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EULER CHARACTERISTICS OF DIGITAL WEDGE SUMS AND THEIR APPLICATIONS

Sang-Eon Han — Wei Yao

Abstract. Many properties or formulas related to the ordinary Euler characteristics of topological spaces are well developed under many mathematical operands, e.g. the product property, fibration property, homotopy axiom, wedge sum property, inclusion-exclusion principle [48], etc. Unlike these properties, the digital version of the Euler characteristic has its own feature. Among the above properties, we prove that the digital version of the Euler characteristic has the wedge sum property which is of the same type as that for the ordinary Euler characteristic. This property plays an important role in fixed point theory for digital images, digital homotopy theory, digital geometry and so forth.

1. Introduction

Based on the research of the Euler characteristic for polyhedral surfaces, finite CW-complexes, more generally, for any topological space, we can define the n-th Betti number b_n as the rank of the n-th singular homology group [48]. The Euler characteristic can then be defined as the alternating sum. This quantity is well-defined if the Betti numbers are all finite and if they are zero beyond

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a certain index n_0. In modern mathematics, this notion arises from homology and, more abstractly, homological algebra. Nowadays it also contributes to the fields of pure and applied mathematics, see e.g. [9], [15], [20], [30]–[32], [42], [47].

In pure mathematics, homology groups are used for introducing the notion of Euler characteristic which naturally has the product property, fibration property, homotopy invariance and so forth. Besides, it is well known that any contractible space has a trivial homology, meaning that the 0-th Betti number is 1 and the others are 0. This case includes Euclidean subspaces of any dimension as well as the nD disk, etc.

Digital topology has a focus on studying digital topological properties of nD digital images, it contributed to the study of some areas of computer sciences, see e.g. [4], [43], [45], [46]. Thus establishment of a digital version of the Euler characteristic can be meaningful [3], [15], [19], [30]–[32], [47]. For a digital image $X \subset \mathbb{Z}^n$ with a k-adjacency, denoted by (X, k) in [45], [46], using the digital homology proposed in the papers [1], [8], the authors of [8] formulated a digital version of the Euler characteristic denoted by $\chi(X, k)$ (see Theorem 4.2 of the paper [8]), and further they studied its various properties. In Section 6 we will discuss some limitations of this quantity. Besides, the recent paper [14] observes that Euler characteristics for digital images do not have the product property, fibration property and homotopy invariance for a k-contractible digital image such as $SC_8^{2,4}$ (see Example 4.1 of [14]) and so forth. The study of non-k-contractible cases will be presented in Section 6 of the present paper.

At this moment we may pose the following query: under what operation do Euler characteristics for digital images have the same features as the ordinary Euler characteristic? Hence, for a digital wedge sum with a k-adjacency [16], [23], denoted by $(X \vee Y, k)$ (see Definition 4.1 of the present paper), we have the following question:

> In digital topology do we have a formula $\chi(X \vee Y, k)$
> which is of the same type as that in algebraic topology?

If we have a positive answer, then we can conclude that

$$\chi(X \vee Y, k) \text{ is equal to } \chi(X, k_1) + \chi(Y, k_2) - 1. \tag{1.1}$$

Property (1.1) can play an important role in fixed point theory for digital images and digital topology. In the present paper all digital images (X, k) are assumed to be non-empty and k-connected. Besides, we point out that the Euler characteristic introduced in [8] is not suitable for studying fixed point theory for digital images.

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 investigates some properties of digital homologies of several digital k-surfaces. Section 4 develops a formula calculating digital homology groups of several digital wedge sums. Section 5 corrects many errors in the papers [10]–[14] and improves them, for this reason the present paper follows the graph-based Rosenfeld model. Section 6 develops the digital wedge sum property of Euler characteristics for digital images and, further, corrects some errors in the papers [10], [11], improving the papers [10], [11], [14]. Besides, we discuss some limitations of Euler characteristics of digital images. Finally, we point out that the Lefschetz number defined in [10], [11] cannot be suitable for studying the fixed point theorem for digital images (for more details see [25]). Section 7 concludes the paper with some remarks.

2. Preliminaries

To study various properties of a digital version of the Euler characteristic of digital image, we need to recall some basic notions from digital topology such as k-adjacency relations of nD integer grids, a digital k-neighbourhood, digital continuity and so forth [16], [37], [45], [46]. Let $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and real numbers, respectively. Let $\mathbb{Z}^n$ be the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbb{N}$.

We will say that two distinct points $p, q \in \mathbb{Z}^n$ are k-(or $k(m,n)$-)adjacent if they satisfy the following property [16] (see also [21] and [22]):

- For a natural number m, $1 \leq m \leq n$, two distinct points

  $$p = (p_1, \ldots, p_n) \quad \text{and} \quad q = (q_1, \ldots, q_n) \in \mathbb{Z}^n,$$

  are $k(m,n)$-adjacent (k-adjacent, for brevity) if

$$\text{at most } m \text{ of their coordinates differ by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

Namely, $k(m,n)$-adjacency relations of $\mathbb{Z}^n$ are determined according to the number $m \in \mathbb{N}$ [16] (see also [21]). The k-adjacency relations of $\mathbb{Z}^n$ are introduced [16] (see also [21], [22]) as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \quad \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

For instance (see [16], [19], [45])

$$(n,m,k) \in \{(2,2,8),(2,1,4);(3,3,26),(3,2,18),(3,1,6);\\ (4,4,80),(4,3,64),(4,2,32),(4,1,8)\}.$$

Rosenfeld [45] called a set $X \subset \mathbb{Z}^n$ with a k-adjacency a digital image, denoted by (X,k). Indeed, to follow a graph theoretical approach of studying nD digital images [28], [46], both the k-adjacency relations of $\mathbb{Z}^n$ of (2.2) and a digital k-neighbourhood are used to study digital images. More precisely, we say that a digital k-neighbourhood of p in $\mathbb{Z}^n$ is the set $N_k(p) := \{q : p \text{ is } k\text{-adjacent to } q\}$ [45]. Furthermore, we often use the notation $N_k^*(p) := N_k(p) \cup \{p\}$ [37]. For

$a, b \in \mathbb{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbb{Z}} = \{n \in \mathbb{Z} : a \leq n \leq b\}$ with 2-adjacency is called a digital interval [37]. Besides, for a k-adjacency relation of $\mathbb{Z}^n$, a simple k-path with $l+1$ elements in $\mathbb{Z}^n$ is assumed to be an injective sequence $(x_i)_{i\in[0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|i-j| = 1$ (see [37]). If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x,y)$, is the number l. We say that a digital image (X,k) is *k-connected* if for any two points in X there is a k-path in X connecting these two points. A simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ [37], [17] (see Figure 1 (b)), is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i-j| = 1 \pmod{l}$ [37] (see Figure 1).

For a digital image (X,k), as a generalization of $N_k^*(p)$ [37], the digital k-neighbourhood of $x_0 \in X$ with radius ε is defined in X to be the following subset [16] of X:

$$N_k(x_0,\varepsilon) := \{x \in X : l_k(x_0,x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.3}$$

where $l_k(x_0,x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$. In particular, for $X \subset \mathbb{Z}^n$ we obtain [18]

$$N_k(x,1) = N_k^*(x) \cap X. \tag{2.4}$$

In Section 3, in relation to the study of Euler characteristic, we use the notion of a digital simplicial complex derived from a digital image (X,k) [1], [8], [17] and [19]. Let us recall it: let S be a set of nonempty subsets of a digital image (X,k). Then the members of S are called simplices of (X,k) if the following hold (see [1], [8]):

(a) If p and q are distinct points of $s \in S$, then p and q are k-adjacent [48].
(b) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

An m-simplex is a simplex S such that $|S| = m+1$. In this case we call the m-simplex a *digital (k,m)-complex* because it is inherited from the k-adjacency of (X,k). Let P be a digital m-simplex (or a digital m-simplex for brevity). If P' is a nonempty proper subset of P, then P' is called a face of P (for more details, see [1], [8]).

Definition 2.1 ([8]). Let (X,k) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following statements hold, then (X,k) is called a finite digital simplicial complex [1]:

(a) If P belongs to X, then every face of P also belongs to X.
(b) If $P,Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q (in case either of P and Q is not a singleton as a subset of the other).

The part (b) of Definition 2.1 implies that not every mutually k-adjacent set of $m+1$ points has to be included as an m-simplex. And this means that, given a digital image (X,k), it is possible to derive many distinct simplicial complexes.

As an example of a digital simplical complex, consider the set $X{:=}\{x_0, x_1, x_2\} \subset \mathbb{Z}^2$ in Figure 1 (a) with $(X, 4)$ or $(X, 8)$. Then we have a digital simplicial complex induced by the given digital images as follows: In case $(X, 4)$, we have digital m-simplices, $m \in \{0, 1\}$, such as $\{\{x_0, x_1\}, \{x_1, x_2\}, \{x_i\} : i \in [0, 2]_{\mathbb{Z}}\}$. In case $(X, 8)$, we obtain digital m-simplices, $m \in \{0, 1, 2\}$, such as $\{X, \{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\}, \{x_i\} : i \in [0, 2]_{\mathbb{Z}}\}$.

The dimension of a digital simplicial complex X is the largest integer m such that X has an m-simplex [1], [8]. It turns out that a digital continuous map induces a simplical map [1]. We say that $C_q^k(X)$ is a free abelian group [1] with a base of all digital (k, q)-complexes in X. Let $(X, k) \subset \mathbb{Z}^n$ be a digital simplical complex of dimension m. Then, for all $q \gneqq m$, $C_q^k(X)$ is a trivial group [1]. The homomorphism $\partial_q \colon C_q^k(X) \to C_{q-1}^k(X)$ defined by

$$\partial_q\langle p_0, \dots, p_q\rangle = \begin{cases} \displaystyle\sum_{i=0}^{q} (-1)^i \langle p_0, \dots, \widehat{p_i}, \dots, p_q\rangle & \text{for } q \le m, \\ 0 & \text{for } q \gneqq m, \end{cases}$$

is called a boundary homomorphism, where $\widehat{p_i}$ means "deleting the point p_i". In [1], $\partial_{q-1} \circ \partial_q = 0$. Besides, from [1], [8], we recall that

(1) $Z_q^k(X) := \operatorname{Ker} \partial_q$ is called the group of digital simplicial q-cycles.

(2) $B_q^k(X) := \operatorname{Im} \partial_{q+1}$ is called the group of digital simplicial q-boundaries.

Based on this approach, the papers [1], [8] introduce the q-th digital simplicial group

$$H_q^k(X) := Z_q^k(X)/B_q^k(X). \tag{2.5}$$

In view of Definition 2.1 and property (2.5), we have $H_q^2([0, l]_{\mathbb{Z}}) = \mathbb{Z}$, $q = 0$ and it is trivial if $q \neq 0$ [1], [8]. Besides, for a singleton $\{x_0\}$ it is obvious that $H_q^k(\{x_0\})$ is isomorphic to $\mathbb{Z}$ if $q = 0$, and it is trivial if $q \neq 0$ [8]. In addition, $H_q^k(SC_k^{n,l}) = \mathbb{Z}$ if $q \in \{0, 1\}$, and it is trivial if $q \notin \{0, 1\}$.

3. Some properties of digital homology groups of closed k-surfaces

To study some properties of digital homology groups associated with the notion of a digital k-homotopy equivalence, we need to recall the notion of digital continuity of a map $f \colon (X, k_0) \to (Y, k_1)$ by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1) [46].

Proposition 3.1 ([16], [17]). *Let (X, k_0) and (Y, k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f \colon (X, k_0) \to (Y, k_1)$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 3.1 in case $n_0 = n_1$ and $k_0 = k_1 := k$, the map f is called a "k-continuous" map. In relation to the classification of nD digital images, we use the term a (k_0,k_1)-isomorphism as in [28] rather than a (k_0,k_1)-homeomorphism as in [5].

Definition 3.2 ([5], see also [28]). Consider two digital images (X,k_0) and (Y,k_1) in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Then a map $h\colon X\to Y$ is called a (k_0,k_1)-isomorphism if h is a (k_0,k_1)-continuous bijection and further, $h^{-1}\colon Y\to X$ is (k_1,k_0)-continuous.

In Definition 3.2, in case $n_0=n_1$ and $k_0=k_1:=k$, we call it a *k-isomorphism* [28]. Furthermore, we denote by $X\approx_k Y$ a k-isomorphism from X to Y [17] and [28].

For a digital image (X,k) and $A\subset X$, (X,A) is called a digital image pair with a k-adjacency [17]. Furthermore, if A is a singleton set $\{x_0\}$, then (X,x_0) is called a pointed digital image [37]. Based on the pointed digital homotopy in [6], [34], the following notion of a k-homotopy relative to a subset $A\subset X$ was used to study a k-homotopic thinning and a strong k-deformation retract of a digital image (X,k) in $\mathbb{Z}^n$ [20], [21].

Definition 3.3 ([16], see also [17]). Let $((X,A),k_0)$ and (Y,k_1) be a digital image pair and a digital image, respectively. Let $f,g\colon X\to Y$ be (k_0,k_1)-continuous functions. Suppose there exist $m\in\mathbf{N}$ and a function $F\colon X\times[0,m]_{\mathbb{Z}}\to Y$ such that

(a) for all $x\in X$, $F(x,0)=f(x)$ and $F(x,m)=g(x)$;
(b) for all $x\in X$, the induced function $F_x\colon[0,m]_{\mathbb{Z}}\to Y$ given by $F_x(t)=F(x,t)$ for all $t\in[0,m]_{\mathbb{Z}}$ is $(2,k_1)$-continuous;
(c) for all $t\in[0,m]_{\mathbb{Z}}$, the induced function $F_t\colon X\to Y$ given by $F_t(x)=F(x,t)$ for all $x\in X$ is (k_0,k_1)-continuous.

Then we say that F is a (k_0,k_1)-homotopy between f and g [6]. Furthermore, if

(d) for all $t\in[0,m]_{\mathbb{Z}}$, $F_t(x)=f(x)=g(x)$ for all $x\in A$,

then we call F a (k_0,k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0,k_1)-homotopic relative to A in Y, $f\simeq_{(k_0,k_1)\mathrm{rel}\,A} g$ in symbols.

In Definition 3.3, if $A=\{x_0\}\subset X$, then we say that F is a pointed (k_0,k_1)-homotopy at $\{x_0\}$ (see [6]). When f and g are pointed (k_0,k_1)-homotopic in Y, we use the notation $f\simeq_{(k_0,k_1)} g$. In addition, if $k_0=k_1$ and $n_0=n_1$, then we say that f and g are pointed k_0-homotopic in Y and we use the notation $f\simeq_{k_0} g$ and $f\in[g]$ which denotes the k_0-homotopy class of g. If, for some $x_0\in X$, 1_X is k-homotopic to the constant map in the space $\{x_0\}$ relative to $\{x_0\}$, then we say that (X,x_0) is pointed k-contractible [6].

In relation to the study of the fixed point property of k-contractible digital images in Sections 4–7, we use the following k-contractibility.

Definition 3.4 ([6]). Let (X, k_0) and (Y, k_1) be digital images. A (k_0, k_1)-continuous map is digitally nullhomotopic if f is (k_0, k_1)-homotopic in (Y, k_1) to a constant map. A digital image (X, k) is a k-contractible if its identity map is digitally nullhomotopic.

Using the trivial extension from [6] and the *Khalimsky operation* in [34], the paper [6] introduces the k-fundamental group of a digital image (X, k) in the algebraic topological approach, called a (digital) k-fundamental group of (X, x_0) [6] and denoted by $\pi^k(X, x_0)$. If X is pointed k-contractible, then $\pi^k(X, x_0)$ is trivial [6]. The following notion of "simply k-connected" in [16] has been often used in digital k-homotopy and digital covering theories (see [16], [20] and [21]).

Definition 3.5 ([16]). A pointed k-connected digital image (X, x_0) is called *simply k-connected* if $\pi^k(X, x_0)$ is a trivial group.

Using both the 8-contractibility of $SC_8^{2,4}$ [6] and the non-8-contractibility of $SC_8^{2,6}$ [16], it was proved in [16] (see also [20]) that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group, where $SC_k^{n,l}$ is not k-contractible.

Definition 3.6 ([6]). For a digital image pair $((X, A), k)$, A is said to be a deformation k-retract of X if there is a k-retraction r of X onto A such that $F\colon i \circ r \simeq_k 1_X$.

The following notion of a digital homotopy equivalence was first introduced in the paper [29] to classify digital images up to a digital k-homotopy equivalence [22].

Definition 3.7 ([29], [22]). Let $f\colon (X, k_0) \to (Y, k_1)$, $g\colon (Y, k_1) \to (X, k_0)$ be (k_0, k_1)- and (k_1, k_0)-continuous maps, respectively, such that

$$g \circ f \simeq_{k_0} 1_X \quad \text{and} \quad f \circ g \simeq_{k_1} 1_Y.$$

Then we say that (X, k_0) and (Y, k_1) have the same (k_0, k_1)-homotopy type (or we say that (X, k_0) is (k_0, k_1)-homotopy equivalent to (Y, k_1)).

To study Euler characteristic of a digital wedge sum of both digital simple closed k-curves and closed k-surfaces, we need to recall basic notions from digital k-surface theory [2], [4], [17]–[19], [43], [44]. A point $x \in X$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [4]. The k-corner x is called *simple* if y, z are not k-corners and if x is the only point k-adjacent to both y, z. X is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve (see [4]). For a k-connected digital image (X, k) in $\mathbb{Z}^n$, we recall $|X|^x = N^*_{3^n-1}(x) \cap X$ [17], [18], where $N^*_{3^n-1}(x) = \{x' : x \text{ and } x'$ are $(3^n - 1)$-adjacent$\}$. Also, we recall that for a digital image (X, k), a *simple* *k-point* is the one whose removal does not change the digital topological property of (X, k) (see [2]).

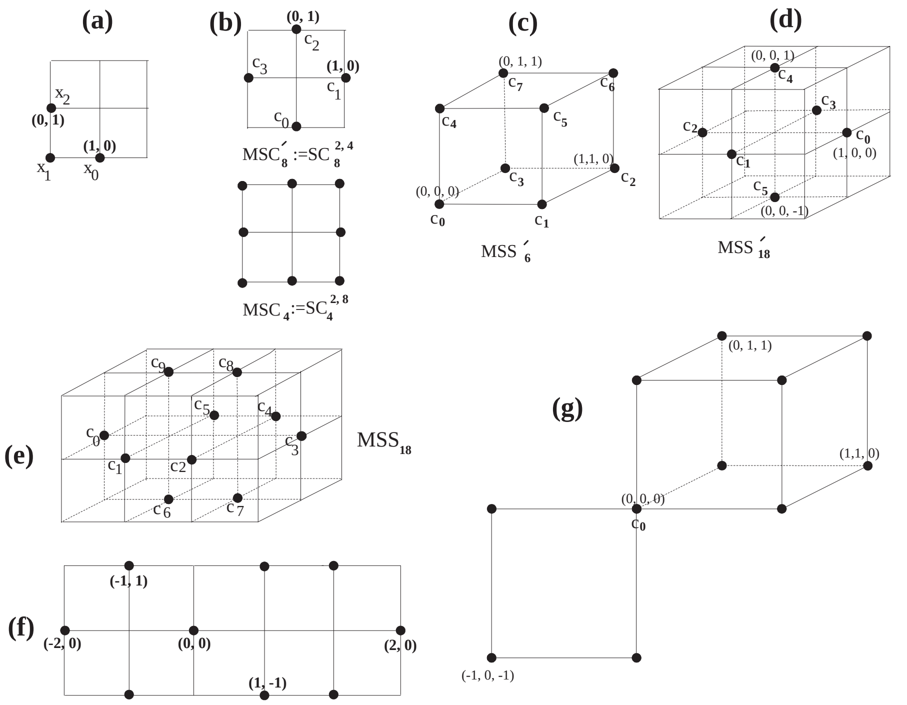

FIGURE 1. (a) $(X, 4)$ or $(X, 8)$; (b) simple closed k-curves, $k \in \{4, 8\}$ ([6] and [16]); (c) MSS'_6 [12]; (d) a simple closed 18-surface with 18-contractibility MSS'_{18} [17]; (e) a simple closed 18-surface without 18-contractibility MSS_{18} [17]; (f) a digital wedge sum $(SC_8^{2,4} \vee SC_8^{2,6}, 8)$; (g) a digital wedge sum $(SC_4^{2,4} \vee MSS'_6, 6)$.

DEFINITION 3.8 ([18], [19]). Let (X, k) and $(\overline{X} = \mathbb{Z}^n \setminus X, \overline{k})$ be digital images in $\mathbb{Z}^n$, $n \geq 3$, $k \neq \overline{k}$. Then X is called a *closed k-surface* if it satisfies the following:

(1) In case $(k, \overline{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$, where the k-adjacency is taken from (2.2) with $k \neq 3^n - 2^n - 1$,

(a) for each point $x \in X$, $|X|^x$ has exactly one k-component k-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{k}$-components $\overline{k}$-adjacent to x; we denote by $C^{x\,x}$ and $D^{x\,x}$ these two components; and

(c) for any point $y \in N_k(x) \cap X$, $N_{\overline{k}}(y) \cap C^{x\,x} \neq \emptyset$ and $N_{\overline{k}}(y) \cap D^{x\,x} \neq \emptyset$.

Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.

(2) In case $(k, \overline{k}) = (3^n - 2^n - 1, 2n)$,
 (a) X is k-connected,
 (b) for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve.
 Furthermore, if the image $|X|^x$ is a simple closed k-curve, then the closed k-surface X is called simple.

In Figure 1, several types of simple closed k-surfaces in $\mathbb{Z}^3$ are shown, see [18] and [19].

REMARK 3.9 ([17], [18]).
(a) MSS'_{18} is 18-contractible.
(b) MSS_{18} is not 18-contractible but simply 18-connected.

REMARK 3.10. In Example 7.2 of the paper [13], the authors said that MSS_{18} is 18-contractible (see page 40 of the paper [13]). In view of Remark 3.9, it is incorrect.

To study digital k-homotopy equivalence axiom related to the digital homology of (2.5), let us recall homology groups of digital image MSS'_{18}, MSS'_6 and MSS_{18} in $\mathbb{Z}^3$ (see Theorems 3.18, 3.19 and 3.20 of the paper [8]).

LEMMA 3.11 ([8]). *For the digital images MSS'_{18}, MSS'_6 and MSS_{18} in $\mathbb{Z}^3$, we obtain*:
(a) $H_q^{18}(MSS'_{18}) = \mathbb{Z}$ *if* $q \in \{0, 2\}$ *and it is trivial if* $q \notin \{0, 2\}$.
(b) $H_q^{6}(MSS'_{6}) = \mathbb{Z}$ *if* $q = 0$*;* $\mathbb{Z}^5$ *if* $q = 1$ *and it is trivial if* $q \notin \{0, 1\}$.
(c) $H_q^{18}(MSS_{18}) = \mathbb{Z}$ *if* $q = 0$*;* $\mathbb{Z}^3$ *if* $q = 1$ *and it is trivial if* $q \notin \{0, 1\}$.

4. Digital homology groups of digital wedge sums

It is well known that in ordinary homotopy theory if two continuous maps $f, g \colon X \to Y$ are homotopy equivalent to each other, they induce the same homology isomorphism $f_* = g_* \colon H_*(X) \to H_*(Y)$, see [48]. However, we need to point out that this kind of approach is valid in digital k-homotopy theory.

The notion of a digital wedge sum was first introduced in [16] where digital images are studied in the field of digital homotopy. For digital images (X_i, k_i) in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$, the notion of digital wedge sum of (X_i, k_i) was introduced in [16] (see also [23]). In relation to the study of digital homology groups of digital images, we need to develop a notion of compatible k-adjacency of a digital wedge sum, as follows.

DEFINITION 4.1 ([16], see also [23]). For pointed digital images $((X, x_0), k_0)$ in $\mathbb{Z}^{n_0}$ and $((Y, y_0), k_1)$ in $\mathbb{Z}^{n_1}$, the wedge sum of (X, k_0) and (Y, k_1), written $(X \vee Y, (x_0, y_0))$, is the digital image in $\mathbb{Z}^n$, $n = \max\{n_0, n_1\}$,

$$\{(x, y) \in X \times Y : x = x_0 \text{ or } y = y_0\} \tag{4.1}$$

with the following compatible $k(m,n)$- (or k)-adjacency relative to both (X, k_0) and (Y, k_1), and the only one point (x_0, y_0) in common such that

(W1) the $k(m,n)$- (or k)-adjacency is determined by the numbers m and n with $m = \max\{m_0, m_1\}$ satisfying (W 1-1) below, where the numbers m_i are taken from the k_i- (or $k(m_i, n_i)$)-adjacency relations of the given digital images $((X, x_0), k_0)$ and $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (4.1), induced from the projection maps, we can consider the natural projection maps

$$W_X\colon (X \vee Y, (x_0, y_0)) \to (X, x_0) \quad \text{and} \quad W_Y\colon (X \vee Y, (x_0, y_0)) \to (Y, y_0).$$

In relation to the establishment of a compatible k-adjacency of the digital wedge sum $(X \vee Y, (x_0, y_0))$, the following restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the below properties, respectively:

(1) $W_X|_{X \times \{y_0\}}\colon (X \times \{y_0\}, k) \to (X, k_0)$ is a (k, k_0)-isomorphism; and
(2) $W_Y|_{\{x_0\} \times Y}\colon (\{x_0\} \times Y, k) \to (Y, k_1)$ is a (k, k_1)-isomorphism.

(W2) Any two distinct elements $x(\neq x_0) \in X \subset X \vee Y$ and $y(\neq y_0) \in Y \subset X \vee Y$ are not $k(m,n)$- (or k)-adjacent to each other.

It is obvious that $(X \vee Y, k)$ is k-isomorphic to $(Y \vee X, k)$.

When establishing a digital wedge sum, we strongly need to examine if there exists a compatible k-adjacency of a digital wedge sum. Hereafter, for convenience we may use a digital wedge sum $(X \vee Y, (x_0, y_0))$ dropping the common point, *i.e.* $X \vee Y$ instead of $(X \vee Y, (x_0, y_0))$ if there is no ambiguity.

Example 4.2 ([16], [23]). We have the following compatible k-adjacencies of digital wedge sums (see Figure 1).

(a) $(SC_k^{2,l_1} \vee SC_k^{2,l_2}, k)$, $k \in \{4, 8\}$.
(b) $(SC_8^{2,l} \vee MSS'_{18}, 18)$.
(c) There is no compatible k-adjacency of $(SC_4^{2,8} \vee SC_8^{2,6}, k)$ otherwise we have a contradiction to property (W1) of Definition 4.1.

Let us now study the digital wedge sum property of digital homology groups.

Theorem 4.3. *Consider a digital wedge sum $(X \vee Y, k)$ in $\mathbb{Z}^n$, where (X, k_1) (resp. (Y, k_2)) is a digital image in $\mathbb{Z}^{n_1}$ (resp. $\mathbb{Z}^{n_2}$) and $n = \max\{n_1, n_2\}$. Then*

$$H_q^k(X \vee Y) \simeq H_q^{k_1}(X) \oplus H_q^{k_2}(Y), \quad q \geq 1.$$

In the present paper we call the property from Theorem 4.3 the *digital wedge sum property of digital homology*. Before proving this theorem, let us recall that for a wedge sum $X \vee Y$ of two connected topological spaces X and Y in algebraic topology, its homology group has the following property:

$$H_q(X \vee Y) \simeq H_q(X) \oplus H_q(Y), \quad q \geq 1.$$

Since digital homology groups have their own feature, we need to calculate digital homology groups of digital images according to property (2.5) instead of homological algebra.

PROOF. Since the digital wedge sum $(X \vee Y, k)$ is assumed to have only one common point (x_0, y_0), the other points in X and Y are disjoint up to k-compatible connectivity of $X \vee Y$ (see property (W2) of Definition 4.1). Besides, recall that $(X \vee Y, k)$ is k-connected. Then for all $q \geq 1$, owing to the definition of $(X \vee Y, k)$, all free abelian groups

$$C_q^k(X \vee Y) \text{ are isomorphic to } C_q^{k_1}(X) \oplus C_q^{k_2}(Y), \quad q \in \mathbb{N}. \tag{4.2}$$

Furthermore, if $z_{X \vee Y} \in Z_q^k(X \vee Y)$, depending on the given digital images (X, k_1) and (Y, k_2), we have

$$\text{either} \quad z_{X \vee Y} = z_X, \quad z_Y \quad \text{or} \quad z_X + z_Y \tag{4.3}$$

because of properties (W1) and (W2) of Definition 4.1, where $z_X \in Z_q^{k_1}(X)$ and $z_Y \in Z_q^{k_2}(Y)$. Hence we see that $Z_q^k(X \vee Y)$ is isomorphic to $Z_q^{k_1}(X) \oplus Z_q^{k_2}(Y)$ and further, $B_q^k(X \vee Y)$ are also isomorphic to $B_q^{k_1}(X) \oplus B_q^{k_2}(Y)$ for all $q \in \mathbb{N}$. As a result, due to (4.3), we obtain that $H_q^k(X \vee Y)$ is isomorphic to $H_q^{k_1}(X) \oplus H_q^{k_2}(Y)$. □

Note, in Theorem 4.3, if $q = 0$, then property (4.2) does not work because each of $X \vee Y$, X and Y has only one k-component.

The paper [12] studied a digital homology group of a digital wedge sum in $\mathbb{Z}^2$ with coefficients as a commutative ring $\mathbb{Z}^2$. However, we can study digital homology groups of any digital wedge sums in $\mathbb{Z}^n$ as follows:

COROLLARY 4.4. *Consider a digital wedge sum* $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$ *in* $\mathbb{Z}^n$, *where* $n := \max\{n_1, n_2\}$. *Then*

$$H_q^k(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^2 & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0, 1\}. \end{cases}$$

As a generalization of Corollary 4.4, we obtain the following:

COROLLARY 4.5. *Consider the digital wedge sum* $(SC_{k_1}^{n_1,l_1} \vee \ldots \vee SC_{k_l}^{n_l,l_l}, k)$ *in* $\mathbb{Z}^n$, *where* $n = \max\{n_i : i \in [1, l]_{\mathbb{Z}}\}$. *Then*

$$H_q^k(SC_{k_1}^{n_1,l_1} \vee \ldots \vee SC_{k_l}^{n_l,l_l}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^l & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0, 1\}. \end{cases}$$

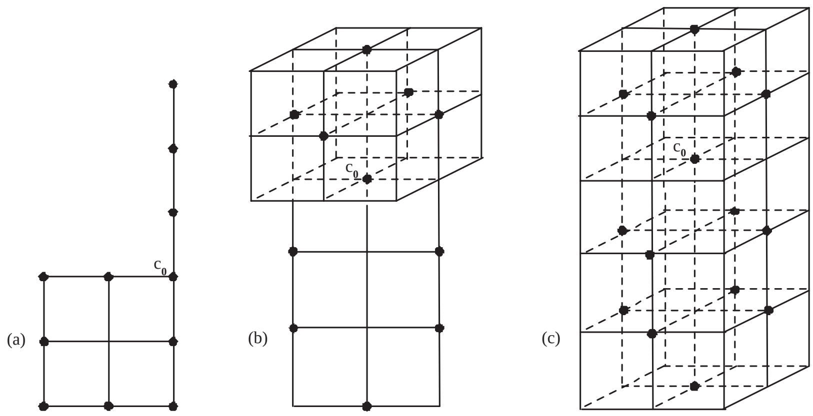

FIGURE 2. Configuration of digital wedge sums: (a) $([0,3]_{\mathbb{Z}} \vee SC_4^{2,8}, 4)$; (b) $(MSS'_{18} \vee SC_8^{2,6}, 18)$; (c) $(MSS'_{18} \vee MSS_{18}, 18)$.

It is obvious that $H_0(X \vee Y)$ is isomorphic to $\mathbb{Z}$ because $(X \vee Y, k)$ is assumed to be k-connected.

By Lemma 3.11 and Theorem 4.3, we obtain the following:

EXAMPLE 4.6.

(a)
$$H_q^6(SC_4^{2,4} \vee MSS'_6) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^6 & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0,1\}. \end{cases}$$

(b)
$$H_q^4([0,3]_{\mathbb{Z}} \vee SC_4^{2,8}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z} & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0,1\}. \end{cases}$$

(c)
$$H_q^{18}(SC_8^{2,6} \vee MSS'_{18}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z} & \text{if } q = 1, \\ \mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \notin \{0,1,2\}. \end{cases}$$

(d)
$$H_q^{18}(MSS'_{18} \vee MSS_{18}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^3 & \text{if } q = 1, \\ \mathbb{Z} & \text{if } q = 2, \\ 0 & \text{if } q \notin \{0,1,2\}. \end{cases}$$

5. Corrections of the papers [10]–[14]

In this section we correct errors in the papers [10]–[14]. Some errors of [10], [11] were corrected in [7].

Motivated by the relative homology [48], the recent paper [14] introduced the relative homology for digital images, and studied its properties. More precisely, let (A,k) be a digital subcomplex of the digital simplicial complex (X,k). Then the chain group $C_k^q(A)$ is a subgroup of the chain group $C_q^k(X)$ (see Section 2). The quotient group $C_q^k(X,A) = C_q^k(X)/C_q^k(A)$ is called the group of relative chains of X modulo A. The boundary operator $\partial_q\colon C_q^k(A) \to C_{q-1}^k(A)$ is the restriction of the boundary operator on $C_q^k(X)$, $q \in \mathbb{N}$. It induces a homomorphism $C_q^k(X,A) \to C_{q-1}^k(X,A)$ of the relative chain groups and this is also denoted by ∂_q.

Definition 5.1 ([48], [14]). Let (A,k) be a digital subcomplex of the digital simplicial complex (X,k).

(a) $Z_q^k(X,A) = \operatorname{Ker}\partial_q$ is called the group of digital relative simplicial q-cycles.
(b) $B_q^k(X,A) = \operatorname{Im}\partial_{q+1}$ is called the group of digital relative simplicial q-boundaries.
(c) $H_q^k(X,A) = Z_q^k(X,A)/B_q^k(X,A)$ is called the qth digital relative simplicial homology group.

[14] states

Theorem 5.2 ([14, Proposition 3.6]). *If $A = \{x_0\}$ is a single point digital image. Then $H_q^k(X,A) = H_q^k(X)$ for any $q \gneq 0$.*

wherefrom the following result is derived:

Theorem 5.3 ([14, Proposition 3.8]). *If $A \subset X$ is a deformation k-retract, then $H_q^k(X,A) = 0$ for any $q \geq 0$.*

Unfortunately, the later is incorrect. Indeed, the proof of this theorem in [14] is based on the following assertion:

$$i_*\colon H_q^k(A) \to H_q^k(X) \quad \text{is an isomorphism} \tag{5.1}$$

for $q \geq 0$. However, we shall see that (5.1) is incorrect (see Remark 5.5). In Remark 5.5 we present a counterexample to Theorem 5.3.

Lemma 5.4. *$SC_8^{2,6}$ is a deformation 8-retract of $(SC_8^{2,6} \vee SC_8^{2,4}, 8)$.*

Proof. Since $SC_8^{2,4}$ is 8-contractible [6], we obtain that $(SC_8^{2,6} \vee SC_8^{2,4}, 8)$ is 8-homotopy equivalent to $(SC_8^{2,6} \vee \{*\}, 8)$, where the point "$*$" is the common point of the digital wedge sum $(SC_8^{2,6} \vee SC_8^{2,4}, 8)$. Besides, since $(SC_8^{2,6} \vee \{*\}, 8)$ is 8-isomorphic to $SC_8^{2,6}$, the proof is completed. □

Remark 5.5. Let us show that property (5.1) is incorrect if $q \gneqq 0$. Consider the digital image $SC_8^{2,4}$ which is 8-contractible [6] (see also [16]). Then, although the singleton $\{c_0\}$ is a deformation 8-retract of $SC_8^{2,4} := (c_i)_{i\in[0,3]_{\mathbb{Z}}}$, it is clear that the induced homomorphism $i_*\colon H_q^8(\{c_0\}) \to H_q^8(SC_8^{2,4})$ is not an isomorphism for $q = 1$ contrary to property (5.1).

In view of Remark 5.5, we obtain the following:

Proposition 5.6. (a) *Let (X, k) be a digital image and let (A, k) be a deformation k-retract of (X, k). Then $H_q^k(X)$ need not be isomorphic to $H_q^k(A)$, $q \in \mathbb{N}$.*

(b) *If $A \subset X$ is a deformation k-retract, then $H_q^k(X, A)$ need not be trivial, for $q \in \mathbb{N}$.*

Papers [10], [11] study the fixed point theorem for digital images in terms of a digital version of the ordinary Lefschetz number. However, the works contain many errors. To be specific, the authors of [10], [11] introduced a digital version of the Lefschetz number in terms of the digital continuity and some properties of the digital homology group proposed in [1], [10], [11]. Recall that the Lefschetz number gives a method for recognizing the existence of a fixed point of a continuous mapping from a compact topological space X to itself. Lefschetz [40] introduced the Lefschetz number of a map and proved that if the number is nonzero, then the map has a fixed point. For a formal statement of the theorem, let $f\colon X \to X$ be a continuous self-map of a compact triangulable space X. Define the Lefschetz number $\lambda(f)$ of f by

$$\lambda(f) := \sum_{k\geq 0} (-1)^k \operatorname{Tr}(f_* \mid H_k(X, \mathbb{Q})) \tag{5.2}$$

the alternating (finite) sum of the matrix traces of the linear maps induced by f on $H_k(X, \mathbb{Q})$, the singular homology of X with rational coefficients. A simple version of the Lefschetz fixed point theorem states: if $\lambda(f) \neq 0$, then f has at least one fixed point, i.e. there exists at least one point $x \in X$ such that $f(x) = x$ (see [40]).

Motivated by this approach, the authors of [10] consider the following formula (see Definition 3.1 of [10] and Definition 3.3 of [11]).

Definition 5.7 ([10]). For a map $f\colon (X, k) \to (X, k)$, where (X, k) is a digital image whose digital homology groups are finitely generated and vanish above some dimension, the Lefschetz number $\lambda(f)$ is defined as follows:

$$\lambda(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_*), \tag{5.3}$$

where $f_*\colon H_i^k(X) \to H_i^k(X)$ is the induced homomorphism by the given map f, where X is the digital simplex inherited from the digital image (X, k).

Recall that a non-empty k-connected digital image (X, k) has the *fixed point property* (FPP for short) [46] if every k-continuous self-map $f: (X, k) \to (X, k)$ has a point $x \in X$ such that $f(x) = x$. Then the point x is called a fixed point of the map f.

It is obvious that only a singleton has the fixed point property from the viewpoint of digital topology in the graph theoretical approach (for more details, see [24]–[27]). Hence the author of [46] studied the almost fixed point property of digital images instead of the FPP.

The authors of the papers [10, 11] asserted that if the number $\lambda(f) \neq 0$, then the map f has at least a fixed point in X (see Theorem 3.3 of the paper [10] and Theorem 3.4 of the paper [11]). But it is incorrect due to the following example.

Example 5.8. Consider the above bijective 2-continuous self-map f of ($[0, 2l-1]_{\mathbb{Z}}, 2)$ given by $f(t) = 2l - t - 1$, $t \in [0, 2l-1]_{\mathbb{Z}}$. Then we have

$$\lambda(f) = 1 - 0 + \ldots = 1. \tag{5.4}$$

In spite of property (5.4), the map f does not have any fixed point, which proves the invalidity of Theorem 3.4 of the paper [11].

6. Euler characteristics of digital wedge sums and corrections of the papers [10], [11]

Before starting this section, we need to recall some properties of the Euler characteristic of a topological space in algebraic topology, denoted by $\chi(X)$, namely: let X and Y be any topological spaces, and $\{p\}$ be a singleton set, then $\chi(\{p\}) = 1$. If X is homeomorphic to Y then $\chi(X) = \chi(Y)$. For any homotopic compact spaces X and Y, we have $\chi(X) = \chi(Y)$. For every closed subset $C \subset X$, we have $\chi(X) = \chi(C) + \chi(X \setminus C)$. This property has a dual form $\chi(X) = \chi(U) + \chi(X \setminus U)$ for every open subset $U \subset X$ and $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. However, these analogues to digital topology do not satisfy [14]. Besides, a digital version of the Euler characteristic does not satisfy the digital homotopy axiom [12]: Indeed, the paper [12] suggested an 8-contractible digital image such as $SC_8^{2,4}$ to guarantee the non-homotopy axiom of a digital version of the ordinary Euler characteristic. Let us now study the non-homotopy axiom of a digital version of the ordinary Euler characteristic for a non-k-contractible digital image.

As a generalization of the Euler characteristic in [19], the paper [8] introduced the following notion:

Definition 6.1 ([8]). Let (X, k) be a digital image in $\mathbb{Z}^n$, and for each $q \geq 0$, let α_q be the number of digital (k, q)-simplices in X. The Euler characteristic

$\chi(X)$ is defined as follows:

$$\chi(X,k)=\sum_{q=0}^{m}(-1)^q\alpha_q. \tag{6.1}$$

Using the digital simplex (see Definition 2.1) from a digital image (X,k), the authors of [8] established the following formula.

Theorem 6.2 (see [8, Theorem 4.2]). *If (X,k) is a digital image of dimension m, then*

$$\chi(X,k)=\sum_{q=0}^{m}(-1)^q \operatorname{rank} H_q^k(X). \tag{6.2}$$

The authors of [11] studied the FPP by using Theorem 6.2. Let us also recall Theorem 3.7 of the paper [11].

Theorem 6.3 ([11, Theorem 3.7]). *Let (X,k) be a digital image. Let f be any k-continuous self-map f of (X,k) and f be k-homotopic to the identity 1_X. If $\chi(X,k)\neq 0$, then the map f has a fixed point.*

Unfortunately, this assertion is obviously incorrect [7], [25], [26] as the two examples show.

Example 6.4. (a) Let (X,k) be a k-path in $\mathbb{Z}^n$, $n\geq 1$, which is $(k,2)$-isomorphic to the digital line interval $([0,3]_{\mathbb{Z}},2)$ (see Figure 3 (a)). Then put $X:=(x_i)_{i\in[0,3]_{\mathbb{Z}}}$. Let us now consider the self-map f of (X,k) given by $f(x_i)=x_{3-i}$. It is clear that f is a k-continuous map which is k-homotopic to the identity map 1_X. Besides we see that $\chi(X,k)=1\neq 0$. While this situation satisfies the hypothesis of Theorem 6.3, we see that f does not have any fixed point, which leads to contradiction.

(b) Let us consider the bijective self-map f of $MSS'_{18}:=\{c_i : i\in[0,5]_{\mathbb{Z}}\}$, i.e. given by (see Figure 3 (b))

$$\begin{aligned} f(c_1)&=c_5,\quad f(c_5)=c_3,\quad f(c_3)=c_4,\quad f(c_4)=c_1\\ \text{and}\quad f(c_0)&=c_2,\quad f(c_2)=c_0. \end{aligned}$$

Then we see that the map f is an 18-continuous bijection and f is 18-homotopic to the identity $1_{MSS'_{18}}$ in terms of the processing $H\colon MSS'_{18}\times[0,2]_{\mathbb{Z}}\to MSS'_{18}$ such that

- $H(x,0)=1_{MSS'_{18}}(x)$, $x\in MSS'_{18}$;
- $H(x,1)$ implies two-clicks rotation of the set $X_1:=\{c_1,c_4,c_3,c_5\}$ and the other points c_0,c_2 are fixed; and
- $H(x,2)$ means a reflection relative to the set X_1, $x\in MSS'_{18}$.

Furthermore, by Lemma 3.11 (a), we see that

$$\chi(MSS'_{18}, 18) = 1 - 0 + 1 - 0 + \ldots = 2 \neq 0.$$

Let us apply this situation to Theorem 6.3. It is clear that the 18-continuous bijection f does not have any fixed point, which leads to contradiction.

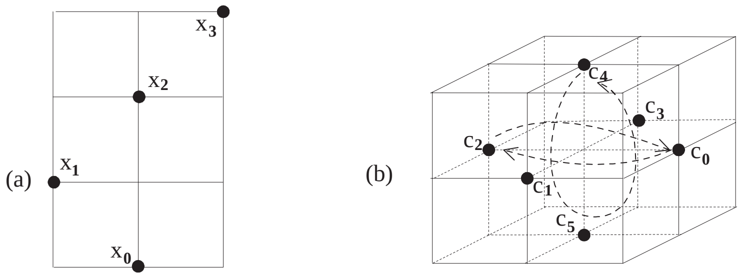

FIGURE 3. Invalidity of the digital version of the Lefschetz theorem and its applications in the papers [10], [11].

Although the paper [8] studied a digital version of the Euler characteristic, we point out that the formula (6.2) is very restrictive as it does not satisfy the digital homotopy equivalence property (for more details, see Section 7). But we shall show that it has the digital wedge sum property.

THEOREM 6.5. *Let* (X, k_1) *(resp.* (Y, k_2)*) in* $\mathbb{Z}^{n_1}$ *(resp.* $\mathbb{Z}^{n_2}$*) be a pointed* k_1*- (resp.* k_2*-)connected digital image. Assume the digital wedge sum with a* k*-compatible* k*-adjacency* $(X \vee Y, k)$*. Then we have*

$$\chi(X \vee Y, k) = \chi(X, k_1) + \chi(X, k_2) - 1.$$

PROOF. In view of the digital homology of digital image and the notion of a digital wedge sum, since we glue together the k_1-(resp. k_2-)path components of X (resp. Y) containing their respective base point, we see that

$$\text{rank}\, H_0^k(X \vee Y) = \text{rank}\, H_0^{k_1}(X) + \text{rank}\, H_0^{k_2}(Y) - 1, \tag{6.3}$$

because the given two digital images (X, k_1) (resp. (Y, k_2)) are originally assumed to be separated up to k-connectivity.

Next, by Theorem 4.3, for $q \geq 1$ we may use the isomorphism

$$H_q^k(X \vee Y) = H_q^{k_1}(X) \oplus H_q^{k_2}(Y), \tag{6.4}$$

because in the process of the formation of digital wedge sum the other digital (k, n)-complexes except the common one point (x_0, y_0) are remained without any change. By (6.3) and (6.4), after combining (6.3) and (6.4), by Theorem 6.2, we obtain

$$\chi(X \vee Y, k) = \chi(X, k_1) + \chi(X, k_2) - 1. \qquad \square$$

Example 6.6. By Examples 4.2 and 4.6, and Theorem 4.3, since

$$H_q^4(SC_4^{2,8} \vee SC_4^{2,4}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}^2 & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0,1\}, \end{cases}$$

we obtain

$$\chi(SC_4^{2,8} \vee SC_4^{2,4}, 4) = 1 - 2 + 0 - 0 + 0 - \ldots = -1. \tag{6.5}$$

Furthermore, we see

$$\begin{aligned} \chi(SC_4^{2,8}, 4) &= 1 - 1 + 0 - \ldots = 0, \\ \chi(SC_4^{2,4}, 4) &= 1 - 1 + 0 - \ldots = 0. \end{aligned} \tag{6.6}$$

Thus

$$\chi(SC_4^{2,8} \vee SC_4^{2,4}, 4) = \chi(SC_4^{2,8}, 4) + \chi(SC_4^{2,4}, 4) - 1.$$

Example 6.7. By Examples 4.2 and 4.6, and Theorem 4.3, since

$$H_q^6(SC_4^{2,4} \vee MSS_6') = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}^6 & \text{if } q = 1, \\ 0 & \text{if } q \notin \{0,1\}, \end{cases}$$

we obtain

$$\chi(SC_4^{2,4} \vee MSS_6', 6) = 1 - 6 + 0 - 0 + \ldots = -5. \tag{6.7}$$

Besides, by Lemma 3.11 (b),

$$\begin{aligned} \chi(MSS_6', 6) &= 1 - 5 + 0 - \ldots = -4, \\ \chi(SC_4^{2,4}, 4) &= 0. \end{aligned} \tag{6.8}$$

In view of (6.7) and (6.8), we have

$$\chi(SC_4^{2,4} \vee MSS_6', 6) = \chi(SC_4^{2,4}, 4) + \chi(MSS_6', 6) - 1.$$

In view of Example 4.6 (d), we obtain the following:

Example 6.8.

$$\chi(MSS_{18}' \vee MSS_{18}, 18) = 1 - 3 + 1 - 0 + \ldots = -1. \tag{6.9}$$

Besides, by Lemma 3.11 (a)–(b), we have

$$\begin{aligned} \chi(MSS_{18}', 8) &= 1 - 0 + 1 + 0 - \ldots = 2, \\ \chi(MSS_{18}, 18) &= 1 - 3 + 0 - \ldots = -2. \end{aligned} \tag{6.10}$$

By (6.9) and (6.10), we have

$$\chi(MSS_{18}' \vee MSS_{18}, 18) = \chi(MSS_{18}', 8) + \chi(MSS_{18}', 18) - 1.$$

The paper [10] tried to study the FPP of a digital wedge sum (see Proposition 3.12 of the paper [10]), as follows: let $(A \vee B, k)$ be a digital wedge of two digital images (A, k_1) and (B, k_2), then $(A \vee B, k)$ has the FPP if and only if both (A, k_1) and (B, k_2) have the FPP. However, in view of digital wedge sum property, Theorem 6.5, this assertion has no sense because every non-trivial digital wedge sum $(A \vee B, k)$ does not have the FPP.

7. Further remarks and works

We have developed the digital wedge product property of the Euler characteristics for digital images. Using various properties of digital wedge sums in terms of the digital homology, we have established a formula for calculating Euler characteristics for digital images which is of the same type of that for the ordinary Euler characteristics in pure topology. As mentioned in Sections 4, 5 and 6, making use of the digital wedge product property of the Euler characteristics, we have studied both Euler characteristics and Lefschetz numbers for even non-k-contractible digital images. Besides, due to the digital wedge sum property (Theorem 6.5), we can further investigate the digital homotopy invariance property for non-k-contractible digital images.

In general, the study of digital images has been undertaken in the framework of Khalimsky topology [35], [43], Marcus–Wyse topology [49], cellular complex structures [38], axiomatic locally finite spaces [39], cubical homology developed by Kaczynski, Mischaikow and Mrozek [33]. However, the present paper follows the graph-based Rosenfeld model with the aim to correct papers mentioned in Sections 5–6.

As a further work, one can study fixed point theory for digital spaces in the Khalimsky topological model [27]. Also, motivated by [36], using fixed point theory for digital images, one can study equilibrium of market economy and game theory in terms of a discrete dynamic system.

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SANG-EON HAN
Department of Mathematics Education
Institute of Pure and Applied Mathematics
Chonbuk National University
Jeonju-City Jeonbuk, 54896, REPUBLIC OF KOREA

E-mail address: sehan@jbnu.ac.kr

WEI YAO
Department of Mathematics
Hebei University of Science and Technology
P.R. CHINA

E-mail address: yaowei0516@163.com