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TRAJECTORY ATTRACTOR AND GLOBAL ATTRACTOR FOR KELLER–SEGEL–STOKES MODEL WITH ARBITRARY POROUS MEDIUM DIFFUSION

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Abstract. We investigate long-time behavior of weak solutions for the Keller–Segel–Stokes model with arbitrary porous medium diffusion in 2D bounded domains. We first prove the existence of the trajectory attractor $\mathcal{A}^{\rm tr}$ for the translation semigroup in the trajectory space. Further, we construct the global attractor $\mathcal{A}$ in a generalized sense. The results are shown by the definition of trajectory attractor and global attractor, and energy estimates.

1. Introduction

When bacteria of the species Bacillus subtilis are suspended in water, it can be observed experimentally that spatial patterns may spontaneously emerge from initially almost homogeneous distributions of bacteria [10]. A mathematical model for such processes was proposed in [21], where it is assumed that the essentially responsible mechanisms are a chemotactic movement of bacteria towards oxygen which they consume, a gravitational effect on the motion of the fluid by the heavier bacteria, and a convective transport of both cells and oxygen

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through the water (cf. also [10], [15]). This leads to a PDE model of the form

$$
\begin{cases}
n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\
c_t + u \cdot \nabla c = \Delta c - nf(c), \\
u_t + \kappa(u \cdot \nabla)u + \nabla p - \eta\Delta u + n\nabla\phi = 0, \\
\nabla \cdot u = 0,
\end{cases}
\tag{1.1}
$$

where n and c represent the bacterial density and oxygen concentration, respectively; u, p, η and $\nabla\phi$ denote the velocity field, pressure, viscosity and gravitation force of the fluid, respectively. The function χ is the chemotactic sensitivity, f measures the consumption rate of the oxygen by the bacteria. The fixed number $\kappa \in \mathbb{R}$. When the fixed number κ in (1.1) is nonzero, the fluid motion is governed by the full Navier–Stokes equations involving nonlinear convection, whereas if $\kappa = 0$ we consider the simplified Stokes evolution for u which appears to be justified if the fluid flow remains small [15]. For more details we refer to [10], [15], [21], etc.

Recently, there were some results about well-posedness of solutions for the chemotaxis-(Navier–)Stokes system (1.1). More precisely, Lorz [15] constructed certain local-in-time weak solutions of the boundary value problem for (1.1) in the three-dimensional setting under the assumptions that $\chi \equiv \text{const}$ and f be nondecreasing such that $f(0) = 0$. Duan, Lorz and Markowich [11] studied the Cauchy problem for (1.1) on the basis of a priori estimates involving energy type functionals. It is asserted there that when $\Omega = \mathbb{R}^2$, appropriate smallness assumptions on either $\nabla\phi$ or the initial data for c ensure global existence of weak solutions to the chemotaxis-Stokes system (1.1) with $\kappa = 0$, provided that some further technical structural conditions on κ and f are satisfied. Liu and Lorz [14] improved a priori estimation of [11], which allows for the construction of global weak solutions to the Navier–Stokes version of (1.1) with $\kappa = -1$ and arbitrarily large initial data in $\Omega = \mathbb{R}^2$, under basically the same assumptions on χ and f as made in [11]. Winkler [26] obtained global large-data solutions of the initial boundary value problem for the two- and three-dimensional chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops. We also note that Di Francesco, Lorz and Markowich [9] extended system (1.1) to the one with a porous medium-type diffusion of bacteria, which is represented in the following form:

$$
\begin{cases}
n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n\chi(c)\nabla c), \\
c_t + u \cdot \nabla c = \Delta c - nf(c), \\
u_t + \nabla p - \eta\Delta u + n\nabla\phi = 0, \\
\nabla \cdot u = 0,
\end{cases}
\tag{1.2}
$$

here m is the adiabatic exponent, and discussed global existence and asymptotic behavior of weak solutions to the chemotaxis-fluid coupled model for swimming bacteria with some nonlinear diffusion. Tao and Winkler [20] obtained global existence and boundedness of weak solutions for the two-dimensional Keller–Segel–Stokes model with arbitrary porous medium diffusion. In this paper, we are going to investigate trajectory attractor and global attractor for the two-dimensional Keller–Segel–Stokes model (1.2) with arbitrary porous medium diffusion.

Since we study system (1.2) in 2D bounded smooth domain $\Omega \subseteq \mathbb{R}^2$, we append equation (1.2) with the following initial data:

$$n(0,x) = n_0(x) \geq 0, \quad c(0,x) = c_0(x) \geq 0, \quad u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

and boundary conditions:

$$\partial_\nu n^m(t,x) = \partial_\nu c(t,x) = 0 \quad \text{and} \quad u(t,x) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.4}$$

where $m > 1$ is a constant, and ν is the outward normal unit vector to $\partial\Omega$. The main purpose is concerned with investigating the long time behaviors to the initial boundary value problem (1.2)–(1.4). To this end, the following assumption is necessary:

$$\begin{cases} n_0 \in L^\infty(\Omega) \text{ and } c_0 \in W^{1,\infty}(\Omega) \text{ are nonnegative,} \\ u_0 \in D(A^\theta) \text{ for some } \theta > 1/2, \text{ and} \\ \|n_0\|_{L^\infty(\Omega)} \leq K,\ \|c_0\|_{W^{1,\infty}(\Omega)} \leq K,\ \|A^\theta u_0\|_{L^2(\Omega)} \leq K \\ \hfill \text{for some } K > 0, \end{cases} \tag{1.5}$$

where A^θ represents the—possibly fractional—power of the usual Stokes operator A in the Hilbert space $L^2_\sigma(\Omega) := \{u \in L^2(\Omega) \mid \nabla\cdot u = 0 \text{ in } \mathcal{D}'(\Omega)\}$ of all solunoidal vector fields over Ω, with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ (see [19]). In addition, we also suppose

$$\begin{cases} \chi \in \mathcal{C}^1([0,+\infty)) \text{ is nonnegative,} \\ f \in \mathcal{C}^1([0,+\infty)) \text{ satisfies } f(0) = 0 \text{ and } f(c) > 0 \text{ for all } c > 0, \\ \phi \in W^{1,\infty}(\Omega). \end{cases} \tag{1.6}$$

Under assumptions (1.5) and (1.6), Tao and Winkler [20] proved that the initial boundary value problem (1.2)–(1.4) possesses at least one global weak solution (n,c,u). Since the global weak solution of (1.2)–(1.4) is not unique, here we only study the trajectory attractor and the global attractor of the initial boundary value problem (1.2)–(1.4). In fact, some notions were introduced to overcome the difficulties associated to possible non-uniqueness of solutions in the study of dynamical systems generated from partial differential equations. We can refer to [2]–[7], [12], [16], [17], [22], [25] and the references therein. In this paper, we borrow the notations and arguments of [6], [7], [22] to study the asymptotic behavior of weak solutions for the Keller–Segel–Stokes model (1.2). That is, we will prove that the initial boundary value problem (1.2)–(1.4) possesses a trajectory attractor. Further, we verify that the initial boundary value problem (1.2)–(1.4) also has a global attractor.

Now, we state the results of this paper in the following theorems.

Theorem 1.1. *Assume* (1.5)–(1.6) *hold and* $m > 1$, *then the translation semigroup* $\{S(t)\}_{t\geq 0}$ *defined by* (3.1) *possesses a trajectory attractor* $\mathcal{A}^{\mathrm{tr}} \subseteq \mathcal{T}^+$ *with respect to the topology* Θ_+^{loc}, *which satisfies*:

(a) $\mathcal{A}^{\mathrm{tr}}$ *is bounded in* $\mathfrak{F}_+^b$*-norm and compact in* Θ_+^{loc};
(b) $\mathcal{A}^{\mathrm{tr}}$ *is strictly invariant:* $S(t)\mathcal{A}^{\mathrm{tr}} = \mathcal{A}^{\mathrm{tr}}$ *for any* $t \geq 0$;
(c) $\mathcal{A}^{\mathrm{tr}}$ *is an attracting set in the topology* Θ_+^{loc}, *i.e. for any bounded set* $\mathcal{B} \subseteq \mathcal{T}^+$ *and any neighborhood* $\mathcal{O}(\mathcal{A}^{\mathrm{tr}})$ *in* Θ_+^{loc}, *there exists* $t^* = t^*(\mathcal{B}, \mathcal{O}) \geq 0$ *such that* $S(t)\mathcal{B} \subseteq \mathcal{O}(\mathcal{A}^{\mathrm{tr}})$ *for all* $t \geq t^*$.

Theorem 1.2. *Assume* (1.5)–(1.6) *hold and* $m > 1$, *then system* (1.2)–(1.4) *possesses a global attractor* $\mathcal{A} = \mathcal{A}^{\mathrm{tr}}(0) \subseteq \mathcal{N}_+$ *in the following sense*:

(a) $\mathcal{A}$ *is bounded in* $\mathcal{N}_+$ *and compact in the topology* $\mathcal{M}_+$;
(b) *for any bounded* (*in* $\mathfrak{F}_+^b$*-norm*) *set* $\mathcal{B} \subseteq \mathcal{T}^+$, $\lim\limits_{t\to+\infty} \mathrm{dist}_{\mathcal{M}_+}(S(t)\mathcal{B}, \mathcal{A}) = 0$;
(c) $\mathcal{A}$ *is the minimal set* (*for the inclusion relation*) *among those satisfying* (a) *and* (b).

Remark 1.3. According to [20], there exists at least one weak solution, while the uniqueness of the solution cannot be obtained under condition (1.5). That is why we only study the trajectory attractor and global attractor instead of the classical global attractor here.

Remark 1.4. The global attractor obtained in Theorem 1.2 is strictly invariant under the acting of the translation semigroup $\{S(t)\}_{t\geq 0}$. Indeed, since $\mathcal{A}^{\mathrm{tr}}(t)$ is independent of t, it holds that $\mathcal{A} = \mathcal{A}^{\mathrm{tr}}(0) = \mathcal{A}^{\mathrm{tr}}(t) = S(t)\mathcal{A}^{\mathrm{tr}}(0) = S(t)\mathcal{A}$, for all $t \geq 0$.

The outline of the proofs is as follows. First, we construct the trajectory space $\mathcal{T}^+$ and consider the translation semigroup $\{S(t)\}_{t\geq 0}$ acting on it. Then, we show that the translation semigroup possesses an absorbing set Λ for the family $\{\mathcal{T}^+\}$. Finally, we prove the absorbing set Λ is compact in the topology Θ_+^{loc}. To investigate the existence of the absorbing set Λ, the key point is to establish estimate (3.2) (i.e. Lemma 3.6). Due to the structural characteristics of the Keller–Segel–Stokes model, we cannot get Lemma 3.6 by processing n, c and u separately. Therefore, a more precise calculation is needed. To show the absorbing set Λ is compact in the topology Θ_+^{loc}, the key point is to verify that the trajectory space $\mathcal{T}^+$ is closed in the topology Θ_+^{loc}, i.e. Lemma 3.8, which will be proved by using the estimate obtained in Lemma 3.6 and the embedding between function spaces, and combining the Aubin–Lions compactness theory (see Lemma 2.4). Based on Theorem 1.1, the proof of Theorem 1.2 will be completed by the way of a direct analysis and verification.

Throughout this paper, we denote the usual Lebesgue space and Sobolev space (see [1], [28]) by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. For example, $\|\varphi\|_{L^p} = (\int_\Omega |\varphi|^p\,dx)^{1/p}$ and $\|\varphi\|_{m,p} := \Big(\sum_{|\beta|\le m}\int_\Omega |D^\beta\varphi|^p\,dx\Big)^{1/p}$. Especially, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and by $H_0^1(\Omega)$ the closure of $\{\varphi\in\mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$-norm. Then, we introduce the following function spaces:

$$
\begin{aligned}
L^p(I;X) :=\ & \text{strongly measurable functions on the closed interval } I,\\
& \text{with values in a Banach space } X, \text{ endowed with norm}\\
& \qquad \|\varphi\|_{L^p(I;X)} := \Big(\int_I \|\varphi\|_X^p\,dt\Big)^{1/p}, \quad \text{for } 1\le p<\infty,\\
\mathcal{C}(I;X) :=\ & \text{continuous functions on the interval } I, \text{ with values}\\
& \text{in the Banach space } X, \text{ endowed with the usual norm},\\
L^2_{\mathrm{loc}}(I;X) :=\ & \text{locally square integrable functions on the interval } I, \text{ with}\\
& \text{values in the Banach space } X, \text{ endowed with the usual norm.}
\end{aligned}
$$

In the subsequent, we simplify the notations $\|\cdot\|_{L^2(\Omega)}$ and $\int_I\int_\Omega \cdot\,dx\,dt$ by $\|\cdot\|$ and $\int_I\int_\Omega \cdot$, respectively, if there is no confusion. In addition, we denote by $(\,\cdot\,,\,\cdot\,)$ the inner product in $L^p(\Omega)$ or $W^{m,p}(\Omega)$, and by $\langle\,\cdot\,,\,\cdot\,\rangle$ the dual pairing between spaces X and X', where X' represents the dual space of X. We also denote the compact embedding between spaces by $\hookrightarrow\hookrightarrow$, and use $\mathrm{dist}_M(X,Y)$ to represent the Hausdorff semidistance between $X\subseteq M$ and $Y\subseteq M$ with $\mathrm{dist}_M(X,Y) = \sup_{x\in X}\inf_{y\in Y}\mathrm{dist}_M(x,y)$.

The rest of the paper is organized as follows. In Section 2, we make some necessary preliminaries. Section 3 is committed to the proof of Theorem 1.1. First, we construct the trajectory space $\mathcal{T}^+$ on solutions and consider the natural translation semigroup $\{S(t)\}_{t\ge0}$ acting on $\mathcal{T}^+$. Then, we show that the semigroup $\{S(t)\}_{t\ge0}$ possesses an absorbing set, which is bounded in the space $\mathfrak{F}_+^b$ and compact in the topology space Θ_+^{loc}. In Section 4, we prove the existence of the global attractor, i.e. Theorem 1.2.

2. Preliminaries

In this section, we will recall the global existence of weak solutions for the initial boundary value problem (1.2)–(1.4) and introduce some useful results. First, we give the definition of weak solutions of the initial boundary value problem (1.2)–(1.4).

Definition 2.1. Let $T \in (0,\infty)$. A triple (n,c,u) is said to be a weak solution of the initial boundary value problem (1.2)–(1.4) in $(0,T)\times\Omega$ if

$$\begin{gathered} n \in L^\infty(0,T;L^\infty(\Omega)), \quad \nabla n^m \in L^2(0,T;L^2(\Omega)), \\ c \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)), \\ u \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)), \end{gathered}$$

and the following equalities:

$$\begin{gathered} \int_0^T\int_\Omega n_t\psi - \int_0^T\int_\Omega \nabla\psi\cdot un + \int_0^T\int_\Omega \nabla n^m\cdot\nabla\psi = \int_0^T\int_\Omega n\chi(c)\nabla c\cdot\nabla\psi, \\ \int_0^T\int_\Omega c_t\psi - \int_0^T\int_\Omega \nabla\psi\cdot uc + \int_0^T\int_\Omega \nabla c\cdot\nabla\psi = -\int_0^T\int_\Omega nf(c)\psi, \\ -\int_0^T\int_\Omega \widetilde{\psi}_t\cdot u - \int_\Omega \widetilde{\psi}(0)\cdot u_0 - \eta\int_0^T\int_\Omega u\cdot\Delta\widetilde{\psi} + \int_0^T\int_\Omega n\nabla\phi\cdot\widetilde{\psi} = 0 \end{gathered}$$

hold for all $\psi \in L^2(0,T;W^{1,2}(\Omega))$ and any $\widetilde{\psi} \in L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$ with values in $\mathbb{R}^2$, $\nabla\cdot\widetilde{\psi} = 0$ and $\widetilde{\psi}|_{\partial\Omega} = 0$. If (n,c,u) is a weak solution of the initial boundary value problem (1.2)–(1.4) in $(0,T)\times\Omega$ for any $T\in(0,\infty)$, then we call (n,c,u) a global weak solution.

Based on the above definition, we have

Lemma 2.2 (see [20]). *Suppose $m > 1$ and the triple (n_0,c_0,u_0) satisfies* (1.5)*, then the initial boundary value problem* (1.2)–(1.4) *has at least one global weak solution (n,c,u). Moreover, (n,c,u) is bounded in $(L^\infty(0,\infty;L^\infty(\Omega)))^4$ and $n\ge 0, c\ge 0$ in $(0,\infty)\times\Omega$. In addition,*

$$\|c(t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \textit{for all } t\ge 0. \tag{2.1}$$

Further, one can check that the solution (n,c,u) satisfies the following energy inequality or equality:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|n\|^2 + \langle\nabla n^m,\nabla n\rangle \le \langle n\chi(c)\nabla c,\nabla n\rangle, \\ \dfrac{1}{2}\dfrac{d}{dt}\|c\|^2 + \|\nabla c\|^2 + \langle nf(c),c\rangle = 0, \\ \dfrac{1}{2}\dfrac{d}{dt}\|u\|^2 + \eta\|\nabla u\|^2 = -\langle n\nabla\phi,u\rangle, \end{cases} \tag{2.2}$$

in the sense of distribution $\mathcal{D}'(0,T)$ for any $T\in(0,\infty)$. Moreover, it holds that

Lemma 2.3. *If (n,c,u) is a weak solution of the initial boundary value problem* (1.2)–(1.4)*, then, for any $T\in(0,\infty)$,*

$$n_t \in L^2(0,T;(W^{1,2}(\Omega))'),\ c_t \in L^2(0,T;(W^{1,2}(\Omega))'),\ u_t \in L^2(0,T;(W_0^{1,2}(\Omega))').$$

PROOF. First, it is not difficult to check that the functions $u\cdot\nabla n$, Δn^m, $\nabla\cdot(n\chi(c)\nabla c)$, $u\cdot\nabla c$, Δc, $nf(c)$, $\eta\Delta u$ and $n\nabla\phi$ are measurable in $(0,T)\times\Omega$. Let

$$c_M := \|c\|_{L^\infty(0,T;L^\infty(\Omega))}, \quad \alpha_1 := \max_{0\le c\le c_M}\chi(c) \quad \text{and} \quad \alpha_2 := \max_{0\le c\le c_M} f(c). \tag{2.3}$$

Since

$$W^{2,2}(\Omega)\hookrightarrow W^{1,2}(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow (W^{1,2}(\Omega))',$$

for any $\varphi_1\in L^2(0,T;W^{1,2}(\Omega))$, $\varphi_2\in L^2(0,T;W_0^{1,2}(\Omega))$ with $\nabla\cdot\varphi_2=0$, we have

$$\begin{aligned}
|\langle u\cdot\nabla n,\varphi_1\rangle| &= |\langle un,\nabla\varphi_1\rangle| \\
&\le \|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))}\|\varphi_1\|_{L^2(0,T;W^{1,2}(\Omega))},\\
|\langle\Delta n^m,\varphi_1\rangle| &= |\langle\nabla n^m,\nabla\varphi_1\rangle| \le \|\nabla n^m\|_{L^2(0,T;L^2(\Omega))}\|\varphi_1\|_{L^2(0,T;W^{1,2}(\Omega))},\\
|\langle\nabla\cdot(n\chi(c)\nabla c),\varphi_1\rangle| &\le \alpha_1\|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|c\|_{L^2(0,T;W^{2,2}(\Omega))}\|\varphi_1\|_{L^2(0,T;W^{1,2}(\Omega))},\\
|\langle u\cdot\nabla c,\varphi_1\rangle| &= |\langle uc,\nabla\varphi_1\rangle| \\
&\le \|c\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))}\|\varphi_1\|_{L^2(0,T;W^{1,2}(\Omega))},\\
|\langle nf(c),\varphi_1\rangle| &\le \alpha_2\|n\|_{L^2(0,T;L^2(\Omega))}\|\varphi_1\|_{L^2(0,T;L^2(\Omega))},\\
|\langle n\nabla\phi,\varphi_2\rangle| &\le \|n\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{W^{1,\infty}(\Omega)}\|\varphi_2\|_{L^2(0,T;L^2(\Omega))},\\
|\langle\nabla p,\varphi_2\rangle| &= |\langle p,\nabla\cdot\varphi_2\rangle| = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|u\cdot\nabla n\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega)},\\
\|\Delta n^m\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \|\nabla n^m\|_{L^2(0,T;L^2(\Omega)},\\
\|\nabla\cdot(n\chi(c)\nabla c)\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \alpha_1\|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|c\|_{L^2(0,T;W^{2,2}(\Omega))},\\
\|u\cdot\nabla c\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \|c\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))},\\
\|nf(c)\|_{L^2(0,T;L^2(\Omega))} &\le \alpha_2\|n\|_{L^2(0,T;L^2(\Omega))},\\
\|n\nabla\phi\|_{L^2(0,T;L^2(\Omega))} &\le \|\phi\|_{W^{1,\infty}(\Omega)}\|n\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

which together with (1.2) and (1.6) yields

$$\begin{aligned}
\|n_t\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \|u\cdot\nabla n\|_{L^2(0,T;(W^{1,2}(\Omega))')}\\
&\quad + \|\Delta n^m\|_{L^2(0,T;(W^{1,2}(\Omega))')} + \|\nabla\cdot(n\chi(c)\nabla c)\|_{L^2(0,T;(W^{1,2}(\Omega))')}\\
&\le \|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))} + \|\nabla n^m\|_{L^2(0,T;L^2(\Omega))}\\
&\quad + \alpha_1\|n\|_{L^\infty(0,T;L^\infty(\Omega))}\|c\|_{L^2(0,T;W^{2,2}(\Omega))},
\end{aligned}\tag{2.4}$$

$$\begin{aligned}
\|c_t\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\le \|u\cdot\nabla c\|_{L^2(0,T;(W^{1,2}(\Omega))')}\\
&\quad + \|\Delta c\|_{L^2(0,T;L^2(\Omega))} + \|nf(c)\|_{L^2(0,T;L^2(\Omega))}\\
&\le \|c\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))}\\
&\quad + \|c\|_{L^2(0,T;W^{2,2}(\Omega))} + \alpha_2\|n\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}(2.6)\quad \|u_t\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\leq \eta\|\Delta u\|_{L^2(0,T;L^2(\Omega))} + \|n\nabla\phi\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \eta\|u\|_{L^2(0,T;W^{2,2}(\Omega))} + \|\phi\|_{W^{1,\infty}(\Omega)}\|n\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

This completes the proof. □

Finally, we end this section with the following useful lemma.

Lemma 2.4 (see [7], [13]). *Let X_1, X_2, X_3 be Banach spaces such that X_1, X_3 are reflexive and $X_1 \hookrightarrow\hookrightarrow X_2 \hookrightarrow X_3$. For $0 < T < \infty$ and $1 < r_0$, $r_1 < \infty$, define*

$$X := \left\{ w \;\middle|\; w \in L^{r_0}(0,T;X_1),\ \frac{dw}{dt} \in L^{r_1}(0,T;X_3) \right\}.$$

Then X is a Banach space equipped with the norm

$$\|w\|_X := \|w\|_{L^{r_0}(0,T;X_1)} + \|w'\|_{L^{r_1}(0,T;X_3)}.$$

Furthermore, $X \hookrightarrow\hookrightarrow L^{r_0}(0,T;X_2)$.

3. Existence of a trajectory attractor

In this section, we will show the existence of the trajectory attractors for the initial boundary value problem (1.2)–(1.4). For the general theory and applications about the trajectory attractor, one can refer to [6], [8], [22], [24], [27] and the references therein. In the sequel, we denote by Π_+ the restriction operator with respect to the semi-infinite interval $\mathbb{R}_+$. Similarly, Π_T represents the restriction operator to the interval $[0,T]$. Let us begin with the following definition.

Definition 3.1. A space $\mathcal{T}^+$ consisting of the triple (n,c,u) is called a trajectory space of the initial boundary value problem (1.2)–(1.4) if

$$\begin{gathered} n \in L^\infty(0,T;L^\infty(\Omega)), \quad \nabla n^m \in L^2(0,T;L^2(\Omega)), \\ c \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)), \\ u \in L^2(0,T;W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)), \end{gathered}$$

such that for all $T > 0$, the triple $\Pi_T(n,c,u)$ is a weak solution of the initial boundary value problem (1.2)–(1.4) on the interval $(0,T)$ and $\Pi_T(n,c,u)$ satisfies equations (2.2).

With Definition 3.1 at hand, the natural translation semigroup $\{S(t)\}_{t\geq 0}$ acting on the trajectory space $\mathcal{T}^+$ is defined by

$$S(t)(n(\,\cdot\,),c(\,\cdot\,),u(\,\cdot\,)) = (n(t+\,\cdot\,),c(t+\,\cdot\,),u(t+\,\cdot\,)), \tag{3.1}$$

for all $t \geq 0$, $(n(\,\cdot\,),c(\,\cdot\,),u(\,\cdot\,)) \in \mathcal{T}^+$. Then, we have

LEMMA 3.2. (a) *For any* (n_0, c_0, u_0) *satisfying* (1.5), *there exists at least one trajectory* $(n, c, u) \in \mathcal{T}^+$ *satisfying* $(n(0), c(0), u(0)) = (n_0, c_0, u_0)$.

(b) $\mathcal{T}^+$ *is translation invariant under the action of* $\{S(t)\}_{t\geq 0}$, *that is*

$$S(t)\mathcal{T}^+ \subseteq \mathcal{T}^+ \quad \textit{for all } t \geq 0.$$

PROOF. Observe that (a) is a direct consequence of Lemma 2.2. To prove (b), we first set $(n(r), c(r), u(r)) \in \mathcal{T}^+$ with $r \in \mathbb{R}_+$. Then, it is clear that the function $S(t)(n(r), c(r), u(r)) = (n(t+r), c(t+r), u(t+r))$, for all $t \geq 0$, is a weak solution of the initial boundary value problem (1.2)–(1.4) and has the same property with $(n(r), c(r), u(r))$. $\square$

Now, we construct the spaces $\mathfrak{F}_+^{\mathrm{loc}}$ and $\Pi_T\mathfrak{F}_+^{\mathrm{loc}}$ as follows:

$$\begin{aligned}
\mathfrak{F}_+^{\mathrm{loc}} := \big\{&(n, c, u) \mid n \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\Omega)), \nabla n^m \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega)),\\
&n_t \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (W^{1,2}(\Omega))'), c \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; W^{2,2}(\Omega)),\\
&c_t \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (W^{1,2}(\Omega))'), u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)),\\
&u_t \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (W^{1,2}(\Omega))')\big\},\\
\Pi_T\mathfrak{F}_+^{\mathrm{loc}} := \big\{&(n, c, u) \mid n \in L^\infty(0, T; L^\infty(\Omega)), \nabla n^m \in L^2(0, T; L^2(\Omega)),\\
&n_t \in L^2(0, T; (W^{1,2}(\Omega))'), c \in L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)),\\
&c_t \in L^2(0, T; (W^{1,2}(\Omega))'), u \in L^2(0, T; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)),\\
&u_t \in L^2(0, T; (W^{1,2}(\Omega))')\big\}.
\end{aligned}$$

Let $\{(n_k(t, x), c_k(t, x), u_k(t, x))\}_{k\geq 1}$ be a sequence of $\Pi_T\mathfrak{F}_+^{\mathrm{loc}}$, if the following convergence relations hold as $k \to \infty$:

$$\begin{aligned}
n_k(t, x) &\rightharpoonup^* n(t, x) && \text{weakly star in } L^\infty(0, T; L^\infty(\Omega)),\\
\nabla n_k^m(t, x) &\rightharpoonup \nabla n^m(t, x) && \text{weakly in } L^2(0, T; L^2(\Omega)),\\
c_k(t, x) &\rightharpoonup^* c(t, x) && \text{weakly star in } L^\infty(0, T; L^\infty(\Omega)),\\
c_k(t, x) &\rightharpoonup c(t, x) && \text{weakly in } L^2(0, T; W^{2,2}(\Omega)),\\
u_k(t, x) &\rightharpoonup u(t, x) && \text{weakly in } L^2(0, T; W^{1,2}_0(\Omega)),\\
u_k(t, x) &\rightharpoonup u(t, x) && \text{weakly in } L^2(0, T; W^{2,2}(\Omega)),
\end{aligned}$$

and

$$\begin{aligned}
(n_k)_t(t, x) &\rightharpoonup n_t(t, x) && \text{weakly in } L^2(0, T; W^{1,2}(\Omega)'),\\
(c_k)_t(t, x) &\rightharpoonup c_t(t, x) && \text{weakly in } L^2(0, T; (W^{1,2}(\Omega))'),\\
(u_k)_t(t, x) &\rightharpoonup u_t(t, x) && \text{weakly in } L^2(0, T; (W^{1,2}(\Omega))'),
\end{aligned}$$

then we say that $\{(n_k(t, x), c_k(t, x), u_k(t, x))\}_{k\geq 1}$ converges to $(n(t, x), c(t, x), u(t, x))$ in the topology of $\Pi_T\mathfrak{F}_+^{\mathrm{loc}}$.

In addition, we say that the sequence $\{(n_k(t,x), c_k(t,x), u_k(t,x))\}_{k\geq 1} \subseteq \mathfrak{F}_+^{\mathrm{loc}}$ converges to $(n(t,x), c(t,x), u(t,x)) \in \mathfrak{F}_+^{\mathrm{loc}}$ in the topology of $\mathfrak{F}_+^{\mathrm{loc}}$ as $k \to \infty$ if, for any $T > 0$,

$$(n_k(t,x), c_k(t,x), u_k(t,x)) \to (n(t,x), c(t,x), u(t,x))$$

in the topology of $\Pi_T \mathfrak{F}_+^{\mathrm{loc}}$. We denote by Θ_+^{loc} the space $\mathfrak{F}_+^{\mathrm{loc}}$ with this topology. Note that $\mathcal{T}^+ \subseteq \mathfrak{F}_+^{\mathrm{loc}}$. Then, for $S(t)$ defined by (3.1), we have

Lemma 3.3. *The translation semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{T}^+$ is continuous in the topological space Θ_+^{loc}.*

Proof. Let $G_k(s) := (n_k(s), c_k(s), u_k(s)) \to (n(s), c(s), u(s)) =: G(s)$ in Θ_+^{loc} as $k \to \infty$. Then, for any $T \in \mathbb{R}_+$, $\Pi_T G_k(s) \to \Pi_T G(s)$ in Θ_+^{loc} on the interval $[0,T]$ as $k \to \infty$. In particular, $\Pi_{T+t} G_k(s) \to \Pi_{T+t} G(s)$ in Θ_+^{loc} on the interval $[0, T+t]$ for any $t \geq 0$. Hence, $\Pi_T S(t) G_k(s) \to \Pi_T S(t) G(s)$ in Θ_+^{loc} for any $t \geq 0$, i.e. $S(t)G_k(s) \to S(t)G(s)$ in Θ_+^{loc} as $k \to \infty$. Therefore, $S(t)$ is continuous in the topological space Θ_+^{loc}. This ends the proof. $\square$

Further, we define another Banach space $\mathfrak{F}_+^b$ as

$$\mathfrak{F}_+^b := \big\{(n(t,x), c(t,x), u(t,x)) \in \mathfrak{F}_+^{\mathrm{loc}} \mid \|(n,c,u)\|_{\mathfrak{F}_+^b} < \infty\big\},$$

where the norm in $\mathfrak{F}_+^b$ is defined by

$$\begin{aligned}\|S(t)(n,c,u)\|_{\mathfrak{F}_+^b} := {} & \|S(t)n\|_{L^\infty(0,1;L^\infty(\Omega))} \\ & + \|S(t)(\nabla n^m)\|_{L^2(0,1;L^2(\Omega))} + \|S(t)c\|_{L^\infty(0,1;L^\infty(\Omega))} \\ & + \|S(t)c\|_{L^2(0,1;W^{2,2}(\Omega))} + \|S(t)u\|_{L^2(0,1;W_0^{1,2}(\Omega))} \\ & + \|S(t)u\|_{L^2(0,1;W^{2,2}(\Omega))} + \|S(t)n_t\|_{L^2(0,1;(W^{1,2}(\Omega))')} \\ & + \|S(t)c_t\|_{L^2(0,1;(W^{1,2}(\Omega))')} + \|S(t)u_t\|_{L^2(0,1;(W^{1,2}(\Omega))')}.\end{aligned}$$

Also, we need to introduce the following notions.

Definition 3.4. A set $\Lambda \subseteq \Theta_+^{\mathrm{loc}}$ is said to be an absorbing set for the family $\{\mathcal{T}^+\}$ in the topological space Θ_+^{loc} if for any bounded in $\mathfrak{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{T}^+$, there exists $t_0 = t_0(\mathcal{B})$ such that $S(t)\mathcal{B} \subseteq \Lambda$ for all $t \geq t_0$.

Definition 3.5. The set $\mathcal{A}^{\mathrm{tr}} \subseteq \mathcal{T}^+$ is called a trajectory attractor of the translation semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{T}^+$ in the topology Θ_+^{loc} if:

(a) $\mathcal{A}^{\mathrm{tr}}$ is bounded in $\mathfrak{F}_+^b$ and compact in the topology Θ_+^{loc};
(b) $\mathcal{A}^{\mathrm{tr}}$ is strictly invariant: $S(t)\mathcal{A}^{\mathrm{tr}} = \mathcal{A}^{\mathrm{tr}}$ for any $t \geq 0$;
(c) $\mathcal{A}^{\mathrm{tr}}$ is an attracting set in the topology Θ_+^{loc}, i.e. for any bounded in $\mathfrak{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{T}^+$ and any neighbuorhood $\mathcal{O}(\mathcal{A}^{\mathrm{tr}})$ of $\mathcal{A}^{\mathrm{tr}}$ in Θ_+^{loc}, there exists $t^* = t^*(\mathcal{B}, \mathcal{O}) \geq 0$ such that $S(t)\mathcal{B} \subseteq \mathcal{O}(\mathcal{A}^{\mathrm{tr}})$ for all $t \geq t^*$.

Based on the above definitions, we show the following lemma.

LEMMA 3.6. *It holds that $\mathcal{T}^+ \subseteq \mathfrak{F}_+^b$. Furthermore, for any trajectory (n, c, u) in $\mathcal{T}^+$, we have*

$$\|S(t)(n,c,u)\|_{\mathfrak{F}_+^b} \leq M_1 e^{-M_2 t} + M_0, \quad \text{for all } t \geq 0, \tag{3.2}$$

where M_0, M_1, M_2 are positive constants.

PROOF. It is clear that $\mathcal{T}^+ \subseteq \mathfrak{F}_+^{\mathrm{loc}}$. Now, let us prove (3.2). First, testing $(1.2)_2$ by c, we obtain that there exists a positive constant γ such that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|c\|^2 + \gamma\|c\|^2 &\leq \frac{1}{2}\frac{d}{dt}\|c\|^2 + \|\nabla c\|^2 \\ &= \langle c_t, c\rangle + \langle u \cdot \nabla c, c\rangle - \langle \Delta c, c\rangle = -\langle nf(c), c\rangle \leq 0. \end{aligned} \tag{3.3}$$

Further, $d(e^{2\gamma t}\|c(x,t)\|^2)/dt \leq 0$, which implies

$$\|c(t)\|^2 \leq \|c_0\|^2 e^{-2\gamma t}. \tag{3.4}$$

Therefore, we have

$$\|S(t)c\|_{L^\infty(0,1;L^2(\Omega))} \leq \|c_0\|_{L^\infty(0,1;L^2(\Omega))} e^{-\gamma t}. \tag{3.5}$$

It follows again from (3.3) that

$$\frac{1}{2}(\|c(t)\|^2 - \|c(s)\|^2) + \int_s^t \|\nabla c(\theta)\|^2\, d\theta \leq 0, \quad \text{for all } t \geq s,$$

which together with (3.4) gives

$$\begin{aligned} \|S(t)(\nabla c)\|_{L^2(0,1;L^2(\Omega))} &= \bigg(\int_t^{t+1} \|\nabla c(\theta)\|^2\, d\theta\bigg)^{1/2} \\ &\leq \bigg(\frac{\|c(t)\|^2}{2}\bigg)^{1/2} \leq \frac{\|c_0\|}{\sqrt{2}}\, e^{-\gamma t}. \end{aligned} \tag{3.6}$$

Now, testing $(1.2)_2$ by $-\Delta c$ and using Hölder inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\nabla c|^2\, dx &+ 2\int_\Omega |\Delta c|^2\, dx = 2\int_\Omega nf(c)\Delta c\, dx + 2\int_\Omega (u\cdot\nabla c)\Delta c\, dx \\ &\leq c_2\alpha_2^2 \int_\Omega n^2\, dx + c_3\int_\Omega |u|^2|\nabla c|^2\, dx + \overline{c}\int_\Omega |\Delta c|^2\, dx \\ &\leq c_2\alpha_2^2 \int_\Omega n^2\, dx + c_3\bigg(\int_\Omega |u|^{2p}\, dx\bigg)^{1/p} \\ &\quad \cdot \bigg(\int_\Omega |\nabla c|^{2p/(p-1)}\, dx\bigg)^{(p-1)/p} + \overline{c}\int_\Omega |\Delta c|^2\, dx, \end{aligned}$$

where α_2 comes from (2.3) and $\overline{c}$ will be specified later. By the Gagliardo–Nirenberg inequality (see [18]), we have

$$c_3\bigg(\int_\Omega |u|^{2p}\, dx\bigg)^{1/p} = c_3\|u\|^2_{L^{2p}(\Omega)} \leq c_5\|u\|^{2/p}\|\nabla u\|^{(2p-2)/p}$$

and

$$\bigg(\int_\Omega |\nabla c|^{2p/(p-1)}\,dx\bigg)^{(p-1)/p} = \|\nabla c\|^2_{L^{2p/(p-1)}(\Omega)} \\ \le c_5\|c\|^{2(p-1)/p}_{L^\infty(\Omega)}\|\Delta c\|^{2/p} + c_5\|c\|^2_{L^\infty(\Omega)},$$

where $p \in (1,2]$. Therefore, by (2.1) and Young's inequality, we can choose appropriate constants $\bar{c}$ and c_6 such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla c|^2\,dx + 2\int_\Omega |\Delta c|^2\,dx &\le c_2\alpha_2^2\int_\Omega n^2\,dx \\ &\quad + c_5^2\|u\|^{2/p}\|\nabla u\|^{(2p-2)/p}(\|c\|^{2(p-1)/p}_{L^\infty(\Omega)}\|\Delta c\|^{2/p} + \|c\|^2_{L^\infty(\Omega)}) + \bar{c}\int_\Omega |\Delta c|^2\,dx \\ &\le c_2\alpha_2^2\int_\Omega n^2\,dx + \frac{m}{m+1}\|\Delta c\|^2 + c_6\|u\|^{2/(p-1)}\|\nabla u\|^2 + c_7,\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega |\nabla c|^2\,dx + \int_\Omega |\Delta c|^2\,dx \le c_2\alpha_2^2\int_\Omega n^2\,dx + c_6\|u\|^{2/(p-1)}\|\nabla u\|^2 + c_7. \tag{3.7}$$

In order to estimate the terms containing u, we introduce the Stokes operator $A = -\eta\mathcal{P}\Delta$, where $\mathcal{P}$ denotes the Helmholtz projection in $L^2(\Omega)$. Then, using the same derivation process with (2.12) in [20], we can get that

$$\|Au(t,\,\cdot\,)\| \le c_8 + c_8\sup_{s\in[0,T]}\|n(s,\,\cdot\,)\|, \quad \text{for all } t\in[0,T], \tag{3.8}$$

which together with the embedding $D(A) \hookrightarrow W^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$, $2 \le r < \infty$, implies

$$\|u(t,\,\cdot\,)\|_{L^r(\Omega)} + \|u(t,\,\cdot\,)\|_{W^{1,r}(\Omega)} \le c_9 + c_9\sup_{s\in[0,T]}\|n(s,\,\cdot\,)\|, \tag{3.9}$$

for all $t\in[0,T]$. Consequently, we have

$$\|S(t)u\|_{L^2(0,1;L^r(\Omega))} + \|S(t)\nabla u\|_{L^2(0,1;L^r(\Omega))} \le c_9 + c_9\sup_{s\in[0,T]}\|n(s,\,\cdot\,)\|. \tag{3.10}$$

Moreover, it follows from (3.7) and (3.9) that

$$\begin{aligned}\|S(t)\Delta c\|^2_{L^2(0,1;L^2(\Omega))} &\le c_2\|S(t)n\|^2_{L^2(0,1;L^2(\Omega))} + \|\nabla c(t)\|^2 \\ &\quad + c_9\sup_{s\in[0,T]}\|n(s,\,\cdot\,)\|^{2/(p-1)}\|S(t)\nabla u\|^2_{L^2(0,1;L^2(\Omega))} \\ &\quad + c_9\|S(t)\nabla u\|^2_{L^2(0,1;L^2(\Omega))} + c_7.\end{aligned} \tag{3.11}$$

Next, testing $(1.2)_1$ by n^m, we have

$$\begin{aligned}\int_\Omega n_t n^m\,dx &+ \int_\Omega (u\cdot\nabla n)n^m\,dx + \int_\Omega |\nabla n^m|^2\,dx\\ &= \frac{1}{m+1}\frac{d}{dt}\int_\Omega n^{m+1}\,dx + \int_\Omega |\nabla n^m|^2\,dx = \int_\Omega n\chi(c)\nabla c\cdot\nabla n^m\,dx\\ &\le c_1\alpha_1^2\|n\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla c|^2\,dx + \frac{m}{m+1}\int_\Omega |\nabla n^m|^2\,dx,\end{aligned}$$

where α_1 comes from (2.3). From the above inequality, we see that

$$\frac{d}{dt}\int_\Omega n^{m+1}\,dx + \int_\Omega |\nabla n^m|^2\,dx \le c_1(m+1)\alpha_1^2\|n\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla c|^2\,dx,$$

which leads to

$$\begin{aligned}\int_\Omega n^{m+1}\,dx &+ \int_0^t\int_\Omega |\nabla n^m|^2\,dx\,ds\\ &\le c_1(m+1)\alpha_1^2\|n\|^2_{L^\infty(0,t;L^\infty(\Omega))}\int_0^t\int_\Omega |\nabla c|^2\,dx\,ds + \int_\Omega n_0^{m+1}\,dx.\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}\|S(t)\nabla n^m\|^2_{L^2(0,1;L^2(\Omega))} &= \int_t^{t+1}\|\nabla n^m(s)\|^2\,ds\\ &\le c_1(m+1)\alpha_1^2\|S(t)n\|^2_{L^\infty(0,1;L^\infty(\Omega))}\|S(t)\nabla c\|^2_{L^2(0,1;L^2(\Omega))}\\ &\qquad + \|n(t,x)\|^{m+1}_{L^{m+1}(\Omega)}.\end{aligned}\tag{3.12}$$

Thanks to Corollaries 2.7 and 2.8 in [20], we can conclude that there exist positive constants $\dot{c}_M$ and $\dot{c}_N$ such that for any $t\in[0,T]$,

$$\begin{cases}\|S(t)n\|_{L^\infty(0,1;L^\infty(\Omega))} \le \dot{c}_M,\\ \displaystyle\int_\Omega n^r(t,x)\,dx \le \dot{c}_N, \quad r > \max\{2, m-1\}.\end{cases}\tag{3.13}$$

Furthermore, according to Lemma 2.6 in [20], we conclude that, for any $m>1$ and $r>\max\{2,m-1\}$,

$$\frac{d}{dt}\Big(\int_\Omega n^r\,dx + \int_\Omega |\nabla c|^2\,dx\Big) + c_4\Big(\int_\Omega n^r\,dx + \int_\Omega |\nabla c|^2\,dx\Big) \le c_{10}.\tag{3.14}$$

Applying the Gronwall inequality to (3.14), we obtain

$$\begin{aligned}\int_\Omega n^r\,dx + \int_\Omega |\nabla c|^2\,dx &\le \Big(\int_\Omega n_0^r\,dx + \int_\Omega |\nabla c_0|^2\,dx\Big)e^{-c_4 t} + c_{10}\int_0^t e^{-c_4(t-s)}\,ds\\ &\le \Big(\int_\Omega n_0^r\,dx + \int_\Omega |\nabla c_0|^2\,dx\Big)e^{-c_4 t} + \frac{c_{10}}{c_4}.\end{aligned}$$

In particular, taking $r=m+1$, there exists a constant $\widetilde{c}$ such that

$$\|n(t,x)\|^{m+1}_{L^{m+1}(\Omega)} + \|\nabla c(t)\|^2 \le \widetilde{c}e^{-c_4 t} + \widetilde{c}, \quad \text{for all } t\in[0,T].\tag{3.15}$$

Hence, for all $t \in [0,T]$,

$$(3.16)\quad \|n(t,\cdot)\| \le \bigg(\int_\Omega |n(t,x)|^{m+1}\,dx\bigg)^{1/(m+1)} \cdot |\Omega|^{(m-1)/(2(m+1))}$$
$$= \|n(t,x)\|_{L^{m+1}(\Omega)} \cdot |\Omega|^{(m-1)/(2(m+1))} \le c_{12}e^{-c_4 t/(m+1)} + c_{12}.$$

Further, we get

(3.17)

$$\begin{aligned}\|S(t)n\|^2_{L^2(0,1;L^2(\Omega))} &= \int_t^{t+1}\int_\Omega |n(s,x)|^2\,dx\,ds \\ &\le \int_t^{t+1}\bigg(\int_\Omega |n(s,x)|^{m+1}\,dx\bigg)^{2/(m+1)} \cdot |\Omega|^{(m-1)/(m+1)}\,ds \\ &\le \int_t^{t+1} c_{13}\,|\Omega|^{(m-1)/(m+1)}(e^{-c_4 s/(m+1)}+1)\,ds \\ &= \frac{c_{13}(m+1)}{c_4}\,|\Omega|^{(m-1)/(m+1)}\bigg[(1-e^{-c_4/(m+1)})e^{-c_4 t/(m+1)} + \frac{c_4}{m+1}\bigg].\end{aligned}$$

In a completely similar way, from (2.4)–(2.6), we have

$$(3.18)\quad \|S(t)n_t\|_{L^2(0,1;(W^{1,2}(\Omega))')} \le \|S(t)n\|_{L^\infty(0,1;L^\infty(\Omega))}\|S(t)u\|_{L^2(0,1;L^2(\Omega))}$$
$$+ \|S(t)(\nabla n^m)\|_{L^2(0,1;L^2(\Omega))} + \alpha_1\|S(t)n\|_{L^\infty(0,1;L^\infty(\Omega))}\|S(t)c\|_{L^2(0,1;W^{2,2}(\Omega))},$$

$$(3.19)\quad \|S(t)c_t\|_{L^2(0,1;(W^{1,2}(\Omega))')} \le \|S(t)c\|_{L^\infty(0,1;L^\infty(\Omega))}\|S(t)u\|_{L^2(0,1;L^2(\Omega))}$$
$$+ \|S(t)c\|_{L^2(0,1;W^{2,2}(\Omega))} + \alpha_2\|S(t)n\|_{L^2(0,1;L^2(\Omega))},$$

$$(3.20)\quad \|S(t)u_t\|_{L^2(0,1;(W^{1,2}(\Omega))')}$$
$$\le \eta\|S(t)u\|_{L^2(0,1;W^{2,2}(\Omega))} + \|\phi\|_{W^{1,\infty}(\Omega)}\|S(t)n\|_{L^2(0,1;L^2(\Omega))}.$$

Finally, taking (1.5), (1.6), (2.1), (2.3), (3.6), (3.8), (3.10)–(3.13), (3.15) and (3.17)–(3.20) into account, we have

$$\begin{aligned}\|S(t)(n,c,u)\|_{\mathfrak{F}^b_+} &\le c_M + c_{11}\|S(t)\nabla c\|_{L^2(0,1;L^2(\Omega))} + 2\|n(t,x)\|^{m+1}_{L^{m+1}(\Omega)} \\ &\quad + \|c_0\|_{L^\infty(\Omega)} + c_{14}\|S(t)n\|_{L^2(0,1;L^2(\Omega))} + c_{15}\|S(t)\nabla u\|_{L^2(0,1;L^2(\Omega))} \\ &\quad + \|\nabla c(t)\| + c_{15}\sup_{s\in[0,T]}\|n(s,x)\|^{1/(p-1)}\|S(t)\nabla u\|_{L^2(0,1;L^2(\Omega))} \\ &\quad + c_{16}\sup_{s\in[0,T]}\|n(s,\cdot)\| + c_{16} \\ &\le \frac{c_{11}}{\sqrt{2}}\,\|c_0\|e^{-\gamma t} + \widetilde{c}e^{-c_4 t} + c_{17}|\Omega|^{(m-1)/(2(m+1))}e^{-c_{18}t} + c_{19}e^{-c_4 t/(m+1)} + c_{20} \\ &\le M_1 e^{-M_2 t} + M_0, \quad \text{for all } t \in [0,T],\end{aligned}$$

where c_i, $i = 1,\ldots,20$, and M_0, M_1, M_2 are positive constants. This completes the proof. $\square$

We immediately have

LEMMA 3.7. *There exists a bounded absorbing set*

$$\Lambda = \big\{(n, c, u) \in \mathcal{T}^+ \mid \|(n, c, u)\|_{\mathfrak{F}^b_+} \le 2M_0\big\}. \tag{3.21}$$

That is, for any bounded (in the norm of $\mathfrak{F}^b_+$) subset $\mathcal{B} \subseteq \mathcal{T}^+$, there exists a time $t_0 = t_0(\mathcal{B})$ such that $S(t)\mathcal{B} \subseteq \Lambda$ for all $t \ge t_0$. M_0 comes from Lemma 3.6.

LEMMA 3.8. *$\mathcal{T}^+$ is closed in the topology $\Theta^{\rm loc}_+$.*

PROOF. Suppose $\{(n_k, c_k, u_k)\}_{k\ge1}$ is a bounded (in the norm of $\mathfrak{F}^b_+$) sequence in $\mathcal{T}^+$ and there exists a triple $(n^*, c^*, u^*) \in \mathfrak{F}^{\rm loc}_+$ such that

$$(c_k, n_k, u_k) \to (n^*, c^*, u^*) \quad \text{in } \Theta^{\rm loc}_+ \text{ as } k \to \infty. \tag{3.22}$$

We should prove $(n^*, c^*, u^*) \in \mathcal{T}^+$. For the sake of clarity, we divide the proof into two steps.

STEP 1. *For any $T > 0$, $\Pi_T(n^*, c^*, u^*)$ is a weak solution of the initial boundary value problem (1.2)–(1.4) on the interval $(0, T)$.*

First, from the boundedness of $\{(n_k(t, x), c_k(t, x), u_k(t, x))\}_{k\ge1} \in \mathcal{T}^+$ in the norm of $\mathfrak{F}^b_+$, using the diagonal procedure, we deduce that there exist functions

$$\begin{gathered} n(t,x) \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\infty(\Omega)) \quad \text{and} \quad \nabla n^m(t,x) \in L^2_{\rm loc}(\mathbb{R}_+; L^2(\Omega)), \\ c(t,x) \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\infty(\Omega)) \cap L^2_{\rm loc}(\mathbb{R}_+; W^{2,2}(\Omega)), \\ u(t,x) \in L^2_{\rm loc}(\mathbb{R}_+; W^{1,2}_0(\Omega)) \cap L^2_{\rm loc}(\mathbb{R}_+; W^{2,2}(\Omega)), \end{gathered}$$

such that (by extracting a subsequence if necessary), for any $T > 0$,

$$\begin{aligned} \Pi_T n_k(t,x) &\rightharpoonup^* \Pi_T n(t,x) && \text{weakly star in } L^\infty(0,T; L^\infty(\Omega)), \\ \Pi_T \nabla n_k^m(t,x) &\rightharpoonup \Pi_T \nabla n^m(t,x) && \text{weakly in } L^2(0,T; L^2(\Omega)), \\ \Pi_T c_k(t,x) &\rightharpoonup^* \Pi_T c(t,x) && \text{weakly star in } L^\infty(0,T; L^\infty(\Omega)), \\ \Pi_T c_k(t,x) &\rightharpoonup \Pi_T c(t,x) && \text{weakly in } L^2(0,T; W^{2,2}(\Omega)), \\ \Pi_T u_k(t,x) &\rightharpoonup \Pi_T u(t,x) && \text{weakly in } L^2(0,T; W^{1,2}_0(\Omega)), \\ \Pi_T u_k(t,x) &\rightharpoonup \Pi_T u(t,x) && \text{weakly in } L^2(0,T; W^{2,2}(\Omega)), \end{aligned}$$

and

$$\begin{aligned} \Pi_T (n_k)_t(t,x) &\rightharpoonup \Pi_T n_t(t,x) && \text{weakly in } L^2(0,T; (W^{1,2}(\Omega))'), \\ \Pi_T (c_k)_t(t,x) &\rightharpoonup \Pi_T c_t(t,x) && \text{weakly in } L^2(0,T; (W^{1,2}(\Omega))'), \\ \Pi_T (u_k)_t(t,x) &\rightharpoonup \Pi_T u_t(t,x) && \text{weakly in } L^2(0,T; (W^{1,2}(\Omega))'). \end{aligned}$$

According to the definition of $\Theta^{\rm loc}_+$, the above convergence relations imply

$$(n_k(t,x), c_k(t,x), u_k(t,x)) \to (n(t,x), c(t,x), u(t,x)) \quad \text{in } \Theta^{\rm loc}_+ \text{ as } k \to \infty.$$

Further, from (3.22) and the uniqueness of limit, it follows that

$$(n(t,x),c(t,x),u(t,x)) = (n^*(t,x),c^*(t,x),u^*(t,x)).$$

Next, we verify that $(\Pi_T n^*, \Pi_T c^*, \Pi_T u^*)$ is a weak solution of the initial boundary value problem (1.2)–(1.4) on the interval $(0,T)$. In fact, from the above convergence relations, it is not difficult to see that

$$\begin{aligned}
&\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(u_k\cdot\nabla n_k)-\Pi_T(u^*\cdot\nabla n^*)]\varphi\,dx\,dt\bigg| \\
&\quad=\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(u_k n_k)-\Pi_T(u^* n^*)]\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\quad=\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(u_k n_k)-\Pi_T(u^* n_k)+\Pi_T(u^* n_k)-\Pi_T(u^* n^*)]\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\quad\le\lim_{k\to\infty}\|\Pi_T n_k\|_{L^\infty(0,T;L^\infty(\Omega))}\bigg|\int_0^T\int_\Omega(\Pi_T u_k-\Pi_T u^*)\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\qquad+\lim_{k\to\infty}\bigg|\int_0^T\int_\Omega(\Pi_T n_k-\Pi_T n^*)u^*\cdot\nabla\varphi\,dx\,dt\bigg|=0,
\end{aligned}$$

which implies

$$\Pi_T(u_k\cdot\nabla n_k)\rightharpoonup\Pi_T(u^*\cdot\nabla n^*)\quad\text{weakly in } L^2(0,T;(W^{1,2}(\Omega))').$$

Similarly, from

$$\begin{aligned}
&\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(\nabla\cdot(n_k\chi(c_k)\nabla c_k))-\Pi_T(\nabla\cdot(n^*\chi(c^*)\nabla c^*))]\varphi\,dx\,dt\bigg| \\
&\quad=\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(n_k\chi(c_k)\nabla c_k)-\Pi_T(n^*\chi(c^*)\nabla c^*)]\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\quad=\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(n_k\chi(c_k)\nabla c_k-n_k\chi(c_k)\nabla c^*) \\
&\qquad+\Pi_T(n_k\chi(c_k)\nabla c^*-n^*\chi(c^*)\nabla c^*)]\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\quad\le\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega(\Pi_T\nabla c_k-\Pi_T\nabla c^*)\cdot\nabla\varphi\,dx\,dt\bigg|\|n_k\chi(c_k)\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\qquad+\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega[\Pi_T(n_k\chi(c_k))-\Pi_T(n^*\chi(c^*))]\nabla c^*\cdot\nabla\varphi\,dx\,dt\bigg| \\
&\quad=\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega(\Pi_T\nabla c_k-\Pi_T\nabla c^*)\cdot\nabla\varphi\,dx\,dt\bigg|\|n_k\chi(c_k)\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\qquad+\bigg|\lim_{k\to\infty}\int_0^T\int_\Omega\Pi_T[n_k\chi'(\widehat{c})(c_k-c^*)+\chi(c^*)(n_k-n^*)]\nabla c^*\cdot\nabla\varphi\,dx\,dt\bigg|=0,
\end{aligned}$$

where $\widehat{c}$ is between c_k and c^*, we can conclude that

$$\Pi_T(\nabla\cdot(n_k\chi(c_k)\nabla c_k))\rightharpoonup\Pi_T(\nabla\cdot(n^*\chi(c^*)\nabla c^*))\quad\text{weakly in } L^2(0,T;(W^{1,2}(\Omega))').$$

In the same way, we can prove the following convergence relations:

$$\begin{aligned}
\Pi_T(\Delta n_k^m) &\rightharpoonup \Pi_T(\Delta(n^*)^m) && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'),\\
\Pi_T(u_k\cdot\nabla c_k) &\rightharpoonup \Pi_T(u^*\cdot\nabla n^*) && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'),\\
\Pi_T(\Delta c_k) &\rightharpoonup \Pi_T(\Delta c^*) && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
\Pi_T(n_k f(c_k)) &\rightharpoonup \Pi_T(n^* f(c^*)) && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
\Pi_T(\Delta u_k) &\rightharpoonup \Pi_T(\Delta u^*) && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
\Pi_T(n_k\nabla\phi) &\rightharpoonup \Pi_T(n^*\nabla\phi) && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
\Pi_T(n_k)_t &\rightharpoonup \Pi_T n_t^* && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'),\\
\Pi_T(c_k)_t &\rightharpoonup \Pi_T c_t^* && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))'),\\
\Pi_T(u_k)_t &\rightharpoonup \Pi_T u_t^* && \text{weakly in } L^2(0,T;(W^{1,2}(\Omega))').
\end{aligned}$$

Then we can pass to the limit and obtain that $(\Pi_T n^*, \Pi_T c^*, \Pi_T u^*)$ is a weak solution of the initial boundary value problem (1.2)–(1.4).

STEP 2. *The triple* $(\Pi_T n^*, \Pi_T c^*, \Pi_T u^*)$ *satisfies* (2.2).

Since $\{(n_k, c_k, u_k)\}_{k\geq 1}$ is bounded in the $\mathfrak{F}_+^b$-norm, by Lemma 2.4, it follows from Lemma 2.3 and the embedding $W^{2,2}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,2}(\Omega))'$ that

$$\Pi_T n_k(t) \rightharpoonup^* \Pi_T n^*(t) \quad \text{weakly star in } L^\infty(0,T;L^\infty(\Omega)) \text{ as } k\to\infty, \tag{3.23}$$

$$\Pi_T(\nabla n_k^m(t)) \rightharpoonup \Pi_T\nabla(n^*)^m(t) \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \text{ as } k\to\infty, \text{ for all } m>1, \tag{3.24}$$

$$\Pi_T c_k(t) \to \Pi_T c^*(t) \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)) \text{ as } k\to\infty, \tag{3.25}$$

$$\Pi_T u_k(t) \to \Pi_T u^*(t) \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)) \text{ as } k\to\infty. \tag{3.26}$$

Therefore, for any $\widetilde{\phi}(s) \in \mathcal{C}_0^\infty(0,T)$ with $\widetilde{\phi}(s)\geq 0$, we have

$$-\frac{1}{2}\int_0^T \|\Pi_T n^*(s)\|^2 \widetilde{\phi}'(s)\,ds = -\frac{1}{2}\lim_{k\to\infty}\int_0^T \|\Pi_T n_k(s)\|^2\widetilde{\phi}'(s)\,ds, \tag{3.27}$$

$$-\frac{1}{2}\int_0^T \|\Pi_T c^*(s)\|^2 \widetilde{\phi}'(s)\,ds = -\frac{1}{2}\lim_{k\to\infty}\int_0^T \|\Pi_T c_k(s)\|^2\widetilde{\phi}'(s)\,ds, \tag{3.28}$$

$$-\frac{1}{2}\int_0^T \|\Pi_T u^*(s)\|^2 \widetilde{\phi}'(s)\,ds = -\frac{1}{2}\lim_{k\to\infty}\int_0^T \|\Pi_T u_k(s)\|^2\widetilde{\phi}'(s)\,ds, \tag{3.29}$$

$$\int_0^T \|\Pi_T \nabla c^*(s)\|^2 \widetilde{\phi}(s)\,ds = \lim_{k\to\infty}\int_0^T \|\Pi_T \nabla c_k(s)\|^2\widetilde{\phi}(s)\,ds, \tag{3.30}$$

$$\eta\int_0^T \|\Pi_T(\nabla u^*(s))\|^2\widetilde{\phi}(s)\,ds=\liminf_{k\to\infty}\eta\int_0^T \|\Pi_T(\nabla u_k(s))\|^2\widetilde{\phi}(s)\,ds. \tag{3.31}$$

Noting that

$$\begin{aligned}
&\langle n_k\chi(c_k)\nabla c_k,\nabla n_k\rangle-\langle n\chi(c)\nabla c,\nabla n\rangle=\frac{1}{2}\langle\chi(c_k)\nabla c_k,\nabla n_k^2\rangle-\frac{1}{2}\langle\chi(c)\nabla c,\nabla n^2\rangle\\
&=\frac{1}{2}\langle\chi(c_k)\nabla c_k-\chi(c)\nabla c,\nabla n_k^2\rangle+\frac{1}{2}\langle\chi(c)\nabla c,\nabla n_k^2-\nabla n^2\rangle\\
&=\frac{1}{2}\langle(\chi(c_k)-\chi(c))\nabla c+\chi(c_k)(\nabla c_k-\nabla c),\nabla n_k^2\rangle+\frac{1}{2}\langle\chi(c)\nabla c,\nabla n_k^2-\nabla n^2\rangle\\
&=\frac{1}{2}\langle(\chi(c_k)-\chi(c))\nabla c,\nabla n_k^2\rangle+\frac{1}{2}\langle\nabla c_k-\nabla c,\chi(c_k)\nabla n_k^2\rangle+\frac{1}{2}\langle\chi(c)\nabla c,\nabla n_k^2-\nabla n^2\rangle\\
&=\frac{1}{2}\langle\chi'(\widehat{c})(c_k-c)\nabla c,\nabla n_k^2\rangle+\frac{1}{2}\langle\nabla c_k-\nabla c,\chi(c_k)\nabla n_k^2\rangle+\frac{1}{2}\langle\chi(c)\nabla c,\nabla n_k^2-\nabla n^2\rangle,
\end{aligned}$$

where $\widehat{c}$ is between c_k and c, and using the Hölder and Gagliardo–Nirenberg inequalities (see [18]), we get from (3.24) and (3.25) that

$$\begin{aligned}
\int_0^T &\langle\Pi_T(n^*(s)\chi(c^*)\nabla c^*(s)),\Pi_T(\nabla n^*(s))\rangle\widetilde{\phi}(s)\,ds\\
&=\lim_{k\to\infty}\int_0^T\langle\Pi_T(n_k(s)\chi(c_k)\nabla c_k(s)),\Pi_T(\nabla n_k(s))\rangle\widetilde{\phi}(s)\,ds.
\end{aligned}\tag{3.32}$$

Similarly, we also have

$$\begin{aligned}
\int_0^T &\langle\Pi_T(n^*(s)f(c^*(s))),\Pi_T c^*(s)\rangle\widetilde{\phi}(s)\,ds\\
&=\lim_{k\to\infty}\int_0^T\langle\Pi_T(n_k(s)f(c_k(s))),\Pi_T c_k(s)\rangle\widetilde{\phi}(s)\,ds,
\end{aligned}\tag{3.33}$$

$$\begin{aligned}
\int_0^T &\langle\Pi_T(n^*(s)\nabla\phi),\Pi_T u^*(s)\rangle\widetilde{\phi}(s)\,ds\\
&=\lim_{k\to\infty}\int_0^T\langle\Pi_T(n_k(s)\nabla\phi),\Pi_T u_k(s)\rangle\widetilde{\phi}(s)\,ds.
\end{aligned}\tag{3.34}$$

Moreover, by (3.24) and the lower semicontinuity of norm, we see that

$$\begin{aligned}
\int_0^T &\langle\Pi_T\nabla n^{*m},\Pi_T\nabla n^*\rangle\widetilde{\phi}(s)\,ds\\
&=\frac{4m}{(m+1)^2}\int_0^T\|\Pi_T\nabla n^{*(m+1)/2}\|^2\widetilde{\phi}(s)\,ds\\
&\le\frac{4m}{(m+1)^2}\liminf_{k\to\infty}\int_0^T\|\Pi_T\nabla n_k^{(m+1)/2}\|^2\widetilde{\phi}(s)\,ds\\
&=\liminf_{k\to\infty}\int_0^T\langle\Pi_T\nabla n_k^m,\Pi_T\nabla n_k\rangle\widetilde{\phi}(s)\,ds.
\end{aligned}\tag{3.35}$$

Since $(n_k(t), c_k(t), u_k(t)) \in \mathcal{T}^+$, there holds

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\Pi_T n_k\|^2 \widetilde{\phi}'(s)\,ds + \int_0^T \langle \Pi_T(\nabla n_k^m), \Pi_T(\nabla n_k)\rangle \widetilde{\phi}(s)\,ds \\
\leq \int_0^T \langle \Pi_T(n_k\chi(c_k)\nabla c_k), \Pi_T(\nabla n_k)\rangle \widetilde{\phi}(s)\,ds, -\frac{1}{2}\int_0^T \|\Pi_T c_k\|^2 \widetilde{\phi}'(s)\,ds \\
+ \int_0^T \|\Pi_T(\nabla c_k)\|^2 \widetilde{\phi}(s)\,ds + \int_0^T \langle \Pi_T(n_k f(c_k)), \Pi_T c_k\rangle \widetilde{\phi}(s)\,ds = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\Pi_T u_k\|^2 \widetilde{\phi}'(s)\,ds + \eta\int_0^T \|\Pi_T(\nabla u_k)\|^2 \widetilde{\phi}(s)\,ds \\
= -\int_0^T \langle \Pi_T(n_k\nabla\phi), \Pi_T u_k\rangle \widetilde{\phi}(s)\,ds.
\end{aligned}
$$

Therefore, from (3.27), (3.32) and (3.35), we conclude that

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\Pi_T n^*\|^2 \widetilde{\phi}'(s)\,ds + \int_0^T \langle \Pi_T(\nabla n^{*m}), \Pi_T(\nabla n^*)\rangle \widetilde{\phi}(s)\,ds \\
\leq \int_0^T \langle \Pi_T(n^*\chi(c^*)\nabla c^*), \Pi_T(\nabla n^*)\rangle \widetilde{\phi}(s)\,ds.
\end{aligned} \tag{3.36}
$$

Similarly, relations (3.28), (3.30) and (3.33) lead to

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\Pi_T c^*\|^2 \widetilde{\phi}'(s)\,ds + \int_0^T \|\Pi_T(\nabla c^*)\|^2 \widetilde{\phi}(s)\,ds \\
+ \int_0^T \langle \Pi_T(n^* f(c^*)), \Pi_T c^*\rangle \widetilde{\phi}(s)\,ds = 0.
\end{aligned} \tag{3.37}
$$

Finally, taking (3.29), (3.31) and (3.34) into account, we obtain

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \|\Pi_T u^*\|^2 \widetilde{\phi}'(s)\,ds + \eta\int_0^T \|\Pi_T(\nabla u^*)\|^2 \widetilde{\phi}(s)\,ds \\
= -\int_0^T \langle \Pi_T(n^*\nabla\phi), \Pi_T u^*\rangle \widetilde{\phi}(s)\,ds.
\end{aligned} \tag{3.38}
$$

Clearly (3.36)–(3.38) imply that $\Pi_T(n^*, c^*, u^*)$ satisfies (2.2) in the distribution sense $\mathcal{D}'(0,T)$. To summarize, we get $(n^*, c^*, u^*) \in \mathcal{T}^+$. □

With Lemma 3.8 at hand, we have

Lemma 3.9. *The absorbing set* Λ *constructed by* (3.21) *is compact in the topology* Θ_+^{loc}.

Proof. Let $\{(n_k, c_k, u_k)\}_{k\geq 1} \subseteq \Lambda$ be a bounded (in the norm of $\mathfrak{F}_+^b$) sequence. Then there exist a subsequence $\{(n_{k_j}, c_{k_j}, u_{k_j})\}_{j\geq 1} \subseteq \{(n_k, c_k, u_k)\}_{k\geq 1}$

and a triple $(n,c,u)\in\mathfrak{F}_+^{\rm loc}$ such that $\{(n_{k_j},c_{k_j},u_{k_j})\}_{j\geq 1}\to(n,c,u)$ in $\Theta_+^{\rm loc}$ as $j\to\infty$. By the lower semicontinuity of norm, we obtain that

$$\|(n,c,u)\|_{\mathfrak{F}_+^b}\leq\liminf_{j\to\infty}\|(n_{k_j},c_{k_j},u_{k_j})\|_{\mathfrak{F}_+^b}\leq 2M_0.$$

According to Lemma 3.8, we get $(n,c,u)\in\mathcal{T}^+$. Therefore, $(n,c,u)\in\Lambda$, which implies the compactness of Λ in $\Theta_+^{\rm loc}$. □

Now let us give the proof of Theorem 1.1.

Proof of Theorem 1.1. According to Definition 3.5, the existence of the trajectory attractor $\mathcal{A}^{\rm tr}$ is a direct consequence of Lemmas 3.2 (b), 3.3, 3.7, 3.9 and Theorem 4.1 in [23]. □

4. The existence of global attractor

In this section, we are devoted to showing the existence of global attractor. That is, we focus on the proof of Theorem 1.2. To begin with, let us define the spaces $\mathcal{M}_+$ and $\mathcal{N}_+$ as follows:

$$\begin{aligned}\mathcal{M}_+:=\big\{(n(t,x),c(t,x),u(t,x))\mid n\in L^\infty(\Omega),\ \nabla n^m\in L^2(\Omega),\\ c\in L^\infty(\Omega)\cap W^{2,2}(\Omega),\ u\in W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)\big\}.\end{aligned}$$

We say that the sequence $\{n_k,c_k,u_k\}_{k\geq 1}\subseteq\mathcal{M}_+$ converges to (n,c,u) in the topology of $\mathcal{M}_+$ if it holds that

$$\begin{aligned}
n_k(t,x)&\rightharpoonup^* n(t,x) && \text{weakly star in } L^\infty(\Omega)\text{as } k\to\infty,\\
\nabla n_k^m(t,x)&\rightharpoonup \nabla n^m(t,x) && \text{weakly in } L^2(\Omega) \text{ as } k\to\infty,\\
c_k(t,x)&\rightharpoonup^* c(t,x) && \text{weakly star in } L^\infty(\Omega) \text{ as } k\to\infty,\\
c_k(t,x)&\rightharpoonup c(t,x) && \text{weakly in} W^{2,2}(\Omega) \text{ as } k\to\infty,\\
u_k(t,x)&\rightharpoonup u(t,x) && \text{weakly in } W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega) \text{ as } k\to\infty.
\end{aligned}$$

$$\mathcal{N}_+:=\big\{(n(t,x),c(t,x),u(t,x))\in\mathcal{M}_+\mid\|(n,c,u)\|_{\mathcal{N}_+}<\infty\big\}$$

with the norm

$$\begin{aligned}\|(n,c,u)\|_{\mathcal{N}_+}&=\|n\|_{L^\infty(\Omega)}+\|\nabla n^m\|_{L^2(\Omega)}\\&\quad+\|c\|_{L^\infty(\Omega)}+\|c\|_{W^{2,2}(\Omega)}+\|u\|_{W_0^{1,2}(\Omega)}+\|u\|_{W^{2,2}(\Omega)}.\end{aligned}$$

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1, we see that the initial boundary value problem (1.2)–(1.4) possesses a trajectory attractor $\mathcal{A}^{\rm tr}$ in $\mathcal{T}^+$. From the strictly invariance of the trajectory attractor, we deduce that $\mathcal{A}^{\rm tr}$ is independent of t. Since $\mathcal{A}^{\rm tr}(0)\subseteq\mathcal{A}^{\rm tr}$, $\mathcal{A}^{\rm tr}$ is compact in $\Theta_+^{\rm loc}$ and bounded in $\mathfrak{F}_+^b$, we obtain $\mathcal{A}=\mathcal{A}^{\rm tr}(0)$ is compact in $\mathcal{M}_+$ and bounded in $\mathcal{N}_+$, that is, property (a) in Theorem 1.2.

Further, it follows from property (c) in Theorem 1.1 that, for any bounded (in the $\mathcal{F}_+^b$-norm) set $\mathcal{B} \subseteq \mathcal{T}^+$,

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{M}_+}(S(t)\mathcal{B}, \mathcal{A}^{\mathrm{tr}}(0)) = 0,$$

that is, property (b) in Theorem 1.2.

Finally, we prove property (c) in Theorem 1.2. Suppose $\mathcal{A}_1$ is compact in the topology $\mathcal{M}_+$ and bounded in $\mathcal{N}_+$, then for any bounded set $\mathcal{B} \subseteq \mathcal{T}^+$ in the $\mathcal{F}_+^b$-norm, it holds that

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{M}_+}(S(t)\mathcal{B}, \mathcal{A}_1) = 0.$$

Taking $\mathcal{B} = \mathcal{A}^{\mathrm{tr}}$, we have

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{M}_+}(\mathcal{A}^{\mathrm{tr}}(t), \mathcal{A}_1) = \lim_{t\to+\infty} \operatorname{dist}_{\mathcal{M}_+}(\mathcal{A}^{\mathrm{tr}}(0), \mathcal{A}_1) = \operatorname{dist}_{\mathcal{M}_+}(\mathcal{A}, \mathcal{A}_1) = 0,$$

which implies $\mathcal{A} \subseteq \bar{\mathcal{A}}_1 \subseteq \mathcal{A}_1$. □

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