Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 2, 2017, 817–833
DOI: 10.12775/TMNA.2017.011

LOCALIZATION OF POSITIVE CRITICAL POINTS IN BANACH SPACES AND APPLICATIONS

Radu Precup — Csaba Varga

Abstract. Two critical point theorems of M. Schechter in a ball of a Hilbert space are extended to uniformly convex Banach spaces by exploiting the properties of the duality mapping. Moreover, the critical points are sought in the intersection of a ball with a wedge, in particular with a cone, making possible applications to positive solutions of variational problems. The extension from Hilbert to Banach spaces not only requires a major refining of reasoning, but also a different statement by adding a third possibility to the original two alternatives from Schechter's results. The theory is applied to positive solutions for p-Laplace equations.

1. Introduction

Fixed point theory offers a large number of useful methods for the study of nonlinear equations. Such a method is the Leray–Schauder continuation principle, see [8], [11], consisting in embedding the original equation in a one-parameter family of equations in a such way that the solution of a simpler equation can be continued inside a given set until a solution of the initial equation. This continuation process is guaranteed by the robustness of the simpler equation and

2010 *Mathematics Subject Classification.* 58E05, 47J30, 35J60.

Key words and phrases. Critical point; mountain pass lemma; positive solution; p-Laplace equation.

The first author, Radu Precup, was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0094. The research of Csaba Varga has been supported by a grant of Babeş-Bolyai University, "Granturi pentru Susţinerea Cercetării de Excelenţă", project number GSCE-30252/2015.

by a condition making closed the boundary of the given set. Here is a simple version of the Leray–Schauder continuation principle.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space, $R > 0$ and let X_R be the closed ball of radius R centered in the origin. Assume that $N\colon X_R \to X$ is a compact map such that the following Leray–Schauder boundary condition is satisfied*:

$$u \neq \lambda N(u) \quad \text{for } \|u\| = R, \text{ and } \lambda \in (0,1). \tag{1.1}$$

Then N has at least one fixed point in X_R.

This theorem allows us to obtain the existence and localization of a solution of the equation

$$N(u) = u, \tag{1.2}$$

in the Banach space X, for a completely continuous operator $N\colon X \to X$, via the so-called "a priori bounds" technique. Indeed, if there exists a number $R > 0$ such that all the solutions $u \in X$ of the equations $\lambda N(u) = u$ for $\lambda \in (0,1)$ are a priori bounded by R, i.e. $\|u\| < R$, then the condition (1.1) is trivially satisfied and thus, according to Theorem 1.1, the equation (1.2) has at least one solution satisfying $\|u\| \leq R$. There is a huge literature devoted to the applications of the Leray–Schauder continuation principle to lots of classes of nonlinear problems, see [11], [12]. Variational versions of the Leray–Schauder principle are due to Schechter [15], [16] (for the role of the Leray–Schauder boundary condition in critical point theory, see also [13]). This kind of results allows to establish the existence and localization of solutions to (1.2), of a precise level of energy, when the equation (1.2) has a variational form, i.e. $N(u) = u - E'(u)$ for some C^1 (energy) functional $E\colon X \to \mathbb{R}$, where X is a Hilbert space identified to its dual and with inner product $(\,\cdot\,;\,\cdot\,)$. Clearly, in this case, the fixed points of the operator N coincide with the critical points of the functional E. In order to recall Schechter's results, we introduce some notions and notations.

We say that a C^1 functional $E\colon X_R \to \mathbb{R}$ satisfies the *Schechter–Palais–Smale condition at the level* λ, $(\mathrm{SPS})_\lambda$ for short, in X_R if any sequence of elements $u_k \in X_R \setminus \{0\}$ for which

$$E(u_k) \to \lambda, \qquad E'(u_k) - \frac{(E'(u_k); u_k)}{\|u_k\|^2}\, u_k \to 0, \qquad (E'(u_k); u_k) \to \nu \leq 0$$

as $k \to \infty$, has a convergent subsequence.

We say that the functional $E\colon X_R \to \mathbb{R}$ satisfies the *mountain pass geometry* in X_R if there are elements $u_0, u_1 \in X_R$ and a number $r > 0$ such that $\|u_0\| < r < \|u_1\|$ and

$$\inf\{\, E(u) : \|u\| = r\} > \max\{E(u_0), E(u_1)\}.$$

Let us introduce the following notations:

$$\Gamma_R = \{\gamma \in C([0,1]; X_R) : \gamma(0) = u_0,\, \gamma(1) = u_1\},$$
$$\xi_R = \inf_{\gamma\in\Gamma_R} \max_{t\in[0,1]} E(\gamma(t)), \qquad m_R = \inf_{u\in X_R} E(u).$$

We say that E is *bounded from below* in X_R, if $m_R > -\infty$.

With these definitions and notations Schechter's theorems read as follows [16]:

THEOREM 1.2. *Let $(X, (\,\cdot\,;\,\cdot\,))$ be a Hilbert space which is identified to its dual, $R > 0$ and let $E\colon X_R \to \mathbb{R}$ be a C^1 functional with $(E'(u); u) \geq -\nu_0$ for $\|u\| = R$ and some $\nu_0 > 0$. If E has the mountain pass geometry in X_R, satisfies the $(\mathrm{SPS})_{\xi_R}$ condition, and the Leray–Schauder boundary condition*

$$E'(u) + \mu u \neq 0 \quad \text{for } \|u\| = R \text{ and } \mu > 0, \tag{1.3}$$

then E has at least one critical point $u \in X_R \setminus \{u_0, u_1\}$ with $E(u) = \xi_R$.

THEOREM 1.3. *Let $(X, (\,\cdot\,;\,\cdot\,))$ be a Hilbert space which is identified to its dual, $R > 0$ and let $E\colon X_R \to \mathbb{R}$ be a C^1 functional with $(E'(u); u) \geq -\nu_0$ for $\|u\| = R$ and some $\nu_0 > 0$. If E is bounded from below in X_R, satisfies the $(\mathrm{SPS})_{m_R}$ condition, and the Leray–Schauder boundary condition* (1.3), *then E has at least one critical point $u \in X_R$ with $E(u) = m_R$.*

In [13] these results were extended to Hilbert spaces not identified to their duals and for the localization of critical points in a wedge (particularly, in the whole space or in a cone). Also, in [14], the results were completed in order to localize critical points in annular conical sets and obtain multiplicity of positive solutions. In the present paper we shall go further, namely we shall extend Theorems 1.2, 1.3 to uniformly convex Banach spaces. As has already been remarked in [9], the extension from Hilbert to Banach spaces is not immediate and requires a major refining of the reasoning based on the use of the duality map. Notice that in [9] only Theorem 1.3 was extended to general Banach spaces, and this extension was done by a completely different method using Ekeland's variational principle. Some related topics can be found in the recent paper [10]. The theory that is developed in the present paper is then applied to positive solutions for elliptic boundary value problems with p-Laplacian. Compared to [9], here we shall localize not only one positive solution but two: a minimum and a mountain pass type critical point. We can anticipate that a similar approach is possible to some other homogeneous operators including the Finsler–Laplace operator, see [1] and [5].

2. Main results

Let X be a real Banach space, X^* its dual, $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the duality between X^* and X. The norm on X and on X^* is denoted by $\|\cdot\|$. By a *wedge* of X we shall understand a convex closed nonempty set $K \subset X$, $K \neq \{0\}$, with $\lambda x \in K$ for every $x \in K$ and $\lambda \geq 0$. For a number $R > 0$, we denote:

$$X_R = \{x \in X : \|x\| \leq R\}, \qquad \partial X_R = \{x \in X : \|x\| = R\},$$
$$K_R = X_R \cap K, \qquad \partial K_R = K \cap \partial X_R.$$

Now we recall some geometric properties of Banach spaces and the notion of duality mapping. For details we refer to [2], [3], [7], [6].

A continuous function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a *normalization function*, if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$.

The *duality mapping on X corresponding to the normalization function φ* is the set-valued mapping $J_\varphi\colon X \to \mathcal{P}(X^*)$ defined by

$$J_\varphi x = \big\{x^* \in X^* : \langle x^*, x\rangle = \varphi(\|x\|)\|x\|,\ \|x^*\| = \varphi(\|x\|)\big\}, \quad x \in X.$$

The Banach space X is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta(\varepsilon))$.

Throughout this paper we suppose that the following assumption holds:

(A1) X and X^* are uniformly convex Banach spaces.

Under this assumption, the duality mapping is single-valued and bijective and both $J_\varphi, J_\varphi^{-1}$ are bounded continuous and monotone operators. In what follows, when there is no confusion, we shall denote J_φ and J_φ^{-1} simply by J and $\overline{J}$. Hence

$$\langle Jx, x\rangle = \varphi(\|x\|)\|x\|, \qquad \|Jx\| = \varphi(\|x\|)$$

and

$$\langle x^\star, \overline{J}x^\star\rangle = \varphi^{-1}(\|x^\star\|)\|x^\star\|, \qquad \|\overline{J}x^\star\| = \varphi^{-1}(\|x^\star\|) \tag{2.1}$$

for every $x \in X$ and $x^\star \in X^*$. Thus $\overline{J}$ appears as the duality mapping on X^* corresponding to the normalization function φ^{-1}. Notice that if $\varphi(t) = t^{p-1}$ $(p > 1)$, then $\varphi^{-1}(t) = t^{q-1}$, where $q = p/(p-1)$.

The following differentiability formula is useful.

Lemma 2.1. *Under assumption* (A1)*, if $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a normalization function, $\psi(t) = \int_0^t \varphi(s)$, and $\sigma \in C^1(\mathbb{R}_+; X)$, then*

$$\frac{d}{dt}\,\psi(\|\sigma(t)\|) = \langle J_\varphi \sigma(t), \sigma'(t)\rangle. \tag{2.2}$$

Proof. The proof is similar to that of Proposition 1.4.9 in [3, p. 29]. Clearly

$$\frac{d}{dt}\,\psi(\|\sigma(t)\|) = \varphi(\|\sigma(t)\|)\,\frac{d}{dt}\,\|\sigma(t)\|. \tag{2.3}$$

Also

$$\langle J_\varphi \sigma(t), \sigma(t)\rangle = \varphi(\|\sigma(t)\|)\|\sigma(t)\|$$

and

$$\langle J_\varphi\sigma(t),\sigma(s)\rangle \le \|J_\varphi\sigma(t)\|\|\sigma(s)\| = \varphi(\|\sigma(t)\|)\|\sigma(s)\|.$$

Substraction gives

$$\langle J_\varphi\sigma(t),\sigma(s)-\sigma(t)\rangle \le \varphi(\|\sigma(t)\|)[\|\sigma(s)\|-\|\sigma(t)\|].$$

If $s > t$, then

$$\left\langle J_\varphi\sigma(t),\frac{\sigma(s)-\sigma(t)}{s-t}\right\rangle \le \varphi(\|\sigma(t)\|)\,\frac{\|\sigma(s)\|-\|\sigma(t)\|}{s-t}$$

and letting $s\downarrow t$ we obtain

$$\langle J_\varphi\sigma(t),\sigma'(t)\rangle \le \varphi(\|\sigma(t)\|)\,\frac{d\|\sigma(t)\|}{dt}.$$

For $s < t$ we obtain the converse inequality. Hence

$$\langle J_\varphi\sigma(t),\sigma'(t)\rangle = \varphi(\|\sigma(t)\|)\,\frac{d\|\sigma(t)\|}{dt}. \tag{2.4}$$

Now (2.3) and (2.4) yield (2.2). □

For the rest of the paper we assume that the duality mapping J on X corresponds to the normalization function $\varphi(t)=t^{p-1}$, where $p>1$. The technical result which follows is a common generalization to Banach spaces of Lemma 5.9.1 in [16] and Lemma 4.1 in [13].

Lemma 2.2. *Assume that the Banach space X satisfies the condition* (A1) *and K is a wedge of X.*

(a) *Let $v\in X\setminus\{0\}$, $w^\star\in X^*\setminus\{0\}$, $\theta>0$ and $0<\alpha<1-\theta$. If*

$$-\langle w^\star,v\rangle \le \theta\|w^\star\|\|v\| \quad\text{and}\quad -\langle Jv,\overline{J}w^\star\rangle \le \theta\|\overline{J}w^\star\|\|Jv\|, \tag{2.5}$$

then there exists $h\in X$ with $\|h\|=1$ such that

$$\langle w^\star,h\rangle \le -\alpha\|w^\star\| \quad\text{and}\quad \langle Jv,h\rangle<0.$$

(b) *If in addition $v\in K$ and $v-\overline{J}w^\star\in K$, then there exists $\lambda^*>0$ such that $v+\mu h\in K$ for every $\mu\in[0,\lambda^*]$.*

(c) *Moreover, if $1-\theta<2\alpha$, $\|v\|\ge\rho>0$ and $\|\overline{J}w^\star\|\ge\omega>0$, then λ^* does not depend on v and $w^\star$.*

Proof. (a) Let

$$h_0 := -\frac{\overline{J}w^\star}{\|\overline{J}w^\star\|} - \beta\,\frac{v}{\|v\|}, \quad \text{where } \beta := \frac{1-\alpha}{\theta+\alpha}. \tag{2.6}$$

Since $\alpha<1-\theta$, one has $\beta>\theta$. Clearly $\|h_0\|\le 1+\beta$. Let us show that $h_0\ne 0$. Assume the contrary. Then from (2.6) and (2.5),

$$-\beta\,\frac{\langle w^\star,v\rangle}{\|v\|} = \frac{\langle w^*,\overline{J}w^\star\rangle}{\|\overline{J}w^\star\|} = \frac{\|w^\star\|^q}{\|w^\star\|^{q-1}} \le \beta\theta\|w^\star\|,$$

where $q = p/(p-1)$. It follows that $1 \leq \beta\theta$, which is false since $\beta\theta < 1$. Hence $h_0 \neq 0$. Next

$$\begin{aligned}\langle w^\star, h_0\rangle &= -\,\|w^\star\| - \beta\,\frac{\langle w^\star, v\rangle}{\|v\|} \leq -\|w^\star\| + \theta\beta\|w^\star\| \\ &= (\theta\beta - 1)\|w^\star\| = -\alpha(1+\beta)\|w^\star\| \leq -\alpha\|w^\star\|\|h_0\|.\end{aligned}$$

Thus, if we take $h := h_0/\|h_0\|$, we have $\langle w^\star, h\rangle \leq -\alpha\|w^\star\|$. Furthermore, since $\beta > \theta$,

$$\begin{aligned}\langle Jv, h_0\rangle &= -\frac{\langle Jv, \overline{J}w^\star\rangle}{\|\overline{J}w^\star\|} - \beta\,\frac{\langle Jv, v\rangle}{\|v\|} \\ &= -\frac{\langle Jv, \overline{J}w^\star\rangle}{\|\overline{J}w^\star\|} - \beta\|v\|^{p-1} \leq (\theta - \beta)\|v\|^{p-1} < 0.\end{aligned}$$

Hence $\langle Jv, h\rangle < 0$ as desired.

(b) We have

$$\begin{aligned}v + \mu h &= v - \frac{1}{\|h_0\|}\,\mu\bigg(\frac{\overline{J}w^\star}{\|\overline{J}w^\star\|} + \beta\,\frac{v}{\|v\|}\bigg) \\ &= \frac{\mu}{\|h_0\|\|\overline{J}w^\star\|}\,(v - \overline{J}w^\star) + \bigg(1 - \frac{\mu}{\|h_0\|\|\overline{J}w^\star\|} - \frac{\mu\beta}{\|h_0\|\|v\|}\bigg)v.\end{aligned}$$

Since $v, v - \overline{J}w^\star \in K$, this shows that $v + \mu h \in K$ for all small enough $\mu > 0$ such that

$$1 - \frac{\mu}{\|h_0\|\|\overline{J}w^\star\|} - \frac{\mu\beta}{\|h_0\|\|v\|} \geq 0, \tag{2.7}$$

that is

$$\mu \leq \frac{\|h_0\|}{1/\|\overline{J}w^\star\| + \beta/\|v\|}.$$

(c) If $1 - \theta < 2\alpha$, then $\beta < 1$. Also $\|h_0\| \geq 1 - \beta > 0$. Then for $\|v\| \geq \rho > 0$ and $\|\overline{J}w^\star\| \geq \omega > 0$, one has

$$\frac{\|h_0\|}{1/\|\overline{J}w^\star\| + \beta/\|v\|} \geq \frac{1-\beta}{1/\omega + \beta/\rho}.$$

Hence the condition (2.7) is guaranteed for every $\mu \in [0, \lambda^*]$, where

$$\lambda^* := \frac{1-\beta}{1/\omega + \beta/\rho}.$$

Remark 2.3. If we denote $\langle u, v\rangle_+ := \langle Jv, u\rangle$ and $[u,v] := \min\{\langle u, v\rangle_+, \langle v, u\rangle_+\}$, then the two inequalities in (2.5) can be put under a single inequality as follows:

$$-\bigg[\frac{v}{\|v\|}, \overline{J}\bigg(\frac{w^*}{\|w^*\|}\bigg)\bigg] \leq \theta.$$

The next technical result extends to Banach spaces Lemma 4.2 from [13].

Lemma 2.4. *Let $G\colon K_R \to X^*$ be continuous and $a > 0$. Consider $\widehat{D} := \{u \in K_R : \|Gu\| \geq a\}$, and a closed set $D_0 \subset \{u \in \widehat{D} : \|u\| = R\}$. Assume that $u - \overline{J}Gu \in K$ for all $u \in K_R$ and there is $\theta \in [0,1)$ such that*

$$-\langle Gu, u\rangle \leq \theta\|u\|\|Gu\| \quad \textit{and} \quad -\langle Ju, \overline{J}Gu\rangle \leq \theta\|Ju\|\|\overline{J}Gu\| \quad \textit{for all } u \in D_0.$$

Then there exists $\alpha > 0$ and a locally Lipschitz map $H\colon \widehat{D} \to X$ such that $\|Hu\| \leq 1$, $u + Hu \in K$,

$$\begin{aligned} \langle Gu, Hu\rangle &\leq -\alpha\|Gu\|, && u \in \widehat{D}, \\ \langle Ju, Hu\rangle &< 0, && u \in D_0. \end{aligned}$$

Proof. Let $0 < \alpha' < 1 - \theta < 2\alpha'$. We look for a mapping $h\colon \widehat{D} \to X$ with $\|h(u)\| = 1$ for all $u \in \widehat{D}$, such that

$$\begin{aligned} h(u) &= -\|\overline{J}Gu\|^{-1}\overline{J}Gu, && u \in \widehat{D} \setminus D_0, \\ \langle Gu, h(u)\rangle &\leq -\alpha'\|Gu\|, && u \in \widehat{D}, \\ \langle Ju, h(u)\rangle &< 0, && u \in D_0, \\ u + \mu h(u) &\in K && \text{for all } u \in \widehat{D} \text{ and } \mu \in [0, \lambda^*], \end{aligned}$$

where $\lambda^* = \lambda^*(\theta, \alpha', a) > 0$. For $u \in D_0$, $h(u)$ is given by Lemma 2.2 applied to $v = u$ and $w^* = G(u)$. Now we check the last two properties for $u \in \widehat{D} \setminus D_0$. Thus, in such case,

$$\langle Gu, h(u)\rangle = -\frac{1}{\|\overline{J}G(u)\|}\,\langle Gu, \overline{J}Gu\rangle = -\|Gu\| \leq -\alpha'\|Gu\|.$$

Also

$$u + \mu h(u) = \left(1 - \frac{\mu}{\|\overline{J}Gu\|}\right)u + \frac{\mu}{\|\overline{J}Gu\|}\,(u - \overline{J}Gu)$$

and since $\|\overline{J}Gu\| \geq a^{q-1}$, we have $u + \mu h(u) \in K$ for all $\mu \in [0, a^{q-1}]$. Thus h has the required properties for some $\lambda^* > 0$. Clearly we may assume that $\lambda^* < 1$. Next, based on this possibly noncontinuous mapping h, we shall construct the desired locally Lipschitz map H. Let $\alpha'' \in (0, \alpha')$ be a fixed number. Because G is continuous, it follows that for every $u \in \widehat{D}$, there exists a neighborhood $V(u) \subset \widehat{D}$ of u, such that

$$\langle Gv, h(u)\rangle \leq -\alpha''\|Gv\|, \quad \text{for all } v \in V(u).$$

If $u \in D_0$, taking into account the continuity of J, we may assume that

$$\langle Jv, h(u)\rangle < 0, \quad \text{for all } v \in V(u).$$

For $u \in \widehat{D} \setminus D_0$, we may take the neighbourhood $V(u)$ of u such that $V(u) \cap D_0 = \emptyset$. Also we may assume that $\operatorname{diam} V(u) \leq r$, for every $u \in \widehat{D}$ and some $r > 0$.

We have that $\{V(u) : u \in \widehat{D}\}$ is an open covering of $\widehat{D}$. Because $\widehat{D}$ is paracompact, it admits a local finite refinement $\{V_\tau\}$. Let $\{\psi_\tau\}$ be a locally

Lipschitz partition of unity subordinated to $\{V_\tau\}$. For every τ, let $u_\tau \in \widehat{D}$ be an element with $V_\tau \subset V(u_\tau)$ and let $b(u_\tau) = u_\tau + \lambda^\star h(u_\tau)$. Clearly $b(u_\tau) \in K$ for every τ. Now we define the locally Lipschitz map $H\colon \widehat{D} \to X$ by

$$Hv = -v + \sum_\tau \psi_\tau(v) b(u_\tau).$$

Clearly

$$v + Hv = \sum_\tau \psi_\tau(v) b(u_\tau) \in K, \quad \text{for all } v \in \widehat{D}.$$

For every $v \in \widehat{D}$ and $r \in (0, 1 - \lambda^\star]$, we have

$$\|Hv\| = \left\| \sum_\tau \psi_\tau(v)(b(u_\tau) - u_\tau) + \sum_\tau \psi_\tau(v)(u_\tau - v) \right\| \le \lambda^\star + r \le 1.$$

Furthermore

$$\begin{aligned}\langle Gv, Hv\rangle &= -\langle Gv, v\rangle + \sum_\tau \psi_\tau(v)\langle Gv, b(u_\tau)\rangle \\ &= \lambda^\star \sum_\tau \psi_\tau(v)\langle Gv, h(u_\tau)\rangle + \sum_\tau \psi_\tau(v)\langle Gv, u_\tau - v\rangle \\ &\le -\lambda^\star\alpha''\|Gv\| + r\|Gv\| \le -(\lambda^\star\alpha'' - r)\|Gv\|.\end{aligned}$$

Hence $\langle Gv, Hv\rangle \le -\alpha\|Gv\|$, where $\alpha := \lambda^*\alpha'' - r > 0$ and $r < \lambda^*\alpha''$.

Next, if $v \in D_0$, then

$$\langle Jv, Hv\rangle = \lambda^* \sum_\tau \psi_\tau(v)\langle Jv, h(u_\tau)\rangle + \sum_\tau \psi_\tau(v)\langle Jv, u_\tau - v\rangle. \tag{2.8}$$

We have $\langle Jv, h(u_\tau)\rangle < 0$ whenever $v \in V(u_\tau)$. Hence the first sum in (2.8) is strictly less than zero. Also, if $v \in V(u_\tau)$, since $v \in D_0$, we have $u_\tau \in D_0$ and so $\|v\| = \|u_\tau\| = R$. Then

$$\langle Jv, u_\tau - v\rangle = \langle Jv, u_\tau\rangle - \varphi(\|v\|)\|v\| \le \varphi(\|v\|)\|u_\tau\| - \varphi(\|v\|)\|v\| = 0.$$

Hence the second sum in (2.8) is less than or equal to zero. Hence

$$\langle Jv, Hv\rangle < 0, \quad \text{for all } v \in D_0. \qquad \square$$

Before stating the main results concerning the existence of $(\mathrm{SPS})_\mu$ sequences, we recall a global existence result for flows in Banach spaces, see [4].

Lemma 2.5. *Let X be a Banach space and let D be a closed convex set in X. Assume that $W\colon D \to X$ is a locally Lipschitz map such that*

$$\|W(u)\| \le C, \qquad \liminf_{\lambda\to 0^+} d(u + \lambda W(u), D) = 0$$

for all $u \in D$ and some constant C. Then, for any $u \in D$, the initial value problem in Banach space

$$\frac{d\sigma}{dt} = W(\sigma), \qquad \sigma(0) = u$$

has a unique solution $\sigma(u,t)$ on $\mathbb{R}_+$ and $\sigma(u,t) \in D$ for every $t \in \mathbb{R}_+$.

Now we are ready to state and prove the extension to general Banach spaces of two bounded critical point theorems of Schechter (Theorems 5.3.1 and 5.3.3 in [16]). We shall work more generally in a wedge K of a Banach space X. This requires that X_R be everywhere replaced by K_R, including the definitions of Γ_R, ξ_R and m_R. In what follows E will be a C^1 functional on X.

Theorem 2.6. *Assume that*

$$u - \overline{J}E'(u) \in K \qquad \text{for all } u \in K, \tag{2.9}$$

$$\min\{\langle E'(u), u\rangle, \langle Ju, \overline{J}E'(u)\rangle\} \geq -\nu_0 \quad \text{for all } u \in \partial K_R, \tag{2.10}$$

and some $\nu_0 > 0$. In addition assume that E has the mountain pass property in K_R. Then there exists a sequence of elements (u_k) such that

$$E(u_k) \to \xi_R \tag{2.11}$$

and one of the following statements holds:

(a) $u_k \in K_R$ *for all k and*

$$E'(u_k) \to 0; \tag{2.12}$$

(b) $u_k \in \partial K_R$ *for all k and*

$$E'(u_k) - \frac{\langle E'(u_k), u_k\rangle}{R^p} Ju_k \to 0, \qquad \langle E'(u_k), u_k\rangle \leq 0; \tag{2.13}$$

(c) $u_k \in \partial K_R$ *for all k and*

$$\overline{J}E'(u_k) - \frac{\langle Ju_k, \overline{J}E'(u_k)\rangle}{R^p} u_k \to 0, \qquad \langle Ju_k, \overline{J}E'(u_k)\rangle \leq 0. \tag{2.14}$$

Proof. Assume that a sequence satisfying (2.11) and (b) does not exist. Then there are $a, \delta > 0$ such that

$$\left\| E'(u) - \frac{(E'(u), u)}{R^p} Ju \right\| \geq a$$

when $u \in \partial K_R$, $|E(u) - \xi_R| \leq \delta$ and $\langle E'(u), u\rangle \leq 0$. Since X^* is assumed uniformly convex, and $\overline{J}$ is the duality mapping on X^* corresponding to the normalization function t^{q-1}, $q = p/(p-1)$, for any $r > 0$, there exists a continuous strictly increasing convex function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$, $g(0) = 0$ such that (see [2, p. 40])

$$\|x + y\|^q - \|x\|^q \geq q\langle y, \overline{J}x\rangle + g(\|y\|) \tag{2.15}$$

for all $x, y \in X^*$ with $\|x\| \leq r$ and $\|y\| \leq r$ (see [2, p. 40]). Take $x := \eta Ju$ and $y := E'(u) - \eta Ju$ with $\eta := \langle E'(u), u\rangle / R^p$. Then

$$\langle y, \overline{J}x\rangle = \langle E'(u) - \eta Ju, \overline{J}(\eta Ju)\rangle = |\eta|^{q-2}\eta\langle E'(u) - \eta Ju, u\rangle = 0.$$

So (2.15) gives

$$\begin{aligned}\|E'(u)\|^q - R^{-p(q-1)}|\langle E'(u),u\rangle|^q &= \|E'(u)\|^q - |\eta|^q R^p\\ &\geq g(\|E'(u)-\eta Ju\|) \geq g(a) > 0.\end{aligned}$$

Thus

$$R^{p(q-1)}\|E'(u)\|^q - |\langle E'(u),u\rangle|^q \geq R^{p(q-1)}g(a).$$

Let $\theta > 0$ be such that

$$\frac{1}{\theta^q} - 1 \leq \nu_0^{-q} R^{p(q-1)} g(a).$$

Then, also using $0 \geq \langle E'(u),u\rangle \geq -\nu_0$, we deduce

$$\begin{aligned}|\langle E'(u),u\rangle|^q\bigg(\frac{1}{\theta^q}-1\bigg) &\leq |\langle E'(u),u\rangle|^q \nu_0^{-q}\big(R^{p(q-1)}\|E'(u)\|^q - |\langle E'(u),u\rangle|^q\big)\\ &\leq R^{p(q-1)}\|E'(u)\|^q - |\langle E'(u),u\rangle|^q.\end{aligned}$$

It follows that

$$|\langle E'(u),u\rangle|^q \leq \theta^q R^{p(q-1)}\|E'(u)\|^q.$$

Hence, since $p(q-1)=q$, for all $u \in \partial K_R$ with $|E(u)-\xi_R| \leq \delta$, $\langle E'(u),u\rangle \leq 0$, we have $-\langle E'(u),u\rangle \leq \theta R\|E'(u)\|$.

Next assume that there are no sequences satisfying (2.11) and (c). We may assume that

$$\bigg\|\overline{J}E'(u) - \frac{\langle Ju,\overline{J}E'(u)\rangle}{R^p}\,u\bigg\| \geq a$$

when $u \in \partial K_R$, $|E(u)-\xi_R| \leq \delta$ and $\langle Ju,\overline{J}E'(u)\rangle \leq 0$. Similarly, since X is uniformly convex, for any $r > 0$, there exists a continuous strictly increasing convex function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$, $h(0)=0$ such that

$$\|x+y\|^p - \|x\|^p \geq p\langle Jx,y\rangle + h(\|y\|),$$

for all $x,y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$. Then, with the choice $x = \eta u$, $y = \overline{J}E'(u) - \eta u$ and $\eta = \langle Ju,\overline{J}E'(u)\rangle/R^p$, we find that $\langle Jx,y\rangle = 0$ and

$$\|\overline{J}E'(u)\|^p - |\eta|^p\|u\|^p \geq h(\|\overline{J}E'(u)-\eta u\|) \geq h(a) > 0.$$

Thus

$$R^{p(p-1)}\|\overline{J}E'(u)\|^p - |\langle Ju,\overline{J}E'(u)\rangle|^p \geq R^{p(p-1)}h(a).$$

Assume that

$$\frac{1}{\theta^p} - 1 \leq \nu_0^{-p} R^{p(p-1)} h(a).$$

Then

$$\begin{aligned}&|\langle Ju,\overline{J}E'(u)\rangle|^p\bigg(\frac{1}{\theta^p}-1\bigg)\\ &\quad\leq |\langle Ju,\overline{J}E'(u)\rangle|^p \nu_0^{-p}\big(R^{p(p-1)}\|\overline{J}E'(u)\|^p - |\langle Ju,\overline{J}E'(u)\rangle|^p\big)\\ &\quad\leq R^{p(p-1)}\|\overline{J}E'(u)\|^p - |\langle Ju,\overline{J}E'(u)\rangle|^p.\end{aligned}$$

It follows that

$$|\langle Ju, \overline{J}E'(u)\rangle|^p \le \theta^p R^{p(p-1)} \|\overline{J}E'(u)\|^p.$$

Thus, for all $u \in \partial K_R$ with $|E(u) - \xi_R| \le \delta$ and $\langle Ju, \overline{J}E'(u)\rangle \le 0$, we have

$$-\langle Ju, \overline{J}E'(u)\rangle \le \theta R^{p-1} \|\overline{J}E'(u)\|.$$

Hence

$$-\langle E'(u), u\rangle \le \theta \|u\| \|E'(u)\| \quad \text{and} \quad -\langle Ju, \overline{J}E'(u)\rangle \le \theta \|Ju\| \|\overline{J}E'(u)\| \tag{2.16}$$

for all $u \in \partial K_R$ with $|E(u) - \xi_R| \le \delta$. Next assume that there are no sequences satisfying (2.11) and (a). Then we may assume that $\|E'(u)\| \ge a$ for all u in

$$Q := \{u \in K_R : |E(u) - \xi_R| \le 3\delta\}.$$

Clearly we may assume that $3\delta < \xi_R - \max\{E(u_0), E(u_1)\}$ and that (2.16) holds in $\widetilde{Q} := Q \cap \partial K_R$. Let

$$\begin{aligned} Q_0 &:= \{u \in K_R : |E(u) - \xi_R| \le 2\delta\}, \\ Q_1 &:= \{u \in K_R : |E(u) - \xi_R| \le \delta\}, \\ Q_2 &:= K_R \setminus Q_0, \end{aligned}$$

and let

$$\eta(u) = \frac{d(u, Q_2)}{d(u, Q_1) + d(u, Q_2)}.$$

Clearly $\eta(u) = 1$ in Q_1, $\eta(u) = 0$ in Q_2 and $0 < \eta(u) < 1$ otherwise.

Applying Lemma 2.4 to $Gu := E'(u)$ and $D_0 := \widetilde{Q}$, we find an $\alpha > 0$ and a locally Lipschitz map $H\colon \widehat{D} \to X$, where $\widehat{D} = \{u \in K_R : \|E'(u)\| \ge a\}$, such that

$$\begin{aligned} \|H(u)\| &\le 1, && u + H(u) \in K \text{ for all } u \in \widehat{D}, \\ \langle E'(u), H(u)\rangle &\le -\alpha \|E'(u)\|, && u \in \widehat{D}, \\ \langle Ju, H(u)\rangle &< 0, && u \in \widetilde{Q}. \end{aligned}$$

Let $W\colon K_R \to X$ be given by

$$W(u) = \begin{cases} \eta(u)H(u) & \text{for } u \in \widehat{D}, \\ 0 & \text{for } u \in K_R \setminus \widehat{D}. \end{cases}$$

This map can be extended to a locally Lipschitz map on the whole K, by

$$W(u) = W\left(\frac{R}{\|u\|}\, u\right) \quad \text{for } u \in K,\ \|u\| > R.$$

Let σ be the flow generated by W as shown by Lemma 2.5. From Lemma 2.1 it follows that

$$\frac{d\|\sigma(u,t)\|^p}{dt} = p\left\langle J\sigma(u,t), \frac{d\sigma(u,t)}{dt}\right\rangle = p\eta(\sigma)\langle J\sigma, H(\sigma)\rangle. \tag{2.17}$$

Now we prove that for every $u \in K_R$, the flow $\sigma(u,\,\cdot\,)$ does not exit K_R. Suppose that $\|\sigma(u,t)\| \le R$ for all $t \in [0,t_0)$ and $u_1 := \sigma(u,t_0) \in \partial K_R$ for some $t_0 \in \mathbb{R}_+$. If $u_1 \in \widetilde{Q}$, then $\langle Ju_1, H(u_1)\rangle < 0$ and (2.17) gives

$$\frac{d\|\sigma(u,t)\|^p}{dt} \le 0$$

in a neighbourhood $[t_0, t_0+\varepsilon)$ of t_0. In consequence, $\|\sigma(u,\,\cdot\,)\|$ is nonincreasing in the interval $[t_0,t_0+\varepsilon)$. If $u_1 \notin \widetilde{Q}$, then $\eta(\sigma(u,t)) = 0$ in some vicinity of t_0, and the conclusion for $\|\sigma(u,\,\cdot\,)\|$ is as above. Therefore $\sigma(u,\,\cdot\,)$ does not exit K_R for $t \in \mathbb{R}_+$.

Using Lemma 2.4 we have

$$\begin{aligned}\frac{dE(\sigma(u,t))}{dt} &= \left\langle E'(\sigma(u,t)), \frac{d\sigma(u,t)}{dt}\right\rangle \\ &= \eta(\sigma(u,t))\langle E'(\sigma(u,t)), H(\sigma(u,t))\rangle \le -\eta(\sigma(u,t))\alpha a,\end{aligned} \tag{2.18}$$

which shows that $E(\sigma(u,\,\cdot\,))$ is a decreasing function.

For any number λ, denote by E_λ the level set $E_\lambda = \{u \in K_R : E(u) \le \lambda\}$. Let $t_1 > 2\delta/(\alpha a)$ and $u \in E_{\xi_R+\delta}$ be an arbitrary element. If there is $t_0 \in [0,t_1]$ with $\sigma(u,t_0) \notin Q_1$, then

$$E(\sigma(u,t_1)) \le E(\sigma(u,t_0)) < \xi_R - \delta.$$

Hence $\sigma(u,t_1) \in E_{\xi_R-\delta}$. Otherwise, $\sigma(u,t) \in Q_1$ for all $t \in [0,t_1]$, and so $\eta(\sigma(u,t)) = 1$. Then (2.18) yields

$$E(\sigma(u,t_1)) \le E(u) - \alpha a t_1 < \xi_R + \delta - 2\delta = \xi_R - \delta.$$

Therefore

$$\sigma(E_{\xi_R+\delta}, t_1) \subset E_{\xi_R-\delta}. \tag{2.19}$$

Now by definition of ξ_R, there is a $\gamma \in \Gamma_R$ with

$$\gamma(t) \in E_{\xi_R+\delta} \quad \text{for all } t \in [0,1]. \tag{2.20}$$

We define a new path γ_1 joining u_0 and u_1, by

$$\gamma_1(t) = \sigma(\gamma(t), t_1), \quad t \in [0,1].$$

Since η vanishes in the neighbourhood of u_0 and u_1, we have that $\sigma(u_0,t) \equiv u_0$ and $\sigma(u_1,t) \equiv u_1$. Hence $\gamma_1(0) = u_0$, $\gamma_1(1) = u_1$ and so $\gamma_1 \in \Gamma_R$. On the other hand, from (2.19) and (2.20) it follows that $E(\gamma_1(t)) \le \xi_R - \delta$ for all $t \in [0,1]$, which contradicts the definition of ξ_R. □

Notice that in case of a Hilbert space identified with its dual, when J is the identity mapping, the statements (b) and (c) coincide and the three alternatives in Theorem 2.6 reduce to only two as in the original theorem of Schechter.

A similar result holds for m_R replacing ξ_R. It extends to Banach spaces Theorem 2.2 in [13].

THEOREM 2.7. *Assume that conditions* (2.9) *and* (2.10) *hold. In addition assume that E is bounded from below in K_R. Then there exists a sequence of elements $u_k \in K_R$ such that*

$$E(u_k) \to m_R \tag{2.21}$$

and one of the statements (a), (b), (c) *holds.*

PROOF. The proof can be reproduced after that of Theorem 2.6 by replacing ξ_R with m_R and making a mirror modification after relation (2.18), as described below. Indeed, using the relation (2.18) i.e.

$$\frac{dE(\sigma(u,t))}{dt} \le -\eta(\sigma(u,t))\alpha a,$$

if we fix any $u \in Q_1 := \{v \in K_R : E(v) \le m_R + \delta\}$ and take $t_1 > 2\delta/(\alpha a)$, one can deduce that $\sigma(u,t) \in Q_1$ for all $t \ge 0$ and

$$E(\sigma(u,t_1)) \le E(\sigma(u,0)) - \alpha a t_1 = E(u) - \alpha a t_1 \le m_R + \delta - \alpha a t_1 < m_R - \delta,$$

contradicting the definition of m_R. □

DEFINITION 2.8. We say that E satisfies the $(\mathrm{SPS})_\mu$ condition in K_R for some $\mu \in \mathbb{R}$, if every sequence of elements $u_k \in K_R$ with $E(u_k) \to \mu$ which satisfies any of the conditions (a), (b), (c), contains a convergent subsequence.

Under the above compactness condition, Theorems 2.6 and 2.7 yield the following critical point results in K_R.

THEOREM 2.9. *Under the assumptions of Theorem* 2.6*, if in addition E satisfies the* $(\mathrm{SPS})_{\xi_R}$ *condition in K_R, and the boundary condition*

$$E'(u) + \mu Ju \ne 0 \quad \textit{for all } u \in \partial K_R \textit{ and } \mu > 0, \tag{2.22}$$

then E has a critical point u in K_R with $E(u) = \xi_R$.

PROOF. Let (u_k) be a sequence as in Theorem 2.6 and let u be the limit of its convergent subsequence guaranteed by the $(\mathrm{SPS})_{\xi_R}$ condition. If we are in the case (a) we are finished. Assume the case (b). Since $\langle E'(u_k), u_k\rangle \in [-\nu_0, 0]$, passing if necessary to another subsequence we may assume that $-\langle E'(u_k), u_k\rangle/R^p \to \mu \ge 0$. Then $E'(u) + \mu Ju = 0$. The case $\mu > 0$ is excluded by hypothesis. It remains that $\mu = 0$ and we are finished again. Assume (c). As above, we may assume that $-\langle Ju_k, \overline{J}E'(u_k)\rangle/R^p \to \mu \ge 0$. Then $\overline{J}E'(u) + \mu u = 0$. In case that $\mu > 0$, from $\overline{J}E'(u) = -\mu u$ we deduce $E'(u) = J(-\mu u) = -\mu^{p-1}J(u)$. Hence $E'(u) + \mu^{p-1}Ju = 0$, which contradicts the hypothesis. Hence $\mu = 0$, and then $\overline{J}E'(u) = 0$, whence $E'(u) = 0$ as desired. □

THEOREM 2.10. *Under the assumptions of Theorem* 2.7*, if in addition E satisfies the* $(\mathrm{SPS})_{m_R}$ *condition in K_R, and the boundary condition* (2.22)*, then E has a critical point u in K_R with $E(u) = m_R$.*

3. Application

We consider the boundary value problem

$$\begin{cases} \Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Denote by c_q $(1 \le q \le p^*)$ the embedding constant for $W_0^{1,p}(\Omega) \subset L^q(\Omega)$. Using Theorems 2.9 and 2.10 we obtain the following result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function. Assume that the following conditions are satisfied:*

(a) *There exist constants $a, b > 0$ and $q \in [1, p^*)$ such that*

$$f(\tau) \le a\tau^{q-1} + b \quad \textit{for all } \tau \in \mathbb{R}_+;$$

(b) *there exist $\tau_0 > 0$, $\alpha > p - 1$ and $c > 0$ such that*

$$f(\tau) \ge c\tau^{\alpha} \quad \textit{for all } \tau \in [0, \tau_0];$$

(c) *one has*

$$\limsup_{\tau \to 0+} \frac{f(\tau)}{\tau^{p-1}} < \lambda_1;$$

(c) *the following inequality is true:*

$$\frac{1}{p} - \frac{c}{\alpha + 1}\tau_0^{\alpha - p + 1} \int_{(\phi \le 1)} \phi(x)^{\alpha+1} dx \le 0;$$

(e) *there exists $R \ge \tau_0$ with*

$$R^{p-1} \ge a c_q^q R^{q-1} + b c_1.$$

Then (3.1) has at least one solution satisfying $0 < |u|_{W_0^{1,p}(\Omega)} \le R$. In case that $f(0) > 0$, (3.1) has at least two such solutions.

Remark 3.2. (a) If f is such that $f(\tau) \ge c\tau^{\alpha}$ for $\tau \in [0, \tau_0]$ and some $\alpha \ge 0$ and $c > 0$, and conditions (c) and (d) in Theorem 3.1 hold, then necessarily $\alpha > p - 1$. Indeed, if $\alpha < p - 1$, then from $c\tau^{\alpha - p + 1} \le f(\tau)/\tau^{p-1}$ we deduce $\limsup_{\tau \to 0+} f(\tau)/\tau^{p-1} = \infty$, which contradicts (c). Also, if $\alpha = p - 1$, then (c) gives $c < \lambda_1$, while (d) shows that $c \ge 1/\int_{(\phi \le 1)} \phi(x)^p \, dx$. Since $1 = |\phi|_{W_0^{1,p}(\Omega)}$ and $\int_{(\phi \le 1)} \phi(x)^p \, dx \le |\phi|_{L^p(\Omega)}^p$, we have $1/\int_{(\phi \le 1)} \phi(x)^p \, dx \ge |\phi|_{W_0^{1,p}(\Omega)}^p / |\phi|_{L^p(\Omega)}^p = \lambda_1$. Hence $c \ge \lambda_1$, which is contradictory.

(b) The condition (d) in Theorem 3.1 requires that for a given $c > 0$, the length τ_0 of the interval $[0, \tau_0]$ of "p-superliniarity" of f is large enough.

(c) If $q < p$, then the condition (e) in Theorem 3.1 holds with any sufficiently large R.

EXAMPLE 3.3. The typical example of a function satisfying the assumptions (a)–(d) is the following:

$$f(\tau) = \begin{cases} c\tau^\alpha & \text{for } \tau \in [0, \tau_0], \\ a(\tau - \tau_0)^{q-1} + c\tau_0^\alpha & \text{for } \tau > \tau_0, \end{cases}$$

where $a, c > 0$, $p - 1 < \alpha < p^*$, $1 \leq q < p^*$ and $\tau_0 > 0$ is large enough. Thus f is "p-superlinear" on a large enough interval $[0, \tau_0]$ and has a p-subcritical behavior on $[0, \infty)$.

PROOF OF THEOREM 3.1. Let $E\colon W_0^{1,p}(\Omega) \to \mathbb{R}_+$,

$$E(u) := \int_\Omega \left(\frac{1}{p} |\nabla u|^p - F(u) \right) dx$$

be the energy functional associated to (3.1), where $F(\tau) = \int_0^\tau f(s)\,ds$, and let $K := \{u \in W_0^{1,p}(\Omega) : u \geq 0\}$.

(1) E is bounded from below on the intersection of K with any ball of $W_0^{1,p}(\Omega)$. Indeed, if $u \in K$, then

$$E(u) = \frac{1}{p} |u|^p_{W_0^{1,p}(\Omega)} - \int_\Omega F(u)\,dx \geq \frac{1}{p} |u|^p_{W_0^{1,p}(\Omega)} - \frac{a}{q} |u|^q_{L^q(\Omega)} - b|u|_{L^1(\Omega)}.$$

Since $q \leq p^*$, the embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is continuous as well as $W_0^{1,p}(\Omega) \subset L^1(\Omega)$. Hence

$$E(u) \geq \frac{1}{p} |u|^p_{W_0^{1,p}(\Omega)} - \widetilde{a}|u|^q_{W_0^{1,p}(\Omega)} - \widetilde{b}|u|_{W_0^{1,p}(\Omega)}.$$

The function $x^p/p - \widetilde{a}x^q - \widetilde{b}x$ is bounded on each compact interval $[0, R]$, whence our claim.

(2) The mountain pass condition: Choose a number d such that

$$\frac{1}{p} \limsup_{\tau \to 0^+} \frac{f(\tau)}{\tau^{p-1}} < d < \frac{\lambda_1}{p}.$$

From (a) (here we use the strict inequality $q < p^*$) and (c), we deduce that

$$F(\tau) \leq d\tau^p + c_d\tau^{p^*}, \quad \text{for all } \tau \in \mathbb{R}_+.$$

Then, for every $u \in K$,

$$E(u) \geq \frac{|u|^p_{W_0^{1,p}(\Omega)}}{p} - \int_\Omega (du^p + c_d u^{p^*})\,dx \geq |u|^p_{W_0^{1,p}(\Omega)} \left(\frac{1}{p} - \frac{d}{\lambda_1} - \widetilde{c}_d |u|^{p^*-p}_{W_0^{1,p}(\Omega)} \right).$$

We have $1/p - d/\lambda_1 > 0$ and $p^* > p$. Then there exist $r \in (0, \tau_0)$ and $\gamma > 0$ such that

$$E(u) \geq \gamma > 0 \quad \text{for } u \in K \text{ with } |u|_{W_0^{1,p}(\Omega)} = r.$$

Furthermore, if we take $u_0 = 0$ and $u_1 = \tau_0\phi$, then $E(u_0) = 0$, while from (b),

$$F(\tau) \geq \frac{c}{\alpha + 1} \tau^{\alpha+1}, \quad \text{for } \tau \in [0, \tau_0]$$

and so

$$E(\tau_0\phi) = \frac{\tau_0^p}{p} - \int_\Omega F(\tau_0\phi)\,dx \le \frac{\tau_0^p}{p} - \frac{c\tau_0^{\alpha+1}}{\alpha+1}\int_{(\phi\le 1)} \phi(x)^{\alpha+1}\,dx \le 0.$$

Thus the mountain pass condition holds.

(3) The boundary condition: Assume that for some $u \in K$, with $|u|_{W_0^{1,p}(\Omega)} = R$, one has $E'(u) + \mu Ju = 0$ for some $\mu > 0$. Then $(1+\mu)\Delta_p u + f(u) = 0$, whence

$$(1+\mu)R^p = \int_\Omega uf(u)\,dx.$$

Using (a), we see that $uf(u) \le au^q + bu$, and so

$$\int_\Omega uf(u)\,dx \le a|u|_{L^q}^q + b|u|_{L^1} \le ac_q^q R^q + bc_1 R.$$

Then

$$R^p < (1+\mu)R^p \le ac_q^q R^q + bc_1 R,$$

which contradicts (e). Thus the boundary condition holds.

(4) The Palais–Smale condition: Let (u_k) satisfy (2.13). Passing to a subsequence we may assume that $-\langle E'(u_k), u_k\rangle / R^p \to \mu \ge 0$. Then

$$v_k := E'(u_k) + \mu Ju_k \to 0.$$

We have $v_k = -(1+\mu)\Delta_p u_k - f(u_k)$, whence

$$u_k = \overline{J}\bigg(\frac{1}{1+\mu}\,(v_k + f(u_k))\bigg).$$

Thus, in order that (u_k) has a convergent subsequence, it is enough to show that the sequence $(f(u_k))$ is relatively compact in $W^{-1,p}(\Omega)$. For this, we first note that according to (a), the superposition operator associated to f maps $L^{p^*}(\Omega)$ into $L^{p^*/(q-1)}(\Omega)$ and is a continuous bounded operator. Since (u_k) is bounded in $W_0^{1,p}(\Omega)$, the embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is bounded and the embedding $L^{p^*/(q-1)}(\Omega) \subset W^{-1,p}(\Omega)$ is compact (since $p^*/(q-1) > (p^*)'$ when $q < p^*$), we may infer that $(f(u_k))$ is relatively compact in $W^{-1,p}(\Omega)$, as we wished.

Let now (u_k) satisfy (2.14). Passing to a subsequence we may assume that $-\langle Ju_k, \overline{J}E'(u_k)\rangle / R^p \to \mu \ge 0$. The case $\mu = 0$ is similar to the previous one. It remains to discuss the case $\mu > 0$. If we let $v_k := \overline{J}E'(u_k) + \mu u_k$, then it is easy to see that

$$u_k = \frac{1}{\mu}\,v_k - \frac{1}{\mu}\,\overline{J}(Ju_k - f(u_k)).$$

Now it is clear that the relatively compactness of the sequence $(f(u_k))$ in $W^{-1,p}(\Omega)$, which follows as above, is enough to conclude about the same property of (u_k). The conclusion now follows from Theorems 2.9 and 2.10. □

REFERENCES

[1] M. BELLONI, V. FERONE AND B. KAWOHL, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*, Z. Angew. Math. Phys. **54** (2003), 771–783.

[2] C. CHIDUME, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Springer, London, 2009.

[3] I. CIORANESCU, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[4] K. DEIMLING, *Ordinary Differential Equations in Banach Spaces*, Springer, Berlin, 1977.

[5] F. DELLA PIETRA AND N. GAVITONE, *Anisotropic elliptic problems involving Hardy-type potential*, J. Math. Anal. Appl. **397** (2013), 800–813.

[6] J. DIESTEL, *Geometry of Banach Spaces — Selected Topics*, Springer, Berlin, 1975.

[7] G. DINCA, P. JEBELEAN AND J. MAWHIN, *Variational and topological methods for Dirichlet problems with p-Laplacian*, Port. Math. **58** (2001), 339–378.

[8] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer, New York, 2003.

[9] H. LISEI, R. PRECUP AND C. VARGA, *A Schechter type critical point result in annular conical domains of a Banach space and applications*, Discrete Contin. Dyn. Syst. **36** (2016), 3775–3789.

[10] R. KAJIKIYA, *Mountain pass theorem in ordered Banach spaces and its applications to semilinear elliptic equations*, Nonlinear Differential Equations Appl. **19** (2012), 159–175.

[11] D. O'REGAN AND R. PRECUP, *Theorems of Leray–Schauder Type*, Gordon and Breach, Amsterdam, 2001.

[12] R. PRECUP, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht, 2002.

[13] ______, *The Leray–Schauder boundary condition in critical point theory*, Nonlinear Anal. **71** (2009), 3218–3228.

[14] ______, *Critical point theorems in cones and multiple positive solutions of elliptic problems*, Nonlinear Anal. **75** (2012), 834–851.

[15] M. SCHECHTER, *A bounded mountain pass lemma without the (PS) condition and applications*, Trans. Amer. Math. Soc. **331** (1992), 681–703.

[16] M. Schechter, *Linking Methods in Critical Point Theory*, Birkhäuser, Boston, 1999.

Manuscript received May 11, 2016

accepted June 27, 2016

RADU PRECUP AND CSABA VARGA
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. Kogălniceanu nr. 1
400084 Cluj-Napoca, ROMANIA

E-mail address: r.precup@math.ubbcluj.ro, csvarga@cs.ubbcluj.ro