 1/n) r$ and, by (2.41),

$$\begin{aligned} |a\lambda - b - \nu| &= |a\lambda - a c_n + a c_n - b - \nu| \\ &\leqslant \max\{|a\lambda - a c_n|, |\nu + b - a c_n|\} < \left(1 + \frac{1}{n}\right) r. \end{aligned}$$

Since this inequality holds for all $n \geqslant n_0$, we derive that $a\lambda - b \in B_{\widehat{\mathbb{K}},r}(\nu)$.

Assume now $|a| > 1$. Then, by (2.40) and (2.41) we get

$$\begin{aligned} \left|\frac{1}{a}\nu + \frac{b}{a} - \lambda\right| &= \left|\frac{1}{a}\nu + \frac{b}{a} - c_n + c_n - \lambda\right| \\ &\leqslant \max\left\{\left|\frac{1}{a}\right| \cdot |\nu + b - a c_n|, |\lambda - c_n|\right\} < \left(1 + \frac{1}{n}\right) r, \end{aligned}$$

and we conclude that $\nu/a + b/a \in B_{\widehat{\mathbb{K}},r}(\lambda)$. $\square$

**2.3.4. Lemma** ([36, Lemma 2.3]). *Let* $\lambda_1, \lambda_2, \ldots \in \widehat{\mathbb{K}} \setminus \mathbb{K}$. *Then, for each* $m \in \mathbb{N}$ *there exists a sequence* $(c_n^m)_n$ *in* $\mathbb{K}$ *with* $\lim_n |\lambda_m - c_n^m| = r_m := \mathrm{dist}(\lambda_m, \mathbb{K})$ *and such that the following holds:*

(1) $(B_{\mathbb{K},|c_n^m - c_{n+1}^m|}(c_n^m))_n$ *is a strictly decreasing sequence of closed balls in* $\mathbb{K}$ *for which*

$$\bigcap_{n\in\mathbb{N}} B_{\mathbb{K},|c_n^m - c_{n+1}^m|}(c_n^m)_n = \emptyset. \tag{2.42}$$

(2) *For every* $n \in \mathbb{N}$,

$$|c_n^m| = |\lambda_m| \quad \textit{and} \quad |\lambda_m - c_n^m| < \left(1 + \frac{1}{n}\right) r_m. \tag{2.43}$$

(3) *For every* $c \in \mathbb{K}$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$|c_n^m - c| = |\lambda_m - c| \quad \textit{for all } n \geqslant n_0. \tag{2.44}$$

*Proof.* Let $m \in \mathbb{N}$. First of all note that, since $r_m = \mathrm{dist}(\lambda_m, \mathbb{K})$, there exists a sequence $(c_n^m)_n$ in $\mathbb{K}$ with $\lim_n |\lambda_m - c_n^m| = r_m$. Since $\mathrm{dist}(\lambda, \mathbb{K})$ is not attained for each $\lambda \in \widehat{\mathbb{K}} \setminus \mathbb{K}$, we can assume that

$$|\lambda_m - c_n^m| > |\lambda_m - c_{n+1}^m| \tag{2.45}$$

for all $n \in \mathbb{N}$. Next, let us prove (1)–(3).

(1) It follows from (2.45) that

$$|c_n^m - c_{n+1}^m| = |\lambda_m - c_n^m| \tag{2.46}$$

for all $n \in \mathbb{N}$. Now, from (2.45) and (2.46) it follows that

$$|c_n^m - c_{n+1}^m| > |c_{n+1}^m - c_{n+2}^m| \tag{2.47}$$

for all $n \in \mathbb{N}$, for which we get that $(B_{\mathbb{K},|c_n^m - c_{n+1}^m|}(c_n^m))_n$ is a strictly decreasing sequence of closed balls in $\mathbb{K}$.

Suppose the intersection of these balls is nonempty i.e. there exists a $c \in \mathbb{K}$ with $|c - c_n^m| \leqslant |c_n^m - c_{n+1}^m|$ for all $n$. Then, by (2.46), $|\lambda_m - c| \leqslant |\lambda_m - c_n^m|$ for all $n \in \mathbb{N}$. Hence, $|\lambda_m - c| = r_m$ and we imply that $\mathrm{dist}(\lambda_m, \mathbb{K})$ is attained, a contradiction.

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(2) It is obvious that the $c_n^m$ $(m, n \in \mathbb{N})$ can be chosen satisfying the second part of (2.43). To prove the first part observe that, $r_m < |\lambda_m|$, from which we have that $|\lambda_m - c_n^m| < |\lambda_m|$ for large $n$, so $|c_n^m| = |\lambda_m|$. Therefore, the $c_n^m$ also can be chosen satisfying the first part of (2.43).

(3) Fix $c \in \mathbb{K}$. By (2.42), there exists $n_0 \in \mathbb{N}$ such that

$$c \notin B_{\mathbb{K},|c_{n_0}^m - c_{n_0+1}^m|}(c_{n_0}^m).$$

Also, by (2.47), $|c_{n_0}^m - c_{n_0+1}^m| \geqslant |c_{n_0}^m - c_n^m|$ for all $n \geqslant n_0$, hence we obtain

$$\begin{aligned}|c_n^m - c| &= |c_n^m - c_{n_0}^m + c_{n_0}^m - c| \\ &= \max\left\{|c_n^m - c_{n_0}^m|, |c_{n_0}^m - c|\right\} = |c_{n_0}^m - c|.\end{aligned}$$

Finally, by (2.46),

$$\begin{aligned}|\lambda_m - c| &= |\lambda_m - c_{n_0}^m + c_{n_0}^m - c| \\ &= \max\left\{|\lambda_m - c_{n_0}^m|, |c_{n_0}^m - c|\right\} = |c_{n_0}^m - c|,\end{aligned}$$

and we are done. □

**2.3.5. Lemma** ([36, Lemma 2.4]). *If* $E$ *has the OFDDP, then every one-dimensional linear subspace of* $E$ *is contained in a finite-dimensional orthocomplemented subspace of* $E$.

*Proof.* We may assume that $E$ is infinite-dimensional. Let $E = \bigoplus_{i \in \mathbb{N}} E_i$, where each $E_i$ is a finite-dimensional subspace of $E$. Let $[x]$ be a one-dimensional subspace of $E$. We can write $x = \sum_{i \in \mathbb{N}} x_i$, with $x_i \in E_i$ for each $i \in \mathbb{N}$. Fix $i_0 \in \mathbb{N}$ and $t < 1$. Then, applying [47, Theorem 2.3.13], we can select a subspace $D_{i_0} \subset E_{i_0}$ such that $E_{i_0} = D_{i_0} + [x_{i_0}]$ and $\|x_{i_0} + d\| \geqslant t \cdot \max\{\|x_{i_0}\|, \|d\|\}$ for all $d \in D_{i_0}$. Let $I_0 = \{i \in \mathbb{N} : \|x_i\| \geqslant t \cdot \|x_{i_0}\|, i \neq i_0\}$. Since $x_i \to 0$ if $i \to \infty$, $I_0$ is finite. Define

$$F := [x] + D_{i_0} + \left(\bigoplus_{i \in I_0} E_i\right).$$

One can easily verify that

$$E = F \bigoplus \Big( \bigoplus_{i \in \mathbb{N} \setminus (I_0 \cup \{i_0\})} E_i \Big).$$

Hence, $F$ is a finite-dimensional orthocomplemented subspace of $E$ containing $[x]$. $\square$

The construction of the space presented in Theorem 2.3.6 is based on some properties of sequences of elements of $\widehat{\mathbb{K}}$.

Choose $\lambda_1, \lambda_2, \ldots \in \widehat{\mathbb{K}} \setminus \mathbb{K}$ such that $|\lambda_k| = 1$ $(k \in \mathbb{N})$ and $\mathrm{dist}(\lambda_k, \mathbb{K}) = \mathrm{dist}(\lambda_1, \mathbb{K})$ for all $k \geqslant 2$. Set $r := \mathrm{dist}(\lambda_1, \mathbb{K})$ and $\Lambda := \{\lambda_1, \lambda_2, \ldots\}$. Hence, for all $c \in \mathbb{K}$,

$$r < |\lambda_k - c|, \text{ so } r < |\lambda_k| \text{ and } r < 1. \tag{2.48}$$

In $\Lambda$ we define a relation $\sim$ as follows

$$\lambda_i \sim \lambda_j \text{ if there exist } a, b \in \mathbb{K} \text{ such that } a\lambda_i + b \in B_{\widehat{\mathbb{K}},r}(\lambda_j).$$

Let $E_\Lambda := \overline{[\{e_1, \lambda_1 e_1, e_2, \lambda_2 e_2, \ldots\}]}$ be the closure in $l^\infty(\mathbb{N}, \widehat{\mathbb{K}})$ of the $\mathbb{K}$-linear subspace spanned by $\{e_1, \lambda_1 e_1, e_2, \lambda_2 e_2, \ldots\}$ ($e_1, e_2, \ldots$ are standard unit vectors; note that $\lambda_k e_k \notin [e_k]$ for every $k \in \mathbb{N}$, since $\lambda_k \in \widehat{\mathbb{K}} \setminus \mathbb{K}$). Then, $E_\Lambda$ is a Banach space of countable type with the OFDDP, since we can write $E_\Lambda = \bigoplus_k D_k$, where $D_k := [e_k, \lambda_k e_k]$, $k \in \mathbb{N}$.

Define $X_1 := \{e_1, e_2, \ldots\}$, $X_2 := \{\lambda_1 e_1 + \lambda_2 e_2, \lambda_1 e_1 + \lambda_3 e_3, \ldots\}$ and $D_\Lambda := \overline{[X_1 \cup X_2]}$. Then, $D_\Lambda$ is a one-codimensional (hence closed) subspace of $E_\Lambda$, since $\lambda_1 e_1 \notin D_\Lambda$ and $E_\Lambda = D_\Lambda + [\lambda_1 e_1]$. One can easily verify that $X_1$ and $X_2$ are orthogonal sets, hence every $x \in D_\Lambda$ can be uniquely written as

$$x = \sum_{i=1}^{\infty} a_i e_i + \sum_{i=2}^{\infty} A_i(\lambda_1 e_1 + \lambda_i e_i), \quad a_i, A_i \in \mathbb{K},\ i \in \mathbb{N}. \tag{2.49}$$

For such $\Lambda$, $E_\Lambda$ and $D_\Lambda$ we prove the following:

**2.3.6. Theorem** ([36, Theorem 3.1])**.** *$D_\Lambda$ has the OFDDP if and only if $\Lambda$ has finitely many equivalence classes with respect to the relation $\sim$.*

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*Proof.* ($\Rightarrow$) Assume for a contradiction that $\Lambda$ has infinitely many equivalence classes with respect to $\sim$. For each $m \in \mathbb{N}$, let $(c_n^m)_n$ be as in Lemma 2.3.4. By the OFDDP for $D_\Lambda$ and Lemma 2.3.5, there exists a finite-dimensional subspace $F \subset D_\Lambda$, containing $e_1$, and a closed subspace $G \subset D_\Lambda$ such that $D_\Lambda = F \oplus G$.

Since by assumption F is finite-dimensional, there exists $m_0 \in \mathbb{N}$ such that for every $x \in F$, $\|x\| \leqslant 1$, written as

$$x = \sum_{i=1}^{\infty} a_i e_i + \sum_{i=2}^{\infty} A_i(\lambda_1 e_1 + \lambda_i e_i)$$

(see (2.49)), $|a_i| < r$ and $|A_i| < r$ for all $i > m_0$.

Choose $m > m_0$ such that $\lambda_m \not\sim \lambda_i$ for all $i \in \{1, \dots, m_0\}$. There exist $u, w \in F$ for which

$$\begin{aligned} e_m &= u - (u - e_m), \\ \lambda_1 e_1 + \lambda_m e_m &= -w + (\lambda_1 e_1 + \lambda_m e_m + w) \end{aligned}$$

and $(u - e_m), (\lambda_1 e_1 + \lambda_m e_m + w) \in G$. Since $F \perp G$, $\|u\|, \|w\| \leqslant 1$. We can write

$$u = \sum_{i=1}^{\infty} a_i e_i + \sum_{i=2}^{\infty} A_i(\lambda_1 e_1 + \lambda_i e_i), \quad |a_i|, |A_i| < r \text{ if } i > m_0, \quad (2.50)$$

$$w = \sum_{i=1}^{\infty} a_i' e_i + \sum_{i=2}^{\infty} A_i'(\lambda_1 e_1 + \lambda_i e_i), \quad |a_i'|, |A_i'| < r \text{ if } i > m_0. \quad (2.51)$$

By Lemma 2.3.4 (second part of (2.43)), for every $n \in \mathbb{N}$ we get

$$\begin{aligned} &\|\lambda_1 e_1 + \lambda_m e_m - c_n^1 e_1 - c_n^m e_m\| \\ &\qquad\qquad = \max\left\{|\lambda_1 - c_n^1|, |\lambda_m - c_n^m\right\} < \left(1 + \frac{1}{n}\right) r. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\|\lambda_1 e_1 + \lambda_m e_m - c_n^1 e_1 - c_n^m e_m\| \\ &= \|\lambda_1 e_1 + \lambda_m e_m + w - w - c_n^1 e_1 - c_n^m e_m - c_n^m u + c_n^m u\| \\ &= \max\left\{\|\lambda_1 e_1 + \lambda_m e_m + w + c_n^m (u - e_m)\|, \|c_n^1 e_1 + c_n^m u + w\|\right\}, \end{aligned}$$

since $e_1, u, w \in F$, $(u - e_m)$, $(\lambda_1 e_1 + \lambda_m e_m + w) \in G$ and $F \perp G$. Hence,

$$\|\lambda_1 e_1 + \lambda_m e_m + w + c_n^m(u - e_m)\| < \left(1 + \frac{1}{n}\right) r. \tag{2.52}$$

Now, applying (2.50) and (2.51), for every $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\|\lambda_1 e_1 &+ \lambda_m e_m + w + c_n^m(u - e_m)\| \\
&= \left\| \lambda_1 e_1 + \lambda_m e_m + \sum_{i=1}^{\infty} (a_i' + c_n^m a_i) e_i \right. \\
&\qquad \left. + \sum_{i=2}^{\infty} (A_i' + c_n^m A_i)(\lambda_1 e_1 + \lambda_i e_i) - c_n^m e_m \right\| \\
&= \max \left\{ \left| a_1' + c_n^m a_1 + \lambda_1 \left(1 + \sum_{i=2}^{m_0} (A_i' + c_n^m A_i)\right) \right|, \right. \\
&\qquad \left. \max_{i=2,\dots,m_0} |a_i' + c_n^m a_i + (A_i' + c_n^m A_i)\lambda_i|, |\lambda_m - c_n^m| \right\},
\end{aligned} \tag{2.53}$$

since $|a_i|, |a_i'| < r$ and $|A_i|, |A_i'| < r$ for all $i > m_0$ (see (2.50) and (2.51)), $|c_n^m| = 1$ (see (2.43)) and $|\lambda_m - c_n^m| > r$ (see (2.48)). Thus, by (2.52) and (2.53), for every $i \in \{2, \dots, m_0\}$ and every $n \in \mathbb{N}$, we obtain

$$|a_i' + c_n^m a_i + (A_i' + c_n^m A_i)\lambda_i| < \left(1 + \frac{1}{n}\right) r. \tag{2.54}$$

We deduce that

$$|A_i' + c_n^m A_i| \leqslant 1 \tag{2.55}$$

for every $i \in \{2, \dots, m_0\}$ and large $n$. Indeed, by (3) of Lemma 2.3.4, for every $i \in \{2, \dots, m_0\}$ there exists $n_i \in \mathbb{N}$ and $d_i \geqslant 0$ such that $|A_i' + c_n^m A_i| = d_i$ for all $n \geqslant n_i$. Assuming that $d_i > 1$, we can choose $k > n_i$ for which $d_i > (1 + 1/k)$. Then, by (2.54),

$$\left| \frac{a_i' + c_k^m a_i}{A_i' + c_k^m A_i} + \lambda_i \right| < \frac{1}{d_i} \left(1 + \frac{1}{k}\right) r < r,$$

a contradiction with $r = \mathtt{dist}(\lambda_i, \mathbb{K})$.

Let $i \in \{2, \dots, m_0\}$. It follows from (2.55) that, if $A_i \neq 0$,

$$|A_i| \left| \frac{A_i'}{A_i} + c_n^m \right| \leqslant 1$$

for large $n$. Also, since by Lemma 2.3.4 (see (2.44)) and (2.48),

$$r < \left|\frac{A_i'}{A_i} + \lambda_m\right| = \lim_n \left|\frac{A_i'}{A_i} + c_n^m\right|,$$

we have

$$r < \left|\frac{A_i'}{A_i} + c_n^m\right| \leqslant \frac{1}{|A_i|}$$

again for large $n$. We derive that $|A_i| < 1/r$ (it is trivially true when $A_i = 0$). Choose $p \in \mathbb{N}$ for which

$$|A_i| < \frac{1}{\left(1+\frac{1}{p}\right)^2 r}. \tag{2.56}$$

From Lemma 2.3.4 (see (2.43), (2.46)) and (2.47) we know that

$$|\lambda_i - c_p^i| < \left(1+\frac{1}{p}\right)r \quad \text{and} \quad |c_q^m - c_p^m| < \left(1+\frac{1}{p}\right)r$$

for all $q > p$. Hence, by (2.56)

$$|A_i(\lambda_i - c_p^i)(c_q^m - c_p^m)| < r. \tag{2.57}$$

As $A_i(\lambda_i - c_p^i)(c_q^m - c_p^m) = A_i c_q^m \lambda_i - A_i c_p^m \lambda_i - A_i c_p^i c_q^m + A_i c_p^i c_p^m$, from (2.54) and (2.57) we obtain

$$\begin{aligned}&\big|a_i' + c_q^m a_i + (A_i' + c_q^m A_i)\lambda_i \\ &\qquad - (A_i c_q^m \lambda_i - A_i c_p^m \lambda_i - A_i c_p^i c_q^m + A_i c_p^i c_p^m)\big| < \left(1+\frac{1}{q}\right)r\end{aligned}$$

for large $q$. Thus, for those $q$ we get

$$\begin{aligned}&\big|a_i' + c_q^m a_i + A_i'\lambda_i + A_i c_p^m \lambda_i + A_i c_p^i c_q^m - A_i c_p^i c_p^m\big| \\ &= \big|\lambda_i(A_i' + A_i c_p^m) + c_q^m(a_i + A_i c_p^i) - A_i c_p^i c_p^m + a_i'\big| < \left(1+\frac{1}{q}\right)r.\end{aligned}$$

Assume that $|A_i' + A_i c_p^m| = 1$. Then we have

$$\left|\lambda_i + c_q^m \frac{a_i + A_i c_p^i}{A_i' + A_i c_p^m} - \frac{A_i c_p^i c_p^m - a_i'}{A_i' + A_i c_p^m}\right| < \left(1+\frac{1}{q}\right)r$$

for large $q$. Since $|\lambda_m - c_q^m| < (1+1/q)r$ for all $q$ (see Lemma 2.3.4, (2.43)), applying Lemma 2.3.3, we conclude that $\lambda_m \sim \lambda_i$, a contradiction with the choice of $m$.

By (2.55), $|A_i' + A_i c_p^m| < 1$ for all $i \in \{2, \dots, m_0\}$. Observe that, according to the construction of $p$, we can take the same $p$ for all $i \in \{2, \dots, m_0\}$. Hence,

$$\left|1 + \sum_{i=2}^{m_0} (A_i' + A_i c_p^m)\right| = 1. \tag{2.58}$$

Further, by (2.52) and (2.53) we get, for all $q \in \mathbb{N}$, that

$$\left|a_1' + c_q^m a_1 + \lambda_1 \left(1 + \sum_{i=2}^{m_0} (A_i' + c_q^m A_i)\right)\right| < \left(1 + \frac{1}{q}\right) r.$$

Proceeding as in (2.57) we obtain that

$$\left|A_i(\lambda_1 - c_p^1)(c_q^m - c_p^m)\right| < r$$

for every $i \in \{2, \dots, m_0\}$ and large $q$. Then, in this case we arrive at

$$\begin{aligned}\Bigg|a_1' + c_q^m a_1 + \lambda_1 \left(1 + \sum_{i=2}^{m_0} (A_i' + c_q^m A_i)\right) \\ - \sum_{i=2}^{m_0} (A_i c_q^m \lambda_1 - A_i c_p^m \lambda_1 - A_i c_p^1 c_q^m + A_i c_p^1 c_p^m)\Bigg| < \left(1 + \frac{1}{q}\right) r\end{aligned}$$

for large $q$. Thus, for those $q$ we have

$$\begin{aligned}&\left|a_1' + c_q^m a_1 + \lambda_1 \left(1 + \sum_{i=2}^{m_0} (A_i' + A_i c_p^m)\right) + \sum_{i=2}^{m_0} (A_i c_p^1 c_q^m - A_i c_p^1 c_p^m)\right| \\ &= \left|\lambda_1 \left(1 + \sum_{i=2}^{m_0} (A_i' + A_i c_p^m)\right) + c_q^m \left(a_1 + \sum_{i=2}^{m_0} A_i c_p^1\right) + a_1' - \sum_{i=2}^{m_0} A_i c_p^1 c_p^m\right| \\ &< \left(1 + \frac{1}{q}\right) r.\end{aligned}$$

By (2.58) we get

$$\left|\lambda_1 + c_q^m \frac{a_1 + \sum_{i=2}^{m_0} A_i c_p^1}{1 + \sum_{i=2}^{m_0} (A_i' + A_i c_p^m)} + \frac{a_1' - \sum_{i=2}^{m_0} A_i c_p^1 c_p^m}{1 + \sum_{i=2}^{m_0} (A_i' + A_i c_p^m)}\right| < \left(1 + \frac{1}{q}\right) r$$

for large $q$. Using Lemma 2.3.3 again, we imply $\lambda_m \sim \lambda_1$, a contradiction with the choice of $m$.

($\Leftarrow$) Suppose now $\Lambda$ has finitely many, say $s$, equivalence classes with respect to $\sim$. We will show that $D_\Lambda$ has the OFDDP.

Define $S := \{1, \dots, s\}$. Next, form $\{M_k\}_{k \in S}$, a partition of $\mathbb{N}$ such that $i, j \in M_k$ ($k \in S$) if and only if $\lambda_i \sim \lambda_j$. Assume $1 \in M_1$. We will construct closed subspaces $H_1, \dots, H_s, H_{s+1}$ of $D_\Lambda$ as follows.

If $M_1 = \{1\}$, set $H_1 := \{0\}$. Otherwise, for $n \in M_1 \setminus \{1\}$, let

$$D_n^1 := \left[\lambda_1 e_1 + \lambda_n e_n + \frac{b_n^1}{a_n^1} e_n, e_1 + \frac{1}{a_n^1} e_n\right],$$

where $a_n^1, b_n^1 \in \mathbb{K}$ satisfy $a_n^1 \lambda_n + b_n^1 \in B_{\widehat{\mathbb{K}},r}(\lambda_1)$ (which implies $|a_n^1| = 1$, see the comments before Theorem 2.3.6). Then, for every $x \in D_n^1$, which can be written as

$$x = \alpha\left(\lambda_1 e_1 + \lambda_n e_n + \frac{b_n^1}{a_n^1} e_n\right) + \beta\left(e_1 + \frac{1}{a_n^1} e_n\right)$$

for some $\alpha, \beta \in \mathbb{K}$, we obtain

$$\|x\| = \max\left\{|\alpha\lambda_1 + \beta|, \left|\alpha\lambda_n + \alpha\frac{b_n^1}{a_n^1} + \beta\frac{1}{a_n^1}\right|\right\}.$$

Also,

$$\begin{aligned}\left|\alpha\lambda_n + \alpha\frac{b_n^1}{a_n^1} + \beta\frac{1}{a_n^1}\right| &= |\alpha a_n^1 \lambda_n + \alpha b_n^1 + \beta| \\ &= |\alpha a_n^1 \lambda_n + \alpha b_n^1 - \alpha\lambda_1 + \alpha\lambda_1 + \beta| = |\alpha\lambda_1 + \beta|,\end{aligned}$$

since $|\lambda_1 + c| > r$ for each $c \in \mathbb{K}$ (see (2.48)) and $|a_n^1 \lambda_n + b_n^1 - \lambda_1| \leqslant r$. Thus,

$$\|x\| = \left|\alpha\lambda_n + \alpha\frac{b_n^1}{a_n^1} + \beta\frac{1}{a_n^1}\right|. \tag{2.59}$$

From (2.59) we conclude that, for all $n \in M_1 \setminus \{1\}$,

$$D_n^1 \perp \sum_{m \in M_1 \setminus \{1,n\}} D_m^1,$$

and from (2.49) and (2.59) that, for those $n$,

$$\begin{aligned} D_\Lambda = D_n^1 \oplus \overline{[Y_1 \cup Y_2]}, \qquad & Y_1 := \{e_i : i \neq n\}, \\ & Y_2 := \{\lambda_1 e_1 + \lambda_i e_i : i \geqslant 2, i \neq n\}. \end{aligned} \tag{2.60}$$

Set $H_1 := \bigoplus_{n \in M_1 \setminus \{1\}} D_n^1$. Applying (2.60) recurrently on $n \in M_1 \setminus \{1\}$, we have

$$\begin{aligned} D_\Lambda = H_1 + \overline{[W_1^1 \cup W_2^1]}, \qquad & W_1^1 := \{e_1\} \cup \{e_i : i \notin M_1\}, \\ & W_2^1 := \{\lambda_1 e_1 + \lambda_i e_i : i \notin M_1\}. \end{aligned}$$

Now, for $k \in S \setminus \{1\}$, choose $n_k \in M_k$. If $M_k = \{n_k\}$, set $H_k := \{0\}$. Otherwise, for each $n \in M_k \setminus \{n_k\}$ define

$$D_n^k := \left[\lambda_{n_k} e_{n_k} - \lambda_n e_n - \frac{b_n^k}{a_n^k} e_n, e_{n_k} - \frac{1}{a_n^k} e_n\right],$$

where $a_n^k, b_n^k \in K$ with $a_n^k \lambda_n + b_n^k \in B_{\widehat{\mathbb{K}},r}(\lambda_{n_k})$ (again $|a_n^k| = 1$). Similarly as above we obtain that for every

$$x = \alpha\left(\lambda_{n_k} e_{n_k} - \lambda_n e_n - \frac{b_n^k}{a_n^k} e_n\right) + \beta\left(e_{n_k} - \frac{1}{a_n^k} e_n\right) \in D_n^k,$$

$$\|x\| = \left|\alpha \lambda_n + \alpha \frac{b_n^k}{a_n^k} + \beta \frac{1}{a_n^k}\right| \tag{2.61}$$

and also that for every $n \in M_k \setminus \{n_k\}$, $D_n^k \perp \sum_{m \in M_k \setminus \{n_k, n\}} D_m^k$.

Set $H_k := \bigoplus_{n \in M_k \setminus \{n_k\}} D_n^k$. We have

$$\begin{aligned} D_\Lambda &= H_1 + H_k + \overline{[W_1^k \cup W_2^k]}, \\ W_1^k &:= \{e_1, e_{n_k}\} \cup \{e_i : i \notin M_1 \cup M_k\}, \\ W_2^k &:= \{\lambda_1 e_1 + \lambda_{n_k} e_{n_k}\} \cup \{\lambda_1 e_1 + \lambda_i e_i : i \notin M_1 \cup M_k\}. \end{aligned} \tag{2.62}$$

Next, define

$$\begin{aligned} Z_1 &:= \{e_1\} \cup \{e_{n_k} : k \in S \setminus \{1\}\}, \\ Z_2 &:= \{\lambda_1 e_1 + \lambda_{n_k} e_{n_k} : k \in S \setminus \{1\}\} \end{aligned}$$

and $H_{s+1} := \overline{[Z_1 \cup Z_2]}$. Using (2.59) and (2.61) one can easily verify that, for each $i \in J$, where $J = \{1, \ldots, s, s+1\}$, $H_i \perp \sum_{j \in J, j \neq i} H_j$. Therefore, applying (2.62) recurrently on $k \in S \setminus \{1\}$, we finally get

$$D_\Lambda = \bigoplus_{1 \leqslant i \leqslant s+1} H_i = H_{s+1} \bigoplus_{k \in S, n \in M_k \setminus \{n_k\}} D_n^k \quad (n_1 := 1). \tag{2.63}$$

As $S$ is finite, $H_{s+1}$ is finite-dimensional and from (2.63) we conclude that $D_\Lambda$ has the OFDDP. □

As an application of Theorem 2.3.6 we derive the following.

**2.3.7. Proposition** ([36, Example 3.2]). *Let* $K = \mathbb{C}_p$. *There exist infinite sets* $\Lambda_1, \Lambda_2$ *and the one-codimensional subspaces* $D_{\Lambda_1}, D_{\Lambda_2}$ *of* $E_{\Lambda_1}$ *and* $E_{\Lambda_2}$ *respectively such that*

(1) $D_{\Lambda_1}$ *does not have the OFDDP;*

(2) $D_{\Lambda_2}$ *has the OFDDP.*

*Proof.* (1) By [30, Corollary 2.14], $\widehat{\mathbb{C}}_p \setminus \mathbb{C}_p$ contains infinitely many elements $\lambda_1, \lambda_2, \ldots$ with $|\lambda_i| = |\lambda_j|$ and $\mathrm{dist}(\lambda_i, \mathbb{C}_p) = \mathrm{dist}(\lambda_j, \mathbb{C}_p)$ for all $i, j$, such that $\lambda_i \not\sim \lambda_j$ for all $i \neq j$. As $|\widehat{\mathbb{C}}_p| = |\mathbb{C}_p|$, by scalar multiplication we may assume that $|\lambda_i| = 1$ for all $i$. So, $\Lambda_1 := \{\lambda_1, \lambda_2, \ldots\}$ has infinitely many equivalence classes with respect to $\sim$. Now, by Theorem 2.3.6, $D_{\Lambda_1}$ does not have the OFDDP.

(2) Choose $\lambda \in \widehat{\mathbb{C}}_p \setminus \mathbb{C}_p$ with $|\lambda| = 1$ and $c \in \mathbb{C}_p$ with $0 < |c| \leqslant r := \mathrm{dist}(\lambda, \mathbb{C}_p)$ (as in (2.48) we have $r < 1$). Then define $\lambda_1 := \lambda$ and $\lambda_i := \lambda_{i-1} + c$ $(\in \widehat{\mathbb{C}}_p \setminus \mathbb{C}_p)$ for $i \geqslant 2$. It is straightforward to verify that $|\lambda_i| = 1$, $\mathrm{dist}(\lambda_i, \mathbb{C}_p) = \mathrm{dist}(\lambda_j, \mathbb{C}_p)$ for all $i, j$ and that the infinite set $\Lambda_2 := \{\lambda_1, \lambda_2, \ldots\}$ has only one equivalence class with respect to $\sim$. Applying again Theorem 2.3.6 we deduce that $D_{\Lambda_2}$ has the OFDDP. □

The next result shows that for certain class of non-Archimedean Banach spaces of countable type over non-spherically complete $\mathbb{K}$, the OFDDP is preserved by taking finite-codimensional subspaces.

**2.3.8. Theorem** ([36, Theorem 4.1]). *Let* $E$ *be a non-Archimedean Banach space over non-spherically complete* $\mathbb{K}$*. Assume that* $E = F_E \oplus G_E$*, where* $F_E$ *and* $G_E$ *are closed linear subspaces of* $E$ *and* $G_E$ *has an orthogonal base. Let* $D$ *be a* $n$*-codimensional subspace of* $E$ *for some* $n \in \mathbb{N}$*. Then, there exist* $u_1, \dots, u_n \in E$ *and closed linear subspaces* $F_D, G_D \subset E$ *such that* $F_D \subset F_E + [u_1, \dots, u_n]$*,* $G_D$ *has an orthogonal base and* $D = F_D \oplus G_D$*.*

*Proof.* It suffices to prove the result when D is one-codimensional. For $n > 1$, take (closed) subspaces $D_1, \dots, D_n$ with $D = D_n \subset D_{n-1} \subset \dots \subset D_1 \subset E$ and $\dim(E/D_1) = \dim(D_{k-1}/D_k) = 1$, $k \in \{2, \dots, n\}$. Then apply recurrently the one-codimensional case to get the conclusion.

So, let us assume that $D$ is a one-codimensional (hence closed) subspace of $E$. If $F_E \subset D$, then $D = F_E \oplus (G_E \cap D)$ and, since $G_E \cap D$ has an orthogonal base by Theorem 1.1.4, we are done. Similarly, if $G_E \subset D$ we have $D = (F_E \cap D) \oplus G_E$.

Hence, additionally we may assume that $F_E \setminus D$ and $G_E \setminus D$ are nonempty. Let $\{z_j : j \in J\}$ be an algebraic base of $F_E$ and let $\{x_i : i \in I\}$ be an orthogonal base of $G_E$ (where I is a finite set if $G_E$ is finite-dimensional and $I := \mathbb{N}$ if $G_E$ is infinite-dimensional). Choose $f \in E'$ for which $D = \ker f$. Let $b_i = f(x_i)$ $(i \in I)$ and $a_j = f(z_j)$ $(j \in J)$. By assumption, $I_0 := \{i \in I : b_i \neq 0\}$ and $J_0 := \{j \in J : a_j \neq 0\}$ are nonempty sets. Define $F = \overline{[\{a_j z_i - a_i z_j : i, j \in J, i \neq j\}]}$. Clearly, $F \subset D \cap F_E$.

Consider the following two cases:

1. There exist $i_0 \in I_0$ and $j_0 \in J_0$ such that

$$\operatorname{dist}(b_{i_0} z_{j_0}, F) \geqslant \|a_{j_0} x_{i_0}\|. \tag{2.64}$$

Define $u := b_{i_0} z_{j_0} - a_{j_0} x_{i_0}$ $(\in D)$, $F_D := F + [u]$, $X = \{b_{i_0} x_i - b_i x_{i_0} : i \in I \setminus \{i_0\}$ $(\subset D)$ and $G_D := \overline{[X]}$. Clearly $F_D \subset F_E + [u]$. Also, by Theorem 1.1.4, $G_D$ has an orthogonal base as a closed subspace of $G_E$. Let us show that $D = F_D \oplus G_D$.

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First we prove that $F_D \perp G_D$. Take $z \in F_D$, $x \in G_D$. It suffices to see that $\|z + x\| \geqslant \|z\|$. We can write $z = z_0 + c(b_{i_0} z_{j_0} - a_{j_0} x_{i_0})$, where $z_0 \in F$, $c \in K$. If $c = 0$ then $z \perp x$ since $F \subset F_E$ and $F_E \perp G_E$; otherwise, we get

$$\begin{aligned}
\|z + x\| &= \|z_0 + c(b_{i_0} z_{j_0} - a_{j_0} x_{i_0}) + x\| \\
&= \|z_0 + c b_{i_0} z_{j_0} - c a_{j_0} x_{i_0} + x\| \\
&= \max\left\{\|z_0 + c b_{i_0} z_{j_0}\|, \|- c a_{j_0} x_{i_0} + x\|\right\} \geqslant \|z\|,
\end{aligned}$$

since

$$\begin{aligned}
\|z\| &= \max\left\{\|z_0 + c b_{i_0} z_{j_0}\|, \|c a_{j_0} x_{i_0}\|\right\} \\
&= |c| \max\left\{\left\|\frac{z_0}{c} + b_{i_0} z_{j_0}\right\|, \|a_{j_0} x_{i_0}\|\right\} = \|z_0 + c b_{i_0} z_{j_0}\|,
\end{aligned}$$

where the last equality follows from (2.64).

Next we prove that $D = F_D + G_D$ (then, $D = F_D \oplus G_D$ and we are done). The inclusion $F_D + G_D \subset D$ is obvious. For the other inclusion, let $d \in D$. It can be written as

$$d = \sum_{j \in J_d} \alpha_j z_j + \sum_{i \in I} \beta_i x_i, \tag{2.65}$$

where $\alpha_j, \beta_i \in K$, $J_d \subset J$, $J_d$ finite, $j_0 \in J_d$, with $\alpha_{j_0}$ eventually null. Since $f(d) = 0$, we have

$$\alpha_{j_0} = -\frac{1}{a_{j_0}}\left(\sum_{j \in J_d, j \neq j_0} \alpha_j a_j + \sum_{i \in I} \beta_i b_i\right),$$

from which we get

$$a_{j_0} d = \sum_{j \in J_d, j \neq j_0} \alpha_j (a_{j_0} z_j - a_j z_{j_0}) + \sum_{i \in I} \beta_i (a_{j_0} x_i - b_i z_{j_0}). \tag{2.66}$$

Also, one can easily see that, for each $i \in I$,

$$a_{j_0} x_i - b_i z_{j_0} = \frac{a_{j_0}}{b_{i_0}}(b_{i_0} x_i - b_i x_{i_0}) + \frac{b_i}{b_{i_0}}(a_{j_0} x_{i_0} - b_{i_0} z_{j_0}). \tag{2.67}$$

Putting together (2.66) and (2.67) we conclude that $a_{j_0}d \in F_D + G_D$, i.e. $d \in F_D + G_D$, as $a_{j_0} \neq 0$.

2. For each $i \in I_0$ and $j \in J_0$, $\mathrm{dist}(b_i z_j, F) < \|a_j x_i\|$ . Let $j_0 \in J_0$. For every $i \in I_0$ choose $w_i \in F$ such that

$$\|b_i z_{j_0} + w_i\| < \|a_{j_0} x_i\|, \tag{2.68}$$

from which

$$\|b_i z_{j_0} + w_i - a_{j_0} x_i\| = \|a_{j_0} x_i\|. \tag{2.69}$$

Let $X := \left\{b_i z_{j_0} + w_i - a_{j_0} x_i : i \in I_0\right\} \cup \{x_i : i \in I \setminus I_0\}$ ($\subset D$). Then, using that $F_E \perp G_E$, orthogonality of $\{x_i : i \in I\}$ and (2.69), one can easily verify that $X$ is an orthogonal set. Set $F_D := F$ and $G_D := \overline{[X]}$. Clearly, $X$ is an orthogonal base of $G_D$. Let us show that $D = F_D \oplus G_D$.

First we prove that $F_D \perp G_D$. Take $z \in F_D$, $x \in G_D$. It suffices to see that $\|z + x\| \geqslant \|z\|$ (see the Preliminaries). We can write

$$x = \sum_{i \in I_0} c_i(b_i z_{j_0} + w_i - a_{j_0} x_i) + \sum_{i \in I \setminus I_0} c_i x_i \quad (c_i \in K).$$

Then, as $F_E \perp G_E$,

$$\|z+x\| = \max\left\{\left\|z + \sum_{i \in I_0} c_i(b_i z_{j_0} + w_i)\right\|, \left\|-\sum_{i \in I_0} c_i a_{j_0} x_i + \sum_{i \in I \setminus I_0} c_i x_i\right\|\right\}.$$

Now, by (2.68) and orthogonality of $X$, it follows from the above that

$$\|z + x\| = \max\left\{\|z\|, \max_{i \in I_0} \|c_i a_{j_0} x_i\|, \max_{i \in I \setminus I_0} \|c_i x_i\|\right\} \geqslant \|z\|.$$

Next we prove that $D = F_D + G_D$ (then, $D = F_D \oplus G_D$ and we are done). The inclusion $F_D + G_D \subset D$ is obvious. For the other inclusion, let $d \in D$ be as in (2.65). It follows from (2.66) that

$$a_{j_0} d = \sum_{j \in J_d, j \neq j_0} \alpha_j (a_{j_0} z_j - a_j z_{j_0}) + \sum_{i \in I_0} \beta_i (a_{j_0} x_i - b_i z_{j_0}) + \sum_{i \in I \setminus I_0} \beta_i a_{j_0} x_i.$$

Hence,

$$\begin{aligned} a_{j_0} d = &\sum_{j \in J_d, j \neq j_0} \alpha_j (a_{j_0} z_j - a_j z_{j_0}) + \sum_{i \in I_0} \beta_i w_i \\ &+ \sum_{i \in I_0} \beta_i (a_{j_0} x_i - b_i z_{j_0} - w_i) + \sum_{i \in I \setminus I_0} \beta_i a_{j_0} x_i \end{aligned}$$

(observe that, by (2.69), $\{\beta_i(a_{j_0}x_i - b_i z_{j_0} - w_i) : i \in I_0\}$ is summable and hence so is $\{\beta_i w_i : i \in I_0\}$). This implies that $a_{j_0}d \in F_D + G_D$ i.e. $d \in F_D + G_D$, as $a_{j_0} \neq 0$. $\square$

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The following conclusion, concerning the heredity of OFDDP by closed, finite-codimensional linear subspaces, is a direct consequence of Theorem 2.3.8

**2.3.9. Corollary.** *Assume* $E = F_E \oplus G_E$, *where* $F_E$ *is finite-dimensional and* $G_E$ *is a closed subspace of* $E$ *with an orthogonal base. Then* $E$ *has the OFDDP and for every finite-codimensional subspace* $D$ *of* $E$,

(1) *there exist closed subspaces* $F_D, G_D \subset E$, *where* $F_D$ *is finite-dimensional and* $G_D$ *has an orthogonal base, such that* $D = F_D \oplus G_D$;

(2) $D$ *has the OFDDP.*

*Proof.* (1) follows directly from Theorem 2.3.8. Also, since Banach spaces of countable type with an orthogonal base and finite-dimensional Banach spaces clearly have the OFDDP, the orthogonal direct sums of these two kinds of spaces, e.g. $E$ and $D$, again have the OFDDP, which finishes the proof. $\square$

# Measures of weak noncompactness. Non-Archimedean quantitative compactness theorems. 3



Measures of noncompactness are commonly used in functional analysis. Usually there are defined as mapping $B(E) \to [0, \infty)$, where $B(E)$ denotes the family of all nonempty and bounded subsets of $E$, and they are equal to zero on every relatively compact subset of $E$. The value of a measure of noncompactness taken on a given $M \in B(E)$ inform us, loosely speaking, how far is it from being relatively compact.

There are several applications of noncompactness measures. One of them are quantitative compactness theorems. Using suitable inequalities involving distances we can substantially strengthen the original, classical results about compactness.

In this chapter we present some basic properties of a few selected noncompactness measures defined on a non-Archimedean Banach space $E$ equipped with the weak topology $\sigma(E, E^*)$. As an application, we provide quantitative versions of Grothendieck, Gantmacher and Krein's theorems

Recall that a subset $M$ of a locally convex space $X$ is called *precompact* if, for every zero neighbourhood $U$ there is a finite set $F \subset X$ such that $M \subset U + F$. Among non-Archimedean valued fields, not all are locally compact. Since any nonempty convex set in a Hausdorff locally convex space X which contains at least two points contains the homeomorphic image of $B_{\mathbb{K}}$; hence, if $\mathbb{K}$ is not locally compact, the only possible convex precompact sets are singletons. It is the reason to restrict our considerations only to locally compact fields.

Throughout this chapter, we will additionally assume that $\mathbb{K}$ is locally compact.



## 3.1 Basic properties of noncompactness measures. Non-Archimedean quantitative Krein's theorem

For a set $A \subset E$, we define *the absolutely convex hull* of A as

$$\mathrm{aco}A := \left\{ \sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A \right\}.$$

We say that A is *absolutely convex* if $A = \mathrm{aco}A$.

A subset of a topological space is called *relatively compact* if its closure is compact. Let $M \subset E$ be a bounded set. Then, M is relatively weakly compact if and only if $\overline{M}^{\sigma(E^{**},E^*)} \subset E$. Using this observation, we can introduce some more general definition. Let $\varepsilon > 0$. We say that M is $\varepsilon$*—-weakly relatively compact* if $\overline{M}^{\sigma(E^{**},E^*)} \subset E + B_{E^{**},\varepsilon}$. In this context, it is natural to define the following noncompactness measure

$$k(M) := \sup_{x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}} \mathrm{dist}(x^{**}, E). \tag{3.1}$$

Clearly, $k(M) = 0$ if and only if $\overline{M}^{\sigma(E^{**},E^*)} \subset E$ that is equivalent to the fact that M is relatively weakly compact.

We say that M *$\varepsilon$-interchanges limits* with $B_{E^*}$ if for any two sequences $(x_n) \subset M$ and $(z_n^*) \subset B_{E^*}$, assuming that the both limits $\lim_m \lim_n z_m^*(x_n)$ and $\lim_n \lim_m z_m^*(x_n)$ exist, we have

$$\left| \lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n) \right| \leqslant \varepsilon.$$

Applying this concept we can define other noncompactness measure, setting

$$\gamma(M) := \sup\Big\{\Big|\lim_m \lim_n z^*_m(x_n) - \lim_n \lim_m z^*_m(x_n)\Big| : (z^*_m) \subset B_{E^*}, (x_n) \subset M\Big\}. \quad (3.2)$$

Note, as we show in Corollary 3.1.5, $M$ is weakly relatively compact if and only if $\gamma(M) = 0$.

We will also consider the *Hausdorff measure* of noncompactness given by

$$h(M) := \inf\{\varepsilon > 0 : M \subset F_\varepsilon + B_{E,\varepsilon};\ F_\varepsilon \text{ is finite}\}, \quad (3.3)$$

and *de Blasi measure* defined as

$$\omega(M) := \inf\{\varepsilon > 0 : M \subset K_\varepsilon + B_{E,\varepsilon};\ K_\varepsilon \text{ is } \sigma(E, E^*)\text{-compact}\}. \quad (3.4)$$

Since by Theorem 1.1.14 in every non-Archimedean normed space $E$ over locally compact $\mathbb{K}$ any compact set of $E$ is weakly compact, the measure of weak noncompactness $\omega$ introduced by De Blasi compares with the Hausdorff measure $h$ on every bounded subset of $E$.

The problem of the equivalence of other, considered measures of weak noncompactness will be studied later in this Chapter (see Corollary 3.1.5 and Proposition 3.1.6).

First, we check interrelationships between $k$ and $\gamma$. To do it we define the function $\phi_{\mathbb{K}}\colon [0,\infty) \to [0,\infty)$ as follows

$$\phi_{\mathbb{K}}(\varepsilon) := \max\{|\lambda| : \lambda \in \mathbb{K}, |\lambda| \leqslant \varepsilon\}.$$

Clearly, $\phi_{\mathbb{K}}(\varepsilon) \in |\mathbb{K}|$ and $\phi_{\mathbb{K}}(\varepsilon) = \varepsilon$ if $\varepsilon \in |\mathbb{K}|$. We say that $t > 0$ is an *upper accumulation point of* $\|E\|$ if there exists $(x_n)_n \subset E$ such that $\|x_1\| < \|x_2\| < \ldots < t$ and $\lim_n \|x_n\| = t$.

Since $0$ is the only accumulation point of $|\mathbb{K}|$, we observe that for each bounded $M \subset E$ we have $\gamma(M) \in |\mathbb{K}|$. If $\|E\| \neq |\mathbb{K}|$, the defined functions $\omega$, $k$ and $\gamma$ may have different sets of values, as Example 3.1.12 shows.

**3.1.1. Theorem** ([4, Theorem 3.3]). *Let* $M \subset E$ *be a bounded set and let* $\varepsilon > 0$. *Then,*

(1) *if* $M$ *is* $\varepsilon$*-weakly relatively compact, then* $M$ $\phi_{\mathbb{K}}(\varepsilon)$*-interchanges limits with* $B_{E^*}$;

(2) *if* $M$ $\varepsilon$*-interchanges limits with* $B_{E^*}$, *then there exists* $\delta_\varepsilon \leqslant \phi_{\mathbb{K}}(\varepsilon)/|\rho|$ (*where* $\rho$ *is an uniformizing element of* $\mathbb{K}$ *with* $|\rho| < 1$) *such that* $M$ *is* $\delta_\varepsilon$*-weakly relatively compact. If* $1$ *is not an upper accumulation point of* $\|E\|$, *then we can select such* $\delta_\varepsilon$ *with* $\delta_\varepsilon < \phi_{\mathbb{K}}(\varepsilon)/|\rho|$.

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For the proof of Theorem 3.1.1, we need the following two lemmas.

**3.1.2. Lemma** ([4, Proposition 3.1]).

(1) $\overline{j_{E^*}(B^-_{E^*})}^{\sigma(E^{***},E^{**})} = B^-_{E^{***}}$.

(2) *If* $1$ *is not an upper accumulation point of* $\|E\|$, *then*

$$\overline{j_{E^*}(B^-_{E^*})}^{\sigma(E^{***},E^{**})} = B_{E^{***}}.$$

*Proof.* (a) follows from [47, Corollary 7.4.8].

(b) Since $j_{E^*}$ is an isometry and $B_{E^{***}}$ is $\sigma(E^{***}, E^{**})$-closed, we have

$$\overline{j_{E^*}(B_{E^*})}^{\sigma(E^{***},E^{**})} \subset B_{E^{***}}.$$

Assume for a contradiction that there exists

$$f \in B_{E^{***}} \setminus \overline{j_{E^*}(B_{E^*})}^{\sigma(E^{***},E^{**})}.$$

It follows from [47, Theorem 7.4.6] that there is $x^{**} \in E^{**}$ such that $|f(x^{**})| \geqslant 1/|\rho|$ and $|x^{**}(z^*)| \leqslant 1$ for all $z^* \in B_{E^*}$; thus, we can easily deduce that $\|x^{**}\| \leqslant 1/|\rho|$. Since

$$1 \geqslant \|f\| \geqslant \frac{|f(x^{**})|}{\|x^{**}\|},$$

we obtain

$$\frac{1}{|\rho|} \geqslant \|x^{**}\| \geqslant |f(x^{**})| \geqslant \frac{1}{|\rho|} \Longrightarrow \|x^{**}\| = |f(x^{**})| = \frac{1}{|\rho|}. \qquad (3.5)$$

On the other hand,

$$\|x^{**}\| = \sup_{z^* \in E^* \setminus \{0\}} \frac{|x^{**}(z^*)|}{\|z^*\|}.$$

Hence, we can select $(z_n^*)_n \subset B_{E^*}$, assuming that $\|z_n^*\| > |\rho|$ for all $n \in \mathbb{N}$, such that $\|x^{**}\| = \lim_n |x^{**}(z_n^*)|/\|z_n^*\|$. Suppose that there is $n_0 \in \mathbb{N}$ for which
$$\|x^{**}\| = \frac{|x^{**}(z_{n_0}^*)|}{\|z_{n_0}^*\|}.$$
Then, by (3.5), we get
$$\|z_{n_0}^*\| = \frac{|x^{**}(z_{n_0}^*)|}{\|x^{**}\|} = |x^{**}(z_{n_0}^*)| \cdot |\rho| \leqslant |\rho|,$$
which contradicts with the assumption that $\|z_n^*\| > |\rho|$ for all $n \in \mathbb{N}$. So, we can assume, selecting a subsequence, if necessary, that
$$1 < \frac{|x^{**}(z_n^*)|}{\|z_n^*\|} < \frac{|x^{**}(z_{n+1}^*)|}{\|z_{n+1}^*\|} < \|x^{**}\| = \frac{1}{|\rho|}$$
for every $n \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$ we can choose $n_k \in \mathbb{N}$ such that
$$\frac{|x^{**}(z_{n_k}^*)|}{\|z_{n_k}^*\|} > \frac{1}{|\rho|} - \frac{1}{k}.$$
Thus,
$$\frac{k}{k-|\rho|}|\rho| > \frac{\|z_{n_k}^*\|}{|x^{**}(z_{n_k}^*)|} \geqslant \|z_{n_k}^*\| > |\rho|.$$
But then, for every $k \in \mathbb{N}$ we can choose $x_k \in E$ satisfying
$$\frac{k}{k-|\rho|}|\rho| > \frac{|z_{n_k}^*(x_k)|}{\|x_k\|} > |\rho|.$$
Without loss of generality, we can assume that $|z_{n_k}^*(x_k)| = |\rho|$ for all $k \in \mathbb{N}$. Then, we obtain
$$\frac{k-|\rho|}{k} < \|x_k\| < 1.$$
Hence, $\lim_k \|x_k\| = 1$ and we deduce that 1 is an upper accumulation point of $\|E\|$, a contradiction. □

**3.1.3. Remark.** Note that the part (1) of Lemma 3.1.2 follows from p-adic Goldstine theorem which says that if $E$ is normpolar, then $j_E(B_E)$ is a $\sigma(E^{**}, E^*)$-dense subset of $B_{E^{**}}$ (see [47, Corollary 7.4.8]).

**3.1.4. Lemma** ([4, Lemma 3.2]). *Let* $x^{**} \in E^{**}$ *and assume that* $d = \operatorname{dist}(x^{**}, E) > 0$. *For every* $x_1, \dots, x_n \in E$, *a non-zero* $\lambda \in \mathbb{K}$ *and* $\varepsilon > 0$ *with* $0 < \varepsilon < |\lambda| < d$ *there exist* $z^* \in B_{E^*}$ *and* $\lambda_0 \in \mathbb{K}$, $|\lambda_0| < \varepsilon$, *such that* $x^{**}(z^*) = \lambda + \lambda_0$ *and* $|x_i(z^*)| < \varepsilon$ *for each* $i \in \{1, \dots, n\}$. *If* 1 *is not an upper accumulation point of* $\|E\|$ *then we can even assume that* $|\lambda| \leqslant d$.

*Proof.* First, define a linear functional $f_0 \colon E + [x^{**}] \to \mathbb{K}$ for which $f_0(x^{**}) = \lambda$ and $f_0(y) = 0$ for all $y \in E$. Then, $\|f_0\| = |\lambda|/d < 1$ if we assume that $|\lambda| < d$ ($\|f_0\| \leqslant 1$ if we assume that $|\lambda| \leqslant d$ ). Applying Ingleton's theorem (Theorem 1.1.12) we get $f \in E^{***}$ such that $\|f\| = \|f_0\|$ and $f|_{E+[x^{**}]} = f_0$. Let

$$V = \{g \in E^{***} : |(g - f)(x^{**})| < \varepsilon,\ |(g - f)(x_i)| < \varepsilon,\ i = 1, \dots, n\}.$$

Then, since $j_{E^*}(B^-_{E^*})$ is $\sigma(E^{***}, E^{**})$-dense in $B^-_{E^{***}}$ (and if 1 is not an upper accumulation point of $\|E\|$, then $j_{E^*}(B_{E^*})$ is dense in $B_{E^{***}}$ with respect to the topology $\sigma(E^{***}, E^{**})$), applying Proposition 3.1.2, we can find $z^* \in V \cap B_{E^*}$. Let $\lambda_0 = (z^* - f)(x^{**})$. Then

$$x^{**}(z^*) = f(x^{**}) - f(x^{**}) + z^*(x^{**}) = f(x^{**}) + (z^* - f)(x^{**}) = \lambda + \lambda_0.$$

Since $f|_E = 0$, we obtain

$$|x_i(z^*)| = |z^*(x_i) - f(x_i)| = |(z^* - f)(x_i)| < \varepsilon$$

for each $i = \{1, \dots, n\}$. $\square$

*Proof of Theorem 3.1.1.* (1) Assume that M is $\varepsilon$-weakly relatively compact. Let $(x_n)_n \subset M$ and $(z^*_n)_n \subset B_{E^*}$ be sequences such that

$$\lim_n \lim_m x_n(z^*_m), \qquad \lim_m \lim_n x_n(z^*_m)$$

exist. We prove that

$$\left| \lim_n \lim_m x_n(z^*_m) - \lim_m \lim_n x_n(z^*_m) \right| \leqslant \phi_{\mathbb{K}}(\varepsilon).$$

Let $x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}$ be a $\sigma(E^{**}, E^*)$-cluster point of the sequence $(x_n)_n$. Clearly, $\operatorname{dist}(x^{**}, E) \leqslant \varepsilon$. Fix $\delta > 0$ and choose $x \in E$ for which

$$\|x - x^{**}\| \leqslant \operatorname{dist}(x^{**}, E) + \delta.$$

Next, take $z^* \in E^*$, a $\sigma(E^*, E)$-cluster point of $(z_m^*)_m$. Since $x$ and $x_1, x_2, \dots$ are in $E$, $x(z^*)$ and $x_n(z^*)$ $(n = 1, 2, \dots)$ are cluster points of $(x(z_m^*))_m$ and $(x_n(z_m^*))_m$, respectively. Thus, we can select a subsequence of $(z_m^*)_m$, denoted again by $(z_m^*)_m$, such that $\lim_m x(z_m^*)$ exists. Hence, we obtain

$$\lim_m x(z_m^*) = x(z^*), \tag{3.6}$$

and $\lim\limits_m x_n(z_m^*) = x_n(z^*)$ for every $n \in \mathbb{N}$. Clearly,

$$\lim_n x_n(z_m^*) = x^{**}(z_m^*) \tag{3.7}$$



for every $m \in \mathbb{N}$ and

$$\lim_n \lim_m x_n(z_m^*) = \lim_n x_n(z^*) = x^{**}(z^*). \tag{3.8}$$

Thus, by (3.7), (3.6) and (3.8) we have

$$\begin{aligned}
\left|\lim_n \lim_m x_n(z_m^*) - \lim_m \lim_n x_n(z_m^*)\right| &= \left|x^{**}(z^*) - \lim_m x^{**}(z_m^*)\right| \\
&= \left|x^{**}(z^*) - \lim_m x(z_m^*) + \lim_m x(z_m^*) - \lim_m x^{**}(z_m^*)\right| \\
&= \left|x^{**}(z^*) - x(z^*) + \lim_m (x - x^{**})(z_m^*)\right| \\
&\leqslant \max\left\{|(x^{**} - x)(z^*)|, \left|\lim_m (x - x^{**})(z_m^*)\right|\right\} \leqslant \|x^{**} - x\|.
\end{aligned}$$

Since

$$\left|\lim_m (x - x^{**})(z_m^*)\right|, |(x - x^{**})(z^*)| \in |\mathbb{K}|,$$

$\|x^{**} - x\| \leqslant \varepsilon + \delta$ and $\delta > 0$ is arbitrary, we conclude that

$$\max\left\{|\lim_m (x - x^{**})(z_m^*)|, |(x - x^{**})(z^*)|\right\} \leqslant \phi_{\mathbb{K}}(\varepsilon).$$

So, the proof of (1) is finished.

(2) Suppose that $M$ $\varepsilon$-interchanges limits with $B_{E^*}$; i.e. for any two sequences $(x_n)_n \subset M$ and $(z_n^*)_n \subset B_{E^*}$ we have

$$\left|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)\right| \leqslant \varepsilon, \tag{3.9}$$

assuming that the involved limits exist. Clearly,

$$\left|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)\right| \in |\mathbb{K}|,$$

as $\mathbb{K}$ is discretely valued. Hence, we get

$$\left|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)\right| \leqslant \phi_{\mathbb{K}}(\varepsilon). \tag{3.10}$$



Take $x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}$ and suppose that $d_0 = \mathrm{dist}(x^{**}, E) > 0$. Set $x_1 \in M$ and $\lambda_0 \in \mathbb{K}$ such that $|\lambda_0| = |\rho| \cdot d_0$ if $d_0 \in |\mathbb{K}|$ and 1 is an upper accumulation point of $\|E\|$, and $|\lambda_0| = \phi_{\mathbb{K}}(d_0)$, otherwise. Applying Lemma 3.1.4, we select $\lambda_1 \in \mathbb{K}$, $|\lambda_1| < |\lambda_0|/2$, and $z_1^* \in B_{E^*}$ for which $x^{**}(z_1^*) = \lambda_0 + \lambda_1$ and $|x_1(z_1^*)| < |\lambda_0/2|$. Let

$$V = \left\{u \in E^{**} : |(x^{**} - u)(z_1^*)| < \frac{|\lambda_0|}{3}\right\}.$$

Taking $x_2 \in M \cap V$, and applying Lemma 3.1.4 again, we choose $\lambda_2 \in \mathbb{K}$ with $|\lambda_2| < |\lambda_0|/3$ and $z_2^* \in B_{E^*}$ for which $x^{**}(z_2^*) = \lambda_0 + \lambda_1 + \lambda_2$ and $|x_i(z_2^*)| < |\lambda_0|/3$ for $i = 1, 2$. Continuing on this direction in the $n$-th step we choose $x_n \in M$ for which

$$|(x^{**} - x_n)(z_i^*)| < \frac{|\lambda_0|}{n+1}, \quad i = 1, \ldots, n-1. \tag{3.11}$$

Next, using Lemma 3.1.4, we select $\lambda_n \in \mathbb{K}$ with $|\lambda_n| < |\lambda_0|/(n+1)$ and $z_n^* \in B_{E^*}$ for which $x^{**}(z_n^*) = \lambda_0 + \lambda_1 + \ldots + \lambda_n$ and

$$|x_i(z_n^*)| < \frac{|\lambda_0|}{n+1} \tag{3.12}$$

for $i = 1, \ldots, n$. This procedure enables us to form sequences $(x_n)_n \subset M$, $(\lambda_n)_n \subset \mathbb{K}$ and $(z_n^*)_n \subset B_{E^*}$ such that for every $n \in \mathbb{N}$ we have

$$x^{**}(z_n^*) = \lambda_0 + \ldots + \lambda_n, \quad |x^{**}(z_n^*)| = |\lambda_0|,$$
$$|x_i(z_n^*)| < \frac{|\lambda_0|}{n+1} \quad \text{for } i \in \{1, \ldots, n\}.$$

Clearly, by (3.11), for every $m \in \mathbb{N}$ we have $x_n(z_m^*) \to x^{**}(z_m^*)$ if $n \to \infty$; hence,

$$\lim_m \lim_n x_n(z_m^*) = \lim_m x^{**}(z_m^*) = \sum_{i=0}^{\infty} \lambda_i.$$

On the other hand, it follows from (3.12) that for every $n \in \mathbb{N}$ one has $|x_n(z_m^*)| \to 0$ if $m \to \infty$; thus, $\lim_n \lim_m x_n(z_m^*) = 0$. Hence, we conclude that

$$\left|\lim_n \lim_m x_n(z_m^*) - \lim_m \lim_n x_n(z_m^*)\right| = \left|\lim_m \lim_n x_n(z_m^*)\right| = |\lambda_0|.$$

Thus, $|\lambda_0| \leqslant \phi_{\mathbb{K}}(\varepsilon)$ by (3.10).

Assume that $d_0 \in |\mathbb{K}|$ ($d_0 = \mathtt{dist}(x^{**}, E)$) and 1 is an upper accumulation point of $\|E\|$; recall that in this case $|\lambda_0| = |\rho| \cdot d_0$, so $d_0 \leqslant \phi_{\mathbb{K}}(\varepsilon)/|\rho|$. Suppose now that $d_0 \notin |\mathbb{K}|$. Then, $\phi_{\mathbb{K}}(d_0) = |\lambda_0|$ and $|\lambda_0| < d_0$. Hence, $d_0 \in (|\lambda_0|, |\lambda_0|/|\rho|)$ and $d_0 < \phi_{\mathbb{K}}(\varepsilon)/|\rho|$. Setting $\delta_\varepsilon := \sup_{x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}} d(x^{**}, E)$, we obtain $\delta_\varepsilon \leqslant \phi_{\mathbb{K}}(\varepsilon)/|\rho|$.

Assume now that 1 is not an upper accumulation point of $\|E\|$. Then $\phi_{\mathbb{K}}(\varepsilon)/|\rho|$ is not an accumulation point of $\|E\|$, either. Thus, we can choose $r > 0$ such that $\mathtt{dist}(x^{**}, E) < \phi_{\mathbb{K}}(\varepsilon)/|\rho| - r$ for every $x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}$. Defining $\delta_\varepsilon := \sup_{x^{**} \in \overline{M}^{\sigma(E^{**},E^*)}} d(x^{**}, E)$ similarly as above, we get promised $\delta_\varepsilon < \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$. $\square$

**3.1.5. Corollary.** *Let $M$ be a bounded subset of $E$. Then $M$ is weakly relatively compact if and only if $\gamma(M) = 0$.*

Next proposition deals with the measure $\omega$.

**3.1.6. Proposition** ([4, Proposition 3.5])**.** *Let $M \subset E$ be a bounded set. Then*

(1) *for every $\varepsilon > \omega(M)$ there exist $y_1, \ldots, y_k \in E$ such that*

$$\begin{aligned} M \subset \{y_1, \ldots, y_k\} + B_{E,\varepsilon} &\subset \mathrm{aco}\{y_1, \ldots, y_k\} + B_{E,\varepsilon} \\ &\subset [y_1, \ldots, y_k] + B_{E,\varepsilon}; \end{aligned}$$

(2) $\omega(M) = \inf\{\varepsilon > 0 : M \subset [F_\varepsilon] + B_{E,\varepsilon}$ *where* $F_\varepsilon \subset E$ *is finite* $\}$;

(3) $\omega(M) = \omega(\mathrm{aco} M)$;

(4) $\omega(M) = \sup\left\{\overline{\lim_m} \mathtt{dist}(x_m, [x_1, \ldots, x_{m-1}]) : (x_m)_m \subset M\right\}$.

(5) $k(M) \leqslant \omega(M)$.

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*Proof.* (1) Let $\varepsilon > 0$. If $\varepsilon > \omega(M)$, then, by definition, there exists a weakly compact set $K_\varepsilon$ (in fact compact by Theorem 1.1.14), for which $M \subset K_\varepsilon + B_{E,\varepsilon}$. By compactness of $K_\varepsilon$ we can select $y_1, \dots, y_k \in E$ such that $K_\varepsilon \subset \bigcup_{i=1}^{k} U_i$, where

$$U_i = \{x \in E : \|x - y_i\| \leqslant \varepsilon\} = \{y_i\} + B_{E,\varepsilon}, \quad i = 1, \dots, k.$$

Since $B_{E,\varepsilon} + B_{E,\varepsilon} = B_{E,\varepsilon}$ by Lemma 1.1.2, we get

$$M \subset \bigcup_{i=1}^{k} (\{y_i\} + B_{E,\varepsilon}) + B_{E,\varepsilon} \subset \{y_1, \dots, y_k\} + B_{E,\varepsilon}.$$

Other inclusions in (1) are obvious.

(2) Denote

$$\omega_0 := \inf\{\varepsilon > 0 : M \subset [F_\varepsilon] + B_{E,\varepsilon} \text{ where } F_\varepsilon \subset E \text{ is finite}\}.$$

To prove $\omega_0 \geqslant \omega(M)$, take $\varepsilon > 0$ and assume that there exists a finite set $F_\varepsilon \subset E$ such that $M \subset [F_\varepsilon] + B_{E,\varepsilon}$. Since $M$ is bounded, there exists $r > \varepsilon > 0$ for which $M \subset B_{E,r}$. Then, $K'_\varepsilon = [F_\varepsilon] \cap B_{E,r}$ is compact. Set $x \in M$. Then, $x = x_F + x_\varepsilon$, where $x_F \in [F_\varepsilon]$ and $x_\varepsilon \in B_{E,\varepsilon}$. Clearly,

$$x_F \in [F_\varepsilon] \cap (M + B_{E,\varepsilon}) \subset [F_\varepsilon] \cap (B_{E,r} + B_{E,\varepsilon}) = [F_\varepsilon] \cap B_{E,r}$$

by Lemma 1.1.2. Thus, $x \in K'_\varepsilon + B_{E,\varepsilon}$ and we imply $M \subset K'_\varepsilon + B_{E,\varepsilon}$. Hence, $\omega(M) \leqslant \omega_0$. The inequality $\omega_0 \leqslant \omega(M)$ follows directly from (1).

(3) Clearly $\omega(M) \leqslant \omega(\mathrm{aco}M)$. Assume that $M \subset F + B_{E,\varepsilon}$ for some finite-dimensional subspace $F \subset E$ and $\varepsilon > 0$. Take $z \in \mathrm{aco}M$. Then $z = \sum_{i=1}^{n} \lambda_i x_i$ for some $\lambda_i \in B_{\mathbb{K}}$ and $x_i \in M$, $i = 1, \dots, n$. Since $x_i \in M$, for every $i \in \{1, \dots, n\}$, we can choose $x'_i \in F$ and $x_i^\varepsilon \in B_{E,\varepsilon}$ such that $x_i = x'_i + x_i^\varepsilon$. Then we have

$$z = \sum_{i=1}^{n} \lambda_i (x'_i + x_i^\varepsilon) = \sum_{i=1}^{n} \lambda_i x'_i + \sum_{i=1}^{n} \lambda_i x_i^\varepsilon,$$

and conclude that $z \in F + B_{E,\varepsilon}$, since $\sum_{i=1}^{n} \lambda_i x_i' \in F$ and $\sum_{i=1}^{n} \lambda_i x_i^{\varepsilon} \in B_{E,\varepsilon}$. Hence, $\operatorname{aco} M \subset F + B_{E,\varepsilon}$ and $\omega(M) \geqslant \omega(\operatorname{aco} M)$.

(4) Denote

$$\omega_{NA} := \sup\left\{\overline{\lim_{m}}\ \operatorname{dist}(x_m, [x_1, \ldots, x_{m-1}]) : (x_m) \subset M\right\}.$$

Let $\varepsilon_0 = \omega(M)$. Fix $\varepsilon > \varepsilon_0$ and assume that there exists a sequence $(x_n)_n \subset M$ for which

$$\overline{\lim_{n}}\ \operatorname{dist}(x_n, [x_1, \ldots, x_{n-1}]) > \varepsilon.$$

Then we can choose a subsequence $(x_{n_k})_k$ of $(x_n)_n$ for which

$$\lim_{k} \operatorname{dist}(x_{n_k}, [x_1, \ldots, x_{n_k - 1}]) > \varepsilon$$

and even, removing finitely many elements, such that

$$\operatorname{dist}(x_{n_k}, [x_1, \ldots, x_{n_k - 1}]) > \varepsilon \tag{3.13}$$

for all $k \in \mathbb{N}$. Clearly, $\|x_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. By (1), we can select $y_1, \ldots, y_p \in E$ such that $M \subset \{y_1, \ldots, y_p\} + B_{E,\varepsilon}$; we can assume that $\|y_i - y_j\| > \varepsilon$ for all $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Since $x_{n_1} \in M$, we find $j_1 \in \{1, \ldots, p\}$ for which

$$\|x_{n_1} - y_{j_1}\| \leqslant \varepsilon. \tag{3.14}$$

By (3.13), $\operatorname{dist}(x_{n_2}, [x_1, \ldots, x_{n_2 - 1}]) > \varepsilon$, hence, we have $\|x_{n_2} - x_{n_1}\| > \varepsilon$. Applying (3.14), we obtain

$$\|x_{n_2} - y_{j_1}\| = \|x_{n_2} - x_{n_1} + x_{n_1} - y_{j_1}\| = \|x_{n_2} - x_{n_1}\| > \varepsilon.$$

Thus, we can choose $j_2 \in \{1, \ldots, p\} \setminus \{j_1\}$ for which $\|x_{n_2} - y_{j_2}\| \leqslant \varepsilon$. Continuing on this direction, we show that $\|x_{n_i} - y_{j_i}\| \leqslant \varepsilon$ for each $i = 1, \ldots, p$, where $\{j_1, \ldots, j_p\} = \{1, \ldots, p\}$. Hence, $M \subset \left\{x_{n_1}, \ldots, x_{n_p}\right\} + B_{E,\varepsilon}$. Then, $\|x_{n_{p+1}} - x_{n_i}\| \leqslant \varepsilon$ for some $i \in \{1, \ldots, p\}$. But, by (3.13)

$$\operatorname{dist}(x_{n_{p+1}}, [x_1, \ldots, x_{n_p}]) > \varepsilon,$$

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providing a contradiction. Thus $\omega_{NA} \leqslant \varepsilon$, and we conclude $\omega_{NA} \leqslant \omega(M)$.

In order to show $\omega_{NA} \geqslant \omega(M)$ take $\varepsilon < \omega(M)$. Since, by (1), $M \not\subseteq F + B_{E,\varepsilon}$ for every finite-dimensional subspace $F \subset E$, setting $x_1 \in M$, we get $M \not\subseteq [x_1] + B_{E,\varepsilon}$. Hence, there exists $x_2 \in M$ such that $\mathrm{dist}(x_2, [x_1]) > \varepsilon$. Continuing on this direction, inductively, we select a sequence $(x_n)_n \subset M$ for which $\mathrm{dist}(x_n, [x_1, \dots, x_{n-1}]) > \varepsilon$. Thus

$$\overline{\lim_n}(\mathrm{dist}(x_n, [x_1, \dots, x_{n-1}])) \geqslant \varepsilon,$$

and the proof of this part is completed.

(5) Observe that for $\varepsilon > 0$ and a weakly compact set $K_\varepsilon \subset E$ such that $M \subset K_\varepsilon + B_{E,\varepsilon}$ we have

$$\overline{M}^{\sigma(E^{**},E^*)} \subset K_\varepsilon + B_{E^{**},\varepsilon} \subset E + B_{E^{**},\varepsilon}.$$

Hence $k(M) \leqslant \omega(M)$. $\square$

Note that for any set I, $\|c_0(I)\| = |\mathbb{K}|$; thus $\omega(M) \in |\mathbb{K}|$ for any bounded set $M \subset c_0(I)$. For the case E being the space $c_0(I)$ we have the following.

**3.1.7. Lemma** ([4, Lemma 3.6]). *Let $\varepsilon > 0$ and $\varepsilon \in |\mathbb{K}|$. If $(w_n)_n \subset c_0(I)$, $w_n = (w_n^i)_{i \in I}$ $(n \in \mathbb{N})$, is a bounded sequence for which there exists an infinite subset $J \subset I$ such that $\max_n |w_n^i| = \varepsilon$ for all $i \in J$ then*

(1) *there exists $(u_n)_n \in \mathrm{aco}\{w_1, w_2, \dots\}$ and $\{k_1, k_2, \dots\} \subset J$ such that for every $n \in \mathbb{N}$ $|u_n^{k_n}| = \varepsilon$, $u_n^{k_m} = 0$ if $m \in \{1, \dots, n-1\}$ and $|u_n^{k_m}| < \varepsilon$ for all $m > n$,*

(2) $\omega(\{w_1, w_2, \dots\}) \geqslant \varepsilon$.

*Proof.* Take $n_1 \in \mathbb{N}$ and $k_1 \in J$ for which $|w_{n_1}^{k_1}| = \varepsilon$. Note that $J_1 = \{i \in I : |w_{n_1}^i| \geqslant \varepsilon\}$ is finite, since $w_{n_1}$ is an element of $c_0(I)$. Thus, we can find $n_2 > n_1$ and $k_2 \in J \setminus J_1$ such that $|w_{n_2}^{k_2}| = \varepsilon$; then, clearly $|w_{n_1}^{k_2}| < \varepsilon$. Next, we find $n_3 > n_2$ and $k_3 \in J \setminus (J_1 \cup J_2)$, where $J_2 = \{i \in I : |w_{n_2}^i| \geqslant \varepsilon\}$, such that $|w_{n_3}^{k_3}| = \varepsilon$. Continuing on this direction we select sequences $(k_j)_j \subset J$ and $(n_j)_j \subset \mathbb{N}$ such that $|w_{n_j}^{k_j}| = \varepsilon$ and $|w_{n_i}^{k_j}| < \varepsilon$ for each $i \in \{1, \dots, j-1\}$.

Define $u_1 := w_{n_1}$. Suppose that we have specified $u_1, \dots, u_{m-1}$ satisfying the required properties. Then we define

$$\begin{aligned} u_{m,1} &:= w_{n_m} - \frac{w_{n_m}^{k_1}}{u_1^{k_1}} u_1, \\ u_{m,n} &:= u_{m,n-1} - \frac{u_{m,n-1}^{k_{n-1}}}{u_n^{k_n}} u_n \quad \text{for } n = 2, \dots, m-1. \end{aligned}$$

Set $u_m := u_{m,m-1}$. We can easily verify that $(u_m)_m \subset \operatorname{aco}\{w_1, w_2, \dots\}$ and $(u_m)_m$ satisfies the required properties, i.e. for every $i \in \mathbb{N}$ $|u_i^{k_i}| = \varepsilon$, $u_i^{k_j} = 0$ for each $j < i$, and $|u_i^{k_j}| < \varepsilon$ for all $j > i$. Let $P\colon c_0(I) \to c_0(J_0)$, where $J_0 = \{k_1, k_2, \dots\}$, be the natural orthoprojection. Clearly, $\|P(u_n)\| = \varepsilon$ for every $n \in \mathbb{N}$. Denoting $P(u_n) = (v_n^1, v_n^2, \dots)$, $n \in \mathbb{N}$, we see that for every $n \in \mathbb{N}$, $|v_n^n| = \varepsilon$, $|v_n^k| = 0$ if $k < n$ and $|v_n^n| < \varepsilon$ if $k > n$; hence, $\{P(u_n) : n \in \mathbb{N}\}$ is orthogonal. Thus, for fixed $m \in \mathbb{N}$, we have

$$\operatorname{dist}(P(u_m), [P(u_1), \dots, P(u_{m-1})]) = \varepsilon.$$

But $\operatorname{dist}(u_m, [u_1, \dots, u_{m-1}]) \geqslant \operatorname{dist}(P(u_m), [P(u_1), \dots, P(u_{m-1})])$, since $P$ is orthoprojection. Hence, using Proposition 3.1.6 (3) and (4), we finally obtain $\omega(\{w_1, w_2, \dots\}) = \omega(\operatorname{aco}\{w_1, w_2, \dots\}) \geqslant \varepsilon$. □

**3.1.8. Proposition** (see [4, Proposition 3.7] and [24, Proposition 3.1 and Corollary 3.2]). *Let $E = c_0(I)$, $\varepsilon > 0$ and $M \subset E$ be a bounded and infinite subset. Then,*

(1) $\omega(M) = \varepsilon$ *if and only if there exists* $x = (x^i)_{i \in I} \in l^\infty(I)$ *such that the following conditions hold*:
   (a) $|w^i| \leqslant |x^i|$ *for every* $w = (w^j)_{j \in I} \in M$ *and* $i \in I$, *and* $\{i \in I : |x^i| \neq \varepsilon\}$ *is finite;*
   (b) *there exist* $(w_n)_n \subset M$ *and infinite* $J = \{k_1, k_2, \dots\} \subset I$ *such that* $|x^{k_n}| = |w_n^{k_n}|$ *for every* $n \in \mathbb{N}$.

(2) $\gamma(M) = \omega(M)$.

(3) $M$ *is weakly relatively compact if and only if there exists* $x = (x^i)_{i \in I} \in c_0(I)$ *such that* $|w^i| \leqslant |x^i|$ *for every* $w = (w^j)_{j \in I} \in M$ *and* $i \in I$.

(4) *If* $\omega(M) = \varepsilon$, *then* $\operatorname{aco} M$ *contains an orthogonal sequence* $(u_n)_n$ *for which* $\|u_n\| = \varepsilon$.

(5) *Let* $M \subset c_0(I)$ *be a bounded subset. Then* $\omega(M) = \max\{\varepsilon :$ *there exists an orthogonal sequence* $(u_n)_n \subset \operatorname{aco} M$ *with* $\|u_n\| = \varepsilon$ *for all* $n \in \mathbb{N}\}$.

*Proof.* (1) Suppose that $x = (x^i)_{i \in I} \in l^\infty(I)$ is such that (a) and (b) are satisfied. Let $M_0 = \{x^i e_i : i \in I\} \subset c_0(I)$. Then $M_0$ is an orthogonal set. Using Proposition 3.1.6 (4) we deduce that $\omega(M_0) = \varepsilon$. Clearly, $M \subset \overline{\operatorname{aco} M_0}$ since $|w^i| \leqslant |x^i|$ for every $w = (w^j)_{j \in I} \in M$ and $i \in I$. Hence, by Proposition 3.1.6 (3) we note

$$\omega(M) \leqslant \omega(\operatorname{aco} M_0) = \omega(M_0) = \varepsilon.$$

On the other hand, taking a sequence $(w_n)_n \in M$ defined as in (b), Lemma 3.1.7 implies $\omega(M) \geqslant \varepsilon$, so we conclude that $\omega(M) = \varepsilon$.

Now, suppose that $\omega(M) = \varepsilon$. Since $\mathbb{K}$ is discretely valued and $M$ is bounded, for every $i \in I$ we can choose $\lambda_i \in \mathbb{K}$ such that

$$|\lambda_i| = \max\left\{|w^i| : w = (w^j)_{j \in I} \in M\right\}.$$

Take $\lambda_0 \in \mathbb{K}$ for which $|\lambda_0| = \varepsilon$. Next, define $x = (x^i)_{i \in I} \in l^\infty(I)$, setting $x^i = \lambda_i$ if $|\lambda_i| \geqslant \varepsilon$ and $x^i = \lambda_0$, otherwise. Assume that we can select an infinite set $\{n_1, n_2, \ldots\} \subset I$ such that $|x^{n_j}| > \varepsilon$, $j \in \mathbb{N}$. But then, for every $j \in \mathbb{N}$ we can find $w_j \in M$ for which $|w_j^{n_j}| = |x^{n_j}|$. Choosing a subsequence $(j_k)_k$ such that $|w_{j_k}^{n_{j_k}}| = |x^{n_{j_k}}| = \varepsilon_0$ for some $\varepsilon_0 > \varepsilon$ and applying Lemma 3.1.7, we deduce that $\omega(\{w_1, w_2, \ldots\}) \geqslant \varepsilon_0 > \varepsilon$. This yields $\omega(M) > \varepsilon$, a contradiction. Hence, the set $J_0 = \left\{i : |x^i| > \varepsilon,\ i \in I\right\}$ is finite and (a) is established.

To prove (b) it is enough to show that the set $J_1 := \{i : |\lambda_i| = \varepsilon,\ i \in I\}$ is infinite. Having this one can easily form a required sequence $(w_n)_n \subset M$. Indeed, assume that $J_1$ is finite. Then, $|w^i| < \varepsilon$ for every $w = (w^j)_{j \in I} \in M$ and $i \in I \setminus (J_0 \cup J_1)$. But then, we can easily deduce that $M \subset [\{e_i : i \in J_0 \cup J_1\}] + B_{E, \varepsilon|\rho|}$, a contradiction.

(2) $\gamma(M) \leqslant \omega(M)$ by Theorem 3.1.1 and Proposition 3.1.6 (5). Let $\varepsilon = \omega(M)$. Applying (1), we can select a finite $I_0 \subset I$, $(w_n)_n \subset M$ and infinite $J = \{k_1, k_2, \ldots\} \subset I \setminus I_0$ such that $|w_n^j| \leqslant \varepsilon$ for every $n \in \mathbb{N}$ and every $j \in I \setminus I_0$, and $|w_n^{k_n}| = \varepsilon$ for all $n \in \mathbb{N}$. Additionally, since for every $n \in \mathbb{N}$, $|w_n^j| = \varepsilon$ only for finitely many $j \in I \setminus I_0$, passing

to a subsequence, if necessary, we can assume that $|w_m^{k_n}| < \varepsilon$ for all $n \in \mathbb{N}$ and each $m < n$. Let $T\colon c_0(I) \to c_0(I \setminus I_0)$ be the natural orthoprojection. Denote $v_n = T(w_n)$, $n \in \mathbb{N}$. Then, $\|v_n\| = \varepsilon$ for all $n \in \mathbb{N}$. We prove that $(v_n)_n$ is orthogonal. Take any $\{p_1, \dots, p_l\} \subset \mathbb{N}$, $p_1 < \dots < p_l$, and $a_1, \dots, a_l \in \mathbb{K}$ with $|a_i| = 1$ for each $i \in \{1, \dots, l\}$. Then, since, by assumption, $|v_{p_i}^{k_{p_l}}| < \varepsilon$ for each $i < l$, we get

$$\left\| \sum_{i=1}^{l} a_i v_{p_i} \right\| \geqslant \left| \sum_{i=1}^{l} a_i v_{p_i}^{k_{p_l}} \right| = |v_{p_n}^{k_{p_l}}| = \varepsilon = \max_{i=1,\dots,l} \|a_i v_{p_i}\|.$$

Thus, $(v_n)_n$ is orthogonal. Fix $\lambda_0 \in \mathbb{K}$ with $|\lambda_0| = \varepsilon$. Let $v_n^*$ $(n \in \mathbb{N})$ denotes the linear functional defined on $[v_1, v_2, \dots]$by setting $v_n^*(v_m) = 0$ if $n \neq m$ and $v_n^*(v_n) = \lambda_0$; since $(v_n)_n$ is orthogonal, $\|v_n^*\| = 1$ for all $n \in \mathbb{N}$. Using Ingleton's theorem (Theorem 1.1.12), for every $n \in \mathbb{N}$ we find a preserving norm extension of $v_n^*$ on the whole of $c_0(I \setminus I_0)$, denoted again by $v_n^*$, and define $z_n^* = \sum_{i=1}^{n} v_i^* \circ T$, a linear functional on $c_0(I)$. Clearly, $\|z_n^*\| \leqslant 1$ $(n \in \mathbb{N})$. Observe that $z_m^*(w_n) = 0$ if $n > m$ and $z_m^*(w_n) = \lambda_0$ if $n \leqslant m$. Hence, $\lim_m z_m^*(w_n) = \lambda_0$ for any $n \in \mathbb{N}$ and $\lim_n z_m^*(w_n) = 0$ for every $m \in \mathbb{N}$. Thus,

$$\left| \lim_n \lim_m z_m^*(w_n) - \lim_m \lim_n z_m^*(w_n) \right| = |\lambda_0| = \varepsilon$$

and we conclude $\gamma(M) \geqslant \omega(M)$.

(3) Suppose that $M$ is weakly relatively compact. For every $i \in I$ choose $a_i \in \mathbb{K}$ such that $|a_i| = \max\left\{|w^i| : w = (w^j)_{j \in I} \in M\right\}$, and define $M_0 = \{a_i e_i : i \in I\}$. Assume that there exists $\varepsilon > 0$ and an infinite $J \subset I$ such that $|a_i| > \varepsilon$ for all $i \in J$. Then, we can select $(w_n)_n \subset M$ and $\{n_1, n_2, \dots\} \subset J$ for which $|w_j^{n_j}| = |a_{n_j}| = \varepsilon_0$ for some $\varepsilon_0 > \varepsilon$. But applying Lemma 3.1.7, we conclude that $\omega(M) > \varepsilon$, a contradiction. Hence, setting $y^i := a_i$, $i \in I$, we obtain $(y^i)_{i \in I} \in c_0(I)$. Now assume that there exists $x = (x^i)_{i \in I} \in c_0(I)$ such that $|w^i| \leqslant |x^i|$ for every $w = (w^i)_{i \in I} \in M$ and $i \in I$. Define $M_0 = \left\{x^i e_i : i \in I\right\} \subset c_0(I)$. Using Proposition 3.1.6 (4) we deduce that $\omega(M_0) = 0$. Since $M \subset \overline{\operatorname{aco} M_0}$, we imply $\omega(M) \leqslant \omega(\operatorname{aco} M_0) = \omega(M_0) = 0$, thus, $M$ is weakly relatively compact.

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(4) Applying (1) we choose a finite $J_0 \subset I$, a countable $J = \{k_1, k_2, \ldots\} \subset I \setminus J_0$, and a sequence $(w_m)_m \subset M$, where $w_m = (w_m^i)_{i \in I}$ $(m \in \mathbb{N})$, such that for every $m \in \mathbb{N}$ we have $|w_m^{k_m}| = \varepsilon$ and $|w_m^i| \leqslant \varepsilon$ for all $i \in I \setminus J_0$.

First, we form a sequence $(z_n)_n \subset \operatorname{aco}\{w_1, w_2, \ldots\}$ and $\{l_1, l_2, \ldots\} \subset I$ such that $\|z_n\| = |z_n^{l_n}| = \varepsilon$ for all $n \in \mathbb{N}$.

If $\|w_m\| = \varepsilon$ for infinitely many $m$, we can choose a subsequence $(w_{m_n})_n \subset (w_m)$ with $\|w_{m_n}\| = \varepsilon$, and then set $z_n := w_{m_n}$ and $l_n := k_{m_n}$. Suppose now, that $\|w_m\| = \varepsilon$ only for finitely many $m$. Set $J_0 = \{j_1, \ldots, j_s\}$. Without loss of generality, we may assume that there exists $n_0 \in \mathbb{N}$ with $|w_{n_0}^{j_1}| > \varepsilon$.

In the first step, fix $m_1 \in \{m : |w_m^{j_1}| = \max_{n \in \mathbb{N}} |w_n^{j_1}|\}$ and define $L_1 := \{n \in \mathbb{N} : |w_{m_1}^{k_n}| < \varepsilon\}$. Clearly, $L_1$ is infinite. Next, for every $n \in L_1$ define

$$w_{1,n} := w_n - \frac{w_n^{j_1}}{w_{m_1}^{j_1}} w_{m_1}.$$

Then $w_{1,n}^{j_1} = 0$ and $|w_{1,n}^{k_n}| = \varepsilon$ for every $n \in L_1$.

In the p-th step of the construction, when $1 < p \leqslant s$, and if

$$\max_{n \in L_{p-1}} |w_{p-1,n}^{j_p}| > \varepsilon,$$

we fix $m_p \in \left\{m \in L_{p-1} : |w_{p-1,m}^{j_p}| = \max_{n \in L_{p-1}} |w_{p-1,n}^{j_p}|\right\}$. Then we define $L_p := \left\{n \in L_{p-1} : n \neq m_p \text{ and } |w_{p-1,m_p}^{k_n}| < \varepsilon\right\}$ and

$$w_{p,n} := w_{p-1,n} - \frac{w_{p-1,n}^{j_p}}{w_{p-1,m_1}^{j_p}} w_{p-1,m_1} \ (n \in L_p);$$

otherwise, we set $L_p := L_{p-1}$ and $w_{p,n} := w_{p-1,n}$, $n \in L_p$. Then, following the construction of the $s$-th step, for $L_p = \{s_1, s_2, \ldots\}$ and defining $z_n := w_{p,s_n}$ for all $n \in \mathbb{N}$ we obtain the required sequence $(z_n)_n$.

Finally, using $(z_n)_n$ defined previously, we form a sequence $(u_n)_n$. Set $u_1 := z_1$. Suppose, that we already selected orthogonal elements $u_1, \ldots, u_m$. The set $\{i \in I : |u_k^i| = \varepsilon$ for some $k = 1, \ldots, m\}$ is finite.

Hence, we can choose $n_{m+1} \in \mathbb{N}$ for which $|u_p^{k_{n_{m+1}}}| < \varepsilon$ for every $p \in \{1, \dots, m\}$. Then we set $u_{m+1} := z_{n_{m+1}}$. Note that we can easily check that the set $\{u_1, \dots, u_{m+1}\}$ is orthogonal. Continuing on this direction we obtain the required orthogonal sequence $(u_n)_n$ as we wanted.

(5) It follows directly from (4) and Proposition 3.1.6 (4). □

**3.1.9. Corollary** (see [4, Corollary 3.8]). *Let* $M$ *be a bounded set of* $E$. *Then,* $\gamma(M) \geqslant |\rho| \cdot \omega(M)$, *where* $\rho \in \mathbb{K}$ *is an uniformizing element.*

*Proof.* By Lemma 1.3.1, there exist a set I and a linear homeomorphism $T \colon E \to c_0(I)$ such that $|\rho| \cdot \|Tx\| < \|x\| \leqslant \|Tx\|$. Hence, we have $\omega(M) \leqslant \omega(T(M))$. Observe that for $z^* \in B_{c_0(I)^*}$ we derive

$$\|z^* \circ T\| = \sup_{x \in E} \frac{|(z^* \circ T)(x)|}{\|x\|} \leqslant \frac{1}{|\rho|} \sup_{x \in E} \frac{|z^*(T(x))|}{\|T(x)\|} \leqslant \frac{1}{|\rho|}.$$

Hence $(\rho z^* \circ T) \in B_{E^*}$, and then $\gamma(M) \geqslant |\rho| \cdot \gamma(T(M))$. Applying Proposition 3.1.8(2) we finally obtain

$$\frac{1}{|\rho|} \cdot \gamma(M) \geqslant \gamma(T(M)) \geqslant \omega(T(M)) \geqslant \omega(M).$$ □

Now we present the following quantitative versions of Krein's theorem.

**3.1.10. Theorem.** (see [4, Corollary 3.9]) *For a bounded set* $M \subset E$ *we have*

$$\gamma(M) \leqslant \gamma(\mathrm{aco} M) \leqslant \frac{1}{|\rho|} \gamma(M).$$

*If* $|\mathbb{K}| = \|E\|$ *then* $\gamma(M) = \gamma(\mathrm{aco} M)$.

*Proof.* Clearly, $\gamma(M) \leqslant \gamma(\mathrm{aco} M)$. To complete the proof, observe that

$$\begin{aligned} \gamma(M) &\leqslant \gamma(\mathrm{aco} M) \leqslant k(\mathrm{aco} M) \\ &\leqslant \omega(\mathrm{aco} M) = \omega(M) \leqslant \frac{1}{|\rho|} \gamma(M) \end{aligned} \tag{3.15}$$

by Theorem 3.1.1, Proposition 3.1.6(5), (3) and Corollary 3.1.9. By Theorem 1.3.1, if $|\mathbb{K}| = \|E\|$ then $E$ is isometrically isomorphic to $c_0(I)$ for some I. Thus, $\omega(M) = \gamma(M)$ by Proposition 3.1.8(2), and $\gamma(M) = \gamma(\mathrm{aco} M)$ by (3.15). □

**3.1.11. Theorem** (see [4, Theorem 3.10]). *If* $M \subset E$ *is bounded, then*

$$\gamma(M) \leqslant k(M) \leqslant k(\mathrm{aco}M) \leqslant \omega(M) = \omega(\mathrm{aco}M) \leqslant \frac{1}{|\rho|}\gamma(M). \quad (3.16)$$

*If additionally* $|\mathbb{K}| = \|E\|$, *then*

$$\gamma(M) = \gamma(\mathrm{aco}M) = k(M) = k(\mathrm{aco}M) = \omega(M) = \omega(\mathrm{aco}M). \quad (3.17)$$

*Proof.* Clearly, $k(M) \leqslant k(\mathrm{aco}M)$. The rest of (3.16) follows directly from Theorem 3.1.1, Proposition 3.1.6 (3) and (5), and Proposition 3.1.8 (2). Now, assume that $|\mathbb{K}| = \|E\|$. Since, by Theorem 1.3.1, $E$ is isometrically isomorphic to $c_0(I)$ for some $I$, we can apply Proposition 3.1.8 (2) obtaining $\gamma(M) = \omega(M)$. Thus, using (3.16) and Corollary 3.1.10 we reach (3.17). □

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In general, if $|\mathbb{K}| \neq \|E\|$, the equality (3.17) does not hold, as the following example shows.

**3.1.12. Example** (see [4, Example 3.12]). Set a real $r_0$ such that $|\rho| < r_0 < 1$. Let $E = (c_0(I), \|\cdot\|')$, where the norm $\|\cdot\|'$ is defined by the formula

$$\|(x^1, x^2, \dots)\|' = \max\left\{|x^1|, |x^2|\cdot r_0, |x^3|\cdot r_0, \dots\right\}$$

Then, $M = \{e_2, e_3, \dots\}$ is a bounded subset of $E$. We prove that $\gamma(M) = |\rho|$. First, note that $\|x\| = r_0$ for every $x \in M$; thus, for every $z^* \in B_{E^*}$ we get $|z^*(e_i)| \leqslant |\rho|$, $i = 2, 3, \dots$; otherwise, assuming that $|z^*(e_j)| > |\rho|$ for some $j \in \{2, 3, \dots\}$ and $z^* \in B_{E^*}$ we get $|z^*(e_j)| \geqslant 1$, since $|\mathbb{K}| \cap (|\rho|, \infty) = \{1, |\rho|^{-1}, |\rho|^{-2}, \dots\}$. Thus, $\|x^*\| \geqslant |x^*(e_j)|/\|e_j\| \geqslant 1/r_0 > 1$, a contradiction. Hence, $\gamma(M) \leqslant |\rho|$.

Now, let $(e_n^*)_n$ denotes the sequence of functionals such that $e_n^*(e_m) = \rho$ if $n = m$ and $e_n^*(e_m) = 0$ if $n \neq m$. For every $n \in \mathbb{N}$ define $z_n^* = e_1^* + \dots + e_n^*$; clearly, $(z_n^*)_n \subset B_{E^*}$. Then, $\lim_m z_n^*(e_m) = 0$ for every $n \in \mathbb{N}$, and $\lim\limits_n z_n^*(e_m) = \rho$ for every $m \in \{2, 3, \dots\}$. Hence

$$\left|\lim_n \lim_m z_n^*(e_m) - \lim_m \lim_n z_n^*(e_m)\right| = |\rho|,$$

and we conclude that $\gamma(M) = |\rho|$.

On the other hand, $\|x - y\| = r_0$ for any $x, y \in M$, $x \neq y$. This easily yields $\omega(M) = r_0$.

## 3.2 Non-Archimedean quantitative Grothendieck's theorem

Grothendieck proved that an uniformly bounded set $H$ in the Banach space $C(X, \mathbb{R})$, where $X$ is a compact topological space, is relatively compact in the pointwise topology $\tau_p$ if and only if it is relatively compact in the weak topology $w$ of $C(X, \mathbb{R})$, see [17], [14, Theorem 4.2] and [3, Theorem 3.5]. The non-Archimedean version of Grothendieck's theorem about weakly compact sets for $C(X, \mathbb{K})$, the spaces of continuous maps on $X$ with values in a locally compact non-trivially valued non-Archimedean field $\mathbb{K}$, fails in general (see Theorem 3.2.4). However, it works with some additional assumptions (see Theorem 3.2.5 and Corollary 3.2.7).

Let $X$ be a nonempty, zero-dimensional compact Hausdorff topological space. Then, the structure of the space $C(X, \mathbb{K})$ as a Banach space is significantly different than $C(X, \mathbb{R})$, as shown by the following results.

**3.2.1. Theorem** (see [47, Theorems 2.5.22, 2.5.24 and 2.5.27])**.** *Let $X$ be compact zero-dimensional space. Then $C(X, \mathbb{K})$ has an orthonormal base.*

(1) *If $\mathcal{U}$ is a maximal collection of clopen sets for which $\{\xi_U : U \in \mathcal{U}\}$ is orthonormal, then $\{\xi_U : U \in \mathcal{U}\}$ is an orthonormal base of $C(X, \mathbb{K})$.*
(2) *$C(X, \mathbb{K})$ is of countable type if and only if $X$ is ultrametrizable.*
(3) *If $X$ is a compact ultrametric space, then $C(X, \mathbb{K})$ has an orthonormal base consisting of characteristic functions of balls. Each maximal system of balls whose characteristic functions are linearly independent is an orthonormal base of $C(X, \mathbb{K})$.*

To prove main results of this section, we need two, more general lemmas

**3.2.2. Lemma** ([23, Lemma 4]))**.** *Let $Y = (Y, d)$ be a compact ultrametric space and let $(B_{Y,r_n}(y_n))_n$ be a sequence of pairwise different closed balls. Then $\lim_n r_n = 0$.*

*Proof.* Denote $B_n = B_{Y,r_n}(y_n), n \in \mathbb{N}$. Assume for a contradiction that for some $r > 0$ the set $M_r = \{i \in \mathbb{N} : r_i > r\}$ is infinite. Set

$\mathcal{B}_r = \{B_i : i \in M_r\}$ an denote by $\mathcal{M}_r$ the family of all maximal totally ordered subsets of $(\mathcal{B}_r, \subseteq)$. Then, consider two cases:

(a) Any element of $\mathcal{M}_r$ is finite. For every $M \in \mathcal{M}_r$ denote by $B_{i(M)}$ the minimal element of $M$. Then, the balls $B_{i(M)}$, $M \in \mathcal{M}_r$, are pairwise disjoint. Thus, for each $M, M' \in \mathcal{M}_r$ with $M \neq M'$ we get $d(y_{i(M)}, y_{i(M')}) > r$ . By compactness of $Y$ we infer that $\mathcal{M}_r$ is finite; so $M_r$ is finite, a contradiction.

(b) there exists an infinite $M_0 \in \mathcal{M}_r$. Let $N_0 = \{i \in \mathbb{N} : B_i \in M_0\}$. Since, for $i, j \in N_0$ we have $B_i \subsetneq B_j$ if and only if $r_i < r_j$, we can choose a subsequence $(i_k)_k$ of elements of $N_0$ such that $(B_{i_k})_k$ is strictly monotonic. Suppose that $(B_{i_k})_k$ is strictly decreasing. Then, for every $k \in \mathbb{N}$ we can select $x_k \in B_{i_k} \setminus B_{i_{k+1}}$; hence,

$$d(x_k, x_{k+1}) > r_{k+1} \geqslant d(x_{k+1}, x_{k+2}) > r_{k+2} \geqslant \ldots > r,$$

and we conclude that $(x_k)_k$ has no convergent subsequence.

Similarly, assuming that $(B_{i_k})_k$ is strictly increasing, we can choose a sequence $(x_k)_k$ with the same property. This contradicts with compactness of $Y$. So, the both cases yield that $\lim_i r_i = 0$. $\square$

**3.2.3. Lemma** (see [23, Lemma 5]). *Let $Y$ be an ultrametric, compact space. Then, there exists a sequence of closed balls $(U_n)_n$ in $Y$ such that*

$$U_1 = Y, \quad U_n \not\subseteq \bigcup_{j=n+1}^{\infty} U_j \quad (n \in \mathbb{N})$$

*and $(\xi_{U_n})$, where $\xi_{U_n}$ denotes the characteristic function of $U_n$ $(n \in \mathbb{N})$, is a maximal orthonormal sequence in $C(Y, \mathbb{K})$.*

*Proof.* Let $B(Y)$ be the family of all closed balls in $Y$. Denote by $\mathcal{M}$ the family of all $M \subset B(Y)$ with $Y \in M$ such that $\{\xi_B : B \in M\}$ is linearly independent in $C(Y, \mathbb{K})$. By Kuratowski–Zorn Lemma, $(\mathcal{M}, \subseteq)$ has a maximal element $M_0 = \{B_i : i \in I\}$. It is easy to see that $I$ is infinite and countable by Lemma 3.2.2; so, we can assume that $I = \mathbb{N}$. Denote $B_i := B_{Y,r_i}(y_i)$, $i \in \mathbb{N}$. By Lemma 3.2.2, $\lim_i r_i = 0$. Let $\pi$ be a permutation of $\mathbb{N}$ such that $(r_{\pi(i)})_i$ is decreasing. Set $U_i = B_{\pi(i)}$, $i \in \mathbb{N}$. Clearly, for $i, j \in \mathbb{N}$ with $i > j$ we have $U_i \subsetneq U_j$

or $U_i \cap U_j = \emptyset$. Moreover, $U_i \not\subseteq \bigcup_{j=i+1}^{\infty} U_j$ for any $i \in \mathbb{N}$. Indeed, otherwise, there exist $i_0, k \in \mathbb{N}$ and $j(1), \ldots, j(k) \in \{i_0+1, i_0+2, \ldots\}$ such that $\{U_{j(1)}, \ldots, U_{j(k)}\}$ is a partition of $U_{i_0}$. Then $\xi_{U_{i_0}} = \sum_{n=1}^{k} \xi_{U_{j(n)}}$; so $(\xi_{U_i})_i$ is linearly dependent, a contradiction. $\square$

**3.2.4. Theorem** ([22, Theorem 2.1]). *Let* $X$ *be an infinite compact zero-dimensional space. Then there exists a* $\tau_p$*-relatively compact set* $H := \{g_n : n \in \mathbb{N}\}$*, which is not relatively weakly compact in* $C(X, \mathbb{K})$*, such that all* $\|g_n\| = 1$ *and* $\gamma(H) > 0$.

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*Proof.* Since $X$ is compact and infinite, there exists $x \in X$ which is not isolated. Let $U_1 := U$ be a clopen neighbourhood of $x$. Since $U \neq \{x\}$, there are $x_1 \in U \setminus \{x\}$ and a clopen neighbourhood $U_2$ of $x$ such that $U_2 \subset U$ and $x_1 \in U \setminus U_2$. Then $U_2 \neq \{x\}$ and we find a clopen neighbourhood $U_3$ of $x$ with $U_3 \subset U_2$ and an $x_2 \in U_2 \setminus U_3$. Continuing this procedure we construct a sequence $x_1, x_2, \ldots$ in $X$ and a decreasing sequence $(U_n)_n$ of clopen subsets of $X$ such that $x_n \in U_n \setminus U_{n+1}$ for all $n \in \mathbb{N}$.

Since each set $U_n$ is clopen, for each $n \in \mathbb{N}$ the function $f_n : X \to \mathbb{K}$ defined by $f_n(x) := \chi_{U_n}(x)$, $x \in X$, is continuous. If $x \in \bigcap_n U_n$, then $f_n(x) \to 1$. If $x \notin \bigcap_n U_n$, then $f_n(x) \to 0$. For every $n \in \mathbb{N}$ set $g_n(x) := f_n(x) - f_{n+1}(x)$, $x \in X$. Then $g_n \to 0$ for each $x \in X$. Moreover, $1 \geqslant \|g_n\| \geqslant |f_n(x_n) - f_{n+1}(x_n)| = 1$, so $\|g_n\| = 1$ for all $n \in \mathbb{N}$. Set $H := \{g_n : n \in \mathbb{N}\}$. The only cluster point of $H$ in $\mathbb{K}^X$ (equipped with the product topology) is a zero function, obviously continuous; hence, $H$ is $\tau_p$-relatively compact. But, $H$ is not relatively compact in the weak topology of $C(X, \mathbb{K})$. Indeed, otherwise $g_n \to 0$ in the weak topology of $C(X, \mathbb{K})$. Since in $C(X, \mathbb{K})$ every weakly converging sequence converges in the norm (see Corollary 1.1.15), we reach a contradiction as $\|g_n\| = 1$ for each $n \in \mathbb{N}$.

Let $D$ be the linear span of $H$ in $C(X, \mathbb{K})$. Define $g_n^* \in D^*$ by $g_n^*(g_m) := 1$ if $n = m$ and $g_n^*(g_m) := 0$ if $n \neq m$. Using Ingleton's theorem (Theorem 1.1.12) for every $n \in \mathbb{N}$ we extend $g_n^*$ to the whole of $C(X, \mathbb{K})$. For every $n \in \mathbb{N}$ define a continuous linear functional $h_n^* := g_1^* + \ldots + g_n^*$ on $C(X,\mathbb{K})$. Observe that $h_n^*(g_m) = 1$ if $m \leqslant n$ and $h_n^*(g_m) = 0$ if $m > n$, so for every $m \in \mathbb{N}$ we have $\lim_n h_n^*(g_m) = 1$ and $\lim_m h_n^*(g_m) = 0$ for each $n \in \mathrm{N}$. Thus $\lim_n \lim_m h_n^*(g_m) \neq \lim_m \lim_n h_n^*(g_m)$, so $\gamma(H) > 0$. $\square$

Let H be a bounded subset of $C(X,\mathbb{K})$, where X is a zero-dimensional compact space. Define the map

$$\gamma_X(H) := \sup\Big\{\Big|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)\Big| : (f_m) \subset H, (x_n) \subset X\Big\},$$

provided the iterated limits exist. Clearly, $\gamma_X(H) = 0$ if and only if H is relatively $\tau_p$-compact (i.e. compact with respect to the topology of the pointwise convergence $\tau_p$). Considering weak topology and $\tau_p$ defined on $C(X,\mathbb{K})$ we get the following variant of quantitative Grothendieck's theorem.

**3.2.5. Theorem** ([24, Theorem 3.3]). *Let X be an infinite zero-dimensional, metrizable compact space and let H be an uniformly bounded absolutely convex subset of $C(X,\mathbb{K})$. Then $\gamma_X(H) = \gamma(H)$.*

*Proof.* First we prove that $\gamma_X(H) \leqslant \gamma(H)$. Define the map $\delta\colon X \to B_{C(X,\mathbb{K})^*}$ by the formula $\delta(x)(f) = f(x)$. Then, since $\delta(X) \subset B_{C(X,\mathbb{K})^*}$, we conclude $\gamma_X(H) \leqslant \gamma(H)$.

Next we show that $\gamma(H) \leqslant \gamma_X(H)$. Assume that $\gamma(H) = \varepsilon > 0$. We prove $\gamma_X(H) \geqslant \varepsilon$. Applying Propositions 3.1.8 (4), we select an orthogonal sequence $(u_n)_n \subset H$ such that $\|u_n\| = \varepsilon$. Since, by assumption X is metrizable, $C(X,\mathbb{K})$ is a non-Archimedean Banach space of countable type by Theorem 3.2.1. Applying Lemma 3.2.3 and Theorem 3.2.1, we choose a sequence of closed balls $(U_n)_n \subset X$ such that

$$U_1 = X, \qquad U_n \not\subseteq \bigcup_{j=n+1}^{\infty} U_j \tag{3.18}$$

and $(\chi_{U_n})_n$, the sequence of characteristic functions of $U_n, n \in \mathbb{N}$, forms an orthonormal base of $C(X,\mathbb{K})$.

Denote $g_{n,0} := u_n$, $n \in \mathbb{N}$. Consequently, for any $n \in \mathbb{N}$ we have the following form

$$g_{n,0} = \sum_{m=1}^{\infty} \lambda_{n,0}^{m} \chi u_m$$

for some $(\lambda_{n,0}^{m})_m \subset \mathbb{K}$; then, $\|g_{n,0}\| = \max_m |\lambda_{n,0}^{m}|$ for all $n \in \mathbb{N}$.

Now, set $i_1 := \min\{k : \lambda_{n,0}^{k} \neq 0 \text{ for some } n \in \mathbb{N}\}$. Choose $n_1 \in \mathbb{N}$ such that $|\lambda_{n_1,0}^{i_1}| = \max\{|\lambda_{n,0}^{i_1}| : n \in \mathbb{N}\}$ and for every $n > n_1$ define

$$g_{n,1} := g_{n,0} - \frac{\lambda_{n,0}^{i_1}}{\lambda_{n_1,0}^{i_1}} g_{n_1,0}. \tag{3.19}$$

Clearly, $g_{n,1} \in H$ and, since $(g_{n,0})_n$ is orthogonal, $\|g_{n,1}\| = \varepsilon$ for all $n > n_1$.

Take $c_1, \dots, c_{p-1} \in B_{\mathbb{K}}$ and $k_1, \dots, k_p > n_1$. Then, we get

$$\begin{aligned} &\|c_1 g_{k_1,1} + \ldots + c_{p-1} g_{k_{p-1},1} + g_{k_p,1}\| \\ =&\left\| c_1 g_{k_1,0} + \ldots + c_{p-1} g_{k_{p-1},0} + g_{k_p,0} - \frac{c_1 \lambda_{k_1,0}^{i_1} + \ldots + \lambda_{k_p,0}^{i_1}}{\lambda_{n_1,0}^{i_1}} g_{n_1,0} \right\| = \varepsilon; \end{aligned}$$

hence, $(g_{n,1})_{n>n_1}$ is orthogonal. For every $n > n_1$ we can choose $\lambda_{n,1}^{m} \in \mathbb{K}$, $m \in \mathbb{N}$, and write

$$g_{n,1} = \sum_{m=1}^{\infty} \lambda_{n,1}^{m} \chi u_m.$$

Then, from (3.19) we deduce that $\lambda_{n,1}^{m} = 0$ for each $m \leqslant i_1$.

Continuing on this direction in the k-th step, having defined $n_{k-1}$, $i_{k-1}$ and $\{g_{n,k-1} : n = n_{k-1}, n_{k-1}+1, \dots\} \subset H$, where

$$g_{n,k-1} = \sum_{m=1}^{\infty} \lambda_{n,k-1}^{m} \chi u_m, \text{quadwhere } \lambda_{n,k-1}^{m} \in \mathbb{K},$$

we set$i_k := \min\{i : \lambda_{n,k-1}^{i} \neq 0 \text{ for some } n > n_{k-1}\}$. Next, we select $n_k$ such that $|\lambda_{n_k,k-1}^{i_k}| = \max\{|\lambda_{n,k-1}^{i_k}| : n > n_{k-1}\}$ and for every $n > n_k$ define

$$g_{n,k} := g_{n,k-1} - \frac{\lambda_{n,k-1}^{i_k}}{\lambda_{n_k,k-1}^{i_k}} g_{n_k,k-1}.$$

Then, $g_{n,k} \in H$ for all $n > n_k$, either. Applying the same argumentation as above we deduce that $\|g_{n,k}\| = \varepsilon$ for all $n > n_k$ and $(g_{n,k})_{n>n_k}$ is orthogonal. Choosing $\lambda^m_{n,k} \in \mathbb{K}$, $m \in \mathbb{N}$ such that

$$g_{n,k} = \sum_{m=1}^{\infty} \lambda^m_{n,k} \chi_{U_m} \ (n > n_k),$$

we imply that $\lambda^m_{n,k} = 0$ for $m \leqslant i_k$. We see that the sequences $(n_k)_k$ and $(i_k)_k$ are strictly increasing.

Now, consider the sequence $(g_{n_k,k-1})_k$. Set

$$z_k := \max\left\{m : |\lambda^m_{n_k,k-1}| = \varepsilon\right\}, \quad k \in \mathbb{N}.$$

Observe that $z_{k+1} > i_k$ for every $k \in \mathbb{N}$. Next, we select a strictly increasing sequence $(k_p)_p \subset \mathbb{N}$, setting $k_1 = 1$, such that the condition $i_{k_{p+1}-1} > z_{k_p}$ holds for every $p \in \mathbb{N}$. Now, define $f_p := g_{n_{k_p},k_p-1}$, $p \in \mathbb{N}$. Consequently, for every $p \in \mathbb{N}$ we can write

$$f_p = \sum_{m=1}^{\infty} \mu^m_p \chi_{U_m}$$

for some $\mu^m_p \in \mathbb{K}$, $m \in \mathbb{N}$. Then, $(f_p)_p$ is orthogonal, $\|f_p\| = \max_m |\mu^m_p| = \varepsilon$ and

$$\begin{aligned} \min\{m : |\mu^m_p| = \varepsilon\} &\leqslant \max\{m : |\mu^m_p| = \varepsilon\} \\ &< \max\{m : |\mu^l_{p+1}| = 0 \text{ for all } l \in \{1, \dots, m\}\}. \end{aligned} \tag{3.20}$$

Set $t_p := \min\{m : |\mu^m_p| = \varepsilon\}$, $p \in \mathbb{N}$. Applying (3.18), for every $p \in \mathbb{N}$ choose

$$x_p \in U_{t_p} \setminus \bigcup_{j>t_p} U_j. \tag{3.21}$$

Next, select a convergent subsequence $(x_{k_m})_m \subset (x_k)_k$. Let $x_0 := \lim_m x_{k_m}$. Set $f'_m := f_{k_m}$, $x'_m := x_{k_m}$ and $d_m := \lim_n f'_m(x'_n)$ for every $m \in \mathbb{N}$. Clearly $|d_m| \leqslant \varepsilon$ for all $m \in \mathbb{N}$. Set $M := \{m : |d_m| = \varepsilon\}$.

Assume that $M$ is infinite. Then, we can choose a sequence $(m_k)_k$ of elements of $M$ such that $\lim_k \lim_n f'_{m_k}(x'_n)$ exists. Since, by (3.20) and

(3.21), $\lim_k f'_{m_k}(x'_n) = 0$ for every $n \in \mathbb{N}$, we obtain

$$\left|\lim_k \lim_n f'_{m_k}(x'_n) - \lim_n \lim_k f'_{m_k}(x'_n)\right| = \left|\lim_k \lim_n f'_{m_k}(x'_n)\right| = \varepsilon. \quad (3.22)$$

Suppose now that $M$ is finite. Removing the first few elements of $(f'_m)_m$ and $(x'_m)_m$ we can assume that $|d_m| < \varepsilon$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ define$h_m := f'_1 + \ldots + f'_m$; obviously $h_m \in H$. Applying (3.20) and (3.21) again, we get $f'_m(x'_n) = 0$ if $m > n$. Since, by assumption $|d_m| < \varepsilon$ for all $m \in \mathbb{N}$, for every $k \in \mathbb{N}$ we can find $k' \in \mathbb{N}$ such that $k' > k$ and $|f'_k(x'_n)| < \varepsilon$ if $n \geqslant k'$. Hence, passing to subsequences, we can assume that $|f'_m(x'_n)| < \varepsilon$ if $m < n$. It follows from (3.21) that $|f'_n(x'_n)| = \varepsilon$, $n \in \mathbb{N}$. Hence, for each $m \geqslant n$ we obtain

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$$h_m(x'_n) = f'_1(x'_n) + \ldots + f'_n(x'_n)$$

and conclude that $\lim_m h_m(x'_n)$ exists. Moreover, $\left|\lim_m h_m(x'_n)\right| = \varepsilon$. So, we can choose a sequence $(n_k)_k$ such that $\lim_k \lim_m h_m(x'_{n_k})$ exists.

On the other hand, for every $m \in \mathbb{N}$ set $\beta_m := \lim_k h_m(x'_{n_k})$. Then,

$$\beta_m = \lim_n h_m(x'_n) = d_1 + \ldots + d_m$$

and, by assumption, $|\beta_m| < \varepsilon$ for all $m \in \mathbb{N}$. Choose a convergent subsequence $(\beta_{m_l})_l$. Then, $\left|\lim_l \beta_{m_l}\right| < \varepsilon$. Therefore, we obtain

$$\left|\lim_k \lim_l h_{m_l}(x'_{n_k}) - \lim_l \lim_k h_{m_l}(x'_{n_k})\right| = \left|\lim_k \lim_l h_{m_l}(x'_{n_k})\right| = \varepsilon. \quad (3.23)$$

Thus, by (3.22) and (3.23), $\gamma_X(H) \geqslant \varepsilon = \gamma(H)$. □

**3.2.6. Remark.** Note that [24, Theorem 3.3] gives the formulation of Theorem 3.2.5 without the assumption about metrizability of X. However, the proof of [22, Theorem 2.7] which is used to get [24, Theorem 3.3] is not quit correct. Hence, the question if the assumption about metrizability of X can be omitted should be specified as an open problem.

**3.2.7. Corollary.** *Let* $X$ *be an infinite zero-dimensional, metrizable compact space and let* $H$ *be an uniformly bounded subset of* $C(X,\mathbb{K})$. *Then*

$$\gamma_X(\mathrm{aco}H) = \gamma(H).$$

*Proof.* Since $\|C(X,\mathbb{K})\| = |\mathbb{K}|$, the equality $\gamma(H) = \gamma(\mathrm{aco}H)$ follows from Theorem 3.1.11. Applying Theorem 3.2.5 for the set $\mathrm{aco}H$ we complete the proof. □

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## 3.3 Non-Archimedean quantitative versions of Gantmacher and Schauder's theorems

For any Banach space $X$, its unit ball $B_X$ is weakly compact if and only if $X$ reflexive. If $\mathbb{K}$ is locally compact, then $E$ is reflexive if and only if $E$ is finite-dimensional (see Proposition 1.1.9). Hence, $B_E$ is weakly compact if and only if $E$ is finite-dimensional. It is worthwhile to remark that (similarly like in the real case), applying Proposition 3.1.8, we imply $\omega(B_{c_0(I)}) = 1$ for any infinite set $I$. However, there exist infinite-dimensional Banach spaces over locally compact $\mathbb{K}$ for which the value of de Blasi measure defined on its unit ball is less than unity (see Example 3.3.1).

**3.3.1. Example.** Let $(r_n)_n \subset (|\rho|,1]$, $r_1 = 1$, be a strictly decreasing sequence (where $\rho$ is an uniformizing element of $\mathbb{K}$ with $|\rho| < 1$). Define $s\colon \mathbb{N} \to (|\rho|,1]$ by $s\colon n \mapsto r_n$. Then, using Proposition 3.1.6, we imply $\omega(B_{c_0(I:s)}) = \lim_n r_n < 1$.

We obtain the following non-Archimedean counterpart of the Gantmacher's (Schauder's) quantitative theorem (recall that in this case weak compactness coincide with compactness).

**3.3.2. Theorem** ([24, Theorem 3.5]). *Let* $E$, $F$ *be Banach spaces with* $\|E\| = \|F\| = |\mathbb{K}|$, $T\colon E \to F$ *be a continuous operator and* $T^*\colon F^* \to E^*$ *be its adjoint. Then,*

$$\omega(TB_E) = \omega(T^*B_{F^*}) \quad \textit{and} \quad \gamma(TB_E) = \gamma(T^*B_{F^*}).$$

*Proof.* Assume that $\omega(TB_E) = \varepsilon > 0$. Then, since $TB_E$ is absolutely convex, by Proposition 3.1.8 there exists a sequence $(x_n)_n \subset B_E$ such that $(Tx_n)_n$ is orthogonal and $\|Tx_n\| = \varepsilon$ for all $n \in \mathbb{N}$. Take $\lambda \in \mathbb{K}$ with $|\lambda| = \varepsilon$. For every $n \in \mathbb{N}$ define a linear functional $f_n$ on $D := [(Tx_n)_n]$, setting $f_n(Tx_n) = \lambda$ and $f_n(Tx_m) = 0$ if $n \neq m$. Since $(Tx_n)_n$ is orthogonal, $\|f_n\| = 1$ for each $n \in \mathbb{N}$; applying Ingleton's theorem (Theorem 1.1.12), we extend, preserving norm, each $f_n$ on the whole of $F$. Observe that for every $k \in \mathbb{N}$ and every $a_1, \dots, a_{k-1} \in \mathbb{K}$

$$\begin{aligned}&\|a_1T^*f_1 + \ldots + a_{k-1}T^*f_{k-1} + T^*f_k\| \\ &\quad\geqslant \frac{|(a_1f_1 + \ldots + a_{k-1}f_{k-1} + f_k)(Tx_k)|}{\|x_k\|} = \frac{|f_k(Tx_k)|}{\|x_k\|} = \frac{\varepsilon}{\|x_k\|} \geqslant \varepsilon.\end{aligned}$$

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Hence, $\operatorname{dist}\big(T^*f_k, [T^*f_1, \dots, T^*f_{k-1}]\big) \geqslant \varepsilon$ for every $k \in \mathbb{N}$ and by Proposition 3.1.6(4), $\omega(T^*B_{F^*}) \geqslant \varepsilon$. Hence, $\omega(TB_E) \leqslant \omega(T^*B_{F^*})$.

Let $\omega(T^*B_{F^*}) = \varepsilon > 0$. Then, by Proposition 3.1.8, there exists a sequence $(f_n^0)_n \subset B_{F^*}$ such that $(T^*f_n^0)_n$ is orthogonal and $\|T^*f_n^0\| = \varepsilon$ for each $n \in \mathbb{N}$. Choose $x_1 \in B_E$ for which $\|T^*f_1^0\| = |f_1^0(Tx_1)| = \varepsilon$. Next, for every $k = 2, 3, \dots$, define

$$f_k^1 := f_k^0 - \frac{f_k^0(Tx_1)}{f_1^0(Tx_1)}\, f_1^0.$$

Then, $f_k^1 \in B_{F^*}$ and $f_k^1(Tx_1) = 0$ for every $k = 2, 3, \dots$. Since $(T^*f_n^0)_n$ is orthogonal, for each $k = 2, 3, \dots$, we get

$$\|T^*f_k^1\| = \max\left\{\|T^*f_k^0\|, \left\|\frac{f_k^0(Tx_1)}{f_1^0(Tx_1)}\, T^*f_1^0\right\|\right\} = \varepsilon.$$

Taking $\lambda_1, \dots, \lambda_{m-1} \in B_{\mathbb{K}}$ and $k_1, \dots, k_m \in \mathbb{N} \setminus \{1\}$ we obtain

$$\begin{aligned}&\|\lambda_1T^*f_{k_1}^1 + \ldots + \lambda_{m-1}T^*f_{k_{m-1}}^1 + T^*f_{k_m}^1\| = \Big\|\lambda_1T^*f_{k_1}^0 + \ldots \\ &+ \lambda_{m-1}T^*f_{k_{m-1}}^0 + T^*f_{k_m}^0 - \frac{\lambda_1f_{k_1}^0(Tx_1) + \ldots + f_{k_m}^0(Tx_1)}{f_1^0(Tx_1)}\, T^*f_1^0\Big\| = \varepsilon;\end{aligned}$$

hence, $(T^*f_n^1)_{n>1}$ is orthogonal. Now, we choose $x_2 \in B_E$ for which $|f_2^1(Tx_2)| = \varepsilon$.

Continuing on this direction and using the same argumentation as above, for every $n = 2, 3, \ldots$ and, for every $k = n+1, n+2, \ldots$, define

$$f_k^n := f_k^{n-1} - \frac{f_k^{n-1}(Tx_n)}{f_n^{n-1}(Tx_n)} f_n^{n-1}.$$

For every $n=2,3,\ldots$ we select $x_{n+1} \in B_E$ such that $|f_{n+1}^n(Tx_{n+1})| = \varepsilon$.

Now, set $g_n := f_n^{n-1}$, $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, $g_n \in B_{F^*}$, $|g_n(Tx_n)| = \varepsilon$ and $|g_n(Tx_m)| = 0$ if $m < n$. Fix $k \in \mathbb{N}$. Then, for every $a_1, \ldots, a_{k-1} \in \mathbb{K}$, we get

$$\begin{aligned} 1 \geqslant \|g_k\| &\geqslant \frac{|g_k(a_1Tx_1 + \ldots + a_{k-1}Tx_{k-1} + Tx_k)|}{\|a_1Tx_1 + \ldots + a_{k-1}Tx_{k-1} + Tx_k\|} \\ &= \frac{|g_k(Tx_k)|}{\|a_1Tx_1 + \ldots + a_{k-1}Tx_{k-1} + Tx_k\|} \\ &= \frac{\varepsilon}{\|a_1Tx_1 + \ldots + a_{k-1}Tx_{k-1} + Tx_k\|}. \end{aligned}$$

Thus, $\operatorname{dist}(Tx_k, [Tx_1, \ldots, Tx_{k-1}]) \geqslant \varepsilon$. Applying Proposition 3.1.6(4) again, we imply $\omega(TB_E) \geqslant \varepsilon$; hence, $\omega(T^*B_{F^*}) \leqslant \omega(TB_E)$. The equality $\gamma(TB_E) = \gamma(T^*B_{F^*})$ follows directly from Theorem 3.1.11. □

The following Example shows that the conclusion of Theorem 3.3.2 fails if we remove the assumption $\|E\| = \|F\| = |\mathbb{K}|$.

**3.3.3. Example** ([24, Example 3.6]). Choose $s_1, s_2 < 1$ such that $s_1 \cdot s_2 > |\rho|$. Let $s_1', s_2'$ be maps defined on $\mathbb{N}$ such that $s_1'(n) = s_1$ and $s_2'(n) = s_2$ for each $n \in \mathbb{N}$. Let $E := c_0 \oplus c_0(\mathbb{N} : s_1')$ and $F := c_0 \oplus c_0(\mathbb{N} : s_2')$; then, every $x \in E$ can be written as $x = x_1 + \sum\limits_n \lambda_n e_n$ where $x_1 \in c_0$, $(\lambda_n)_n \subset \mathbb{K}$ and $(e_n)_n$ is a standard base of $c_0(\mathbb{N} : s_1')$; similarly for $y \in F$ we can write $y = y_1 + \sum\limits_n \beta_n f_n$, $y_1 \in c_0$, $(\beta_n)_n \subset \mathbb{K}$, $(f_n)_n$ is a standard base of $c_0(\mathbb{N} : s_2')$.

Define

$$T \colon E \to F, \quad x_1 + \sum_n \lambda_n e_n \mapsto \sum_n \lambda_n f_n.$$

Then, $TB_E = \{0\} \oplus \left\{x \in c_0(\mathbb{N} : s_2') : \|x\| \leqslant s_2\right\}$. Hence, by Proposition 3.1.6, $\omega(TB_E) \geqslant s_2 > |\rho|/s_1$.

Now, assume that $\|T^*f\| > |\rho|/s_1$ for some $f \in B_{F^*}$. Then, there exists $x \in B_E$, ($x = x_1 + x_2$, $x_1 \in c_0$ and $x_2 \in c_0(\mathbb{N} : s'_1)$) such that

$$\frac{|f(Tx)|}{\|x\|} > \frac{|\rho|}{s_1}.$$

But then, since $Tx = T(x_1 + x_2) = T(x_2)$,

$$|f(Tx_2)| > \|x_2\| \cdot \frac{|\rho|}{s_1}.$$

Suppose that $\|x_2\| = s_1$. Then, $|f(Tx_2)| > |\rho|$; hence, $|f(Tx_2)| = 1$. Since $\|Tx_2\| = s_2 < 1$, we get

$$\|f\| \geqslant \frac{|f(Tx_2)|}{\|Tx_2\|} = \frac{1}{s_2} > 1$$

and conclude that $f \notin B_{F^*}$, a contradiction. Thus,

$$\omega(T^*B_{F^*}) \leqslant \frac{|\rho|}{s_1} < \omega(TB_E).$$

Since, by Theorem 1.3.1, for every $(E, \|\cdot\|)$ there exists an isomorphism $S\colon E \to c_0(I)$ such that $|\rho| \cdot \|Sx\| < \|x\| \leqslant \|Sx\|$ ($\rho$ is an uniformizing element), defining $\|x\|_K := \|S(x)\|$, $x \in E$, we introduce a norm on $E$, equivalent with $\|\cdot\|$ such that

$$|\rho| \cdot \|x\|_K < \|x\| \leqslant \|x\|_K, \quad x \in E. \tag{3.24}$$

Clearly, $(E, \|\cdot\|_K)$ is isometrically isomorphic with $c_0(I)$. Furthermore, $\|x\|_K = \inf\{r : r \in |\mathbb{K}|, \|x\| \leqslant r\}$, $\|E\|_K = |\mathbb{K}|$, $B_E = \{x \in E : \|x\|_K \leqslant 1\}$. Define for a bounded set $M \subset E$

$$\omega_K(M) := \inf\big\{\varepsilon > 0 : M \subset K_\varepsilon + \{x \in E : \|x\|_K \leqslant \varepsilon\}; \\ K_\varepsilon \text{ is } \sigma(E, E^*)\text{-compact}\big\}.$$

Then, we obtain the following generalization of Theorem 3.3.2.

**3.3.4. Corollary** ([24, Corollary 3.7]). *Let* $E, F$ *be Banach spaces,* $T\colon E \to F$ *be a continuous operator and* $T^*\colon F^* \to E^*$ *be its adjoint. Then*

$$|\rho| \cdot \omega(TB_E) \leqslant \omega(T^*B_{F^*}) \leqslant \frac{1}{|\rho|}\omega(TB_E), \tag{3.25}$$

$$|\rho|^2 \cdot \gamma(TB_E) \leqslant \gamma(T^*B_{F^*}) \leqslant \frac{1}{|\rho|^2}\gamma(TB_E). \tag{3.26}$$

*Proof.* Since, by Proposition 3.1.6,

$$\omega(TB_E) = \sup\left\{\overline{\lim_m}\operatorname{dist}(x_m, [x_1, \dots, x_{m-1}]) : (x_m) \subset TB_E\right\},$$

it follows from (3.24) that

$$\omega(TB_E) \leqslant \omega_K(TB_E) \leqslant \frac{1}{|\rho|}\omega(TB_E). \tag{3.27}$$

Let

$$\|x^*\|_K^* := \sup_{x\neq 0} \frac{|x^*(x)|}{\|x\|_K} \quad (x^* \in F^*),$$

$$V_{F^*} := \{x^* \in F^* : \|x^*\|_K^* \leqslant 1\} \quad \text{and} \quad V_{F^*,r} := \{x^* \in F^* : \|x^*\|_K^* \leqslant r\}.$$

Take $x^* \in B_{F^*}$. Then,

$$1 \geqslant \frac{|x^*(x)|}{\|x\|} \geqslant \frac{|x^*(x)|}{\|x\|_K}$$

for every $x \in F$, $x \neq 0$; hence, $x^* \in V_{F^*}$ and $B_{F^*} \subset V_{F^*}$. If $x^* \in V_{F^*,|\rho|}$ then for every $x \in F$, $x \neq 0$

$$|\rho| \geqslant \frac{|x^*(x)|}{\|x\|_K}. \tag{3.28}$$

Using (3.28) and (3.24), we get

$$1 \geqslant \frac{|x^*(x)|}{|\rho| \cdot \|x\|_K} \geqslant \frac{|x^*(x)|}{\|x\|}$$

and conclude $V_{F^*,|\rho|} \subset B_{F^*}$. Thus, $T^*V_{F^*,|\rho|} \subset T^*B_{F^*} \subset T^*V_{F^*}$ and

$$|\rho| \cdot \omega_K(T^*V_{F^*}) \leqslant \omega(T^*B_{F^*}) \leqslant \omega_K(T^*V_{F^*}). \tag{3.29}$$

By Theorem 3.3.2, $\omega_K(TB_E) = \omega_K(T^*V_{F^*})$. Hence, by (3.27) and (3.29), we get (3.25). The inequalities (3.26) follow directly from (3.25) and Theorem 3.1.11. □

Recall that $T \in L(E, F)$ is called (*weakly compact*) *compact* if $TB_E$ is (relatively weakly compact) relatively compact.

**3.3.5. Corollary.** *Let $E, F$ be Banach spaces, $T\colon E \to F$ be a continuous operator and $T^*\colon F^* \to E^*$ be its adjoint. Then, $T$ is weakly compact (compact) if and only if $T^*$ is weakly compact (compact).*

*Proof.* It follows directly from Corollary 3.3.4 that $\omega(TB_E) = 0$ if and only if $\omega(T^*B_{F^*}) = 0$. □

## 3.4 Remarks

The results of this chapter, obtained for non-Archimedean Banach spaces, were strongly motivated by recent studies about quantitative compactness theorems carried out for real Banach spaces by many authors (see [2], [3], [5], [9], [13], [15], [16] and [28], among others; see also [21, Chapter 4]).

The concept of $\varepsilon$-weakly relatively compact sets (for $\varepsilon > 0$) was considered by several authors (see for instance [2], [13], [15], [9] and [16]). Theorem 3.1.1 for real Banach spaces was proved by Fabian, Hajek, Montesinos and Zizler, see [13, Theorems 2 and 13]. They demonstrated that whenever $M$ is $\varepsilon$-weakly relatively compact for some $\varepsilon > 0$, then $\mathrm{co}M$ is $2\varepsilon$-weakly relatively compact. Moreover if $B_{E^*}$ is $\sigma(E^*, E)$-angelic (recall that a Hausdorff topological space $X$ is called *angelic* if every relatively countably compact set $K$ in $X$ is relatively compact and for every $x \in K$ there exists a sequence in $K$ converging to $x$), then $\mathrm{co}M$ is $\varepsilon$-weakly relatively compact.

In the corresponding real case, for any bounded set $M$ of a real Banach space we have $k(M) \leqslant \gamma(M) \leqslant 2k(M)$, see [3, Theorem 2.3], and the equality $k(M) = k(\mathrm{co}M)$ fails in general, see [15, Theorem 7]. Although $\gamma$ and $\omega$ are equivalent on the real space $c_0$ (see [28, Theorem 2.9]), in contrast to the non-Archimedean case, there exist real Banach spaces for which $\gamma$ and $\omega$ are not equivalent (see [3, Remark 3.3 and Corollary 3.4] and [5, p. 372]).

A quantitative versions of Gantmacher and Grothendieck's theorems were proved by Angosto and Cascales, see [3, Theorems 3.1 and 3.5]. For an uniformly bounded subset $H$ of $C(K, \mathbb{R})$, where $K$ is a compact set, they obtained the inequalities$\gamma_X(H) \leqslant \gamma(H) \leqslant 2\gamma_X(H)$. For real Banach spaces $E, F$ and for an operator $T \in L(E, F)$ they provided also $\gamma(TB_E) \leqslant \gamma(T^*B_{F^*}) \leqslant 2\gamma(TB_E)$.

# Isometrics in finite-dimensional non-Archimedean normed spaces

# 4



Chapter 4 is devoted to selected properties of isometric maps defined on finite-dimensional non-Archimedean spaces. First section concerns the Aleksandrov problem, i.e. the question under what conditions is a mapping of a normed space into itself preserving unit distance an isometry. Next, we refer to the remarkable Mazur–Ulam theorem, examining its fulfilling in non-Archimedean setting. The last, third section, is related to the problem whether every isometric map defined on a finite-dimensional non-Archimedean space is surjective.

Recall that a map (not necessary linear) $T\colon X \to Y$, where $X, Y$ are normed spaces, is *isometric* (an *isometry*) if $\|T(x)-T(y)\| = \|x-y\|$ for all $x, y \in X$.

## 4.1 The distance preserving mappings. Aleksandrov problem

We will say that a map $T\colon X \to Y$, where $X, Y$ are normed spaces, is *non-expansive* if $\|T(x) - T(y)\| \leqslant \|x - y\|$ for all $x, y \in X$; $T$ has *the strong distance one preserving property* (SDOPP) if for all $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|T(x) - T(y)\| = 1$ and conversely.

The problem, under what conditions is a mapping of a metric space into itself preserving unit distance an isometry, known as Aleksandrov problem, has been intensively studied by many specialists in the real and complex case (see [51]–[53], [71], [19] and [54], among others). For example, Rassias and Semrl (see [51, Theorem 5]) proved that every non-expansive, surjective mapping with SDOPP $T\colon X \to Y$ between real normed spaces $X$, $Y$ such that one of them has dimension greater than one is isometric. In non-Archimedean setting, this topic was studied in [39] and [29].

We get the following non-Archimedean counterpart of Rassias and Semrl's result.

**4.1.1. Theorem** ([29, Theorem 5 and Corollary 11]). *Let $E$ be finite-dimensional. Then, every surjective, non-expansive map $T\colon E \to E$ which has SDOPP is isometric if and only if $\mathbb{K}$ is locally compact.*

The proof of Theorem 4.1.1 needs a couple of lemmas.



**4.1.2. Lemma** ([29, Lemma 6]). *Let $E$ be finite-dimensional, $x_0 \in E$ and let $r_0 > r > 0$. If there exist $x_1, \ldots, x_n \in E$ such that $B_{E,r}(x_i)$ $(i = 1, \ldots, n)$ form a finite partition of $B_{E,r_0}(x_0)$ $(B^-_{E,r_0}(x_0))$, then for every $y \in E$ there exist $y_1, \ldots, y_n \in E$ such that $B_{E,r}(y_i)$ $(i = 1, \ldots, n)$ form a finite partition of $B_{E,r_0}(y)$ $(B^-_{E,r_0}(y))$.*

*Proof.* Observe that the map $h\colon E \to E$ given by $h(x) := x + y - x_0$ is isometric; thus, we can easily verify that $B_{E,r}(h(x_1)), \ldots, B_{E,r}(h(x_n))$ form a finite partition of $B_{E,r_0}(y)$. The proof for $B^-_{E,r_0}(y)$ is the same. $\square$

**4.1.3. Lemma** ([29, Lemma 7]). *If $\mathbb{K}$ is discretely valued, $E$ is finite-dimensional and $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2$, then $\|E^\times\| \cap [r_1, r_2]$ has at most finitely many elements and $0$ is only an accumulation point of $\|E^\times\|$.*

*Proof.* Since $\mathbb{K}$ is discretely valued, $|\mathbb{K}^\times| = \{s^n : n \in \mathbb{Z}\}$ for some $s < 1$. Hence, $|\mathbb{K}^\times| \cap [r_1, r_2]$ is at most finite. By [57, Lemma 5.5], $E$ has an orthogonal base, say $\{x_1, \ldots, x_n\}$. Then, for every $x \in E$ there are $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $x = \sum_{i=1}^{n} \lambda_i x_i$. Thus,

$$\|x\| = \max_{i=1,\ldots,n} \{|\lambda_i| \cdot \|x_i\|\}.$$

Therefore, $\|E^\times\|$ contains at most $n$ cosets of $|\mathbb{K}^\times|$ and $\|E^\times\| \cap [r_1, r_2]$ has at most finitely many elements $\square$

**4.1.4. Lemma** ([29, Lemma 8]). *Let* $E$ *be locally compact and* $T\colon E \to E$ *be a surjective, non-expansive map with SDOPP such that* $T(0) = 0$*. Then, for every* $x_0 \in E$ *with* $\|x_0\| > 1$ *we have*

(1) $T^{-1}(B_E(T(x_0))) \subset B_E(x_0)$,

(2) $T^{-1}(B^-_{E,\|T(x_0)\|}(T(x_0))) \subset B^-_{E,\|T(x_0)\|}(x_0)$.

*Proof.* First, we prove that $\|T(x_0)\| > 1$. Assuming that $\|T(x_0)\| = 1$, since $T(0) = 0$, we get $\|T(x_0) - T(0)\| = 1$. But $T$ has SDOPP, thus

$$1 = \|T(x_0) - T(0)\| = \|x_0 - 0\| = \|x_0\|,$$

a contradiction. Suppose $\|T(x_0)\| < 1$. Taking $x_1 \in E$ with $\|x_1\| = 1$, we obtain

$$\|x_1 - x_0\| = \|x_0\| > 1 \tag{4.1}$$

and $1 = \|x_1\| = \|x_1 - 0\| = \|T(x_1) - T(0)\| = \|T(x_1)\|$, hence, $\|T(x_1)\| > \|T(x_0)\|$. But then

$$\|T(x_1) - T(x_0)\| = \max\{\|T(x_1)\|, \|T(x_0)\|\} = \|T(x_1)\| = 1,$$

a contradiction with (4.1) and SDOPP.

(1) Suppose that $y \in E$ and $T(y) \in B_E(T(x_0))$. We prove that $y \in B_E(x_0)$.

If $\|T(x_0) - T(y)\| = 1$, then $\|x_0 - y\| = 1$ by SDOPP, thus $y \in B_E(x_0)$. If $\|T(x_0) - T(y)\| < 1$, taking $z \in B_E(x_0)$ for which $\|z - x_0\| = 1$, we imply, by SDOPP,

$$\|T(z) - T(x_0)\| = 1.$$

From

$$\|T(z) - T(y)\| = \|T(z) - T(x_0) + T(x_0) - T(y)\| = \|T(z) - T(x_0)\| = 1,$$

applying SDOPP again, we get $\|z - y\| = 1$; hence,

$$\|x_0 - y\| = \|x_0 - z + z - y\| \leqslant \max\{\|x_0 - z\|, \|z - y\|\} = 1;$$

thus, $y \in B_E(x_0)$, either.

(2) Choose $x_1, \dots, x_m \in E$ such that balls $B_E(x_j)$, $j = 1, \dots, m$, form a finite partition of $B^-_{E,\|T(x_0)\|}(x_0)$ (since, by assumption $E$ is locally compact, $B^-_{E,\|T(x_0)\|}(x_0)$ is compact). Then, since $T$ is non-expansive and $T(0) = 0$, we get$\|x_0 - x_j\| < \|T(x_0)\| \leqslant \|x_0\|$. Hence, $\|x_j\| = \|x_0\| > 1$ for every $j \in \{1, \dots, m\}$. By (1)

$$T^{-1}(B_E(T(x_j))) \subset B_E(x_j) \quad \text{for every } j \in \{1, \dots, m\}. \tag{4.2}$$

Therefore, to finish the proof, it remains to show that $B_E(T(x_j))$, for $j = 1, \dots, m$, form a finite partition of $B^-_{E,\|T(x_0)\|}(T(x_0))$. Taking $y \in E$ such that $T(y) \in B^-_{E,\|T(x_0)\|}(T(x_0))$ and choosing $j \in \{1, \dots, m\}$ for which $T(y) \in B_E(T(x_j))$, by (4.2) we get $y \in B^-_{E,\|T(x_0)\|}(x_0)$. Observe, that

$$\|T(x_i) - T(x_j)\| > 1 \quad \text{if } \ i \neq j \ (i, j \in \{1, \dots, m\}). \tag{4.3}$$

Indeed, clearly $\|x_i - x_j\| > 1$; hence, $\|T(x_i) - T(x_j)\| \neq 1$ by SDOPP. Suppose that $\|T(x_i) - T(x_j)\| < 1$ and take $z_0 \in B_E(T(x_j))$ such that $\|z_0 - T(x_j)\| = 1$. By surjectivity of $T$, there exists $y \in E$ for which $T(y) = z_0$; hence, by SDOPP we get

$$\begin{aligned} \|T(y) - T(x_j)\| &= \|y - x_j\| = 1, \\ \|T(x_i) - T(y)\| &= \|T(x_i) - T(x_j) + T(x_j) - T(y)\| \\ &= \|T(y) - T(x_j)\| = 1; \end{aligned}$$

thus, $\|y - x_i\| = 1$ by SDOPP and $y \in B_E(x_i) \cap B_E(x_j)$, a contradiction, since, by assumption, $B_E(x_i) \cap B_E(x_j) = \emptyset$ if $i \neq j$.

Using Lemma 4.1.2, we select $y_1, \dots, y_m \in E$ for which balls $B_E(y_i)$, $i = 1, \dots, m$, form a finite partition of $B^-_{E,\|T(x_0)\|}(T(x_0))$. It follows from (4.3) that there exists a bijective map

$$h\colon \{1, \dots, m\} \to \{1, \dots, m\}$$

such that $T(x_i) \in B_E(y_{h(i)})$, thus $B_E(y_{h(i)}) = B_E(T(x_i))$, for each $i \in \{1, \dots, m\}$. This yields that $B_E(T(x_i))$, $i = 1, \dots, m$, form a finite partition of $B^-_{E,\|T(x_0)\|}(T(x_0))$. $\square$

*Proof of Theorem* 4.1.1. First, observe that we can assume $T(0) = 0$. Indeed, for surjective, non-expansive map $T_0 \colon E \to E$ with SDOPP, the map $T(x) := T_0(x) - T_0(0)$ is also surjective, non-expansive map, with SDOPP, and additionally, $T(0) = 0$. Since $\|T_0(x) - T_0(y)\| = \|T(x) - T(y)\|$ for all $x, y \in E$, proving that $T$ is isometric, we get the same conclusion for $T_0$.

($\Leftarrow$) Let $\mathbb{K}$ be locally compact and $T \colon E \to E$ be a surjective, non-expansive map which has SDOPP such that $T(0) = 0$. Assume for a contradiction that for some $y_1, y_2 \in E$

$$\|y_1 - y_2\| > \|T(y_1) - T(y_2)\| \tag{4.4}$$

In the first part of the proof we show that

$$\text{there exists } x_0 \in E \text{ for which } \|x_0\| > \|T(x_0)\|. \tag{4.5}$$

Next, using (4.5), in the second part we provide a contradiction with SDOPP.

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Part I. $\|y_1\| > \|y_2\|$ implies $\|y_1\| > \|T(y_1)\|$. Indeed, assuming $\|y_1\| = \|T(y_1)\|$, since $T$ is non-expansive and $T(0) = 0$ we get $\|T(y_1)\| = \|y_1\| > \|y_2\| \geqslant \|T(y_2)\|$. Hence,

$$\|T(y_1)\| = \|y_1\| = \|y_1 - y_2\| > \|T(y_1) - T(y_2)\| = \|T(y_1)\|,$$

a contradiction. Therefore, we set $x_0 := y_1$.

Assume now that $r := \|y_1\| = \|y_2\| > 0$. If $\|T(y_1)\| \neq \|T(y_2)\|$ or $\|T(y_1)\| = \|T(y_2)\| < r$, then we are done. So, suppose $\|T(y_1)\| = \|T(y_2)\| = r$. Since $S = B_{E,r} \setminus B^-_{E,r}$ is compact, we can select balls

$$B^-_{E,\|y_1 - y_2\|}(z_j), \quad j = 1, \ldots, m, \; z_1, \ldots, z_m \in S,$$

which form a finite partition of $S$. Additionally, we can assume that $y_1 \in B^-_{E,\|y_1 - y_2\|}(z_1)$ and $y_2 \in B^-_{E,\|y_1 - y_2\|}(z_2)$.

Suppose that $\|T(z_i)\| = r$ for each $i \in \{1, \ldots, m\}$; otherwise we are done. Since $T$ is surjective, for every $i \in \{1, \ldots, m\}$ there exists $j \in \{1, \ldots, m\}$ such that

$$T(B^-_{E,\|y_1 - y_2\|}(z_i)) \subset B^-_{E,\|y_1 - y_2\|}(z_j). \tag{4.6}$$

We can find $k \in \{1, \ldots, m\}$ with $T(y_1), T(y_2) \in B^{-}_{E,\|y_1-y_2\|}(z_k)$ using (4.4). Hence, by (4.6),

$$T(B^{-}_{E,\|y_1-y_2\|}(z_1) \cup B^{-}_{E,\|y_1-y_2\|}(z_2)) \subset B^{-}_{E,\|y_1-y_2\|}(z_k).$$

Thus, applying (4.6) again, we conclude that there exists $l \in \{1, \ldots, m\}$ such that there is no $z_0 \in S$ for which $T(z_0) \in B^{-}_{E,\|y_1-y_2\|}(z_l)$. $T$ is surjective, hence, we can find $x_0 \in E$ with $T(x_0) \in B^{-}_{E,\|y_1-y_2\|}(z_l)$. Clearly, $\|x_0\| > \|T(x_0)\|$ as $T$ is non-expansive and $T(0) = 0$. Hence, we get (4.5).

Part II. We will consider three cases:

(1) Suppose that $\|x_0\| > 1 \geqslant \|T(x_0)\|$. Assume $\|T(x_0)\| < 1$. Taking $x_1 \in E$ with $\|x_1\| = 1$, applying SDOPP we get $\|T(x_1)\| = 1$ since $T(0) = 0$. Then, $\|T(x_1) - T(x_0)\| = \max\{\|T(x_1)\|, \|T(x_0)\|\} = 1$. Thus

$$1 = \|T(x_1) - T(x_0)\| = \|x_1 - x_0\| = \|x_0\|,$$

a contradiction. Suppose that $\|T(x_0)\| = 1$, then, by SDOPP, we get

$$1 = \|T(x_0) - T(0)\| = \|x_0 - 0\| = \|x_0\|,$$

respectively, a contradiction.

(2) Let $\|x_0\| > \|T(x_0)\| > 1$ and $S_0 := \{z \in E : \|z\| = \|T(x_0)\|\}$. First, we show that there exists $x_1 \in S_0$ for which $\|x_1\| > \|T(x_1)\|$. Assume the contrary and suppose that $\|T(x)\| = \|T(x_0)\|$ for every $x \in S_0$. Choose $z_1, \ldots, z_n \in S_0$ for which balls $B^{-}_{E,\|T(x_0)\|}(z_j,)$, $j = 1, \ldots, n$, form a finite partition of $S_0$ (recall that $S_0$ is compact). By Lemma 4.1.4,

$$T^{-1}(B^{-}_{E,\|T(x_0)\|}(T(z_j))) \subset B^{-}_{E,\|T(x_0)\|}(z_j), \tag{4.7}$$

thus

$$B^{-}_{E,\|T(x_0)\|}(T(z_i)) \cap B^{-}_{E,\|T(x_0)\|}(T(z_j)) = \emptyset \tag{4.8}$$

if $i \neq j$ $(i, j \in \{1, \ldots, n\})$ (assuming that $T(z_j) \in B^{-}_{E,\|T(x_0)\|}(T(z_j))$ $(i \neq j)$, by (4.7) we imply that $z_i \in B^{-}_{E,\|T(x_0)\|}(z_j)$, a contradiction). Hence, for every $i \in \{1, \ldots, n\}$ there exists $j \in \{1, \ldots, n\}$ such that $T(z_j) \in B^{-}_{E,\|T(x_0)\|}(z_i)$. Obviously, $B^{-}_{E,\|T(x_0)\|}(z_i) = B^{-}_{E,\|T(x_0)\|}(T(z_j))$; thus, we conclude that $B^{-}_{E,\|T(x_0)\|}(T(z_j))$, $j = 1, \ldots, n$, form a finite partition of $S_0$. In particular $T(x_0) \in B^-_{E,\|T(x_0)\|}(T(z_k))$ for some $k \in \{1,\dots,n\}$. Since,

$$B^-_{E,\|T(x_0)\|}(T(x_0)) = B^-_{E,\|T(x_0)\|}(T(z_k)),$$

applying Lemma 4.1.4 again, we obtain

$$T^{-1}(B^-_{E,\|T(x_0)\|}(T(z_k))) \subset B^-_{E,\|T(x_0)\|}(x_0) \tag{4.9}$$

and conclude that $z_k \in B^-_{E,\|T(x_0)\|}(x_0)$, thus $\|z_k\| = \|x_0\| > \|T(x_0)\|$, a contradiction. This way, we deduce that there exists $x_1 \in S_0$ such that $\|x_1\| > \|T(x_1)\|$.

Continuing on this direction and applying Lemma 4.1.3, we select inductively a sequence $x_1,\dots,x_p \in E$ satisfying

$$\|x_k\| > \|T(x_k)\| = \|x_{k+1}\|, \quad k = 1,\dots,p,$$

and $\|x_p\| > 1 \geqslant \|T(x_p)\|$. By (1), applied for $x_p$, we get a contradiction.

(3) Suppose that $1 \geqslant \|x_0\| > \|T(x_0)\|$. Set $S_1 := \{z \in E : \|z\| = \|x_0\|\}$ and choose $z_1,\dots,z_n \in S_1$ for which $B^-_{E,\|x_0\|}(z_j)$, $j = 1,\dots,n$, form a finite partition of $S_1$.

Fix $j \in \{1,\dots,n\}$. Then $T(B^-_{E,\|x_0\|}(z_j)) \subset B^-_{E,\|x_0\|}(z_i)$ for some $i \in \{1,\dots,n\}$ or $T(B^-_{E,\|x_0\|}(z_j)) \cap S_1 = \emptyset$. Indeed, assume that $T(z_j) \in S_1$, then $T(z_j) \in B^-_{E,\|x_0\|}(z_i)$ for some $i \in \{1,\dots,n\}$. If $x \in B^-_{E,\|x_0\|}(z_j)$, then $T(x) \in B^-_{E,\|x_0\|}(T(z_j))$ since T is non-expansive. Therefore,

$$T(B^-_{E,\|x_0\|}(z_j)) \subset B^-_{E,\|x_0\|}(T(z_j)) = B^-_{E,\|x_0\|}(z_i).$$

Suppose that $y \in S_1$ and $\|T(y)\| < \|x_0\|$. Then, $y \in B^-_{E,\|x_0\|}(z_k)$ for some $k \in \{1,\dots,n\}$. Since T is non-expansive, we get

$$\|T(x') - T(y)\| \leqslant \|x' - y\| < \|x_0\|$$

and

$$\begin{aligned}\|T(x')\| &= \|T(x') - T(y) + T(y)\| \\ &\leqslant \max\left\{\|T(x') - T(y)\|, \|T(y)\|\right\} < \|x_0\|\end{aligned}$$

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for every $x' \in B^-_{E,\|x_0\|}(z_k)$. Hence, if there exists $x \in B^-_{E,\|x_0\|}(z_k)$, $k \in \{1,\dots,n\}$ such that $\|T(x)\| < \|x_0\|$, then $T(B^-_{E,\|x_0\|}(z_k)) \cap S_1 = \emptyset$.

Set $M_0 := \{i \in \{1,\dots,n\} : T(z_i) \in S_1\}$. Then,

$$S_1 \cap \bigcup_{j\in\{1,\dots,n\}} T(B^-_{E,\|x_0\|}(z_j)) \\ = S_1 \cap \bigcup_{j\in M_0} T(B^-_{E,\|x_0\|}(z_j)) \subset \bigcup_{j\in M_0} B^-_{E,\|x_0\|}(z_j).$$

Since, $M_0 \neq \{1,\dots,n\}$ by (4.5), we conclude that $\bigcup_{j\in\{1,\dots,n\}} T(B^-_{E,\|x_0\|}(z_j))$ does not cover $S_1$. But $T$ is surjective, hence, there exists $x_1 \in E$ such that $\|x_1\| > \|T(x_1)\| = \|x_0\|$.

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Applying this observation and Lemma 4.1.3, we can inductively select a sequence $x_1,\dots,x_p \in E$ such that

$$\|x_{k+1}\| > \|T(x_{k+1})\| = \|x_k\|, \quad k = 1,\dots,p,$$

and $\|x_p\| > 1 \geqslant \|T(x_p)\|$. Then, applying case (1) for $x_p$, we get a contradiction.

$(\Rightarrow)$ Assume that $\mathbb{K}$ is not locally compact; then, by [57, 1.B], $\mathrm{card}(\mathbb{k})$ is infinite or $\mathbb{K}$ is densely valued. Considering both cases, we prove that there exists a non-isometric, surjective, non-expansive map $E \to E$ with SDOPP.

First, suppose that $\mathrm{card}(\mathbb{k})$ is infinite. Then, we can select an infinite sequence $(\lambda_n)_n \subset \mathbb{K}$ with $|\lambda_n| = 1$ ($n \in \mathbb{N}$) such that $|\lambda_i - \lambda_j| = 1$ if $i \neq j$. Set $\mu \in \mathbb{K} \setminus \{0\}$ with $|\mu| < 1$, $x_0 \in E \setminus \{0\}$ with $r := \|x_0\| < 1$ and form a sequence $(x_n)_n$ setting $x_n := \lambda_n x_0, n \in \mathbb{N}$. Then, balls $B^-_{E,r}(x_1), B^-_{E,r}(x_2), \dots$ are pairwise disjoint. Define a map $T_1 \colon E \to E$ as follows

$$T_1(x) = \begin{cases} \mu x & \text{if } x \in B^-_{E,r}(x_1), \\ x + x_n - x_{n+1} & \text{if } x \in B^-_{E,r}(x_{n+1}),\ n \in \mathbb{N}, \\ x + \mu x_{n+1} - \mu x_n & \text{if } x \in B^-_{E,r\cdot|\mu|}(\mu x_n),\ n \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then, $T_1(0) = 0$. Note that

$$T_1(B^-_{E,r}(x_1)) = B^-_{E,r\cdot|\mu|}(\mu x_1),$$
$$T_1(B^-_{E,r\cdot|\mu|}(\mu x_n)) = B^-_{E,r\cdot|\mu|}(\mu x_{n+1}) \quad (n \in \mathbb{N})$$

and

$$T_1(B^-_{E,r}(x_n)) = B^-_{E,r}(x_{n-1}) \quad (n = 2, 3, \dots).$$

If $x \notin \bigcup_{n\in\mathbb{N}} (B^-_{E,r\cdot|\mu|}(\mu x_n) \cup B^-_{E,r}(x_n))$ then $T_1(x) = x$; hence, $T_1$ is surjective.

Clearly, $\|T_1(x)\| \leqslant \|x\|$ for all $x \in E$. Take $x, y \in E$, $x \neq y$. We see that $\|x - y\| = 1$ or $\|T_1(x) - T_1(y)\| = 1$ only if $\max\{\|x\|, \|y\|\} \geqslant 1$; hence, we can easily deduce that $T_1$ has SDOPP.

Assume that $x \notin B^-_{E,r}(x_1)$ or $y \notin B^-_{E,r}(x_1)$; then, $\|T_1(x) - T_1(y)\| = \|x - y\|$. Indeed, if $x, y \in B^-_{E,r}(x_n)$ for some $n \in \mathbb{N}$ ($n > 1$), then

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$$\|T_1(x) - T_1(y)\| = \|x + x_n - x_{n+1} - (y + x_n - x_{n+1})\| = \|x - y\|.$$

Similarly, $\|T_1(x) - T_1(y)\| = \|x - y\|$ if $x, y \in B^-_{E,r\cdot|\mu|}(\mu x_n)$ for some $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x \in B^-_{E,r}(x_n)$ ($x \in B^-_{E,r\cdot|\mu|}(\mu x_n)$) and $y \notin B^-_{E,r}(x_n)$ ($y \notin B^-_{E,r\cdot|\mu|}(\mu x_n)$), then, $\|x - y\| \geqslant r$ ($r \cdot |\mu|$) $= \|x\|$. Hence, $\|x - y\| = \max\{\|x\|, \|y\|\}$ and $\|T_1(x) - T_1(y)\| \leqslant \|x - y\|$ since $\max\{\|T(x)\|, \|T(y)\|\} \leqslant \max\{\|x\|, \|y\|\}$.

For $x, y \in B^-_{E,r}(x_1)$ we get

$$\|T_1(x) - T_1(y)\| = |\mu| \cdot \|x - y\| < \|x - y\|.$$

This way we prove that $T_1$ is non-expansive, but it is not an isometry.

Now, suppose that $\mathbb{K}$ is densely valued. Choose reals $r_1$, $r_2$ with $0 < r_1 < r_2 < 1$ and select two sequences $(a_n)_n, (b_n)_n \subset \mathbb{K}$ such that

$$\frac{r_1 + r_2}{2} < |a_n| < |a_{n+1}| < r_2 \quad \text{and} \quad r_1 < |b_{n+1}| < |b_n| < \frac{r_1 + r_2}{2}.$$

for every $n \in \mathbb{N}$. Set $x_0 \in E$ with $\|x_0\| = 1$. Define $A_n := B^-_{E,|a_n|}(a_n x_0)$,

$B_n := B^-_{E,|b_n|}(b_n x_0)$ $(n \in \mathbb{N})$, and the map $T_2 \colon E \to E$ by

$$T_2(x) = \begin{cases} \dfrac{a_n}{a_{n+1}} x & \text{if } x \in A_{n+1},\ n \in \mathbb{N}, \\ \dfrac{b_{n+1}}{b_n} x & \text{if } x \in B_n,\ n \in \mathbb{N}, \\ \dfrac{b_1}{a_1} x & \text{if } x \in A_1, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $T_2(0) = 0$ and $T_2$ is not isometric. However, $T_2$ is a surjective, non-expansive map with SDOPP. Indeed, observe that $T_2(A_1) = B_1$, $T_2(B_n) = B_{n+1}$ $(n \in \mathbb{N})$ and $T_2(A_n) = A_{n-1}$ $(n = 2, 3, \dots)$. If $x \notin \bigcup\limits_{n \in \mathbb{N}} (A_n \cup B_n)$ then $T_2(x) = x$; hence, $T_2$ is surjective.

Take $x, y \in E, x \neq y$. Then, $\|x - y\| = 1$ or $\|T_2(x) - T_2(y)\| = 1$ only if $\max\{\|x\|, \|y\|\} \geqslant 1$; hence, $T_2$ has a SDOPP. If $x, y \notin \bigcup\limits_{n \in \mathbb{N}} (A_n \cup B_n)$ then $\|T_2(x) - T_2(y)\| = \|x - y\|$. If $x \in A_n$ $(x \in B_n)$ for some $n \in \mathbb{N}$ and $y \notin A_n$ $(y \notin B_n)$, then, $\|x - y\| = \max\{\|x\|, \|y\|\}$; thus,

$$\|T_2(x) - T_2(y)\| \leqslant \|x - y\|,$$

since $\|T_2(x')\| \leqslant \|x'\|$ for all $x' \in E$. If $x, y \in A_n$ or $x, y \in B_n$ for some $n \in \mathbb{N}$, then

$$\|T_2(x) - T_2(y)\| < \|x - y\| < 1.$$

Hence, $T_2$ is non-expansive. $\square$

## 4.2 The absence of Mazur–Ulam theorem in non-Archimedean setting

The classical result of Mazur and Ulam states that if $X, Y$ are normed spaces over $\mathbb{R}$ and $T \colon X \to Y$ is a surjective isometry, then $T$ is affine i.e. $T$ is a linear mapping up to translation (see [37]). We show that this conclusion fails when $\mathbb{R}$ is replaced by $\mathbb{K}$.

**4.2.1. Proposition** ([29, Proposition 1]). *If* $E \neq \{0\}$, *then there exists a surjective isometry* $T \colon E \to E$ *with* $T(0) = 0$ *which is not an additive map.*

*Proof.* Let $x_0 \in E$, $x_0 \neq 0$, and let $\lambda \in B_{\mathbb{K}}^{-}$. Define the map $T \colon E \to E$ by

$$T(x) := \begin{cases} (1+\lambda)x & \text{if } \|x\| = \|x_0\|, \\ x & \text{if } \|x\| \neq \|x_0\|. \end{cases}$$

We prove that T is isometric. Let $x, x' \in E$. Consider three cases:

- $\|x\| = \|x'\| = \|x_0\|$. Then,

$$\|T(x) - T(x')\| = |1+\lambda| \cdot \|x - x'\| = \|x - x'\|;$$

- $\|x\| = \|x'\| \neq \|x_0\|$. Then, $T(x) = x$ and $T(x') = x'$; thus,

$$\|T(x) - T(x')\| = \|x - x'\|;$$

- $\|x\| \neq \|x'\|$. Then $\|T(x)\| \neq \|T(x')\|$ and we imply

$$\|T(x) - T(x')\| = \max\left\{\|x\|, \|x'\|\right\} = \|x - x'\|.$$

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Hence, T is an isometry. Obviously, T is surjective and $T(0) = 0$. Let $z_1 := x_0$ and $z_2 := (\lambda - 1)x_0$. Then, we obtain

$$\begin{aligned} T(z_1) + T(z_2) - T(z_1 + z_2) &= (1+\lambda)x_0 + (1+\lambda)(\lambda-1)x_0 - \lambda x_0 \\ &= x_0 + \lambda x_0 + \lambda^2 x_0 - x_0 - \lambda x_0 = \lambda^2 x_0. \end{aligned}$$

Hence, T is not additive. □

Making use of Proposition 4.2.1 we shall prove the following result.

**4.2.2. Proposition** ([29, Theorem 2])**.** *Let* $E, F$ *be non-Archimedean normed spaces. Assume that there exists a surjective isometry* $T \colon E \to F$*. If every surjective isometry* $S \colon E \to F$ *is an additive map up to translation, then* $E = F = \{0\}$.

*Proof.* Let $T \colon E \to F$ be a surjective isometry. Assume for a contradiction that $E \neq \{0\}$. Taking a nonzero $x \in E$ we get $\|T(x) - T(0)\| = \|x\| > 0$; thus, $F \neq \{0\}$. Conversely, $F \neq \{0\}$ implies $E \neq \{0\}$.

Applying Proposition 4.2.1, we can construct $T_1 \colon F \to F$, $T_1(0) = 0$, a surjective isometry which is not additive. Hence, there exist $x_1, x_2 \in F$ for which

$$T_1(x_1 + x_2) - T_1(x_1) - T_1(x_2) \neq 0. \tag{4.10}$$

Next, define $T_2 \colon E \to F$ by $T_2(x) := T(x) - T(0)$. By assumption, $T_2$ is additive and surjective. Choose $z_1, z_2 \in E$ with $T_2(z_1) = x_1$, $T_2(z_2) = x_2$ and define $T' = T_1 \circ T_2 \colon E \to F$. Then, $T'$ is a surjective isometry and $T'(0) = 0$. By (4.10), we obtain

$$\begin{aligned} T'(z_1 + z_2) - T'(z_1) - T'(z_2) &= T_1(T_2(z_1 + z_2)) - T_1(x_1) - T_1(x_2) \\ &= T_1(T_2(z_1) + T_2(z_2)) - T_1(x_1) - T_1(x_2) \\ &= T_1(x_1 + x_2) - T_1(x_1) - T_1(x_2) \neq 0. \end{aligned}$$

Hence, $T' \colon E \to F$ is not additive, a contradiction. □

Let us note that some other results with respect to this topic are obtained in [25], [42] and [39].

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## 4.3 Surjective isometrics

In this chapter we continue the study of isometric maps defined on a finite-dimensional non-Archimedean spaces. Namely, we prove Theorem 4.3.1, extending Schikhof's result obtained for $\mathbb{K}$ (see [59, Theorem 2]), where we characterize the class of finite-dimensional non-Archimedean spaces for which every isometric map defined on the member of this class into itself is surjective.

**4.3.1. Theorem.** *Let $E$ be finite-dimensional. Then, every isometric map $T \colon E \to E$ is surjective if and only if $\mathbb{K}$ is spherically complete and $\mathbb{k}$ is finite.*

To prove Theorem 4.3.1 we need the following lemmas.

**4.3.2. Lemma** (see [29, Lemma 13])**.** *Let $E$ be finite-dimensional and $\mathbb{K}$ be spherically complete. Then, every ball $B_{E,r}(x)$, $x \in E$, has a finite partition consisting of balls $B^-_{E,r}(x_i)$ $(i = 1, \dots, n)$ for some $x_1, \dots, x_n \in B_{E,r}(x)$ if and only if $\mathbb{k}$ is finite.*

*Proof.* First, assume that $\mathbb{k}$ is finite. If $r \notin \|E^\times\|$, the conclusion is straightforward since $B_{E,r}(x) = B^-_{E,r}(x)$. Suppose now that $r \in \|E^\times\|$. Since, by assumption, $\mathbb{K}$ is spherically complete, by [57, Lemma 5.5],

$E$ has an orthogonal base, say $\{z_1, \ldots, z_m\}$. Without loss of generality we can assume that $\|z_i\| = r$ if $i \leqslant m_0$ and $\|z_i\| \notin |\mathbb{K}^\times|$ if $i > m_0$ for some $m_0 \in \{1, \ldots, m\}$.

Since $\Bbbk$ is finite, we can choose $M_\lambda = \left\{\lambda_1, \ldots, \lambda_{\mathrm{card}(\Bbbk)}\right\} \subset B_{\mathbb{K}}$ such that $B^-_{\mathbb{K}}(\lambda_i)$, $i = 1, \ldots, \mathrm{card}(\Bbbk)$ form a finite partition of $B_{\mathbb{K}}$; additionally, we can assume that $|\lambda_1| < 1$ and $|\lambda_i| = 1$ if $i > 1$. Denote by $M_V$ the set of all $m_0$-permutations with repetitions of elements of $M_\lambda$. Then, $\mathrm{card}(M_V) = \mathrm{card}(\Bbbk)^{m_0}$.

Next, we show that $\{B^-_{E,r}(y)\}_{y \in M_X}$, where $M_X = \{a_1 z_1 + \ldots + a_{m_0} z_{m_0} : (a_1, \ldots, a_{m_0}) \in M_V\}$, is a finite partition of $B_{E,r}$. Since $\{z_1, \ldots, z_{m_0}\}$ is orthogonal, $\overline{y} := \lambda_1 z_1 + \ldots + \lambda_1 z_{m_0}$ is the only one element of $M_X$ with the norm less than $r$. Take distinct $y, y' \in M_X$ such that $\|y\| = \|y'\| = r$. Then, there exist $(a_1, \ldots, a_{m_0}), (b_1, \ldots, b_{m_0}) \in M_V$ for which

$$y = a_1 z_1 + \ldots + a_{m_0} z_{m_0} \quad \text{and} \quad y' = b_1 z_1 + \ldots + b_{m_0} z_{m_0}.$$

By assumption, there is $j \in M_0$ such that $a_j \neq b_j$. Hence,

$$\|y - y'\| = \max\left\{r \cdot \max_{k \in M_0} |a_k - b_k|\right\} = r \cdot |a_j - b_j| = r.$$

Taking $z \in B^-_{E,r}(y)$ we obtain

$$\|z - y'\| = \|z - y + y - y'\| = \|y - y'\|$$

and conclude that $y \notin B^-_{E,r}(y')$. Hence, the balls $\left\{B^-_{E,r}(y) : y \in M_X\right\}$ are pairwise disjoint.

Take $z \in B_{E,r} \setminus M_X$; then we can write

$$z = \sum_{k=1}^{m} \mu_k z_k$$

for some $\mu_1, \ldots, \mu_m \in \mathbb{K}$. Obviously, $\|\mu_k z_k\| < r$ for each $k > m_0$.

If $\|z\| < r$, then $\|z - \overline{y}\| \leqslant \max\{\|z\|, \|\overline{y}\|\} < r$ and $z \in B^-_{E,r}(\overline{y})$. Let $\|z\| = r$. For every $k \in \{1, \ldots, m_0\}$ we can choose $b_k \in M_\lambda$ with $|\mu_k - b_k| < 1$. Define $y := \sum\limits_{k=1}^{m_0} b_k z_k \in M_X$. Then,

$$\|z - y\| = \max\left\{r \cdot \max_{k \in \{1, \ldots, m_0\}} |\mu_k - b_k|, \max_{k \in \{m_0+1, \ldots, m\}} \|\mu_k z_k\|\right\} < r;$$

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thus, we get $z \in B^-_{E,r}(y)$ and conclude that $\{B^-_{E,r}(y)\}_{y\in M_X}$ is a finite partition of $B_{E,r}$.

To finish the proof we shall consider the following two cases:

- if $\|x\| \leqslant r$, then $B_{E,r} = B_{E,r}(x)$. Hence, $\{B^-_{E,r}(y)\}_{y\in M_X}$ is a required finite partition of $B_{E,r}(x)$;
- if $\|x\| > r$, define the map $h\colon E \to E$ by $h(z) := x + z$. Clearly, $h$ is isometric and $h(B_{E,r}) = B_{E,r}(x)$. Thus, $\{B^-_{E,r}(h(y))\}_{y\in M_X}$ is a finite partition of $B_{E,r}(x)$.

Now, assume that $\Bbbk$ is infinite. Then, we can select an infinite $\{\lambda_1, \lambda_2, \ldots\} \subset B_{\mathbb{K}}$ such that $\left\{B^-_{\mathbb{K}}(\lambda_i)\right\}_i$ is an infinite partition of $B_{\mathbb{K}}$. Take $x_0 \in E$ with $\|x_0\| = 1$. Consider the ball $B_E(\lambda_1 x_0)$ and balls $B^-_E(\lambda_n x_0)$, $n \in \mathbb{N}$. Clearly, $B^-_E(\lambda_n x_0) \subset B_E(\lambda_1 x_0)$ for each $n \in \mathbb{N}$. If $y \in B^-_E(\lambda_i x_0)$ for some $i \in \mathbb{N}$, then, for any $j \in \mathbb{N}$ with $i \neq j$ we get

$$\|y - \lambda_j x_0\| = \|y - \lambda_i x_0 + \lambda_i x_0 - \lambda_j x_0\| = \|\lambda_i x_0 - \lambda_j x_0\| = |\lambda_i - \lambda_j| = 1;$$

hence, balls $B^-_E(\lambda_n x_0)$, $n \in \mathbb{N}$, are pairwise disjoint. □

**4.3.3. Lemma** (see [29, Lemma 13]). *Let $r > 0$, $E$ be finite-dimensional, $T\colon E \to E$ be an isometric map, $x \in E$ and $B^-_{E,r}(x_1), \ldots, B^-_{E,r}(x_n)$ be a finite partition of $B_{E,r}(x)$. Then, for every $y_0 \in E$ for which $T(y_0) \in B_{E,r}(x)$ there exist $y_1, \ldots, y_n \in B_{E,r}(y_0)$ such that $B^-_{E,r}(T(y_i))$, $i = 1, \ldots, n$ form a finite partition of $B_{E,r}(x)$.*

*Proof.* Assume that $y_0 \in E$, $T(y_0) \in B_{E,r}(x)$ (then $B_{E,r}(T(y_0)) = B_{E,r}(x)$) and $B^-_{E,r}(x_1), \ldots, B^-_{E,r}(x_n)$ form a finite partition of $B_{E,r}(x)$. The map $g\colon B_{E,r}(T(y_0)) \to B_{E,r}(y_0)$, defined by $g(y) := y_0 - T(y_0) + y$, is surjective and isometric. Thus, $B^-_{E,r}(y_1), \ldots, B^-_{E,r}(y_n)$, where $y_i := g(x_i)$ $(i = 1, \ldots, n)$, form a finite partition of $B_{E,r}(y_0)$. Then, obviously $\|y_i - y_j\| = r$ for $i \neq j$ $(i, j \in \{1, \ldots, n\})$. Since $T$ is isometric,

$$\begin{aligned}
\|T(y_i) - x\| &= \|T(y_i) - T(y_0) + T(y_0) - x\| \\
&\leqslant \max\{\|T(y_i) - T(y_0)\|, \|T(y_0) - x\|\} \\
&= \max\{\|y_i - y_0\|, \|T(y_0) - x\|\} \leqslant r;
\end{aligned}$$

thus, $T(y_i) \in B_{E,r}(x)$ for each $i \in \{1, \ldots, n\}$ and

$$\|T(y_i) - T(y_j)\| = \|y_i - y_j\| = r \tag{4.11}$$

if $i \neq j$ $(i, j \in \{1, \dots, n\})$.

Choose $n_1$ such that $T(y_1) \in B^-_{E,r}(x_{n_1})$. By (4.11), there is no $m \in \{2, \dots, n\}$ with $T(y_m) \in B^-_{E,r}(x_{n_1})$, thus there is $n_2$, $n_2 \neq n_1$ that $T(y_2) \in B^-_{E,r}(x_{n_2})$. Continuing on this direction we define the bijective map

$$h\colon \{1, \dots, n\} \to \{1, \dots, n\}, \quad h(i) := n_i \quad (i = 1, \dots, n)$$

and conclude that $B^-_{E,r}(T(y_i))$, $i = 1, \dots, n$ form a finite partition of $B_{E,r}(x)$. $\square$

Now, we are ready to prove the main result of this section.

*Proof of Theorem* 4.3.1. First assume that $\mathbb{K}$ is non-spherically complete. By [57, 4.A.], $E$ is non-spherically complete, hence, there exists a sequence of closed balls in $E$ with an empty intersection $(B_{E,r_n}(c_n))_n$. We can assume that $r_n = |c_n - c_{n+1}|$ and $r_n > r_{n+1}$ $(n \in \mathbb{N})$. Obviously, $\inf_{n \in \mathbb{N}} r_n > 0$. Define the map $T\colon E \to E$ by 4

$$T(x) := \begin{cases} x - c_1 & \text{if } x \notin B_{E,r_1}(c_1), \\ x - c_{n+1} & \text{if } x \in B_{E,r_n}(c_n) \setminus B_{E,r_{n+1}}(c_{n+1}). \end{cases}$$

Observe that $T$ is isometric; indeed, take $x, y \in E$, then $T(x) = x - c_i$, $T(y) = y - c_j$ for some $i, j \in \mathbb{N}$. If $i = j$ we are done; so, assume that $i < j$. Then

$$\|x - c_i\| > \|y - c_j\|,$$
$$\|T(x) - T(y)\| = \|(x - c_i) - (y - c_j)\| = \|x - c_i\|.$$

But, $\|x - c_i\| > r_i$, $y \in B_{E,r_i}(c_i)$; hence,

$$\|x - y\| = \|x - c_i + c_i - y\| = \|x - c_i\| = \|T(x) - T(y)\|.$$

However, $T$ is not surjective since $0 \notin T(E)$.

Assume now that $\mathbb{K}$ is spherically complete. First, suppose that $\mathrm{card}(\mathbb{k})$ is infinite. It follows from Lemma 4.3.2 that there exists $x_0 \in E$ and $r > 0$ such that $B_{E,r}(x_0)$ has an infinite partition consisting of balls $B^{-}_{E,r}(x_i)$ $(i \in \mathbb{N})$ for some $x_1, x_2, \ldots \in B_{E,r}(x)$. Define the map $T\colon E \to E$ setting

$$T(x) := \begin{cases} x & \text{if } x \notin B_{E,r}(x_0), \\ x - x_i + x_{i+1} & \text{if } x \in B^{-}_{E,r}(x_i). \end{cases}$$

Then, we can easily verify that $T$ is isometric and $x_1 \notin T(E)$.

Finally, suppose that $\mathtt{card}(\mathbb{k})$ is finite. Let $T\colon E \to E$ be an isometry. Suppose that there is $x_0 \in E$ such that $x_0 \notin T(E)$. Set $r_1 := \mathtt{dist}(x_0, T(E))$. Then $r_1 > 0$. Indeed, otherwise, take a sequence $(y_n)_n \subset E$ such that $T(y_n) \to x_0$ if $n \to \infty$. Since $T$ is isometric and $E$ is complete, we imply that $(y_n)_n$ is convergent to some $y' \in E$; but then

$$\begin{aligned} \|T(y') - x_0\| &= \|T(y') - T(y_n) + T(y_n) - x_0\| \\ &\leqslant \max\left\{\|y' - y_n\|, \|T(y_n) - x_0\|\right\} \end{aligned}$$

for every $n \in \mathbb{N}$; thus, $T(y') = x_0$, a contradiction.

Now, we prove that $\mathtt{dist}(x_0, T(E))$ is not attained, i.e.

$$\|x_0 - x\| > r_1 \text{ for every } x \in T(E). \tag{4.12}$$

Assume for a contradiction that there is $y \in E$ for which $\|x_0 - T(y)\| = r_1$. Using Lemmas 4.3.2 and 4.3.3, we find $z_1, \ldots, z_n \in E$ such that $B^{-}_{E,r_1}(T(z_i))$, $i = 1, \ldots, n$ form a finite partition of $B_{E,r_1}(x_0)$. Choose $n_1 \in \{1, \ldots, n\}$ that $x_0 \in B^{-}_{E,r_1}(T(z_{n_1}))$. But then, $\|x_0 - T(z_{n_1})\| < r_1$, a contradiction.

Select a sequence $(y_n)_n \subset E$ such that

$$\lim_{n\to\infty} \|x_0 - T(y_n)\| = r_1.$$

Assuming that $\|x_0 - T(y_n)\| > \|x_0 - T(y_{n+1})\|$ for every $n \in \mathbb{N}$, we get

$$\begin{aligned} \|y_n - y_{n+1}\| &= \|T(y_n) - T(y_{n+1})\| \\ &= \|T(y_n) - x_0 + x_0 - T(y_{n+1})\| \\ &= \|T(y_n) - x_0\|; \end{aligned} \tag{4.13}$$

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thus, $B_{E,\|y_n-y_{n+1}\|}(y_n)$ is a centered sequence. By [57, 4.A.], $E$ is spherically complete; hence, there is $y' \in \bigcap\limits_{n\in\mathbb{N}} B_{E,\|y_n-y_{n+1}\|}(y_n)$. Then, by (4.13)

$$\begin{aligned}
\|T(y')-x_0\| &= \|T(y')-T(y_n)+T(y_n)-x_0\| \\
&\leqslant \max\left\{\|T(y')-T(y_n)\|, \|T(y_n)-x_0\|\right\} \\
&= \max\left\{\|y'-y_n\|, \|T(y_n)-x_0\|\right\} \\
&\leqslant \max\left\{\|y_n-y_{n+1}\|, \|T(y_n)-x_0\|\right\} = \|T(y_n)-x_0\|
\end{aligned}$$

for every $n \in \mathbb{N}$; thus, $\|T(y')-x_0\| = r_1$. This contradicts with (4.12) and the proof is completed. □

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