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THREE ZUTOT

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Abstract. Three topics in dynamical systems are discussed. First we deal with cascades and solve two open problems concerning, respectively, product recurrence, and uniformly rigid actions. Next we provide a new example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

Introduction

We collect in this paper three short notes ([1]). They are independent of each other and are collected here just because they occurred to us in recent discussions. The first two actually solve some open problems concerning, respectively, product recurrence, and uniformly rigid actions admitting a weakly mixing fully supported invariant probability measure. The third provides a new interesting example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

1. On product recurrence

A *dynamical system* here is a pair (X, T) where X is a compact metric space and T a self-homeomorphism. The reader is referred to [4] for most of the notions used below and for the necessary background.

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([1]) Zuta is minutia (or miniature) in Hebrew; zutot is the plural, minutiae.

In [4, Theorem 9.11, p. 181], Furstenberg has shown that a point x of a dynamical system (X,T) is *product-recurrent* (i.e. has the property that for every dynamical system (Y,S) and a recurrent point $y \in Y$, the pair (x,y) is a recurrent point of the product system $X \times Y$) if and only if it is a distal point (i.e. a point which is proximal only to itself). In [2], Auslander and Furstenberg posed the following question: if (x,y) is recurrent for all *minimal* points y, is x necessarily a distal point? Such a point x is called a *weakly product recurrent point.* This question is answered in the negative in [11].

It turns out (see also [3, Theorem 4.3]) that a negative answer was already at hand for Harry Furstenberg when he and Joe Auslander posed this question. In fact, many years earlier, he proved a theorem according to which an F-flow ([2]) is disjoint from every minimal system [4]. As a direct consequence of this theorem, if X is an F-flow, x a transitive point in X, Y any minimal system and y any point in Y, then the pair (x,y) has a dense orbit in $X \times Y$. In particular, (x,y) is a recurrent point of the product system $X \times Y$. Thus a transitive point x in an F-flow is weakly product recurrent. Since such a point is never distal, one concludes that x is indeed weakly product recurrent but not distal.

In [11, Question 5.3] the authors pose the following natural question:

Problem 1.1. Is every *minimal* weakly product recurrent point a distal point?

(This was also repeated in [3, Question 9.2].)

In this note we show that, here again, the answer is negative. The counter example is based on a result of [5] concerning POD systems and on the existence of doubly minimal systems (see [13] and [14]). A minimal dynamical system (X,T) is called *proximal orbit dense* (POD) if it is totally minimal and for any distinct points u and v in X, there exists $0 \neq n \in \mathbb{Z}$ such that $\Gamma_n = \{(T^n x, x) : x \in X\}$ is contained in $\overline{\mathcal{O}_{T\times T}(u,v)}$, the orbit closure of (u,v) in the product system $X \times X$.

A minimal (X,T) is called *doubly minimal* [14] (or a system having *topologically minimal self-joinings in the sense of del Junco* [13]) if the only orbit closures of $T \times T$ in $X \times X$ are the graphs $\Gamma_m = \{(T^m x, x) : x \in X\}$, $m \in \mathbb{Z}$, and all of $X \times X$. Clearly a doubly minimal system is POD. In [5], the authors prove the following striking property of POD systems:

Theorem 1.2. *If (Y,S) is POD then any minimal (X,T) that is not an extension of (Y,S) is disjoint from it.*

For the reader's convenience we reproduce the short proof:

([2]) Recall that a dynamical system (X,T) is an *F-flow* if it is (i) totally transitive (i.e. every power T^n, $n \neq 0$, is transitive) and (ii) the periodic points are dense in X. E.g. every weakly mixing finite type subshift is an F-flow.

Proof. Suppose Y is not a factor of X and let M be a minimal subset of $Y \times X$. Since X is not an extension of Y, there exist $y, y' \in Y$ with $y \neq y'$ and $x \in X$ such that $(y,x),(y',x) \in M$. From the POD property it follows that for some $z \in X$ and $n \neq 0$ the points (y,z) and $(T^n y, z)$ are both in M. This implies that $(T^n \times \mathrm{id}_X)M \cap M \neq \emptyset$ and, as M is minimal, it follows that $(T^n \times \mathrm{id}_X)M = M$. Finally, since Y is totally minimal we deduce that $M = Y \times X$, as required. □

We will strengthen this property for doubly minimal systems as follows:

Theorem 1.3. *If (Y,S) is doubly minimal and (X,T) is any minimal system then the orbit closure of any point $(y,x) \in Y \times X$ is either all of $Y \times X$ or it is the graph $\Gamma_\pi = \{(\pi(x),x) : x \in X\}$ of some factor map $\pi\colon X \to Y$.*

Proof. Let Y be a doubly minimal system. In particular, Y is weakly mixing and has the POD property. Let X be a minimal system. By [5], either X and Y are disjoint or Y is a factor of X. In the first case the product system $Y \times X$ is minimal.

So we now assume that there is a factor map $\pi\colon X \to Y$. We consider an arbitrary point $(y_0, x_1) \in Y \times X$ and denote $y_1 = \pi(x_1)$. We will denote the acting transformation on both X and Y by the letter T.

Case 1. $y_1 = T^n y_0$ for some $n \in \mathbb{Z}$. In this case the orbit closure $\overline{\mathcal{O}_{T\times T}(y_0,x_1)}$ has the form $\Gamma_{\pi\circ T^{-n}} = \{(\pi(x), T^n x) : x \in X\}$, and is isomorphic to X.

Case 2. $y_1 \notin \mathcal{O}(y_0)$. Recall that by double minimality we have in this case that $\overline{\mathcal{O}_{T\times T}(y_0,y_1)} = Y \times Y$. Also note that, as the union of the graphs $\bigcup_{n\in\mathbb{Z}} \Gamma_n$, where $\Gamma_n = \{(x, T^n x) : x \in X\}$ is dense in $X \times X$, the union of $\{(\pi(x), T^n x) : x \in X\}$ is dense in $Y \times X$.

Let (u,v) be an arbitrary point in $Y \times X$ and fix $\varepsilon > 0$.

(i) Choose a point $w \in X$ and $m \in \mathbb{Z}$ such that $(\pi(w), T^m w) \overset{\varepsilon}{\sim} (u,v)$.

(ii) Choose a sequence $n_i \in \mathbb{Z}$ such that

$$T^{n_i}(y_0, y_1) \to (\pi(w), T^m \pi(w)).$$

We can then assume that for some point $z \in X$

$$T^{n_i}(y_0, x_1) \to (\pi(w), z), \quad \text{with } \pi(z) = T^m \pi(w).$$

(iii) Choose a sequence $k_j \in \mathbb{Z}$ such that

$$T^{k_j} z \to T^m w, \quad \text{whence } T^{k_j}\pi(z) = T^{k_j} T^m \pi(w) \to T^m \pi(w),$$
$$\text{and } T^{k_j}\pi(w) \to \pi(w).$$

Now

$$\lim_j \lim_i T^{k_j} T^{n_i}(y_0, x_1) = \lim_j T^{k_j}(\pi(w), z) = (\pi(w), T^m w) \overset{\varepsilon}{\sim} (u, v).$$

Since $\varepsilon > 0$ is arbitrary we conclude that $(u, v) \in \overline{\mathcal{O}_{T\times T}(y_0, x_1)}$, hence

$$\overline{\mathcal{O}_{T\times T}(y_0, x_1)} = Y \times X. \qquad \square$$

As a corollary of this theorem and the fact that there are weakly mixing doubly minimal systems ([13] and [14]) we get a negative answer to Problem 1.1.

First note that a minimal weakly mixing system does not admit a distal point. One way to see this is via the fact that in a minimal weakly mixing system X, for every point $x \in X$ there is a dense G_δ subset $X_0 \subset X$ such that for every $x' \in X_0$ the pair (x, x') is proximal; see [4, Theorem 9.12], or [1] for an even stronger statement.

Theorem 1.4. *There exists a minimal dynamical system Y which is weakly mixing* (*hence, in particular, does not have distal points*) *yet it has the property that for every minimal system X every pair $(y, x) \in Y \times X$ is recurrent.*

Proof. Let Y be a weakly mixing doubly minimal system and X a minimal system. By [5], either X and Y are disjoint or Y is a factor of X. In the first case the product system $Y \times X$ is minimal and, in particular, every pair (y, x) is recurrent.

In the second case we have, by Theorem 1.3, that either

$$\overline{\mathcal{O}_{T\times T}(y, x)} = \Gamma_\pi = \{(\pi(z), z) : z \in X\},$$

for a factor map $\pi\colon X \to Y$; or, again, $\overline{\mathcal{O}_{T\times T}(y, x)} = Y \times X$. In both cases (y, x) is recurrent. This is clear in the first case, as then $(y, x) = (\pi(x), x)$. In the second case, as the orbit closure is the entire product space $Y \times X$, if the point (y, x) were isolated, it would follow that also y is an isolated point of the nontrivial minimal weakly mixing system Y, which is an absurd claim. $\square$

2. Uniform rigidity

Recall that a topological dynamical system (X, T), where X is a compact Hausdorff space and $T\colon X \to X$ a self-homeomorphism, is called *uniformly rigid* if there is a sequence $n_k \nearrow \infty$ such that the sequence of homeomorphisms T^{n_k} tends uniformly to the identity. This notion was formally introduced in [8] where such systems were analyzed. Building on examples constructed in [9] it was shown in [8] that there exist strictly ergodic (hence minimal) systems which are both topologically weakly mixing and uniformly rigid.

In the paper [12] the authors posed the question whether there are uniformly rigid systems which also admit a weakly mixing measure of full support (Question 3.1).

In the paper [10] we consider the infinite torus $\mathbb{T}^\infty$ and its group of self-homeomorphisms Homeo $(\mathbb{T}^\infty)$, equipped with the uniform convergence topology. We fix a minimal rotation σ on $\mathbb{T}^\infty$ and define a certain subgroup G of Homeo $(\mathbb{T}^\infty)$. Finally we let

$$\mathcal{S} = \operatorname{cls}\{g\sigma g^{-1} : g \in G\}$$

be the closure of the conjugacy class of σ under G. We then show the existence of a residual subset $\mathcal{R}$ of $\mathcal{S}$ such that each T in $\mathcal{R}$ is (i) strictly ergodic and (ii) measure weakly mixing with respect to the Haar measure on $\mathbb{T}^\infty$. See [10] for more details. Now this result, together with the observation in the paper [8, p. 319] that, automatically, a residual subset of $\mathcal{S}$ consists of uniformly rigid homeomorphisms, immediately yield a residual set of uniformly rigid, measure weakly mixing, strictly ergodic homeomorphisms.

A similar construction can be carried out also in the setup of [9] thus producing such examples on the 2-torus $\mathbb{T}^2$.

Theorem 2.1. *There exists a compact metric, strictly ergodic* (*hence minimal*)*, uniformly rigid dynamical system* (X, μ, T) *such that the corresponding measure dynamical system is weakly mixing.*

3. Strict ergodicity

Let G be a topological group. A topological G-system is a pair (X, G), where X is a compact Hausdorff space on which G acts by homeomorphisms in such a way that the map $G \times X \to X$, $(g, x) \mapsto gx$ is continuous. The system is *minimal* if every orbit is dense and it is *uniquely ergodic* if there is on X a unique G-invariant probability measure. Finally the system is *strictly ergodic* if it is uniquely ergodic and the unique G-invariant measure, say μ, has full support, i.e. $\operatorname{supp}(\mu) = X$.

Suppose now that G is an amenable group and that the system (X, G) is strictly ergodic. Then, if $Y \subsetneq X$ is a nonempty, closed, G-invariant, proper subset, it follows by the amenability of G, that there is a G-invariant probability measure on Y, say ν. The measure ν cannot coincide with μ because $\operatorname{supp}(\nu) \subseteq Y$. This however contradicts the unique ergodicity of X and we conclude that X admits no nonempty, closed, G-invariant, proper subsets; i.e. (X, G) is minimal. Thus, *when G is amenable, a strictly ergodic system is necessarily minimal.* A similar argument shows that *a factor of a strictly ergodic G-system is strictly ergodic.* If $\pi\colon X \to Y$ is a factor map from the strictly ergodic system (X, μ, G) onto Y, then one needs to show that for an invariant measure ν on Y there always is an invariant measure on X whose push forward in Y is ν. For this one uses the amenability of G in the disguise of the fixed point property.

In [7, p. 98, line 5], in the first two questions in Exercise 4.8 one is asked to prove the above statements, but the assumption that G be amenable is missing. Now it turns out that these statements need not be true for a general group action. Here is a counterexample for the acting group $G = SL(2,\mathbb{Z})$.

Example 3.1. Consider the topological dynamical system (Y, G), where $Y = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and G acts by automorphisms. It is well known that the only ergodic G-invariant probability measures on Y are the Lebesgue measure λ and finitely supported measures on periodic orbits.

Now, by a well-known procedure, one can "blow-up" a periodic point into a projective line $\mathbb{P}^1$, consisting of all the lines through the origin in $\mathbb{R}^2$. We identify $\mathbb{P}^1$ with the homogeneous space $SL(2,\mathbb{R})/H$, where $H < SL(2,\mathbb{R})$ is the subgroup $H = \{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) : a \neq 0 \}$.

Thus, e.g. the point $(0,0) \in \mathbb{T}^2$ is replaced by $(0,0) \times \mathbb{P}^1$, in such a way that a sequence (x_n, y_n) in $\mathbb{T}^2$ approaches $((0,0),\ell)$ if and only if $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ and the sequence of lines ℓ_n, where ℓ_n is the unique line through the origin and (x_n, y_n), tends to the line $\ell \in \mathbb{P}^1$. The G-action on the larger space is clear. It is easy to see that the action of G on $\mathbb{P}^1$ admits no invariant probability measures. (E.g. the subgroup $\{ \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)^n : n \in \mathbb{Z} \}$ squeezes the whole of $\mathbb{P}^1$ towards the x-axis.) It then follows that in the resulting G-action on the enlarged space the invariant measure $\delta_{(0,0)}$ disappears.

We enumerate the periodic orbits and attach a projective line with diameter ε_n at each point of the n-th orbit. An appropriate choice of a sequence of positive numbers ε_n tending to zero will ensure that the resulting space X is compact and metrizable. Again the action of G on X is naturally defined and we obtain the system (X, G). Finally by collapsing each $\mathbb{P}^1$ back to the point it is attached to we get a natural homomorphism $\pi \colon X \to Y$.

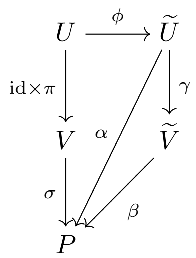
It is easy to check now that X carries a unique invariant measure (the natural lift of the Lebesgue measure on Y) which is full. Thus the system (X, G) is strictly ergodic, but of course it is not minimal. Also, the factor (Y, G) is not uniquely ergodic. This proves the following:

Theorem 3.2. *With $G = SL(2,\mathbb{Z})$ there exists a metric compact strictly ergodic dynamical system (X, G) which is not minimal. Moreover (X, G) admits a factor which is not uniquely ergodic.*

Via a construction of Furstenberg and Weiss, [6] we can obtain this latter phenomenon exhibited in a minimal system.

Theorem 3.3. *There exist a compact metric minimal dynamical system $(\widetilde{U}, G)$, with $G = SL(2,\mathbb{Z})$, and a continuous homomorphism of topological dynamical systems $\gamma \colon (\widetilde{U}, G) \to (\widetilde{V}, G)$ such that the system $(\widetilde{U}, G)$ is strictly ergodic but its factor $(\widetilde{V}, G)$ is not.*

Proof. We apply the Furstenberg–Weiss construction [6] to the above example, as follows. Start with the profinite system (P,G), where P is the inverse limit of the directed set of finite quotients $\{G/\Gamma : \Gamma < G \text{ of finite index}\}$. This is an equicontinuous minimal and strictly ergodic G-system. Consider the product system $(P \times X, G) := (U, G)$, where (X,G) is the example constructed above. Clearly this product system is topologically transitive and it admits $(P \times Y, G) := (V, G)$ as a factor, $\mathrm{id} \times \pi \colon U \to V$. Let $\sigma \colon V = P \times Y \to P$ be the projection map. Now apply (a slight strengthening) of the Furstenberg–Weiss theorem (see [15, Remark 5.3]) to obtain a commutative diagram



where $(\widetilde{U}, G)$ and $(\widetilde{V}, G)$ are minimal systems, the homomorphism α (and hence also β and γ) is an almost one-to-one extension, and the map $\phi \colon U \to \widetilde{U}$ is a Borel isomorphism which induces an affine isomorphism of the simplex of invariant measures on U to that on $\widetilde{U}$ and which moreover induces an isomorphism of the simplex of invariant measures on V to that on $\widetilde{V}$. The map $\gamma \colon \widetilde{U} \to \widetilde{V}$ is then the required map. $\square$

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