Topological **M**ethods in **N**onlinear **A**nalysis
Volume 50, No. 2, 2017, 669–682
DOI: 10.12775/TMNA.2017.036

PROPERTIES OF UNIQUE POSITIVE SOLUTIONS FOR A CLASS OF NONLOCAL SEMILINEAR ELLIPTIC EQUATIONS

Ruiting Jiang — Chengbo Zhai

Abstract. We study a class of nonlocal elliptic equations

$$-M\bigg(\int_\Omega |u|^\gamma\,dx\bigg)\Delta u = \lambda f(x,u)$$

with the Dirichlet boundary conditions in bounded domain. Under suitable assumptions on M and the nonlinear term f, the existence and new properties of a unique positive solutions are obtained via a monotone operator method and a mixed monotone operator method.

1. Introduction

In this article, we consider the following nonlocal elliptic problem:

$$\begin{cases} -M\bigg(\displaystyle\int_\Omega |u|^\gamma\,dx\bigg)\Delta u = \lambda f(x,u) & \text{for } x\in\Omega, \\ u(x)=0 & \text{for } x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subseteq\mathbb{R}^N$ $(N\geq 1)$ is a smooth and bounded domain, $\gamma\in(0,+\infty)$, $\lambda>0$ is a parameter. $M\colon[0,+\infty)\to(0,+\infty)$ and $f\colon\overline{\Omega}\times(0,+\infty)\to[0,+\infty)$ are given functions whose properties will be listed later.

2010 *Mathematics Subject Classification.* 35J57, 35J66, 35D35, 47H10.
Key words and phrases. Nonlocal elliptic equation; positive solution; existence and uniqueness; monotone operator; mixed monotone operator.

The first author was supported in part by the Youth Science Foundation of China (11201272), Shanxi Province Science Foundation (2015011005) and 131 Talents Project of Shanxi Province (2015).

Over the last few decades, motivated by the richness of methods used to establish existence of solutions, the nonlinear elliptic equation (1.1) has attracted attention. For the case $M(t) \equiv 1$, (1.1) becomes

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\Omega \subseteq R^N$ $(N \geq 1)$ is a smooth and bounded domain and $\lambda > 0$. After the pioneering work of Ambrosetti and Rabinowitz [2], there has been a great deal of interest in the equation above, see e.g. [10], [13]. When $\lambda = 1$, in [2], the authors showed the well-known (AR) condition, that is, for some $\rho > 2$, $H > 0$,

$$0 < \rho F(x,t) \leq f(x,t)t, \quad \text{a.e. } x \in \overline{\Omega},\ |t| \geq H,$$

is necessary to obtain the existence of solutions via the variational approach. Their tool was the very famous mountain pass theorem.

For the case $M(t) \neq 1$, there are many results about the existence of solutions obtained by different methods. For example, in [6], [7], [8], the authors established some results about the existence of positive solutions of (1.1) with $\lambda = 1$, the techniques used are the following: the fixed point index theory, sub- and supersolution method and comparison principle.

Another nonlocal elliptic equations are Kirchhoff elliptic problems of the type

$$\begin{cases} -M(\|u\|^2)\Delta u = \lambda f(x,u) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where M is a given function and $\|\cdot\|$ denotes the usual norm in $H_0^1(\Omega)$. As $u \mapsto M(\|u\|^2)\Delta u$ has a variational structure, the existence of solutions can be obtained via variational methods. More details about this class of problems can be found in [3], [4], [5], [11], [12] and reference therein.

Very recently, by applying the change of variables, it was suggested to transform (1.1) into an ordinary differential equation. Yan and Wang [14] proved some results on the multiplicity of positive radial solutions of (1.1) by using the theory of fixed point index. To our knowledge, the existence and multiplicity of positive solutions have been studied by many authors. But there are few results reported on the uniqueness and properties of positive solutions. In our setting, we cannot use the variational method due to the diffusion coefficient M. Unlike the above mentioned works, motivated by [15] and [16], we will establish the fixed point of the operator related to (1.1) by two different methods, and then obtain the existence of a unique positive solution. That is, for any given parameter $\lambda > 0$, we obtain the existence and uniqueness of positive solutions. Moreover, we present some obvious properties of positive solutions to the boundary value problem dependent on the parameter. That is, the unique positive solution u_λ

has the following properties: u_λ is continuous, strictly increasing in λ and

$$\lim_{\lambda\to\infty}\|u_\lambda\| = \infty, \qquad \lim_{\lambda\to 0^+}\|u_\lambda\| = 0.$$

2. Preliminaries

We first list some basic notations and concepts in ordered Banach spaces.

Let $(X, \|\cdot\|)$ be a real Banach space, and X be partially ordered by a cone $K \subset X$, θ be the zero element of X. K is called normal if there is a constant $N > 0$, such that for $x, y \in X$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. For $x, y \in X$, the notation $x \sim y$ means that there are $\lambda, \mu > 0$ such that $\lambda x \le y \le \mu x$. For fixed $h > \theta$ (i.e. $h \ge \theta$ and $h \ne \theta$), we define a set $K_h = \{x \in E : x \sim h\}$. Evidently, $K_h \subseteq K$. We say that an operator $A\colon X \to X$ is increasing if $x \le y$ implies $Ax \le Ay$.

Definition 2.1 (See [9], [16]). An operator $A\colon K \times K \to K$ is said to be mixed monotone if $A(x, y)$ is increasing in x and decreasing in y, i.e. for $u_i, v_i \in K$, $i = 1, 2$, $u_1 \le u_2$, $v_1 \ge v_2$ imply $A(u_1, v_1) \le A(u_2, v_2)$. An element $x \in K$ is called a fixed point of A if $A(x, x) = x$.

Our main tools are the following results on operator equations involving monotone operators and mixed monotone operators.

Lemma 2.2 (See Theorem 2.1 in [15]). *Let K be a normal cone in a real Banach space X, $h > \theta$. Let $A\colon K \to K$ be an increasing operator satisfying*:

(a) *there is $h_0 \in K_h$ such that $Ah_0 \in K_h$;*
(b) *for any $x \in K$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1)$ such that $A(tx) \ge \varphi(t)Ax$.*

Then:

(1) *the operator equation $Ax = x$ has a unique solution x^* in K_h;*
(2) *for any initial point $x_0 \in K_h$ and the sequence $x_n = Ax_{n-1}$, $n = 1, 2, \ldots$, we get $x_n \to x^*$ as $n \to \infty$.*

Lemma 2.3 (See Theorem 2.2 in [15]). *Assume that all the conditions of Lemma 2.1 hold. Let x_λ ($\lambda > 0$) denote the unique solution of the operator equation $Ax = \lambda x$. Then we have the following conclusions*:

(a) *x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*
(b) *if there exists $\gamma \in (0, 1)$ such that $\varphi(t) \ge t^\gamma$ for $t \in (0, 1)$, then x_λ is continuous in λ, that is, $\lambda \to \lambda_0$ ($\lambda_0 > 0$) implies $\|x_\lambda - x_{\lambda_0}\| \to 0$*
(c) *$\lim_{\lambda\to\infty}\|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+}\|x_\lambda\| = \infty$.*

Lemma 2.4 (See Theorem 2.1 in [16]). *Let K be a normal cone of X, $A\colon K \times K \to K$ be a mixed monotone operator satisfying*:

(A_1) *there exists $h \in K$ with $h \ne \theta$ such that $A(h, h) \in K_h$;*

(A_2) *for any* $u,v \in K$ *and* $t \in (0,1)$, *there exists* $\varphi(t) \in (t,1)$ *such that* $A(tu,t^{-}v) \geq \varphi(t)A(u,v)$.

Then the operator equation $A(x,x)=x$ *has a unique solution* x^* *in* K_h. *Moreover, for any points* $x_0,y_0 \in K_h$ *and the sequences*

$$x_n = A(x_{n-1},y_{n-1}), \quad y_n = A(y_{n-1},x_{n-1}), \quad n=1,2,\ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

LEMMA 2.5 (See Theorem 2.3 in [16]). *Assume that the operator* A *satisfies the conditions of Lemma* 2.4. *Let* x_λ *denote the unique solution of the nonlinear eigenvalue equation* $A(x,x)=\lambda x$ *in* K_h. *Then we have the following conclusions*:

(R_1) *if* $\varphi(t) > t^{1/2}$ *for* $t \in (0,1)$, *then* x_λ *is strictly decreasing in* λ, *that is,* $0<\lambda_1<\lambda_2$ *implies* $x_{\lambda_1} > x_{\lambda_2}$;

(R_2) *if there exists* $\beta \in (0,1)$ *such that* $\varphi(t) \geq t^\beta$ *for* $t \in (0,1)$, *then* x_λ *is continuous in* λ, *that is,* $\lambda \to \lambda_0$ $(\lambda_0>0)$ *implies* $\|x_\lambda - x_{\lambda_0}\| \to 0$;

(R_3) *if there exists* $\beta \in (0,1/2)$ *such that* $\varphi(t) \geq t^\beta$ *for* $t \in (0,1)$, *then*

$$\lim_{\lambda\to\infty} \|x_\lambda\| = 0, \qquad \lim_{\lambda\to 0^+} \|x_\lambda\| = \infty.$$

3. A monotone operator method for equation (1.1)

Let $C(\overline{\Omega}) = \{u : \overline{\Omega} \to R | u(x) \text{ is continuous on } \overline{\Omega}\}$ with norm $\|u\| = \max_{x\in\overline{\Omega}} |u(x)|$, it is easy to see that $C(\overline{\Omega})$ is a Banach space. Let $K = \{u \in C(\overline{\Omega}) : u(x) \geq 0$ for all $x \in \overline{\Omega}\}$. Obviously, K is a normal cone in $C(\overline{\Omega})$. For simplicity and convenience, we list some conditions on M, which are necessary to verify our main results.

(M_1) $M \in C([0,+\infty),(0,+\infty))$ with $\inf_{t\geq 0} M(t) = M_1 > 0$.

(M_2) $M(t)$ is strictly decreasing in t and there exists $\varphi_1\colon (0,1) \to (1,+\infty)$ such that $M(\tau t) \leq \varphi_1(\tau)M(t)$, $\tau \in (0,1)$.

(M_1') $M \in C([0,+\infty),(0,+\infty))$ and $\sup_{t\geq 0} M(t) = M_2$ exists.

(M_2') $M(t)$ is strictly increasing in t and there exists $\varphi_1'\colon (0,1) \to (0,1)$ such that $M(\tau t) \geq \varphi_1'(\tau)M(t)$, $\tau \in (0,1)$.

In this part, in order to get the solution of problem (1.1), for $u \in K$, we define

$$(Tu)(x) = \frac{1}{M\left(\displaystyle\int_\Omega |u|^\gamma\,dx\right)} \int_\Omega G(x,y)f(y,u(y))\,dy, \quad x \in \overline{\Omega},$$

where $G(x,y)$ is the Green function for $-\Delta u = h$ in Ω. Let $g(x) = \int_\Omega G(x,y)\,dy$, $x \in \overline{\Omega}$. It is easy to see that $g > \theta$. Let $K_g = \{u \in C(\overline{\Omega}) : u \sim g\}$. Obviously, $K_g \subseteq K$. Our main results are summarized in the following

THEOREM 3.1. *Suppose that* (M_1)–(M_2) *hold. Assume that* $f(x,u) \in C(\overline{\Omega} \times [0,+\infty), [0,+\infty))$ *and it satisfies*:

- (f_1) $f(x,u)$ *is increasing in* u, *and* $a = \min\{f(x,u) : x \in \overline{\Omega},\ u \in [0,b]\} > 0$, *where* $b = \max_{x\in\overline{\Omega}} \int_\Omega G(x,y)\,dy$;
- (f_2) *for* $x \in \overline{\Omega}$, $t \in (0,1)$, *there exists* $\varphi_2\colon (0,1) \to (0,1)$ *satisfying* $t\varphi_1(t^\gamma) < \varphi_2(t)$ *such that* $f(x,tu) \geq \varphi_2(t) f(x,u)$.

Then:

- (a) *for any given* $\lambda > 0$, *equation* (1.1) *has a unique positive solution* u_λ^* *in* K_g;
- (b) *for any given* $\lambda > 0$ *and any initial point* $u_0 \in K_g$ *and the sequence*

$$u_n(x) = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_{n-1}|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y, u_{n-1}(y))\,dy,$$

  $n = 1, 2, \ldots$, $x \in \overline{\Omega}$, *we have* $u_n \to u_\lambda^*$ *as* $n \to \infty$;
- (c) u_λ^* *is strictly increasing in* λ, *that is*, $u_{\lambda_1}^* < u_{\lambda_2}^*$ *for* $0 < \lambda_1 < \lambda_2$;
- (d) $\lim_{\lambda\to\infty} \|u_\lambda^*\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = 0$.

PROOF. Obviously, $T\colon K \to K$. By (M_2) and (f_1), one obtains that T is increasing. Let

$$g_0(x) = g(x) = \int_\Omega G(x,y)\,dy,$$

then g_0 is continuous on $\overline{\Omega}$ due to the continuity of $G(x,y)$. On the one hand, for

$$\int_\Omega |g_0|^\gamma\,dx \geq 0,$$

by (f_1), we have

$$(Tg_0)(x) = \frac{1}{M\bigg(\displaystyle\int_\Omega |g_0|^\gamma dx\bigg)} \int_\Omega G(x,y) f(y, g_0(y))\,dy \geq \frac{a}{M(0)} \int_\Omega G(x,y)\,dy.$$

On the other hand, it is easy to get $f(x, g_0(x)) \leq f(x,b) \leq c := \max_{x\in\overline{\Omega}} f(x,b) > 0$ by (f_1) and the boundedness of $\overline{\Omega}$. Then one gets

$$(Tg_0)(x) = \frac{1}{M\bigg(\displaystyle\int_\Omega |g_0|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y, g_0(y))\,dy \leq \frac{c}{M_1} \int_\Omega G(x,y)\,dy.$$

Therefore, $Tg_0 \subset K_g$. For any $u \in K$ and $t \in (0,1)$, combining with (M_2) and (f_2), we have

$$\begin{aligned} T(tu)(x) &= \frac{1}{M\bigg(\displaystyle\int_\Omega |tu|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,tu(y))\,dy \\ &= \frac{1}{M\bigg(t^\gamma \displaystyle\int_\Omega |u|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,tu(y))\,dy \\ &\geq \frac{\varphi_2(t)}{\varphi_1(t^\gamma)} \frac{1}{M\bigg(\displaystyle\int_\Omega |u|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,u(y))\,dy = \frac{\varphi_2(t)}{\varphi_1(t^\gamma)}\,(Tu)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. Let $\varphi(t) = \varphi_2(t)/\varphi_1(t^\gamma)$. From (f_2), $t < \varphi(t) < 1$, that is, $\varphi(t) \in (t,1)$. Then we obtain that T satisfies the conditions of Lemma 2.2. Therefore, by Lemma 2.3, there exists a unique $u_\lambda^* \in K_g$ such that $Tu_\lambda^* = u_\lambda^*/\lambda$. That is, $\lambda T u_\lambda^* = u_\lambda^*$. Obviously, u_λ^* is a unique positive solution of equation (1.1) for given $\lambda > 0$. Further, by Lemma 2.3 (a), it is easy to check that u_λ^* is strictly increasing in λ, that is, $u_{\lambda_1}^* < u_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$; by Lemma 2.3 (b), one has $\lim\limits_{\lambda\to\infty} \|u_\lambda^*\| = \infty$, $\lim\limits_{\lambda\to 0^+} \|u_\lambda^*\| = 0$.

Obviously, λT also satisfies all the conditions of Lemma 2.2. By Lemma 2.2, for any initial point $u_0 \in K_g$ and the sequence $u_n = Tu_{n-1}$, $n = 1,2,\ldots$, we have $u_n \to u_\lambda^*$ as $n \to \infty$. That is,

$$u_n(x) = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_{n-1}|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,u_{n-1}(y))\,dy \to u_\lambda^*(x), \quad x \in \overline{\Omega},$$

as $n \to \infty$. □

Theorem 3.2. *Suppose that* (M_1) *and* (M_2) *hold. Assume that f satisfies* (f_1) *and*

- (f_2') *for $x \in \overline{\Omega}$, $t \in (0,1)$, there exist $\varphi_2\colon (0,1) \to (0,1)$ and $\beta \in (0,1)$ satisfying $t^\beta \varphi_1(t^\gamma) \leq \varphi_2(t)$ such that $f(x,tu) \geq \varphi_2(t) f(x,u)$.*

Then:

- (a) *for any given $\lambda > 0$, equation* (1.1) *has a unique positive solution u_λ^{**}. Further, for any given $\lambda > 0$ and any initial point $u_0 \in K_g$ and the sequence*
$$u_n(x) = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_{n-1}|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,u_{n-1}(y))\,dy,$$
$n = 1,2,\ldots$, $x \in \overline{\Omega}$, we have $u_n \to u_\lambda^{**}$ as $n \to \infty$;
- (b) *u_λ^{**} is strictly increasing in λ, that is, $u_{\lambda_1}^{**} < u_{\lambda_2}^{**}$ for $0 < \lambda_1 < \lambda_2$;*
- (c) *u_λ^{**} is continuous in λ, that is, $\|u_\lambda^{**} - u_{\lambda_0}^{**}\| \to 0$ as $\lambda \to \lambda_0$ ($\lambda_0 > 0$);*

(d) $\lim_{\lambda\to\infty} \|u_\lambda^{**}\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda^{**}\| = 0$.

PROOF. For any $u \in K$ and $t \in (0,1)$, combining with (M_2) and (f_2'), we have

$$\begin{aligned} T(tu)(x) &= \frac{1}{M\Big(\displaystyle\int_\Omega |tu|^\gamma\,dx\Big)} \int_\Omega G(x,y) f(y, tu(y))\,dy \\ &\geq \frac{\varphi_2(t)}{\varphi_1(t^\gamma)} \frac{1}{M\Big(\displaystyle\int_\Omega |u|^\gamma\,dx\Big)} \int_\Omega G(x,y) f(y,u(y))\,dy = \frac{\varphi_2(t)}{\varphi_1(t^\gamma)}\,(Tu)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. Let $\varphi(t) = \varphi_2(t)/\varphi_1(t^\gamma)$. As $t \in (0,1)$ and $\beta \in (0,1)$, $t^\beta \leq \varphi(t) < 1$, that is $\varphi(t) \in (t,1)$. Similarly to the proof of Theorem 3.1, T satisfies the conditions of Lemma 2.2, then we obtain a unique positive solution u_λ^{**} of equation (1.1) for given $\lambda > 0$ and (a), (b), (d). By (f_2') and Lemma 2.3 (b), one has that u_λ^{**} is continuous in λ. □

REMARK 3.3. Functions which satisfy the conditions of Theorem 3.2 can be found easily. For example, let $M(x) = 1/(1+x^2)+1$ $(x \geq 0)$, $f(x,u) = xu^{\gamma/4}+2$, where $\gamma \in (0,1/3)$. Obviously, $M \in C([0,+\infty),(0,+\infty))$ is strictly decreasing, and $\inf_{x\geq 0} M(x) = 1$. Take $\varphi_1(t) = 1/t^2$, $\varphi_2(t) = t^{\gamma/3}$, $\beta = 3\gamma$. Then, for $\tau \in (0,1)$,

$$M(\tau x) = \frac{1}{1+\tau^2x^2} + 1 \leq \frac{1}{\tau^2}\bigg(\frac{1}{1+x^2} + 1\bigg) = \varphi_1(\tau) M(x),$$

that is, M satisfies (M_1) and (M_2). It is easy to check that $f(x,u) = xu^{\gamma/4} + 2$ is increasing in u with $\min_{x\in\overline{\Omega},\, u\geq 0} f(x,u) = 2$. For $t \in (0,1)$ and $\beta = 3\gamma \in (0,1)$, we have $t^\beta \varphi_1(t^\gamma) = t^\gamma \leq t^{\gamma/3} = \varphi_2(t)$ and

$$f(x,tu) = x(tu)^{\gamma/4} + 2 \geq t^{\gamma/3}(xu^{\gamma/4}+2) = \varphi_2(t) f(x,u),$$

that is, f satisfies (f_1) and (f_2'). Therefore, M and f satisfy the conditions of Theorem 3.2.

4. A mixed monotone operator method for equation (1.1)

In this section, under suitable assumptions on M and f, we will define two different mixed monotone operators and use a mixed monotone operator method to establish the existence of unique positive solutions. First, for $u, v \in K$, we define

$$A(u,v)(x) = \frac{1}{M\Big(\displaystyle\int_\Omega |u|^\gamma\,dx\Big)} \int_\Omega G(x,y) f(y, v(y))\,dy, \quad x \in \overline{\Omega},$$

where $G(x,y)$ is the Green function for $-\Delta u = h$ in Ω. The functions g and K_g are defined in Section 3. In order to get our main results, we list some conditions on f:

- (f_3) $f(x,u)$ is decreasing in u and $a = \min\{f(x,u) : x \in \overline{\Omega},\ u \in [0,b]\} > 0$, where $b = \max_{x\in\overline{\Omega}} \int_\Omega G(x,y)\,dy$;
- (f_4) for $x \in \overline{\Omega}$, $t \in (0,1)$, there exist $\varphi_2\colon (0,1) \to (0,1)$ and $\beta > 0$ satisfying $t^\beta \varphi_1(t^\gamma) \le \varphi_2(t)$ such that $f(x,tu) \le f(x,u)/\varphi_2(t)$.

Theorem 4.1. *Assume that* (M_1)–(M_2) *and* (f_3)–(f_4) *hold. If* $\beta \in (0,1)$, *then*:

- (a) *for any given* $\lambda > 0$, *equation* (1.1) *has a unique positive solution* u_λ *in* K_g, *where* $g = \int_\Omega G(x,y)\,dy$, $x \in \overline{\Omega}$. *Moreover, for any initial points* $u_0, v_0 \in K_g$ *and two sequences*
$$u_{n+1} = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |v_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,u_n(y))\,dy,$$
$$v_{n+1} = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_n|^\gamma dx\bigg)} \int_\Omega G(x,y) f(y,v_n(y))\,dy,$$
$n = 0,1,\ldots$, *we have* $\|u_n - u_\lambda\| \to 0$, $\|v_n - u_\lambda\| \to 0$ *as* $n \to \infty$.
- (b) u_λ *is continuous in* λ, *that is*, $\|u_\lambda - u_{\lambda_0}\| \to 0$ *as* $\lambda \to \lambda_0$ ($\lambda_0 > 0$).

Proof. Obviously, $A\colon K \times K \to K$. Let us check that A satisfies all assumptions of Lemma 2.4. Firstly, we prove that A is a mixed monotone operator. Indeed, for $u_i, v_i \in K$, $i = 1,2$, with $u_1 \ge u_2$, $v_1 \le v_2$, we have $u_1(x) \ge u_2(x)$ and $v_1(x) \le v_2(x)$, $x \in \overline{\Omega}$. By (M_2) and (f_3), one obtains

$$\begin{aligned} A(u_1,v_1)(x) =& \frac{1}{M\bigg(\displaystyle\int_\Omega |u_1|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,tv_1(y))\,dy \\ \ge& \frac{1}{M\bigg(\displaystyle\int_\Omega |u_2|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,tv_2(y))\,dy = A(u_2,v_2)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. That is, $A(u_1,v_1) \ge A(u_2,v_2)$. Let $g_0(x) = g(x) = \int_\Omega G(x,y)\,dy$, then g_0 is continuous on $\overline{\Omega}$ due to continuity of $G(x,y)$. On the one hand, for $\int_\Omega |g_0|^\gamma\,dx \ge 0$, by ($f_3$), we have

$$A(g_0,g_0)(x) = \frac{1}{M\bigg(\displaystyle\int_\Omega |g_0|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y,g_0(y))\,dy \ge \frac{a}{M(0)} \int_\Omega G(x,y)\,dy.$$

On the other hand, it is easy to get that $f(x, g_0(x)) \le f(x, c) \le d := \max_{x\in\overline{\Omega}} f(x, c) > 0$ by (f_3) and the boundedness of $\overline{\Omega}$, where $c = \min_{x\in\overline{\Omega}} \int_\Omega G(x, y)\, dy$. Then one gets

$$A(g_0, g_0)(x) = \frac{1}{M\left(\displaystyle\int_\Omega |g_0|^\gamma\, dx\right)} \int_\Omega G(x, y) f(y, g_0(y))\, dy \le \frac{c}{M_1} \int_\Omega G(x, y)\, dy.$$

Therefore, $A(g_0, g_0) \subset K_g$. For any $u, v \in K$ and $t \in (0, 1)$, by (M_2) and (f_4), one has

$$\begin{aligned} M\left(\int_\Omega |tu|^\gamma\, dx\right) &\le \varphi_1(t^\gamma) M\left(\int_\Omega |u|^\gamma\, dx\right), \\ f(y, t^{-1}v(y)) &\ge \varphi_2(t) f(y, v(y)). \end{aligned}$$

Then we obtain that

$$\begin{aligned} A(tu, t^{-1}v)(x) &= \frac{1}{M\left(\displaystyle\int_\Omega |tu|^\gamma\, dx\right)} \int_\Omega G(x, y) f(y, t^{-1}v(y))\, dy \\ &\ge \frac{\varphi_2(t)}{\varphi_1(t^\gamma)} \frac{1}{M\left(\displaystyle\int_\Omega |u|^\gamma\, dx\right)} \int_\Omega G(x, y) f(y, v(y))\, dy \\ &= \frac{\varphi_2(t)}{\varphi_1(t^\gamma)} A(u, v)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. Let $\varphi(t) = \varphi_2(t)/\varphi_1(t^\gamma)$, then by ($f_4$), one has $t < t^\beta \le \varphi(t) < 1$ for $t \in (0, 1)$ and $\beta \in (0, 1)$, that is, $\varphi(t) \in (t, 1)$. Thus A satisfies the conditions of Lemma 2.4. Therefore, by Lemma 2.5, there exists a unique $u_\lambda \in K_g$ such that $A(u_\lambda, u_\lambda) = u_\lambda/\lambda$. That is, $\lambda A(u_\lambda, u_\lambda) = u_\lambda$. Obviously, u_λ is a unique positive solution of equation (1.1) for given $\lambda > 0$.

Obviously, λA also satisfies all the conditions of Lemma 2.4. By Lemma 2.4, for any initial points $u_0, v_0 \in K_g$ and two sequences $u_{n+1} = \lambda A(u_n, v_n)$, $v_{n+1} = \lambda A(v_n, u_n)$, $n = 0, 1, 2, \ldots$, we have $u_n \to u_\lambda$, and $v_n \to u_\lambda$ as $n \to \infty$. That is,

$$\begin{aligned} u_{n+1}(x) &= \frac{\lambda}{M\left(\displaystyle\int_\Omega |v_n|^\gamma\, dx\right)} \int_\Omega G(x, y) f(y, u_n(y))\, dy \to u_\lambda(x), \quad x \in \overline{\Omega}, \\ v_{n+1}(x) &= \frac{\lambda}{M\left(\displaystyle\int_\Omega |u_n|^\gamma\, dx\right)} \int_\Omega G(x, y) f(y, v_n(y)) dy \to u_\lambda(x), \quad x \in \overline{\Omega}, \end{aligned}$$

as $n \to \infty$. Further, combining $\varphi(t) = \varphi_2(t)/\varphi_1(t^\gamma)$ and (f_4), one has $\varphi(t) \ge t^\beta$. Then combining with (R_2) in Lemma 2.5, we get that u_λ is continuous in λ. □

THEOREM 4.2. *Assume that* (M_1)–(M_2) *and* (f_3)–(f_4) *hold. If* $\beta \in (0, 1/2)$, *then*:

(a) *for any given* $\lambda > 0$, *equation* (1.1) *has a unique positive solution* u_λ *in* K_g, *where* $g = \int_\Omega G(x,y)\,dy$, $x \in \overline{\Omega}$. *Moreover, for any initial points* $u_0, v_0 \in P_g$ *and two sequences*

$$u_{n+1} = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |v_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y, u_n(y))\,dy,$$

$$v_{n+1} = \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y, v_n(y))\,dy,$$

$n = 0, 1, \ldots,$ *we have* $\|u_n - u_\lambda\| \to 0$, $\|v_n - u_\lambda\| \to 0$ *as* $n \to \infty$;

(b) u_λ *is continuous in* λ, *that is*, $\|u_\lambda - u_{\lambda_0}\| \to 0$ *as* $\lambda \to \lambda_0$ ($\lambda_0 > 0$);

(c) $\lim\limits_{\lambda\to\infty} \|u_\lambda\| = \infty$, $\lim\limits_{\lambda\to 0^+} \|u_\lambda\| = 0$;

(d) u_λ *is strictly increasing in* λ, *that is*, $u_{\lambda_1} < u_{\lambda_2}$ *for* $0 < \lambda_1 < \lambda_2$.

PROOF. Conclusions (a) and (b) are a consequence of Theorem 4.1. The function $\varphi(t) = \varphi_2(t)/\varphi_1(t^\gamma)$ is defined in the proof of Theorem 4.1, from (f_3), one has $\varphi(t) \ge t^\beta$. Since $\beta \in (0, 1/2)$, we have $\varphi(t) \ge t^\beta > t^{1/2}$, by ($R_1$) in Lemma 2.5, u_λ is strictly increasing in λ. By (R_3) in Lemma 2.5, it is easy to check that $\lim\limits_{\lambda\to\infty} \|u_\lambda\| = \infty$, $\lim\limits_{\lambda\to 0^+} \|u_\lambda\| = 0$. □

In what follows, in order to get the unique positive solution of problem (1.1), for $u, v \in K$, we define one more operator

$$B(u,v)(x) = \frac{1}{M\bigg(\displaystyle\int_\Omega |v|^\gamma\,dx\bigg)} \int_\Omega G(x,y) f(y, u(y))\,dy, \quad x \in \overline{\Omega},$$

where $G(x,y)$ is the Green function for $-\Delta u = h$ in Ω. The functions K and K_g are defined in Section 3. In order to get our main results, we list some conditions on f:

(f_5) $f(x,u)$ is decreasing in u and $a = \min\{f(x,u) : x \in \overline{\Omega},\ u \in [0,b]\} > 0$, where $b = \max\limits_{x\in\overline{\Omega}} \int_\Omega G(x,y)\,dy$;

(f_6) for all $x \in \overline{\Omega}$, $t \in (0,1)$ and $\beta > 0$, there exists $\varphi_2\colon (0,1) \to (0,1)$ satisfying $\varphi_1'(t^\gamma)\varphi_2(t) \ge t^\beta$ such that $f(x,tu) \le f(x,u)/\varphi_2(t)$.

THEOREM 4.3. *Assume that* (M_1')–(M_2') *and* (f_5)–(f_6) *hold. If* $\beta \in (0,1)$, *then*:

(a) *for any given* $\lambda > 0$, *equation* (1.1) *has a unique positive solution* u_λ' *in* K_g, *where* $g = \int_\Omega G(x,y)\,dy$, $x \in \overline{\Omega}$. *Moreover, for any initial points*

$u_0, v_0 \in K_g$ and two sequences

$$u_{n+1} = \frac{\lambda}{M\left(\displaystyle\int_\Omega |v_n|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, u_n(y))\,dy,$$

$$v_{n+1} = \frac{\lambda}{M\left(\displaystyle\int_\Omega |u_n|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, v_n(y))\,dy,$$

$n = 0, 1, \dots$, we have $\|u_n - u'_\lambda\| \to 0$, $\|v_n - u'_\lambda\| \to 0$ as $n \to \infty$.

(b) *u'_λ is continuous in λ, that is, $\|u'_\lambda - u'_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0$ ($\lambda_0 > 0$).*

Proof. Obviously, $B\colon K \times K \to K$. Let us check that B satisfies all assumptions of Lemma 2.4. Firstly, we prove that B is a mixed monotone operator. Indeed, for $u_i, v_i \in K$, $i = 1, 2$, with $u_1 \geq u_2$, $v_1 \leq v_2$, we have $u_1(x) \geq u_2(x)$ and $v_1(x) \leq v_2(x)$, $x \in \overline{\Omega}$. By (M'_2) and (f_5), one obtains

$$\begin{aligned} B(u_1, v_1)(x) &= \frac{1}{M\left(\displaystyle\int_\Omega |v_1|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, tu_1(y))\,dy \\ &\geq \frac{1}{M\left(\displaystyle\int_\Omega |v_2|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, tu_2(y))\,dy = B(u_2, v_2)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. That is, $B(u_1, v_1) \geq B(u_2, v_2)$. Let

$$g_0(x) = g(x) = \int_\Omega G(x,y)\,dy,$$

then g_0 is continuous on $\overline{\Omega}$ due to continuity of $G(x,y)$. On the one hand, by (f_5), we have

$$B(g_0, g_0)(x) = \frac{1}{M\left(\displaystyle\int_\Omega |g_0|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, g_0(y))\,dy \geq \frac{a}{M_2} \int_\Omega G(x,y)\,dy.$$

On the other hand, it is easy to get $f(x, g_0(x)) \leq f(x, b) \leq c := \max_{x \in \overline{\Omega}} f(x, b) > 0$ by (f_5) and the boundedness of $\overline{\Omega}$. Then one gets

$$B(g_0, g_0)(x) = \frac{1}{M\left(\displaystyle\int_\Omega |g_0|^\gamma\,dx\right)} \int_\Omega G(x,y) f(y, g_0(y))\,dy \leq \frac{c}{M_1} \int_\Omega G(x,y)\,dy.$$

Therefore, $B(g_0, g_0) \subset K_g$. For any $u, v \in P$ and $t \in (0,1)$, by (M'_2) and (f_6), one has

$$M\left(\int_\Omega |t^{-1} v|^\gamma\,dx\right) \leq \frac{1}{\varphi'_1(t^\gamma)} M\left(\int_\Omega |v|^\gamma\,dx\right),$$
$$f(y, tu(y)) \geq \varphi_2(t) f(y, u(y)).$$

Then we obtain that

$$\begin{aligned} B(tu,t^{-1}v)(x) &= \frac{1}{M\bigg(\displaystyle\int_\Omega |t^{-1}v|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,tu(y))\,dy \\ &\geq \varphi_1'(t^\gamma)\varphi_2(t)\,\frac{1}{M\bigg(\displaystyle\int_\Omega |v|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,u(y))\,dy \\ &= \varphi_1'(t^\gamma)\varphi_2(t)B(u,v)(x), \end{aligned}$$

for $x \in \overline{\Omega}$. Let $\varphi(t) = \varphi_1'(t^\gamma)\varphi_2(t)$, then by ($\mathrm{f}_6$), one has $t < t^\beta \leq \varphi(t) < 1$ for $t \in (0,1)$ and $\beta \in (0,1)$, that is, $\varphi(t) \in (t,1)$. Then we obtain that B satisfies the conditions of Lemma 2.4. Therefore, by Lemma 2.5, there exists a unique $u_\lambda' \in K_g$ such that $B(u_\lambda',u_\lambda') = u_\lambda'/\lambda$. That is, $\lambda B(u_\lambda',u_\lambda') = u_\lambda'$. It is easy to check that u_λ' is a unique positive solution of equation (1.1) for given $\lambda > 0$.

Obviously, λB also satisfies all the conditions of Lemma 2.4. By Lemma 2.4, for any points $u_0, v_0 \in K_g$ and two sequences $u_{n+1} = \lambda A(u_n,v_n)$, $v_{n+1} = \lambda A(v_n,u_n)$, $n = 0,1,\ldots$, we have $u_n \to u_\lambda'$, and $v_n \to u_\lambda'$ as $n \to \infty$. That is,

$$\begin{aligned} u_{n+1}(x) &= \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,v_n(y))\,dy \to u_\lambda'(x), \quad x \in \overline{\Omega}, \\ v_{n+1}(x) &= \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |v_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,u_n(y))\,dy \to u_\lambda'(x), \quad x \in \overline{\Omega}, \end{aligned}$$

as $n \to \infty$. Further, combining $\varphi(t) = \varphi_1'(t^\gamma)\varphi_2(t)$ and (f_6), one has $\varphi(t) \geq t^\beta$. Then combining with (R_2) in Lemma 2.5, we get that u_λ' is continuous in λ. □

THEOREM 4.4. *Assume that* (M_1')–(M_2') *and* (f_5)–(f_6) *hold. If* $\beta \in (0,1/2)$, *then*:

(a) *for any given* $\lambda > 0$, *equation* (1.1) *has a unique positive solution* u_λ' *in* K_g, *where* $g = \int_\Omega G(x,y)\,dy$, $x \in \overline{\Omega}$. *Moreover, for any initial points* $u_0, v_0 \in K_g$ *and two sequences*

$$\begin{aligned} u_{n+1} &= \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |v_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,u_n(y))\,dy, \\ v_{n+1} &= \frac{\lambda}{M\bigg(\displaystyle\int_\Omega |u_n|^\gamma\,dx\bigg)} \int_\Omega G(x,y)f(y,v_n(y))\,dy, \end{aligned}$$

$n = 0,1,\ldots$, *we have* $\|u_n - u_\lambda'\| \to 0$, $\|v_n - u_\lambda'\| \to 0$ *as* $n \to \infty$.

(b) u_λ' *is continuous in* λ, *that is,* $\|u_\lambda' - u_{\lambda_0}'\| \to 0$ *as* $\lambda \to \lambda_0$ ($\lambda_0 > 0$).

(c) $\lim\limits_{\lambda\to\infty} \|u_\lambda'\| = \infty$, $\lim\limits_{\lambda\to 0^+} \|u_\lambda'\| = 0$.

(d) u_λ' *is strictly increasing in* λ, *that is,* $u_{\lambda_1}' < u_{\lambda_2}'$ *for* $0 < \lambda_1 < \lambda_2$.

Proof. Conclusions (a)–(b) are a consequence of Theorem 4.3. The function $\varphi(t) = \varphi_1'(t^\gamma)\varphi_2(t)$ is defined in the proof of Theorem 4.3, from (f_6), one has $\varphi(t) \geq t^\beta$. Since $\beta \in (0, 1/2)$, we have $\varphi(t) \geq t^\beta > t^{1/2}$, by ($R_1$) in Lemma 2.5, u'_λ is strictly increasing in λ. By (R_3) in Lemma 2.5, it is easy to check that $\lim_{\lambda\to\infty} \|u'_\lambda\| = \infty$, $\lim_{\lambda\to 0^+} \|u'_\lambda\| = 0$. $\square$

Remark 4.5. When $M(t) \equiv 1$, equation (1.1) becomes

$$\begin{cases} -\Delta u = \lambda f(x, u) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) is a smooth and bounded domain, $\gamma \in (0, +\infty)$, $\lambda > 0$ is a parameter, $f\colon \overline{\Omega} \times (0, +\infty) \to [0, +\infty)$. Assume that f satisfies some similar conditions as in the theorems above, then equation (4.1) has a unique positive solution in K_g. Similarly, we can also get some obvious properties of the unique positive solution depending on the parameter $\lambda > 0$.

Remark 4.6. To our knowledge, there are no results on nonlocal semilinear elliptic equations obtained using our methods: a monotone operator method and a mixed monotone operator method, and the relation between the unique positive solution and the parameter has no clear indication.

5. Conclusion

In this paper, we study a class of nonlocal semilinear elliptic equations with a positive parameter. For any given parameter $\lambda > 0$, we obtain the existence and uniqueness of positive solutions. Further, we showed that the unique positive solution u_λ has some obvious properties: u_λ is continuous, strictly increasing in λ and $\lim_{\lambda\to\infty} \|u_\lambda\| = \infty$, $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$. Our results are based on some recent fixed point theorems for monotone operators or mixed monotone operators in ordered Banach spaces.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] C.O. Alves, F.J.S.A Corrêa and T.F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49** (2005), 85–93.

[4] C.O. Alves and D.P. Covei, *Existence of solution for a class of nonlocal elliptic problem via sub-supersolution method*, Nonlinear Anal. Real World Appl. **23** (2015), 1–8.

[5] C.Y.Chen, Y.C. Kuo and T.F. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differential Equations **250** (2011), 1876–1908.

[6] F.J.S.A. Corrêa, *On positive solutions of nonlocal and nonvariational elliptic problems*, Nonlinear Anal. **59** (2004), 1147–1155.

[7] F.J.S.A. Corrêa and D.C.D.M. Filho, *On a class of nonlocal elliptic problems via Galerkin method*, J. Math. Anal. Appl. **310** (2005), 177–187.

[8] F.J.S.A. Corrêa, D.C.D.M. Filho, S.D.B. Menezes and J. Ferreira, *On a class of problems involving a nonlocal operator*, Appl. Math. Comput. **147** (2004), 475–489.

[9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[10] Y.Y. Lan and C.L. Tang, *Existence of solutions to a class of semilinear elliptic equations involving general subcritical growth*, Proc. Roy. Soc. Edinburgh **144** (2014), 809–818.

[11] Y.H. Li, F.Y. Li and J.P. Shi, *Existence of a posit solutions to Kirchhoff type problems without compactness conditons*, J. Differential Equations **253** (2012), 2285–2294.

[12] T.F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. **63** (2005), 1967–1977.

[13] O.H. Miyagaki and M.A.S. Souto, *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, J. Differential Equations **245** (2008), 3628–3638.

[14] B.Q. Yan and D.C. Wang, *The multiplicity of positive solutions for a class of nonlocal elliptic problem*, J. Math. Anal. Appl. **442** (2016), 72–102.

[15] C.B. Zhai and F. Wang, *Properties of positive solutions for the operator equation $Ax = \lambda x$ and applications to fractional differential equations with integral boundary conditions*, Adv. Differential Equations **2015** (2015), 1–10.

[16] C.B. Zhai, F. Wang and L.L. Zhang, *New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems*, J. Math. Anal. Appl. **382** (2011), 594–614.

Manuscript received December 7, 2016
accepted April 4, 2017

Chengbo Zhai and Ruiting Jiang
School of Mathematical Sciences
Shanxi University
Taiyuan, Shanxi 030006, P.R. CHINA
E-mail address: cbzhai@sxu.edu.cn.