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A HOMOTOPICAL PROPERTY OF ATTRACTORS

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Abstract. We construct a 2-dimensional torus $\mathcal{T} \subseteq \mathbb{R}^3$ having the property that it cannot be an attractor for any homeomorphism of $\mathbb{R}^3$. To this end we show that the fundamental group of the complement of an attractor has certain finite generation property that the complement of $\mathcal{T}$ does not have.

1. Introduction

Given a manifold M and a dynamical system defined on it, we say that a compact set $K \subseteq M$ is an attractor if it is invariant, Lyapunov stable and there is a neighbourhood $U = U(K)$ such that all orbits starting at U converge to the set K. This definition leads to the following question: what compact sets can be realized as attractors of some dynamical system? In the last thirty years several authors have dealt with this question and the known results depend critically on the type of dynamical system and the dimension of the ambient space. For continuous flows we refer to [5], [8], [10], [11], [12], [16], [18] and to [6], [7], [9], [13], [17] for the discrete case.

In the present paper we assume that the ambient space is $M = \mathbb{R}^3$ and the system is discrete, produced by a homeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$. The general (unsolved) problem is to describe the class of compact sets $K \subseteq \mathbb{R}^3$ which are

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attractors for some h. Our more modest goal will be to construct a curious example of a set that cannot be realized as an attractor. In the process we will find an abstract homotopical obstruction that can be of independent interest. To describe our result let us consider one of the most natural attractors in the Euclidean space, the torus of revolution $T \subseteq \mathbb{R}^3$. We aim at constructing a set $\mathcal{T} \subseteq \mathbb{R}^3$ that is homeomorphic to T but cannot be an attractor of any h. At first sight the existence of $\mathcal{T}$ may seem paradoxical but those readers who are familiar with topology in three dimensions will probably agree that $\mathcal{T}$ is conceivable as long as it is a wild surface. Roughly speaking, a surface $S \subseteq \mathbb{R}^3$ is wild if it contains a point p such that S cannot be flattened within $\mathbb{R}^3$ near p. There is nothing particular about the torus in our construction and similar examples of different genus can be constructed. In particular we refer to [17] for a different construction in the case of the sphere.

At this point it seems convenient to discuss the connections of our result with the existing literature. The question posed earlier about the realization of compact sets as attractors can be interpreted in different ways. In our approach the ambient space is fixed ($M = \mathbb{R}^3$) but other authors have considered the problem in different terms: the set K is a given compact metric space and the unknowns are the ambient manifold M (of arbitrary dimension) and the homeomorphism producing an attractor that is homeomorphic to K. The two problems are different but certainly there are links between them. In particular we refer to the approach taken by Günther in [9]. In this interesting paper the very general case of continuous maps $f\colon M \to M$ is considered to show that certain solenoids cannot be realized as attractors on any manifold M. To prove this result Günther considers the Čech cohomology groups of an attractor K and the induced homomorphism $f^*\colon \check{H}^*(K) \to \check{H}^*(K)$, showing that there must exist a finitely generated subgroup $G \subseteq \check{H}^*(K)$ which acts as a sort of algebraic attractor for h^*. This rather vague statement means exactly that

$$\bigcup_{n=1}^{\infty} (f^*)^{-n}(G) = \check{H}^*(K).$$

Our paper is organized as follows. In Section 2 we adapt the idea of Günther to our setting, proving that it still holds after replacing the Čech cohomology group of the attractor by the first homotopy group of its complement $\mathbb{R}^3 - A$. Our construction of the wild torus $\mathcal{T}$ that cannot be an attractor is based on two sets with very surprising topological properties: the Cantor set of Antoine A and the wild sphere of Antoine $\mathcal{A}$. These sets were discovered (invented?) almost one century ago and they seem to be very well adapted for the needs of dynamics. Section 3 reviews how A is constructed and some of its properties. In Section 4 we introduce a number $\delta(\alpha)$ that somehow quantifies the amount of entanglement of a loop $\alpha \subseteq \mathbb{R}^3 - A$ with the set A. Section 5 starts by reviewing how the Antoine sphere $\mathcal{A}$ is defined and then moves on to prove that the torus $\mathcal{T}$ obtained by suitably attaching a handle onto $\mathcal{A}$ cannot be an attractor for any homeomorphism.

We have also included an Appendix that establishes two well known properties of the Antoine set A for which, however, we could not find elementary proofs in the literature.

2. A homotopical property of attractors

2.1. Let X be a path connected metric space with a basepoint $x_0 \in X$, and suppose $h\colon X \to X$ is a homeomorphism such that $h(x_0) = x_0$. Consider the isomorphism $h_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ induced by h on the fundamental group of X.

Definition 2.1. We say that $\pi_1(X, x_0)$ is *finitely generated with respect to* h if there exists a finitely generated subgroup $G \subseteq \pi_1(X, x_0)$ such that

$$\pi_1(X, x_0) = \bigcup_{k \geq 0} h_*^k(G).$$

Notice that when $h = \mathrm{id}$ we recover the notion that $\pi_1(X, x_0)$ be finitely generated.

2.2. Since Definition 2.1 is motivated by our desire to understand what subsets of $\mathbb{R}^3$ can be attractors for homeomorphisms we now turn to dynamics, first recalling some definitions and then establishing a relation with the property described in Definition 2.1.

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism. A compact set $K \subseteq \mathbb{R}^3$ is called *invariant* if $f(K) = K$. A compact set $P \subseteq \mathbb{R}^3$ is said to be *attracted* by K if for every neighbourhood V of K there exists $n_0 \in \mathbb{N}$ such that $f^n(P) \subseteq V$ for every $n \geq n_0$. Finally, an *attractor* for f is a (nonempty) compact invariant set $K \subseteq \mathbb{R}^3$ having a neighbourhood U such that every compact set $P \subseteq U$ is attracted by K. The biggest U with this property is called the *basin of attraction* of K, and it is always an open and invariant subset of $\mathbb{R}^3$.

In the sequel it will be convenient to think of $\mathbb{R}^3$ as the 3-sphere $\mathbb{S}^3$ minus the point at infinity ∞ and extend f to a homeomorphism of $\mathbb{S}^3$ simply by letting ∞ be fixed. Whenever we consider a homotopy group $\pi_1(X, x_0)$ the basepoint will be assumed to be $x_0 = \infty$, even if not explicitly notated. Thus the elements of $\pi_1(X)$ are homotopy classes $[\alpha]$ of loops α based at ∞.

Suppose K is an attractor for a homeomorphism f, and denote C_∞ the connected component of $\mathbb{S}^3 - K$ containing ∞. Clearly $f(C_\infty) = C_\infty$, so $f|_{C_\infty}$ is a homeomorphism of C_∞. Then the following holds:

Proposition 2.2. *$\pi_1(C_\infty)$ is finitely generated with respect to $f|_{C_\infty}$.*

Before proving the proposition let us make the following observation. Let B_1 and B_2 be two disjoint compact subsets of $\mathbb{S}^3$. Cover each point $p \in B_1$ with a closed cube C_p centered at p and disjoint from B_2. Since B_1 is compact, there is a finite subfamily of $\{C_p\}$ whose union N is a neighbourhood of B_1. By construction N is compact and disjoint from B_2. Also, it has a finite triangulation and therefore each of its connected components has a finitely generated fundamental group [20, Corollary 4, p. 141].

Proof. Let U be the region of attraction of f, which is an open subset of $\mathbb{S}^3$. Then $\mathbb{S}^3 - U$ and K are disjoint compact subsets of $\mathbb{S}^3$, so there exists a neighbourhood N of $\mathbb{S}^3 - U$ disjoint from K and such that the fundamental group of each of its connected components is finitely generated. Let C'_∞ be the connected component of N that contains ∞. Denoting $i\colon C'_\infty \to C_\infty$ the inclusion, the group $G := i_*\pi_1(C'_\infty, \infty)$ is then finitely generated.

Let $[\alpha] \in \pi_1(C_\infty, \infty)$. Observe that the image of α is a compact subset of $\mathbb{S}^3 - K$, and notice also that $P := \overline{\mathbb{S}^3 - N}$ is a compact subset of U. Since K is an attractor there exists $k \geq 0$ such that the image of α is disjoint from $f^k(P)$, and consequently also from $f^k(\mathbb{S}^3 - N) = \mathbb{S}^3 - f^k(N)$. Thus $\operatorname{im} \alpha \subseteq f^k(N)$ and letting $\beta := f^{-k} \circ \alpha$ we see that $\operatorname{im} \beta \subseteq N$. Now $\operatorname{im} \beta$ is a connected subset of N which contains ∞, so it is actually contained in C'_∞. Thus $[\beta] \in G$ and so $[\alpha] = f^k_*([\beta]) \in f^k_*(G)$.

In the dynamical situation considered in Proposition 2.2 it is definitely not true in general that $\pi_1(C_\infty, \infty)$ is finitely generated. To clarify this it is illustrative to consider a well known example: the dyadic solenoid.

Example 2.3. Let $T \subseteq \mathbb{R}^3$ be a solid torus of revolution and $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ a homeomorphism such that $f(T) \subseteq \operatorname{int} T$ winds twice around T. The set $K := \bigcap_{k \geq 0} f^k(T)$ is the dyadic solenoid, and by its very construction it is an attractor for f.

For each $k = 0, 1, \ldots$ denote X_k the complement of the torus $T_k := f^k(T)$ in $\mathbb{S}^3$. The X_k form an increasing sequence of open sets whose union is $X := \mathbb{S}^3 - K$. Thus $\pi_1(X)$ is the direct limit of the sequence

$$\mathcal{S}\colon \pi_1(X_0) \longrightarrow \pi_1(X_1) \longrightarrow \pi_1(X_2) \longrightarrow \cdots$$

where the arrows denote the inclusion induced homomorphisms. Since each T_k is an unknotted torus, $\pi_1(X_k) = \mathbb{Z}$ for every k.

Now consider, for instance, the first arrow in this sequence. Figure 1 shows the torus T cut along a meridian disk (thus it looks like a solid cylinder) and T_1 inside it. Clearly $\pi_1(X_0)$ is generated by g_0 and $\pi_1(X_1)$ is generated by g_1. Moreover, $g_0 = 2g_1$ in $\pi_1(X_1)$. The same argument shows that $\pi_1(X_k)$ is generated by a loop g_k such that $g_k = 2g_{k+1}$ in $\pi_1(X_{k+1})$. Hence the sequence $\mathcal{S}$

simply reads

$$\mathcal{S}\colon \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

The direct limit of this sequence is easily seen to be non-finitely generated. However, clearly f_* takes g_k onto g_{k+1}. Thus, letting G be the subgroup of $\pi_1(X)$ generated by g_0, evidently $\pi_1(X) = \bigcup_{k\geq 0} f_*^k(G)$, showing that $\pi_1(X)$ is finitely generated with respect to f.

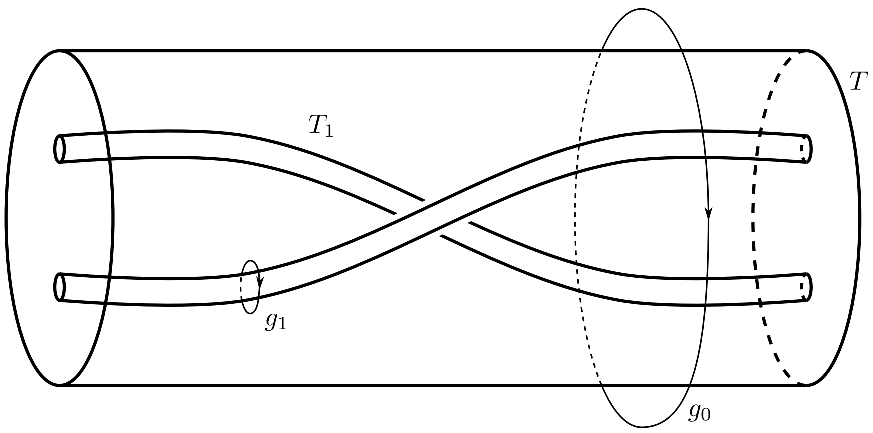

FIGURE 1. The dyadic solenoid.

Observe that Proposition 2.2 holds true for attractors in any $\mathbb{R}^n$.

3. The Cantor set of Antoine

Antoine [1, §78, p. 311 ff.] gave an example of a Cantor set $A \subseteq \mathbb{R}^3$ that is not ambient homeomorphic to the standard Cantor set in $\mathbb{R}^3$. His example, which we shall call the *Antoine necklace*, has many paradoxical properties. Since our construction of the torus $\mathcal{T}$ that cannot be an attractor is based on the set A we now review in some detail how it is defined and enumerate some of its properties. Moise [14] dedicates a whole chapter to this set.

3.1. Consider an unknotted solid torus $T_0 \subseteq \mathbb{R}^3$. Inside T_0 place a chain comprised of $N \geq 4$ smaller solid tori linked as shown in Figure 2 for $N = 5$. Let them be labeled $T_{11}, \ldots, T_{1N}$ and denote $M_1 = T_{11} \cup \ldots \cup T_{1N}$ be their union. These T_{1j} are the *first generation* tori of the process.

Now repeat the same construction at a smaller scale, placing inside each first generation torus T_{1j} a chain of N tori linked again in the pattern of Figure 2. These are the *second generation* tori T_{2j} and there are N^2 of them. We denote their union M_2. This process is then repeated inductively, so for each generation i we construct a family of N^i tori labeled T_{ij}. The set M_i is the union of all

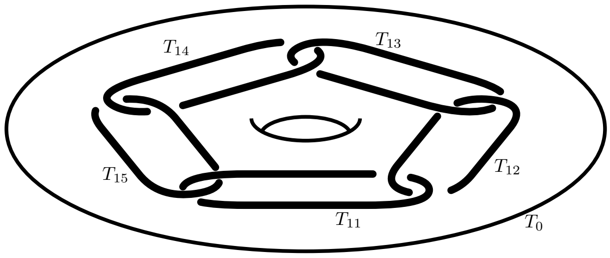

Figure 2.

the tori belonging to generation i and the Antoine necklace is defined as the intersection

$$A = \bigcap_{i=1}^{\infty} M_i.$$

Let us introduce some useful terminology. If T_{ij} is any of the solid tori which constitute the chain M_i, we call the intersection $A_{ij} := T_{ij} \cap A$ a *link* of A of *generation* i. Evidently a link A_{ij} is the (disjoint) union of the N links of generation $i+1$ that it contains, and A is the union of all the ith generation links for any given i. If $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $h(A) = A$, we say that h is *generation preserving* if for each link A_{ij} there exists another link $A_{ij'}$ of the same generation such that $h(A_{ij}) = A_{ij'}$.

3.2. We now enumerate some properties of the Antoine necklace that will play an important role in the sequel. The first one is an easy consequence of the symmetry of the construction of A:

(A1) Given any two A_{ij} and $A_{ij'}$ belonging to the same generation, there exists an ambient homeomorphism $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $g(A) = A$, g is generation preserving, and $g(A_{ij}) = A_{ij'}$.

Now follow two truly paradoxical properties. The first one was already established by Antoine [1, §86, p. 318] and a modern proof can be found in the book by Moise [14, Theorem 3, p. 131]. The second one seems to be also well known but we are not aware of any elementary proofs. Thus we have supplied one in an Appendix (Lemma A.1).

(A2) Let μ_i be a meridian of one of the tori T_{ij}. Then μ_i is not contractible in $\mathbb{S}^3 - A_{ij}$.

(A3) Let μ_0 be a meridian of T_0. Choose a point $p \in A$ and denote $A^* := A - \{p\}$ the result of removing p from A. Then μ_0 is contractible in $T_0 - A^*$, even though it is not contractible in $T_0 - A$.

The next property is easy to prove:

(A4) Let α be a loop in $\mathbb{S}^3 - A$. Then there exists a generation i_0 such that for every $i > i_0$, α is nullhomotopic in $\mathbb{S}^3 - A_{ij}$ (for every j).

Proof. Since $\alpha \cap A = \emptyset$, there exists M_{i_0} such that $A \cap M_{i_0} = \emptyset$. Let T_{ij} be any component of M_i with $i > i_0$. By the construction of the M_i, there is a ball $B \subseteq M_{i_0}$ such that $T_{ij} \subseteq B$, so $\alpha \subseteq \mathbb{S}^3 - B \subseteq \mathbb{S}^3 - T_{ij} \subseteq \mathbb{S}^3 - A_{ij}$. But $\mathbb{S}^3 - B$ is simply connected, so α is contractible in $\mathbb{S}^3 - A_{ij}$. □

One final property will play an important role in the sequel. It seems intuitively reasonable and has been established, in different guises, by many authors such as Sher [19] or Wright [21]. Again, the interested reader can find an elementary proof in the Appendix (Lemma A.5).

(A5) Let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism such that $h(A) = A$. Then h is generation preserving.

4. The depth of a loop

Let α be a loop in $\mathbb{S}^3 - A$, based at ∞ as always. If α is contractible in $\mathbb{S}^3 - A$ let $\delta(\alpha) = -1$; otherwise define $\delta(\alpha)$ as the largest $i = 0, 1, 2, \ldots$ such that the following property holds: there exists a generation i link A_{ij} such that α is not nullhomotopic in $\mathbb{S}^3 - A_{ij}$. This is well defined by property (A4) of the Antoine necklace.

As an example, consider a meridian μ_i of any of the tori T_{ij}. Property (A2) of the Antoine necklace guarantees that $\delta(\mu_i) \geq i$. It is very easy to see that μ_i is contractible in any $\mathbb{S}^3 - T_{k\ell}$, where $k > i$. Therefore it is also contractible in $\mathbb{S}^3 - A_{k\ell}$, and it follows that $\delta(\mu_i) = i$.

Although δ has been defined for loops in $\mathbb{S}^3 - A$, it is actually well defined for homotopy classes of loops in $\mathbb{S}^3 - A$; that is, for elements of $\pi_1(\mathbb{S}^3 - A)$. This is the content of the following proposition:

Proposition 4.1. *If two loops α and β are homotopic in $\mathbb{S}^3 - A$, then $\delta(\alpha) = \delta(\beta)$.*

Proof. By symmetry it suffices to prove $\delta(\alpha) \leq \delta(\beta)$. Denote $i_0 = \delta(\beta)$, so that β is contractible in $\mathbb{S}^3 - A_{ij}$ whenever $i > i_0$. Since α and β are homotopic in $\mathbb{S}^3 - A$, they are also homotopic in $\mathbb{S}^3 - A_{ij}$ and therefore α is contractible in $\mathbb{S}^3 - A_{ij}$ too. Thus $\delta(\alpha) \leq i_0 = \delta(\beta)$. □

The following two propositions describe relevant properties of δ. The first is concerned with its behaviour under inversion and concatenation of loops. The second one proves that δ is invariant under an ambient homeomorphism fixing the Antoine necklace.

Proposition 4.2. *For any loops α, β in $\mathbb{S}^3 - A$,*

(a) *the equality* $\delta(\alpha^{-1}) = \delta(\alpha)$ *holds,*
(b) *the inequality* $\delta(\alpha * \beta) \leq \max\{\delta(\alpha), \delta(\beta)\}$ *holds.*

Proof. Part (a) is trivial, since the definition of $\delta(\alpha)$ is insensitive to the orientation of α. Part (b) is also easy. Assume for definiteness that $\delta(\alpha) \geq \delta(\beta)$ and set $i_0 = \delta(\alpha)$. Then, by the definition of δ, both α and β are nullhomotopic in $\mathbb{S}^3 - A_{ij}$ for every $i > i_0$. Thus the same holds true for $\alpha * \beta$, which implies that $\delta(\alpha * \beta) \leq i_0 = \max\{\delta(\alpha), \delta(\beta)\}$. $\square$

Proposition 4.3. *Let* $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be a homeomorphism that leaves* A *invariant. For any loop* α *in* $\mathbb{S}^3 - A$*, the equality* $\delta(h \circ \alpha) = \delta(\alpha)$ *holds.*

Proof. Clearly $h \circ \alpha$ is a loop in $\mathbb{S}^3 - A$, so it makes sense to consider $\delta(h \circ \alpha)$. Consider any A_{ij}. By property (A5) of the Antoine necklace, $h(A_{ij}) = A_{ij'}$ for some j'. Thus h restricts to a homeomorphism of $\mathbb{S}^3 - A_{ij}$ onto $\mathbb{S}^3 - A_{ij'}$, and so α is contractible in $\mathbb{S}^3 - A_{ij}$ if and only if $h \circ \alpha$ is contractible in $\mathbb{S}^3 - A_{ij'}$. This readily implies that $\delta(h \circ \alpha) \leq \delta(\alpha)$. The same argument applied to the loop $\beta := h \circ \alpha$ and the homeomorphism h^{-1} shows that $\delta(h^{-1} \circ \beta) \leq \delta(\beta)$; that is, $\delta(\alpha) \leq \delta(h \circ \alpha)$. This finishes the proof. $\square$

5. A torus $\mathcal{T} \subseteq \mathbb{R}^3$ that cannot be an attractor

We are finally ready to show how to construct a torus $\mathcal{T}$, or more generally surfaces of any prescribed genus, that cannot be attractors. Our starting point is the wild sphere of Antoine $\mathcal{A}$, which he first introduced in 1921 [2]. It is a 2-sphere embedded in $\mathbb{R}^3$ in such a way that it contains A, the Antoine necklace constructed in Section 3. A modern exposition of his construction can be found in the book by Rolfsen [15, pp. 73 ff.], but we also include a description here.

We use the notation ∂M and $\dot{M}$ for the boundary and interior of a compact manifold with boundary $M \subseteq \mathbb{R}^3$ (these do not necessarily coincide with the frontier and interior of M as a subset of $\mathbb{R}^3$).

5.1. The construction of the Antoine sphere $\mathcal{A}$ builds on that of the Antoine set A. Recall that the first step in defining A was to take a solid torus T_0 and place inside it a chain of linked tori $T_{11}, \ldots, T_{1N}$.

Step 0. Choose a disk $D_0 \subseteq \partial T_0$ and disks $D_{1j} \subseteq \partial T_{1j}$. Draw a surface Σ_0 connecting the curve ∂D_0 to the ∂D_{1j} as shown in grey in Figure 3 (only D_{12} and D_{14} have been labeled to avoid cluttering). More specifically, Σ_0 is a sphere with $N+1$ holes whose boundary $\partial \Sigma_0$ consists precisely of the curves $\partial D_0 \cup \partial D_{11} \cup \ldots \cup \partial D_{1N}$ and whose interior $\dot{\Sigma}_0$ is contained in the interior of $T_0 - \bigcup T_{1j}$.

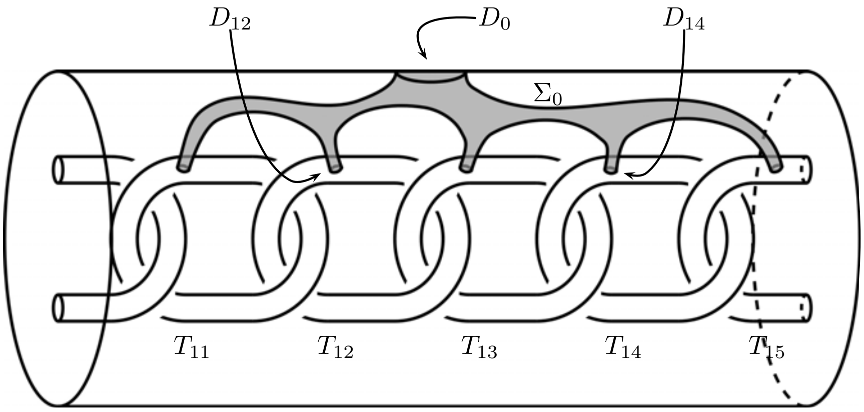

FIGURE 3. Constructing the sphere of Antoine.

Step 1. Now repeat the same construction inside each first generation torus T_{1j}. For instance, suppose that the second generation tori contained in T_{11} are labeled $T_{21}, \ldots, T_{2N}$. Choose disks $D_{21}, \ldots, D_{2N}$ on the boundaries of $T_{21}, \ldots, T_{2N}$ and then find a sphere with holes Σ_{11} connecting ∂D_{11} to the curves $\partial D_{21}, \ldots, \partial D_{2N}$. As in the previous step, $\dot{\Sigma}_{11}$ should be contained in the interior of $T_{11} - \bigcup_{j=1}^{N} T_{2j}$. After doing this in each T_{1j} a total of N spheres with holes $\Sigma_{11}, \ldots, \Sigma_{1N}$ will have been constructed, each Σ_{1j} contained in its T_{1j}.

Repeating this construction inductively it is easily seen that $\mathcal{A} := D_0 \cup \Sigma_0 \cup \bigcup_{i,j} \Sigma_{ij} \cup A$ is homeomorphic to a 2-sphere. We call this the *sphere of Antoine*.

5.2. Let $S \subseteq \mathbb{R}^3$ be a closed surface and consider a point $p \in S$. We say that S is *locally tame* at p if there exist an open neighbourhood U of p in $\mathbb{R}^3$ and a homeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(U \cap S)$ is contained in the $z = 0$ plane. Notice that then S is locally tame at every point in U. For the sake of brevity we shall say that p is a *tame* point of S if S is locally tame at p, and a *wild* point of S otherwise.

If p is a tame point of S then, with the notation of the previous paragraph, every point in U is also tame. Thus the set of tame points is open in S. Also, notice that an ambient homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ carries tame points of S onto tame points of $h(S)$; that is, it preserves the tame or wild character of points.

A deep theorem due independently to Bing [3, Theorem 6, p. 152] and Moise [14, Theorem 4, p. 254] states the following: if every point in S is tame, then there exists a homeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(S)$ is a polyhedral surface. In the particular case where S is a sphere, so that $\varphi(S)$ is a polyhedral sphere, it is a consequence of the polyhedral Schönflies theorem [14, pp. 117 ff.] that each of the two components of $\mathbb{S}^3 - \varphi(S)$ is simply connected. It follows that the same is true of $\mathbb{S}^3 - S$, because it is homeomorphic to $\mathbb{S}^3 - \varphi(S)$ via φ. Thus a sphere with no wild points separates $\mathbb{S}^3$, and also $\mathbb{R}^3$, into two simply connected domains.

Let us consider the particular case of the Antoine sphere $\mathcal{A}$. Notice that the unbounded component of the complement of $\mathcal{A}$ is not simply connected (for instance, the meridian μ_0 is not contractible there because it is not even contractible in the bigger set $\mathbb{S}^3 - A$ by property (A1) of the Antoine necklace). It follows from the previous paragraph that there exists at least a wild point in $\mathcal{A}$. It is clear that every point in $\mathcal{A} - A$ is tame by construction, so we see that there is a wild point in A. We can refine this argument to show that every point in A is a wild point of $\mathcal{A}$.

Proposition 5.1. *The set of wild points of $\mathcal{A}$ is precisely A.*

Proof. Suppose $\mathcal{A}$ were locally tame at some $p \in A$. Then p would have an open neighbourhood U in $\mathcal{A}$ such that every point in U is also tame. Pick a generation i big enough so that some A_{ij} is contained in U and consider the portion of the sphere that is contained in T_{ij}; that is, the intersection $\mathcal{A} \cap T_{ij}$. By the self similar nature of the construction of $\mathcal{A}$, it is clear that $\mathcal{A}' := (\mathcal{A} \cap T_{ij}) \cup D_{ij}$ is ambient homeomorphic to $\mathcal{A}$. Now, the only wild points in $\mathcal{A}'$ could be those belonging to the Antoine necklace, so they must all be contained in $\mathcal{A}' \cap A = A_{ij}$. But every point in $A_{ij} \subseteq U$ is tame in $\mathcal{A}$ and consequently also in $\mathcal{A}'$, so it follows that every point of $\mathcal{A}'$ is tame. Since $\mathcal{A}$ is ambient homeomorphic to $\mathcal{A}'$, every point in $\mathcal{A}$ should also be tame. $\square$

5.3. Now it is easy to construct the torus $\mathcal{T}$ that cannot be an attractor: just take the Antoine sphere $\mathcal{A}$, drill two holes in the interior of the disk D_0, and connect them with a small, hollow polyhedral tube. The tube should intersect T_0 just at its ends. This yields a 2-torus $\mathcal{T}$ whose set of wild points is precisely A.

Theorem 5.2. *$\mathcal{T}$ is not an attractor for a homeomorphism f of $\mathbb{R}^3$.*

Proof. We reason by contradiction. Suppose $\mathcal{T}$ is an attractor for a homeomorphism f, and let C_∞ be the connected component of $\mathbb{S}^3 - \mathcal{T}$ containing ∞. By Proposition 2.2 there exists a finitely generated group $G \subseteq \pi_1(C_\infty)$ such that $\pi_1(C_\infty) = \bigcup_{k \geq 0} f_*^k(G)$.

Any element $g \in G$ is represented by a loop α in C_∞ which is determined only up to homotopy in C_∞. This set is contained in $\mathbb{S}^3 - A$ and so by Proposition 4.1 the depth $\delta(\alpha)$ is independent of the particular representative α of the element g, so we can write $\delta(g) := \delta(\alpha)$.

Let $g_1, \ldots, g_n$ be generators for G and set $\Delta := \max\{\delta(g_i) : 1 \leq i \leq n\}$. Any element $g \in G$ can be written as a product of the g_i or their inverses g_i^{-1}. An inductive application of Proposition 4.2 then implies that $\delta(g) \leq \Delta$. Since f is an ambient homeomorphism it must leave the set of wild points of $\mathcal{T}$ invariant, which is precisely A by Proposition 5.1 (and the construction of $\mathcal{T}$). Thus by Proposition 4.3 we see that $\delta(f_*^k(g)) = \delta(g) \leq \Delta$ for every $g \in G$ and $k \geq 0$. Since G was assumed to satisfy $\bigcup_{k \geq 0} f_*^k(G) = \pi_1(C_\infty)$, it follows that $\delta(\ell) \leq \Delta$ for every $\ell \in \pi_1(C_\infty)$.

The construction of $\mathcal{T}$ is such that, given any torus T_{ij}, it is possible to find a meridian μ_i of T_{ij} contained in C_∞. As mentioned earlier, these meridians satisfy $\delta(\mu_i) = i$ for every $i = 1, 2, \ldots$ but this contradicts the inequality $\delta(\ell) \leq \Delta$ obtained in the previous paragraph. □

The argument of Theorem 5.2 works equally well to show that the Antoine sphere $\mathcal{A}$ itself cannot be an attractor either, and in fact the construction of $\mathcal{T}$ can be easily generalized to obtain surfaces of any prescribed genus that cannot be attractors.

Appendix A

The material in this appendix is well known and has even been established in much more general contexts [19], [21]. However, as an aid to the interested reader we have thought it convenient to provide proofs tailored to our specific situation. In particular, our goal is to establish properties (A3) and (A5) of the Antoine necklace.

A.1. Consider a meridian μ_0 of the torus T_0. Although μ_0 is not contractible in $\mathbb{R}^3 - A$, property (A3) says that μ_0 is contractible not only in $\mathbb{R}^3$ but even in T_0 as soon as a single point is removed from A. More formally, we have:

Lemma A.1. *Let μ_0 be a meridian of T_0. Choose a point $p \in A$ and denote $A^* := A - \{p\}$. Then μ_0 is contractible in $T_0 - A^*$.*

Proof. We need to define a continuous map $F \colon \mathbb{D}^2 \to T_0 - A^*$ such that $F|_{\partial \mathbb{D}^2} = \mu_0$, where $\mathbb{D}^2 \subseteq \mathbb{R}^2$ stands for the closed unit 2-disk. Let us remark that F does not need to be injective.

For each generation $i = 1, 2, \ldots$ let T_{ij_i} be the ith generation torus containing p, so that $T_0 \supseteq T_{1j_1} \supseteq T_{2j_2} \supseteq \ldots$ and the collection $\{T_{ij_i}\}$ is a neighbourhood basis of p.

Step 1. Slide μ_0 along ∂T_0 to a position μ_0' where the meridional disk D_0' that it spans meets T_{1j_1} in precisely two meridional disks D_{11} and D_{12} and is disjoint from every other first generation torus T_{1j}.

Notice that μ_0 and μ_0' cobound an annulus in ∂T_0 and μ_0' bounds a disk with two holes, namely $D_0' - \operatorname{int}(D_{11} \cup D_{12})$. Referring to Figure 4, we then define the map F on $\mathbb{D}^2$ minus the interior of two disks E_{11} and E_{12} in such a way that it takes the outermost, light gray annulus onto the annulus bounded by μ_0

and μ_0' in ∂T_0 and the slightly darker disk with two holes onto the disk with two holes $D_0' - \mathrm{int}\,(D_{11} \cup D_{12})$.

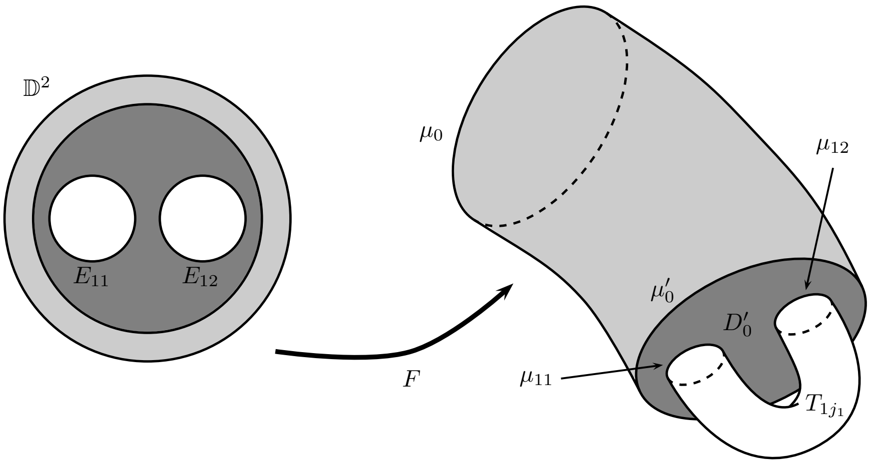

Figure 4.

Step 2. Notice that the curves $\mu_{11} = \partial D_{11}$ and $\mu_{12} = \partial D_{12}$ are meridians of T_{1j_1}. In this second step we perform with each of them the same operation that we did earlier with μ_0. Thus we slide μ_{11} and μ_{12} along ∂T_{1j_1} to positions μ_{11}' and μ_{12}' where the meridional disks they span (D_{11}' and D_{12}', say) are disjoint from all the second generation tori T_{2j} except for T_{2j_2} and D_{11}' and D_{12}' intersect T_{2j_2} in two meridional disks each.

As in the previous step, μ_{11} and μ_{11}' cobound an annulus in ∂T_{1j_1} and μ_{11}' bounds a disk with two holes (shown in very dark gray in the right hand side of Figure 5). Thus the map F can be extended to the disk E_{11} minus the interior of two smaller disks E_{21} and E_{22}. The same goes for μ_{12} and μ_{12}', and exactly in the similar fashion F can also be extended to the disk E_{12} minus the interior of two smaller disks E_{23} and E_{24}.

Continuing in this fashion F can be extended to a continuous map defined on $\mathbb{D}^2$ minus a Cantor set C which is the intersection of the decreasing sequence of sets

$$(E_{11} \cup E_{12}) \supseteq (E_{21} \cup E_{22} \cup E_{23} \cup E_{24}) \supseteq \ldots$$

Notice that F is defined in such a way that $F(E_{ik} - C) \subseteq T_{ij_i}$ and, since the T_{ij_i} are a neighbourhood basis of p, this implies that F can be extended continuously to all $\mathbb{D}^2$ simply letting $F|_C \equiv p$. It is then clear by construction that $F(\mathbb{D}^2) \cap A = \{p\}$, so that $F(\mathbb{D}^2) \subseteq T_0 - A^*$, as required. $\square$

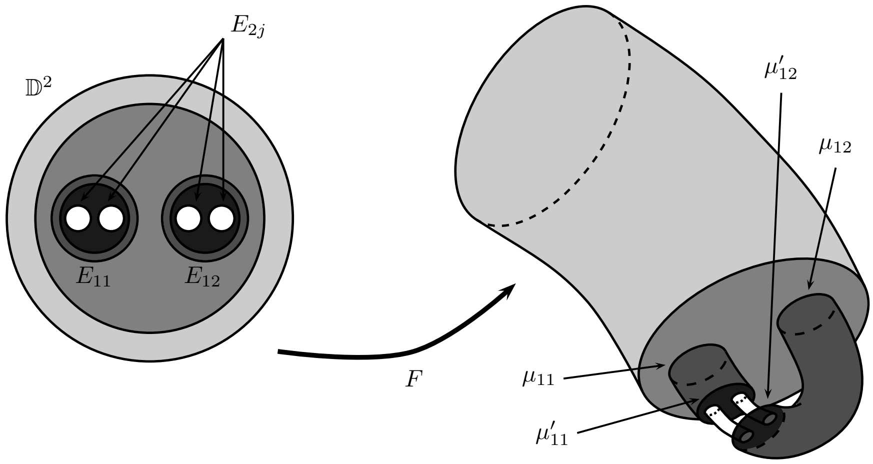

FIGURE 5.

Before moving ahead it is convenient to discuss to what extent the map F described in the proof of Lemma A.1 can be chosen to be injective. Evidently we are always going to have $F|_C \equiv p$, so the best we can hope for is to have F injective on $\mathbb{D}^2 - C$. This requires that the geometric objects that appear in the definition of F be disjoint, as we now explain.

Consider, for instance, the situation just after Step 2. We have four meridians μ_{2k} on the boundary of T_{2j_2}. They are shown as radial lines in Figure 6(a), which is a very schematic representation of the torus T_{2j_2} seen from above. We want to slide the μ_{2k} clockwise to suitable new positions μ'_{2k} and will do so in order: we start with μ_{23}, which is the one closest to T_{3j_3}, then continue with μ_{22} stopping at some μ'_{22} just short of reaching μ'_{23}, and so on until we finish with μ_{24}. As suggested in Figure 6(a) the meridians μ'_{2k} are chosen to be very close to each other and, of course, with the property that the meridional disks D'_{2k} they span meet T_{3j_3} exactly in two meridional disks each.

If at this stage we define F as in the proof of Lemma A.1 it will not be injective because the annuli traced by sliding each μ_{2k} onto μ'_{2k} are not disjoint. However, it is easy to fix this by modifying slightly the construction. Nothing has to be changed regarding μ_{23}. However, instead of sliding μ_{22} along ∂T_{2j_2} we first shrink it slightly so it lies just beneath ∂T_{2j_2} and then slide it parallel to ∂T_{2j_2} but still inside it. Its final position is then a slightly shrinked meridian of T_{2j_2}. The situation is depicted in Figure 6(b). Notice that now the annulus traced by the sliding of μ_{22} is disjoint from the annulus cobounded by μ_{23} and

μ'_{23} as desired. The same can be done with μ_{21} by shrinking it a bit more than μ_{22} and, finally, the same goes for μ_{24} which should be shrunk even further.

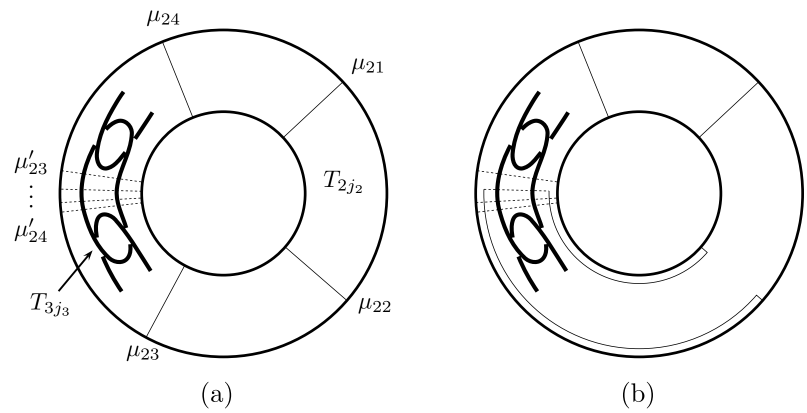

Figure 6.

It should be clear that, if the same precaution is taken at each step of the construction of F, the resulting map F will be injective on $\mathbb{D}^2 - C$.

A.2. Intuition suggests that two consecutive A_{ij} and $A_{ij'}$ of the same generation are, somehow, linked. This can be given a precise definition and it is, in fact, one of the key facts underlying property (A5) of the Antoine necklace.

Let us begin with a standard definition. Recall that a 2-sphere $S \subseteq \mathbb{R}^3$ separates $\mathbb{R}^3$ in two connected components; one bounded and one unbounded. We denote them by Int S and Ext S respectively. Now, two disjoint solid tori $T, T' \subseteq \mathbb{R}^3 \subseteq \mathbb{S}^3$ are *unlinked* if there exists a 2-sphere S such that $T \subseteq \operatorname{Int} S$ and $T' \subseteq \operatorname{Ext} S$ or viceversa; otherwise they are *linked*. In the former case it is always possible to choose S to be a polyhedral sphere by an approximation theorem of Bing [4, Theorem 1, p. 457]. Then the polyhedral Schönflies theorem guarantees that each connected component of $\mathbb{S}^3 - S$ is simply connected and it follows that T is contractible in $\mathbb{R}^3 - T'$ (and viceversa). Thus for instance every pair of adjacent tori in any of the chains M_i used to construct the Antoine necklace A are linked.

The definition translates inmediately to Cantor sets: two disjoint Cantor sets C_1 and C_2 are unlinked if there exists a 2-sphere S such that C_1 and C_2 are contained in different components of $\mathbb{S}^3 - S$; they are linked if they are not unlinked.

Lemma A.2. *Let $C \subsetneq A$ be a compact set. There exists a 2-sphere $S \subseteq T_0$ such that C is contained in* Int S*.*

PROOF. Let μ_0 be a meridian of T_0. First we are going to show that μ_0 bounds a meridional disk D_0 disjoint from C. More precisely, there exists an *embedding* $F\colon \mathbb{D}^2 \to T_0 - C$ such that $F|_{\partial\mathbb{D}^2} = \mu_0$; the meridional disk D_0 is then $F(\mathbb{D}^2)$.

Pick a point $p \in A - C$ and let, as in the proof of Lemma A.1, $T_0 \supseteq T_{1j_1} \supseteq T_{2j_2} \supseteq \dots$ be the sequence of tori containing p. Since C is closed and $\{T_{ij_i}\}$ is a neighbourhood basis of p there exists i_0 such that $C \cap T_{ij_i} = \emptyset$ for every $i \geq i_0$. Perform the construction of Lemma A.1 up to stage i_0 with the required precautions, described above, to render F injective. At that stage there exists a family of disjoint disks $E_{i_0 1}, \dots, E_{i_0 2^{i_0}}$ contained in the interior of $\mathbb{D}^2$ and F is an embedding of $\mathbb{D}^2 - \operatorname{int}(E_{i_0 1} \cup \ldots \cup E_{i_0 2^{i_0}})$ into $T_0 - A$. Now observe that the restrictions $F|_{\partial E_{i_0 k}}$ are, by construction, meridians of $T_{i_0 j_{i_0}}$. Each of them spans a meridional disk $D_{i_0 k}$ contained in $T_{i_0 j_{i_0}}$ and therefore disjoint from C. Thus we can extend F to an embedding of all $\mathbb{D}^2$ into $T_0 - C$ just letting it take each $E_{i_0 k}$ homeomorphically onto $D_{i_0 k}$.

The meridional disk D_0 is a piecewise smooth surface, so it can be "thickened" within T_0. Formally this means that F can be extended to an embedding $\widehat{F}\colon \mathbb{D}^2 \times [-1,1] \to T_0 - C$ such that $\widehat{F}|_{\mathbb{D}^2\times\{0\}} = F$ and $\widehat{F}$ takes $(\partial\mathbb{D}^2) \times [-1,1]$ onto an annulus $V \subseteq \partial T_0$ whose middle circumference is μ_0. Let $D_+ = F(\mathbb{D}^2 \times \{1\})$ and $D_- = F(\mathbb{D}^2 \times \{-1\})$. These are again meridional disks, mutually disjoint and parallel to D. The union of the annulus $(\partial T_0) - V$ and the two disks D_+ and D_- is a 2-sphere S contained in T_0.

The intersection of $\operatorname{int} T_0$ and the sphere S consists only of the interiors of the meridional disks D_+ and D_-. Thus S separates int T_0 in two connected components, one of which contains C. As a consequence C is wholly contained in one of the components of $\mathbb{R}^3 - S$. $\square$

LEMMA A.3. *Let $C \subseteq A_{1j}$ and $C' \subseteq A_{1j'}$ be Cantor sets. Then C and C' are linked if, and only if, the following two conditions hold*:

(a) $C = A_{1j}$ *and* $C' = A_{1j'}$,

(b) T_{1j} *and* $T_{1j'}$ *are linked.*

PROOF. Let us begin with ($\Leftarrow$). Suppose that the Cantor sets $C = A_{1j}$ and $C' = A_{1j'}$ were not linked. Then there would exist a polyhedral sphere S such that $C \subseteq \operatorname{Int} S$ and $C' \subseteq \operatorname{Ext} S$ (or viceversa). An argument of Antoine [1, §84, p. 317] shows that S could be chosen in such a way as to separate also the tori T_{1j} and $T_{1j'}$. However this is impossible, because the tori are linked by hypothesis. Therefore A_{1j} and $A_{1j'}$ must be linked too.

Let us prove ($\Rightarrow$) by contradiction. If T_{1j} and $T_{1j'}$ are unlinked then the same is trivially true of A_{1j} and $A_{1j'}$. Hence it suffices to prove the following: if $C \subsetneq A_{1j}$ then C and C' are unlinked. By Lemma A.2 applied to the Antoine necklace A_{1j} inside the torus T_{1j} there exists a 2-sphere $S \subseteq T_{1j}$ such that C

is contained in Int S, the bounded component of $\mathbb{R}^3 - S$. Since $T_{1j'}$ is disjoint from T_{1j} (and hence from S), it is contained in the component of $\mathbb{R}^3 - S$ that contains the unbounded set $\mathbb{R}^3 - T_{1j}$. Thus $T_{1j'}$, and consequently also C' is contained in Ext S. Therefore C and C' are unlinked. □

Lemma A.4. *Let A be an Antoine necklace and $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ a homeomorphism such that $h(A) = A$. Then for each A_{1j} there exists $A_{1j'}$ such that $h(A_{1j}) = A_{1j'}$. That is, h takes links of generation $i = 1$ onto links of the same generation.*

Proof. Since $h^{-1}(M_1)$ is a neighbourhood of A, there exists an integer $i \geq 0$ such that $M_i \subseteq h^{-1}(M_1)$. Thus for each connected component T_{ij} of M_i there exists a connected component $T_{1j'}$ of M_1 such that $h(T_{ij}) \subseteq T_{1j'}$. As a consequence h takes each A_{ij} into some $A_{1j'}$. Let $m \geq 1$ be the smallest integer with this property, so that:

(i) for each mth generation link A_{mj} there exists a first generation link $A_{1\sigma(j)}$ such that $h(A_{mj}) \subseteq A_{1\sigma(j)}$,
(ii) there exists an $(m-1)$th generation link, which me may assume to be $A_{(m-1)1}$, such that $h(A_{(m-1)1})$ is not contained in any first generation link of A.

Denote $A_{m1}, A_{m2}, \ldots, A_{mN}$ the mth generation links contained in $A_{(m-1)1}$, labeled in such a way that A_{mj} is adjacent (and therefore linked, by Lemma A.3) to $A_{m(j-1)}$ and $A_{m(j+1)}$. Departing from our earlier conventions it will be convenient to use a cyclic notation for j, so that $A_{m(N+1)}$ means A_{m1}, $A_{m(N+2)}$ means A_{m2} and so on. By (i) each one of $h(A_{m1}), h(A_{m2}), \ldots, h(A_{mN})$ is contained in some first generation link of A, but if all of them were contained in the same link, then $h(A_{(m-1)1})$ would also be contained there, contradicting (ii). Thus there exists $1 \leq j_0 \leq N$ such $\sigma(j_0) \neq \sigma(j_0+1)$. The Antoine necklaces A_{mj_0} and $A_{m(j_0+1)}$ are linked, so their images under h are also linked and therefore by Lemma A.3 we must have $h(A_{mj_0}) = A_{1\sigma(j_0)}$, $h(A_{m(j_0+1)}) = A_{1\sigma(j_0+1)}$ and $|\sigma(j_0+1) - \sigma(j_0)| = 1$. Assume for definiteness that $\sigma(j_0+1) = \sigma(j_0)+1$.

Consider $h(A_{m(j_0+2)})$. Since $A_{m(j_0+2)}$ is linked with $A_{m(j_0+1)}$, the same is true of $h(A_{m(j_0+2)})$ and $h(A_{m(j_0+1)}) = A_{1\sigma(j_0+1)}$. Thus again by Lemma A.3 either $h(A_{m(j_0+2)}) = A_{1\sigma(j_0)}$ or $h(A_{m(j_0+2)}) = A_{1(\sigma(j_0)+2)}$. The first case is impossible because h is injective and $A_{1\sigma(j_0)} = h(A_{mj_0})$, so the second must hold. Proceeding in the same way it follows that $\sigma(j) = \sigma(j_0) + (j - j_0)$, so in particular σ is surjective, and $h(A_{mj}) = A_{1\sigma(j)}$ for every $1 \leq j \leq N$. Therefore, since $A_{(m-1)1} = A_{m1} \cup \ldots \cup A_{mN}$ it follows that

$$h(A_{(m-1)1}) = \bigcup_{j=1}^{N} h(A_{mj}) = \bigcup_{j=1}^{N} A_{1\sigma(j)} = A,$$

where in the last equality we have used that σ is surjective. Since h is injective, this means that $A_{(m-1)1}$ must be all of A, so $m = 1$. This proves the proposition.

Lemma A.5. *Let A be an Antoine necklace and $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ a homeomorphism such that $h(A) = A$. Then for each A_{ij} there exists $A_{ij'}$ such that $h(A_{ij}) = A_{ij'}$. That is, h preserves generations.*

Proof. The argument is by induction on i. The case $i = 1$ is settled by Lemma A.4. We now give the inductive step from i to $i + 1$.

Pick any $A_{(i+1)j}$ and denote A_{ik} the previous generation link which contains $A_{(i+1)j}$. By the induction hypothesis there exists k' such that $h(A_{ik}) = A_{ik'}$. According to property (A1) of the Antoine necklace there exists a homeomorphism g of $\mathbb{R}^3$ such that $g(A) = A$, g is generation preserving, and $g(A_{ik'}) = A_{ik}$. The composition gh is a homeomorphism of $\mathbb{R}^3$ which leaves the Antoine necklace A_{ik} invariant. Applying Lemma A.4 to $gh|_{A_{ik}}$, we see that each first generation link of A_{ik} is taken by gh onto a first generation link of A_{ik}. But $A_{(i+1)j}$ is a first generation link of A_{ik}, so there exists $A_{(i+1)j''}$ such that $gh(A_{(i+1)j}) = A_{(i+1)j''}$. Therefore $h(A_{(i+1)j}) = g^{-1}(A_{(i+1)j''}) = A_{(i+1)j'}$, where the last equality follows from the property that g is generation preserving.

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