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MULTIPLICITY OF SOLUTIONS OF ASYMPTOTICALLY LINEAR DIRICHLET PROBLEMS ASSOCIATED TO SECOND ORDER EQUATIONS IN $\mathbb{R}^{2n+1}$

Alessandro Margheri — Carlota Rebelo

Abstract. We present a result about multiplicity of solutions of asymptotically linear Dirichlet problems associated to second order equations in $\mathbb{R}^{2n+1}$, $n \geq 1$. Under an additional technical condition, the number of solutions obtained is given by the gap between the Morse indexes of the linearizations at zero and infinity. The additional condition is stable with respect to small perturbations of the vector field. We show with a simple example that in some cases the size of the perturbation can be explicitly estimated.

1. Introduction

In this paper we are interested on the existence of multiple solutions to the problem

$$\begin{aligned} x'' + A(t,x)x &= 0, \\ x(0) = x(\pi) &= 0, \end{aligned} \tag{1.1}$$

$x \in \mathbb{R}^{2n+1}$, $n \geq 1$, $t \in [0,\pi]$.

We will assume that $A\colon [0,\pi]\times\mathbb{R}^{2n+1} \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ is a continuous function with values in the set of the real symmetric matrices of order $2n+1$, denoted

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by $\mathrm{GL}_s(\mathbb{R}^{2n+1})$, and that there exist $A_i(\,\cdot\,)\colon [0,\pi] \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$, $i = 0, \infty$, also continuous such that

$$\begin{aligned} \lim_{|x|\to 0} A(t,x) &= A_0(t) \quad \text{uniformly in } t \in [0,\pi], \\ \lim_{|x|\to \infty} A(t,x) &= A_\infty(t) \quad \text{uniformly in } t \in [0,\pi], \end{aligned}$$

that is, we assume asymptotically linear conditions at the origin and at infinity. If the indexes of $A_0(\,\cdot\,)$, $i(A_0)$, and of $A_\infty(\,\cdot\,)$, $i(A_\infty)$, are different we prove, under an extra technical assumption, the existence of multiple solutions to the boundary value problem.

The existence of solutions of boundary value problems associated to asymptotically linear problems is a subject which has been studied by many authors for more than 30 years. We mention the seminal papers by Amann and Zehnder [1], [2] as maybe the first in which existence of solutions to asymptotically linear boundary value problems was studied. In those papers the authors introduce an index depending on the linearizations at zero and infinity and prove, when that index is positive, the existence of one or, in nondegenerate cases, two solutions. After those works many authors studied this problem both for Hamiltonian systems and for second order equations, assuming periodic or two-point boundary conditions (see [5] for details on the bibliography). We note that, for scalar second order equations, the multiplicity of solutions for the Dirichlet problem in terms of the gap between the Morse indexes of the linearizations can be easily proved using the link between Morse index and rotation number (see [9], where this approach was used for the Neumann and the periodic BVPs). For higher dimensions and when no symmetry or convexity conditions are assumed usually the existence of at most two solutions is guaranteed. One of the exceptions is [10] where multiple periodic solutions to a planar Hamiltonian system were obtained under no additional conditions by using the Poincaré–Birkhoff theorem. Also, in what concerns second order equations recently, in [5], (1.1) was considered for $x \in \mathbb{R}^2$ and multiple solutions were obtained for the Dirichlet problem assuming sign conditions on the entries of $A(t,x)$. In both of these papers the number of solutions increases when the gap of the indexes of the linearizations at zero and at infinite (Maslov index in the first paper, Morse index in the second) increases.

As far as we know in the case of second order equations in dimension larger than two, multiple solutions were only obtained in [4]. However the number of solutions obtained depends on the number of elements of a set which one has to check to be nonempty.

In this paper we consider the problem in $\mathbb{R}^{2n+1}$, $n \geq 1$ and obtain a result similar to that in [5] by imposing a technical assumption concerning the space of solutions of some linear problems associated to $x'' + A(t,x)x = 0$, see Theorem 2.3

below. Our result works only in a space of odd dimension as a consequence of the result on degree theory that we apply.

The paper is organized as follows: in Section 2 we state the main result. Then in Section 3 we state (and prove some) of the auxiliary results which will be essential for the proof of the main theorem. Also in this section this proof is given. Finally, in the last section, we give an example of application of the theorem.

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2. Statement of the main result

Our main result concerns the multiplicity of solutions to the two-point boundary value problem in $\mathbb{R}^{2n+1}$

$$\begin{cases} x'' + A(t,x)x = 0, & t \in [0,\pi], \\ x(0) = x(\pi) = 0, \end{cases} \tag{2.1}$$

where $A\colon [0,\pi] \times \mathbb{R}^{2n+1} \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ is a continuous function. As pointed out in [4], this formulation of the problem is quite general.

We assume that uniqueness of solutions of Cauchy problems associated to system $x'' + A(t,x)x = 0$ is guaranteed and that there are continuous functions $A_0, A_\infty : [0,\pi] \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ such that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \quad \text{uniformly in } t \in [0,\pi], \tag{2.2}$$

$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \quad \text{uniformly in } t \in [0,\pi]. \tag{2.3}$$

Note that under assumption (2.3) the continuability of the solutions to Cauchy problems associated to $x'' + A(t,x)x = 0$ is guaranteed.

In order to state our main result, we recall the definitions of index and nullity of a path of symmetric matrices. To do this, first we reformulate the proposition proved in [6, Proposition 2.1] and restated for $x \in \mathbb{R}^2$ in [5].

PROPOSITION 2.1. *Given $B(\cdot) \in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^m))$ there exists a sequence of real numbers which we call eigenvalues of $B(\cdot)$, $\lambda_1(B) \le \lambda_2(B) \le \ldots \le \lambda_j(B) \to +\infty$ as $j \to +\infty$ such that, for each j, there exists a space of dimension one of nontrivial solutions (eigenvectors of $B(\cdot)$) of the problem*

$$\begin{cases} x'' + (B(t) + \lambda_j(B) I_m)x = 0, \\ x(0) = x(\pi) = 0, \end{cases}$$

where I_m denotes the identity matrix of order m. Moreover, $H_0^1([0,\pi];\mathbb{R}^m) := \{x\colon [0,\pi] \to \mathbb{R}^m \mid x(\cdot)$ is continuous on $[0,\pi]$, satisfies $x(0) = 0 = x(\pi)$, and $x' \in L^2([0,\pi];\mathbb{R}^m)\}$ admits a basis of eigenvectors of $B(\cdot)$.

Definition 2.2. Given $B(\,\cdot\,) \in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^m))$, its index $i(B)$ is defined as the number of negative eigenvalues and its nullity $\nu(B)$ is defined as the number of zero eigenvalues.

The index of $B(\,\cdot\,) \in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^m))$ as we have just defined coincides with the Morse index (also called Maslov index) of the boundary value problem $x'' + B(t)x = 0$, $x(0) = x(\pi) = 0$ (see [3], [11], [6]).

Note that in the sequence of the eigenvalues of $B(\,\cdot\,)$ we cannot have the same value repeated more than m times. In the case it is repeated s-times we say that the corresponding eigenvalue $\lambda(B) = \lambda_j(B) = \ldots = \lambda_{j+s}(B)$, for some j, has a space of eigenvectors of dimension s.

Now we are in position to state our main result. In [5] the same problem was studied in $\mathbb{R}^2$. Multiplicity of solutions was proved assuming a gap between the indexes of $A_0(\,\cdot\,)$ and $A_\infty(\,\cdot\,)$ and some sign condition on the components of $A(t,x)$. Now we extend this result to $\mathbb{R}^{2n+1}$ replacing the sign condition by a technical one, see (H) below.

Theorem 2.3. *Assume that* $A\colon [0,\pi]\times\mathbb{R}^{2n+1} \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ *satisfies* (2.2) *and* (2.3). *Suppose that the following condition holds*:

(H) *there is an hyperplane* $\mathcal{H}$ *passing through the origin such that for each continuous function* $y\colon [0,\pi] \to \mathbb{R}^{2n+1}$ *the problem* $x''(t)+A(t,y(t))x(t) = 0$, $x(0) = 0 = x(\pi)$ *has no nontrivial solutions with* $x'(0) \in \mathcal{H}$.

Then, if $i(A_0) > i(A_\infty) + \nu(A_\infty)$ (*or* $i(A_0) + \nu(A_0) < i(A_\infty)$), *the problem* (2.1) *has at least* $|i(A_0) - i(A_\infty) - \nu(A_\infty)|$ (*resp.* $|i(A_\infty) - i(A_0) - \nu(A_0)|$) *nontrivial solutions.*

In the following we consider, for each $\alpha \in \mathbb{R}^{2n+1}$, the Cauchy problem

$$\begin{cases} x'' + A(t,x)x = 0, \\ x(0) = 0, \quad x'(0) = \alpha, \end{cases}$$

and denote by x_α its unique solution.

Remark 2.4. (a) It is sufficient to check condition (H) just on the vectors $x'(0) \in \mathcal{H}$ such that $|x'(0)| = 1$. As a consequence, by the continuous dependence of the solution of an ODE on the vetor fields, property (H) will persist under small perturbations of $A(t,x)$ in $L^\infty([0,\pi]\times\mathbb{R}^{2n+1}, \mathrm{GL}_s(\mathbb{R}^{2n+1}))$.

(b) As a consequence of (H), for each continuous function $y\colon [0,\pi] \to \mathbb{R}$, the space of solutions of $x''(t) + A(t,y(t))x(t) = 0$, $x(0) = 0 = x(\pi)$ has at most dimension one. This remark will be important in the proof of the theorem.

(c) In the proof of Theorem 2.3, instead of assumption (H) we will use the slightly weaker assumption:

(H*) there is an hyperplane $\mathcal{H}$ passing through the origin such that for each $\alpha \in \mathbb{R}^{2n+1}$ the problem $x''(t) + A(t, x_\alpha(t))x(t) = 0$, $x(0) = 0 = x(\pi)$ has no nontrivial solutions with $x'(0) \in \mathcal{H}$.

We opted to state the theorem with (H) instead of (H*) as in general it is not easy to verify that this variant is satisfied.

3. Auxiliary results and proof of the main theorem

In this section we prove Theorem 2.3 after stating (and in some cases proving) some auxiliary results. We begin with two lemmas. The first one gives the existence of a suitable open set in which boundary we will look for the initial conditions of the solutions of (2.1). The second one will be applied to prove the existence of those initial conditions.

In the following we denote with $B_r(a)$ and $\overline{B}_r(a)$, respectively, the open and the closed balls with center $a \in \mathbb{R}^m$ and radius $r > 0$. Also, given a set $A \subset \mathbb{R}^m$, ∂A will denote its boundary and $\overline{A}$ its closure.

Lemma 3.1. *Let $0 < r < R$ and let $f \colon \overline{B}_R(0) \setminus B_r(0) \to \mathbb{R}$ be a continuous function. Assume that $f(x) < 0$ in $\partial B_r(0)$ and $f(x) > 0$ in $\partial B_R(0)$. Then there exists an open bounded set Ω containing the origin and such that $f(x) = 0$ in $\partial\Omega$.*

Proof. Consider the continuous extension $\widetilde{f}$ of f to $\overline{B}_R(0)$ defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } r \le |x| \le R, \\ f\left(\dfrac{rx}{|x|}\right)\dfrac{|x|}{r} & \text{if } 0 < |x| < r, \\ 0 & \text{if } x = 0, \end{cases}$$

which is negative in $B_r(0) \setminus \{0\}$. Then let Ω_0 be the connected component of $\widetilde{f}^{-1}(]-\infty, 0[)$ which contains $B_r(0) \setminus \{0\}$ and choose $\Omega = \Omega_0 \cup \{0\}$. □

Next lemma, which will be essential to get our result, follows from degree theory (see [12, Example 13.3]).

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^{2n+1}$ be an open bounded set with $0 \in \Omega$. Let $\psi \colon \overline{\Omega} \to \mathbb{R}^{2n+1}$ be a continuous function such that $\psi(x) \neq 0$ for each $x \in \partial\Omega$. Then there exists $C \in \mathbb{R} \setminus \{0\}$ and $x \in \partial\Omega$ such that $\psi(x) = Cx$.*

We concentrate now on the linear, parameter dependent equation

$$x'' + A(t, x_\alpha(t))x(t) = 0, \tag{3.1}$$

where $\alpha \in \mathbb{R}^{2n+1} \setminus \{0\}$.

We note that if there exists $\alpha \in \mathbb{R}^{2n+1} \setminus \{0\}$ and a solution $x(t)$ of (3.1) satisfying $x(0) = x(\pi) = 0$, and $x'(0) = c\alpha$ for some $c \neq 0$, then $x(t)/c$ is a solution of (1.1). This remark, previously used in [4] and in [5] will be crucial in our proof.

In [4], where equation (3.1) was considered, the following auxiliary result was obtained:

Lemma 3.3 ([4]). *Suppose that the continuous function* $A\colon [0,\pi]\times\mathbb{R}^{2n+1}\to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ *satisfies assumptions* (2.2) *and* (2.3), *then*

$$\begin{aligned} &A(t,x_\alpha(t)) \to A_\infty(t) \quad \text{in } L^1([0,\pi]) \text{ if } |\alpha|\to+\infty,\\ &A(t,x_\alpha(t)) \to A_0(t) \quad \text{in } L^1([0,\pi]) \text{ if } |\alpha|\to 0. \end{aligned}$$

The fact that the eigenvalues of $B_\alpha(\,\cdot\,) := A(\,\cdot\,,x_\alpha(\,\cdot\,))$ will converge to the ones of the corresponding linearizations as $|\alpha|\to 0$ or $|\alpha|\to+\infty$, is a consequence of the result below. Since it is the $\mathbb{R}^m$ version of [5, Proposition 2.4] we do not present its proof. Analogous results can be found in [7] and [8] for the case of a second order equation. In the following, $\mathrm{GL}(\mathbb{R}^m)$ will denote the set of the $m\times m$ matrices with real entries.

Proposition 3.4. *Fixed* $M>0$, *for each* $j\in\mathbb{N}$, $B\to\lambda_j(B)$ *is continuous in* $\{B\in L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^m)) : \|B(t)\|<M$ *for almost every* $t\in(0,\pi)\}$.

Consider now the first order linear system in $\mathbb{R}^{2(2n+1)}$

$$\begin{cases} x' = y, \\ y' = -A(t,x_\alpha(t))x, \quad x,y\in\mathbb{R}^{2n+1},\ t\in[0,\pi], \end{cases} \tag{3.2}$$

associated to the equation (3.1), and denote by $E_\alpha(t)\in\mathrm{GL}(\mathbb{R}^{2(2n+1)})$ the monodromy matrix generated by the matrix

$$\left(\begin{array}{c|c} 0 & I \\ \hline -A(t,x_\alpha(t)) & 0 \end{array}\right).$$

If we write the following block decomposition of $E_\alpha(\pi)$,

$$E_\alpha(\pi) = \left(\begin{array}{c|c} E_\alpha^{1,1} & E_\alpha^{1,2} \\ \hline E_\alpha^{2,1} & E_\alpha^{2,2} \end{array}\right),$$

then one immediately gets that:

Lemma 3.5. *There is an isomorphism between the space* V_α *of solutions of* (3.1) *satisfying* $x(0)=x(\pi)=0$ *and* $\mathrm{Ker}\,E_\alpha^{1,2}$ *given by* $V_\alpha\ni x(\,\cdot\,)\to x'(0)\in \mathrm{Ker}\,E_\alpha^{1,2}$.

The above lemma implies that the nonzero solutions of (3.1) satisfying $x(0)=x(\pi)=0$, are in one-to-one correspondence with the eigenvectors of the zero eigenvalue of $E_\alpha^{1,2}$.

In this connection, the next result concerns the existence of global branches of eigenvectors of the matrix $E_\alpha^{1,2}$ which depend continuously on α.

In general, given a matrix depending continuously on a parameter, it is well known that such branches do not exist. In fact, if the dimension of the eigenspace is not a constant function of the parameter, then, in general, a discontinuity point of its dimension will be also a discontinuity point for any branch of eigenvectors.

Moreover, even if the dimension of the eigenspace is a constant function of a parameter it may not be possible to glue the local continuous branches of eigenvectors to obtain a global one. In the next proposition we show that under an additional assumption, a continuous branch of eigenvectors exists when the dimension of the eigenspace is constant and equal to one.

We state this proposition in the case of the zero eigenvalue, but the result is still valid if we consider eigenvalues depending continuously on a parameter. In the statement below, $\mathbb{P}(\mathbb{R}^m)$ denotes the real projective space of dimension $m-1$, which we describe as the set of all the lines through the origin in $\mathbb{R}^m$.

Proposition 3.6. *Let S be a subset of $\mathbb{R}^m$, $m \geq 2$ and let $E\colon S \to \mathrm{GL}(\mathbb{R}^m)$, $\alpha \to E_\alpha$, be a continuous family of square matrices of order m. Assume that $\dim \operatorname{Ker} E_\alpha = 1$ for any $\alpha \in S$. Then the map $\Phi\colon S \to \mathbb{P}(\mathbb{R}^m)$ defined by $\Phi(\alpha) = \operatorname{Ker} E_\alpha$ is continuous in S. Moreover, if there exists an hyperplane through the origin $\mathcal{H}$ such that $\Phi(S) \cap \mathcal{H} = 0$, then it is possible to define on S a continuous branch of eigenvectors $\alpha \to v_\alpha \in \operatorname{Ker} E_\alpha$.*

Proof. As $\dim(\operatorname{Ker} E_\alpha) = 1$, the function $S \ni \alpha \to \operatorname{Ker} E_\alpha \in \mathbb{P}(\mathbb{R}^m)$ is well defined. Now, for any point in S, we can find a locally continuous branch $\alpha \to v_\alpha \in \operatorname{Ker} E_\alpha \setminus 0$ by solving the system $E_\alpha v = 0$. In fact, without loss of generality we may assume that this solution has the form $v_\alpha = (1, v_{2,\alpha}, \ldots, v_{m,\alpha})$, where the $v_{i,\alpha}$ are continuous functions of α. The first part of the statement then follows since $(v_{2,\alpha}, \ldots, v_{m,\alpha})$ are the affine coordinates of $\operatorname{Ker} E_\alpha$ in the chart Ψ_1 of $\mathbb{P}(\mathbb{R}^m)$ defined by $\Psi([(w_1, \ldots, w_m)]) = (v_2 := w_2/w_1, \ldots, v_m := w_m/w_1)$ with $w_1 \neq 0$.

If there exists an hyperplane $\mathcal{H}$ as in the statement, we can choose coordinates in $\mathbb{R}^m$ such that the affine coordinates above work for any $\alpha \in S$. □

We are finally ready to prove our main result.

Proof of Theorem 2.3. Let us assume that $i(A_0) > i(A_\infty) + \nu(A_\infty)$, the other case can be treated similarly. By the definition of index there are exactly $i(A_0)$ negative eigenvalues of $A_0(\,\cdot\,)$, $\lambda_l(A_0)$, $l \in \{1, \ldots, i(A_0)\}$. Also there are exactly $i(A_\infty)+\nu(A_\infty)$ negative or zero eigenvalues of $A_\infty(\,\cdot\,)$, $\lambda_j(A_\infty)$, $j \in \{1, \ldots, i(A_\infty) + \nu(A_\infty)\}$. Hence $\lambda_j(A_\infty)$ is positive for every $j \in \mathbb{N}$, $j \geq i(A_\infty) + \nu(A_\infty) + 1$.

Fix now any $h \in \mathbb{N}$ satisfying $i(A_0) \geq h \geq i(A_\infty) + \nu(A_\infty) + 1$. We have that

$$\lambda_h(A_0) < 0 < \lambda_h(A_\infty). \tag{3.3}$$

Our aim consists in proving the existence of $\alpha_h \neq 0$ such that $x_{\alpha_h}(\pi) = 0$ as this implies that $x_{\alpha_h}(\,\cdot\,)$ is a solution of (2.1). The intermediate step will be to find $\alpha_h \neq 0$ such that $\lambda_h(A(\,\cdot\,, x_{\alpha_h}(\,\cdot\,))) = 0$ and a solution $v_{\alpha_h}(\,\cdot\,)$ of $x'' + A(t, x_{\alpha_h}(t))x = 0$ satisfying $v_{\alpha_h}(0) = v_{\alpha_h}(\pi) = 0$ and $v'_{\alpha_h}(0) = C\alpha_h$ for some $C \neq 0$, as then $x_{\alpha_h} = v_{\alpha_h}/C$ and the result follows.

We concentrate on the study of the parameter dependent problem

$$\begin{cases} x'' + A(t, x_\alpha(t))x(t) = 0, \\ x(0) = x(\pi) = 0. \end{cases} \tag{3.4}$$

By combining Lemma 3.3 and Proposition 3.4 with the inequalities (3.3) we can choose $0 < R_1 < R_2$ such that $\lambda_h(A(\,\cdot\,, x_\alpha(\,\cdot\,))) < 0$ for every α with $|\alpha| = R_1$ and $\lambda_h(A(\,\cdot\,, x_\alpha(\,\cdot\,))) > 0$ for every α with $|\alpha| = R_2$.

Now from Lemma 3.1 we obtain the existence of an open set Ω containing the origin and such that $\lambda_h(A(\,\cdot\,, x_\alpha(\,\cdot\,))) = 0$ for each $\alpha \in \partial\Omega$.

Using the assumptions of the theorem and Remark 2.4 we are in position to apply Proposition 3.6 and Lemma 3.5 and prove the existence of a continuous branch of vectors $\beta(\alpha) \in \mathbb{R}^{2n+1} \setminus \{(0,0)\}$, $\alpha \in \partial\Omega$ such that $\beta(\alpha) := v'_\alpha(0)$ where, for each α, $v_\alpha(\,\cdot\,)$ is an eigenvector of $A(\,\cdot\,, x_\alpha(\,\cdot\,))$ associated to the zero eigenvalue.

We conclude that, for each α, $v_\alpha(\,\cdot\,)$ is a nontrivial solution of the system

$$\begin{cases} x'' + A(t, x_\alpha(t))x = 0, \\ x(0) = x(\pi) = 0 \end{cases} \tag{3.5}$$

satisfying $x'(0) = \beta(\alpha)$.

By Tietze's theorem we can extend β to $\overline{\Omega}$ as a continuous function and the existence of $\overline{\alpha} \in \partial\Omega$ and $C \neq 0$ such that $\beta(\overline{\alpha}) = C\,\overline{\alpha}$ follows immediately from Lemma 3.2. Hence we obtain $x_{\overline{\alpha}}(\,\cdot\,) = x_{\beta(\overline{\alpha})}(\,\cdot\,)/C$ and, consequently, $x_{\overline{\alpha}}(\pi) = 0$. In particular, we can choose $\alpha_h = \overline{\alpha}$.

To complete the proof it remains to show that all the values α_h that we have found above are mutually different, or, equivalently, that all the solutions of the form x_{α_h} are mutually different.

Assume, by contradiction, that there exist two natural numbers $h, k \in [i(A_\infty) + \nu(A_\infty) + 1, i(A_0)]$ with $h \neq k$ such that $\alpha_h = \alpha_k$. Let us set $\widetilde{\alpha} := \alpha_h = \alpha_k$. In this case $\lambda_h(A(\,\cdot\,, x_{\widetilde{\alpha}}(\,\cdot\,)) = \lambda_k(A(\,\cdot\,, x_{\widetilde{\alpha}}(\,\cdot\,)) = 0$ and this contradicts the fact that under our assumptions the space of eigenvectors associated to the zero eigenvalue has dimension one. $\square$

As an immediate consequence of the continuity of the eigenvalues of the Dirichlet problem stated in Proposition 3.4 and of (a) in Remark 2.4, we get the following:

COROLLARY 3.7. *Assume that $A(t,x)$ is such that the assumptions of Theorem 2.3 are satisfied. Consider $P\colon [0,\pi]\times\mathbb{R}^{2n+1}\to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ such that*

$$\begin{aligned}\lim_{|x|\to 0} P(t,x) &= P_0(t) \quad \text{uniformly in } t\in[0,\pi],\\ \lim_{|x|\to \infty} P(t,x) &= P_\infty(t) \quad \text{uniformly in } t\in[0,\pi],\end{aligned}$$

where $P_i(\,\cdot\,)\colon [0,\pi]\to \mathrm{GL}_s(\mathbb{R}^{2n+1})$, $i=0,\infty$ are also continuous functions. Assume that uniqueness of Cauchy problems associated to $x''+(A(t,x)+P(t,x))x=0$ is guaranteed. Let $||\cdot||$ be a fixed matrix norm. Then, there exists ε^* such that if $0\le\varepsilon\le\varepsilon^*$ and

$$||P(t,x)||\le\varepsilon, \quad \text{for any } (t,x)\in[0,\pi]\times\mathbb{R}^{2n+1},$$

then the same multiplicity results of Theorem 2.3 hold for the Dirichlet problem

$$\begin{cases} x''+(A(t,x)+P(t,x))x=0,\\ x(0)=x(\pi)=0.\end{cases} \tag{3.6}$$

4. An example

Corollary 3.7 in the previous section establishes the robustness of the results stated in Theorem 2.3 with respect to suitable small L^∞ perturbations $P(t,x)$ of the matrix $A(t,x)$. In this Section we present one simple example of application of Corollary 3.7 where it will be actually possible to give an explicit formula to compute ε^*. Essentially, the example consists of a suitable perturbation of three uncoupled equations. Namely, the perturbation $P(t,x)$ is assumed to be such that its limitss at zero and infinity are given by constant diagonal matrices.

Before presenting the example we give an auxiliary lemma. In the following given $s\in\mathbb{R}$ we denote by $\lceil s\rceil$ the smallest integer larger than or equal to s and by $\lfloor s\rfloor$ the largest integer less than or equal to s.

LEMMA 4.1. *Let $a\colon[0,\pi]\to\mathbb{R}$ be a continuous function such that $0<a_-\le a(t)\le a_+$ for each $t\in[0,\pi]$ and consider the differential equation $x''+a(t)x=0$. Let x be a solution of the equation such that $x(0)=0$ and $x'(0)=1$. Then the rotation $\mathrm{rot}_{(x,x')}$ of (x,x') around the origin in the interval $[0,\pi]$ is less or equal to $\lceil 2\sqrt{a_+}\rceil/4$ and larger or equal to $\lfloor 2\sqrt{a_-}\rfloor/4$. Moreover,*

$$\begin{aligned}|x(t)| &\quad \text{is bounded by } \frac{\sqrt{a_+}}{a_-}\left(\frac{a_+}{a_-}\right)^{(\mathrm{rot}_{(x,x')}-1)/2},\\ |x'(t)| &\quad \text{is bounded by } \left(\frac{a_+}{a_-}\right)^{\mathrm{rot}_{(x,x')}/2}\end{aligned}$$

in the interval $[0,\pi]$.

The statements about the rotation are easily obtained using the modified polar coordinates $x = r\cos\theta$ and $y = \sqrt{a_\pm}\, r\sin\theta$ and recalling that the angle coordinate associated to these coordinates and the one associated to the usual polar coordinates coincide in the multiples of $\pi/2$. In what concerns the bounds for $|x|$ and for $|x'|$, they are obtained noticing that, if we let

$$F_\pm(x,y) = a_\pm \frac{x^2}{2} + \frac{y^2}{2},$$

we have that $F_+(x(t),x'(t))$ increases in the odd quadrants and decreases in the even ones while for $F_-(x(t),x'(t))$ the opposite occurs.

In the following, for each matrix $M = [m_{ij}] \in \mathrm{GL}(\mathbb{R}^3)$ we denote by M^* the transpose of M and by $\|M\|_2 = \sqrt{r(M^*M)}$, where $r(M)$ is the spectral radius of M. Also

$$\|M\|_\infty = \max_{1\le i\le 3} \sum_{j=1}^{3} |m_{ij}|.$$

Now we describe our example. Consider a continuous function $a\colon [0,\pi]\times\mathbb{R}^3 \to \mathbb{R}$, Lipschitz continuous in x, such that $\lim_{|x|\to 0} a(t,x) = a_0 > 0$ and $\lim\limits_{|x|\to+\infty} a(t,x) = a_\infty$ uniformly in $t\in[0,\pi]$ and satisfying $a_0 \le a(t,x) \le a_\infty$. Let $A\colon [0,\pi]\times\mathbb{R}^3 \to \mathrm{GL}_s(\mathbb{R}^3)$ be given by

$$A(t,x) = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & a(t,x) \end{bmatrix}.$$

We chose the first two entries in the diagonal of the matrix equal to $1/4$ only to simplify the computations, we could have chosen any values, not necessarily equal, different from squares of integers. As the limits A_0 and A_∞ of $A(t,x)$ as $|x|\to 0$ and $|x|\to+\infty$ are diagonal constant matrices, it is easy to compute the index and the nullities of these matrices and to conclude that with our choice $i(A_\infty)-i(A_0)-\nu(A_0)$ is equal to the number of square of integers strictly between a_0 and a_∞.

Assume now that $P\colon [0,\pi]\times\mathbb{R}^3 \to \mathrm{GL}_s(\mathbb{R}^3)$ is Lipschitz continuous in x and such that

$$\lim_{|x|\to i} P(t,x) = P_i = \begin{bmatrix} p_1^{(i)} & 0 & 0 \\ 0 & p_2^{(i)} & 0 \\ 0 & 0 & p_3^{(i)} \end{bmatrix} \quad \text{uniformly in } t\in[0,\pi],\ i = 0,\infty.$$

We will provide below an explicit way to estimate the value $\varepsilon^* > 0$ such that if

$$\max_{t\in[0,\pi],x\in\mathbb{R}^3} \|P(t,x)\|_\infty < \varepsilon^*$$

then there exist k solutions of the problem (3.6), where k is the number of squares of integers, $n_1^2 < \ldots < n_k^2$, strictly between a_0 and a_∞.

For each $y \in C([0,\pi],\mathbb{R}^3)$ we have that there are no nontrivial solutions of

$$x'' + A(t,y(t))x = 0, \quad x(0) = x(\pi) = 0$$

with $x'(0) \in \mathcal{H}$ where $\mathcal{H} = \{(a,b,0),(a,b) \in \mathbb{R}^2\}$. Considering $s \in [0,\pi]$, let $\psi(t,s) \in \mathrm{GL}(\mathbb{R}^3)$ satisfy

$$x'' + A(t,y(t))x = 0, \quad \psi(s,s) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t}(s,s) = I_3.$$

We have that

$$\psi(t,s) = \begin{bmatrix} 2\sin((t-s)/2) & 0 & 0 \\ 0 & 2\sin((t-s)/2) & 0 \\ 0 & 0 & g(t,s) \end{bmatrix},$$

where $g(t,s)$ is a solution of $x_3'' + a(t,y(t))x_3{=}0$, $x_3(s) = 0$, $x_3'(s) = 1$.

By the previous lemma we have

$$\mathrm{rot}_{(x_3,x_3')} \le \frac{\lceil 2\sqrt{a_\infty}\rceil}{4} := K_1 \quad \text{and} \quad |g(t,s)| \le \frac{\sqrt{a_\infty}}{a_0}\left(\frac{a_\infty}{a_0}\right)^{(K_1-1)/2} := L.$$

Then, it is $\|\psi(t,s)\|_\infty \le \max\{2, L\} := K_2$.

First, we will give an explicit condition on $\varepsilon > 0$ such that if $\|P(t,x)\|_\infty < \varepsilon$ then $A(t,x) + P(t,x)$ verifies condition (H) of Theorem 2.3 (where A is replaced by $A + P$).

As observed in (a) of Remark 2.4, we only need to consider $x'(0) = \ell = (\cos\theta, \sin\theta, 0) \in \mathcal{H}$ for each $\theta \in [0, 2\pi[$.

For each ℓ, the solution of $x'' + A(t,y(t))x = -P(t,y(t))x$ satisfying $x(0) = 0$ and $x'(0) = \ell$ is given by

$$x(t) = \psi(t,0)\ell - \int_0^t \psi(t,s)P(s,y(s))x(s)\,ds. \tag{4.1}$$

Hence we have

$$|x(t)| \le 2 + \int_0^t \|\psi(t,s)\|_2 \|P\|_2 |x(s)|\,ds$$

from which, using Gronwall's inequality, $|x(t)| \le 2\mathrm{e}^{\sqrt{3}K_2\|P\|_\infty t}$.

Now, from (4.1) we obtain $|x(\pi)| \ge 2(1 - K_2\pi\varepsilon\mathrm{e}^{\sqrt{3}K_2\varepsilon\pi})$. We conclude that, if ε is such that

$$K_2\pi\varepsilon\mathrm{e}^{\sqrt{3}K_2\varepsilon\pi} < 1 \tag{4.2}$$

we have that condition (H) holds. We fix a value ε_0 which satisfies inequality (4.2). Finally, since the limits at zero and infinity of $A(t,x) + P(t,x)$ are given by $A_i(t) + P_i(t)$, $i = 0, \infty$, if we take

$$\varepsilon^* < \min\{\varepsilon_0, 3/4, n_1^2 - a_0, a_\infty - n_k^2\}$$

we have that $i(A_\infty + P_\infty) - i(A_0 + P_0) - \nu(A_0 + P_0) \geq i(A_\infty) - i(A_0) - \nu(A_0) = k$, and Theorem 2.3 guarantees the existence of k solutions.

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Alessandro Margheri and Carlota Rebelo
Centro de Matemática e Aplicações Fundamentais
Faculdade de Ciências
Universidade de Lisboa
Campo Grande
Edifício C6, piso 2
P-1749-016 Lisboa, PORTUGAL
E-mail address: amargheri@fc.ul.pt, mcgoncalves@fc.ul.pt