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LOCAL STRONG SOLUTIONS OF THE NONHOMOGENEOUS NAVIER–STOKES SYSTEM WITH CONTROL OF THE INTERVAL OF EXISTENCE

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Abstract. Consider a bounded domain $\Omega \subseteq \mathbb{R}^3$ with smooth boundary $\partial\Omega$, a time interval $[0,T)$, $0 < T \le \infty$, and in $[0,T)\times\Omega$ the nonhomogeneous Navier-Stokes system $u_t - \Delta u + u \cdot \nabla u + \nabla p = f$, $u|_{t=0} = v_0$, $\operatorname{div} u = k$, $u|_{\partial\Omega} = g$, with sufficiently smooth data f, v_0, k, g. In this general case there are mainly known two classes of weak solutions, the class of global weak solutions, similar as in the well known case $k = 0$, $g = 0$ which need not be unique, see [5], and the class of local very weak solutions, see [1], [2], [3], which are uniquely determined but have no differentiability properties and need not satisfy an energy inequality. Our aim is to introduce the new class of local strong solutions in the usual sense for $k \neq 0$, $g \neq 0$ satisfying similar regularity and uniqueness properties as in the well known case $k = 0$, $g = 0$. Further, we obtain precise information through the given data on the interval of existence $[0,T^*)$, $0 < T^* \le T$. Our proof is essentially based on a detailed analysis of the corresponding linear system.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $[0,T)$, $0 < T \le \infty$, be the time interval. Then we consider in $[0,T) \times \Omega$ the general nonhomogeneous Navier–Stokes system

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad u|_{t=0} = v_0, \quad \operatorname{div} u = k, \quad u|_{\partial\Omega} = g, \tag{1.1}$$

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where the vector u denotes the velocity and ∇p the associated pressure gradient. In the physical model the divergence $k = \operatorname{div} u$ is assumed to vanish. However, for mathematical reasons it will be convenient, in particular for linear problems, to consider also the more general case of a prescribed divergence $k \neq 0$, compare [4, Remark 1.9(1)]. We refer to [3] and [5] for very weak and weak solutions of this system, respectively. In particular, a counterpart of this paper on the level of very weak solutions can be found in [3], see also [4] for a general review on very weak solutions. However, the focus of this paper is put on the existence of local in time strong solutions.

For simplicity we use for weak and strong solutions the same data class to exploit both theories simultaneously; see [5] for a more general theory of weak solutions.

Next we describe the general assumptions on the data f, v_0, k and g; here $N(x)$ denotes the outward normal vector at $x \in \partial\Omega$.

Assumptions 1.1.

(a) $f = \operatorname{div} F$, $F \in L^{s/2}(0,T;L^{q/2}(\Omega))$, with $4 \le s \le 8$, $4 \le q \le 6$, $2/s + 3/q = 1$,

(b) for $v_0 \in L^2_\sigma(\Omega)$, $\|v_0\|_{B^{q,s}_T(\Omega)} := \left(\int_0^T \|e^{-tA} v_0\|_q^s \, dt \right)^{1/s} < \infty$,

(c) $k \in L^s(0,T;L^q(\Omega))$, $g \in L^s(0,T;W^{-1/q,q}(\partial\Omega))$ with compatibility condition

$$\int_\Omega k(t)\, dx = \langle g(t), N \rangle_{\partial\Omega}, \quad t\text{-a.e.}$$

Here $L^r(\Omega)$ denotes the usual Lebesgue space of functions (or vector or matrix fields) with norm $\|\cdot\|_r$ and pairing $\langle \cdot\,, \cdot \rangle_\Omega$ with its dual $L^{r'}(\Omega)$, $1 < r < \infty$, $r' = r/(r-1)$. Moreover, $L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$ where $C^\infty_{0,\sigma}(\Omega) := \{v = (v_1, v_2, v_3) \in C^\infty_0(\Omega);\ \operatorname{div} v = 0\}$. Usual Bochner spaces are denoted by $L^s(0,T;L^q(\Omega))$ with norm $\|\cdot\|_{q,s,T}$, $1 < q, s < \infty$, and with pairing $\langle \cdot\,, \cdot \rangle_{\Omega,T}$.

The nonlinear term $u \cdot \nabla u$, defined by $u \cdot \nabla u = (u \cdot \nabla u_1, u \cdot \nabla u_2, u \cdot \nabla u_3)$, can be written, when $\operatorname{div} u = 0$, also as $u \cdot \nabla u = \operatorname{div}(uu) = \nabla \cdot (uu)$ where $uu = (u_i u_j)_{i,j=1,2,3}$. Here we note that the divergence of a matrix-valued function $F = (F_{ij})_{i,j=1,2,3}$ is defined columnwise.

The initial value norm $\|v_0\|_{B^{q,s}_T(\Omega)}$ is a so-called Besov space norm, see [3], [6]–[9], and Section 3 for details. The space $W^{-1/q,q}(\partial\Omega)$ is a Sobolev trace space of negative order $-1/q$, namely the dual of the trace space $W^{1/q,q'}(\partial\Omega)$.

Let $P = P_2 \colon L^2(\Omega) \to L^2_\sigma(\Omega)$ denote the Helmholtz projection and $A = A_2 = -P\Delta$ in Assumption 1.1(b) the Stokes operator on $L^2_\sigma(\Omega)$. Since $A_2 = A_q$ on $C^\infty_{0,\sigma}(\Omega)$, we simply write A for any Stokes operator A_q, $1 < q < \infty$; by analogy, since $P_2 = P_q$ on $C^\infty_0(\Omega)$, we also write P for P_q.

To obtain a precise definition of weak and strong solutions u for (1.1), we use in $[0,T)\times\Omega$ a fixed *very weak solution*

$$E = E_{k,g} \in L^s(0,T;L^q(\Omega)) \tag{1.2}$$

of the linear Stokes system

$$E_t - \Delta E + \nabla h = 0, \quad E|_{t=0} = 0, \quad \operatorname{div} E = k, \quad E|_{\partial\Omega} = g \tag{1.3}$$

with associated pressure ∇h. To be more precise, E is called a very weak solution to (1.3) if for all test functions $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\overline{\Omega}))$ where $C_{0,\sigma}^\infty(\overline{\Omega}) = \{v \in C^\infty(\overline{\Omega});\ v|_{\partial\Omega} = 0,\ \operatorname{div} v = 0\}$,

$$-\langle E, w_t\rangle_{\Omega,T} - \langle E, \Delta w\rangle_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} = 0, \tag{1.4}$$

$$\operatorname{div} E = k,\ E|_{\partial\Omega} = g. \tag{1.5}$$

By [3, Theorem 4] we know that there exists a unique very weak solution $E = E_{k,g}$ of (1.3) satisfying

$$\begin{gathered}(A^{-1}PE)_t \in L^s(0,T;L^q_\sigma(\Omega)),\\ A^{-1}PE \in C([0,T);L^q_\sigma(\Omega)), \quad A^{-1}PE|_{t=0} = 0,\\ \|(A^{-1}PE)_t\|_{q,s,T} + \|E\|_{q,s,T} \le C(\|k\|_{q,s,T} + \|g\|_{-1/q,q,T})\end{gathered} \tag{1.6}$$

with $C = C(\Omega,q) > 0$. The condition $E|_{\partial\Omega} = g$ is well defined in the sense of boundary distributions, see [3, Remarks 3, (2)]. Using (1.6) we see that

$$PE\colon [0,T) \to L^q_\sigma(\Omega) \quad \text{is weakly continuous,} \tag{1.7}$$

and $E|_{t=0} = 0$ means (in a generalized sense) that $PE|_{t=0} = 0$, i.e. $E|_{t=0} = 0$ holds modulo a gradient.

To give the system (1.1) a precise meaning we set

$$u = v + E, \qquad E = E_{k,g} \tag{1.8}$$

and choose a vector field v satisfying in $[0,T)\times\Omega$ the system

$$v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla p^* = f, \quad v|_{t=0} = v_0, \quad v|_{\partial\Omega} = 0, \quad \operatorname{div} v = 0 \tag{1.9}$$

which is called the *perturbed Navier–Stokes system*, see [5], with associated pressure ∇p^*. This yields the following definition.

DEFINITION 1.2 (Weak and strong solutions for (1.1).). Suppose f, v_0, k, g satisfy Assumptions 1.1, and let $E = E_{k,g}$ be as in (1.2)–(1.3).

(1) A vector field v in $[0,T)\times\Omega$ is called a *weak solution of the perturbed system* (1.9) with data f, v_0, and $u := v+E$ is called a *weak solution of the general system* (1.1) with data f, v_0, k, g, if the following conditions are satisfied:

(a) $v \in L^\infty_{\text{loc}}([0,T);L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T);W^{1,2}_0(\Omega))$,

(b) $v\colon [0,T) \to L^2_\sigma(\Omega)$ is weakly continuous and $v|_{t=0} = v_0$,

(c) for each $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$,

$$(1.10)\qquad -\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (v+E)(v+E), \nabla w\rangle_{\Omega,T} - \langle k(v+E), w\rangle_{\Omega,T} = \langle v_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T},$$

(d) for each $t \in [0,T)$,

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F, \nabla v\rangle_\Omega\,d\tau + \int_0^t \langle (v+E)E, \nabla v\rangle_\Omega\,d\tau + \frac{1}{2}\int_0^t \langle k(v+2E), v\rangle_\Omega\,d\tau.$$

(2) Let v be a weak solution of (1.9) with data f, v_0 and let

$$(1.11)\qquad v \in L^s(0,T; L^q(\Omega)).$$

Then v is called a *strong solution of the perturbed system* (1.9) with data f, v_0, and

$$(1.12)\qquad u = v + E_{k,g}$$

is called a *strong solution of the general system* (1.1) with data f, v_0, k, g.

We see that in the well known case $E_{k,g} = 0$, the weak solution v is the Hopf type weak solution, and $u = v$ is the usual strong solution. Condition (1.11) with $2/s + 3/q = 1$ is the so-called Serrin condition ([10]–[12]) which is important to prove regularity and uniqueness of solutions in the nonlinear case.

In the following we will see that the strong solutions v and $u = v + E_{k,g}$ have similar uniqueness and regularity properties as in the case $E_{k,g} = 0$.

2. Main results

We are mainly interested in strong solutions v and $u + v$, given in Definition 1.2, which are also weak solutions. Therefore, for simplicity, we used the same data class in Definition 1.2 for weak and strong solutions. Indeed, the data class in [5, Theorem 1.4], for (global) weak solutions is slightly more general than that in Definition 1.2.

An important aspect in the main Theorem 2.1 below is that the existence of a strong solution in a given interval $[0,T)$ can be precisely determined if the norm $b(T)$ of the data, see (2.1) below or [3, (4.23)] for a similar condition, satisfies a smallness condition $b(T) \le \varepsilon^*(\Omega, q)$. Since $b(T)$ tends to zero for $T \to 0$, we can determine some interval $[0,T^*)$, $0 < T^* \le T$, satisfying $b(T^*) \le \varepsilon^*(\Omega, q)$ and yielding precisely the existence interval for the strong solution, see Corollary 2.2 below. Usually, in the well known case $k = 0$, $g = 0$, the existence of a strong solution has been shown only in some "sufficiently small" subinterval of $[0,T)$.

THEOREM 2.1 (Existence of a strong solution in $[0,T)$). *Let $f = \operatorname{div} F$, v_0, k, g be given as in Assumptions* 1.1(a)–(c), *let $E = E_{k,g}$ be as in* (1.2), *and let*

$$b(T) := \|v_0\|_{B_T^{q,s}(\Omega)} + \|F\|_{q/2,s/2,T} + \|k\|_{q,s,T} + \|g\|_{-1/q,q,s,T} \tag{2.1}$$

be the data norm in $[0,T)$. There exists a constant $\varepsilon^* = \varepsilon^*(\Omega,q) > 0$ such that if

$$b(T) \le \varepsilon^*, \tag{2.2}$$

then there exists a uniquely determined strong solution v of the perturbed system (1.9) *and a uniquely determined strong solution $u = v + E$ of the general system* (1.1).

COROLLARY 2.2 (Interval of existence of strong solutions). *Let $f = \operatorname{div} F$, v_0, k, g be given as in Assumptions 1.1 and let $E = E_{k,g}$ be as in* (1.2). *Then $[0,T^*)$, $0 < T^* \le T$, defined by $b(T^*) \le \varepsilon^*$, is an interval of existence of a uniquely determined strong solution v of* (1.9) *and of a uniquely determined strong solution $u = v + E$ of* (1.1).

Since Theorem 2.1 allows for $T = \infty$, the above mentioned results are not only results local in time but also global in time results if the data are small enough.

The next result yields the regularity of strong solutions.

THEOREM 2.3 (Regularity result for strong solutions). *Let $f = \operatorname{div} F$, v_0, k, g satisfy Assumptions* 1.1, *and let $E = E_{k,g}$ be as in* (1.2). *Assume the following additional regularity properties of the data,*

$$\begin{gathered} F \in L^s(0,T;W^{1,q}(\Omega)),\ k \in L^s(0,T;W^{1,q}(\Omega)),\ k_t \in L^s(0,T;L^q(\Omega)), \\ g \in L^s(0,T;W^{2-1/q,q}(\partial\Omega)),\ g_t \in L^s(0,T;W^{-1/q,q}(\partial\Omega)),\ v_0 \in W^{2,q}(\Omega), \end{gathered} \tag{2.3}$$

and assume that v and $u = v + E$ are strong solutions in $[0,T^*)$, $0 < T^* \le T$, as given in Corollary 2.2. *Then v, E satisfy, additionally to Definition* 1.2(a)–(d) *and* (1.4)–(1.7), *respectively, the following regularity properties*

$$v \in L^\infty_{\text{loc}}([0,T);W_0^{1,2}(\Omega)) \cap L^2_{\text{loc}}([0,T);W^{2,2}(\Omega)), \tag{2.4}$$

$$v_t \in L^2_{\text{loc}}([0,T);L^2_\sigma(\Omega)), \tag{2.5}$$

$$E \in L^s(0,T;W^{2,q}(\Omega)), \quad E_t \in L^s(0,T;L^q(\Omega)), \tag{2.6}$$

and $u = v + E$ satisfies corresponding additional regularity properties. In particular,

$$u \in L^2_{\text{loc}}([0,T);W^{2,2}(\Omega)), \quad u_t \in L^2_{\text{loc}}([0,T);L^2(\Omega)). \tag{2.7}$$

REMARK 2.4. (a) In order to compare the class of strong solutions in Definition 1.2 with the class of very weak solutions (for its definition cf. (1.4) with an additional term due to the nonlinear expression $\operatorname{div}(uu)$) let us restrict, for simplicity, the condition for F in Assumption 1.1(a) and for v_0 in Assumption 1.1(b) as follows:

$$F \in L^s(0,T;L^q(\Omega)), \quad 0 < T < \infty, \quad v_0 = 0. \tag{2.8}$$

Then the data class in Assumptions 1.1, restricted by (2.8), is contained in the data class of very weak solutions in [3, Theorem 1]. Let D denote this restricted data class, let $V_{T'}D$ be the solution class of very weak solutions for data from D according to [3, Theorem 1] where $[0,T')$ with $T' \le T$ denotes the corresponding existence interval, and let $S_T D$ be the solution class of our strong solutions for data from D. Then, by Theorem 2.1, we have $V_{T'}D = S_{T'}D$.

In both cases we obtain the existence of a unique solution in $[0,T')$ if the data in $[0,T')$ satisfy a smallness condition, see (2.1) for strong solutions ($T = T'$) and [3, (4.23)] for very weak solutions. However, the very weak solutions of $V_{T'}D$ need not have any differentiability property in space and time, and need not satisfy any energy inequality. These are weaker conditions as for the usual (possibly non uniquely determined) weak solutions – this is the reason for the notion "very weak".

Since $V_{T'}D = S_{T'}D$ at least for the slightly restricted data set D, our result shows that the very weak solution class $V_{T'}D$ has the regularity properties of the class of strong solutions $S_{T'}D$. Thus in this case the notion "very weak" is no longer justified.

(b) Let u be a strong solution as in Theorem 2.1. Then we can use similar arguments as in [13, V. Theorem 1.8.2] and obtain for smooth data $f, k, g, v_0 \in C^\infty$ that v and $u = v + E_{k,g}$ satisfy $v, u \in C^\infty((0,T)\times\Omega)$.

3. Preliminaries

In Assumptions 1.1 we define for $0 < T \le \infty$ the Besov-type space

$$B_T^{q,s}(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B_T^{q,s}} := \left(\int_0^T \|e^{-\tau A} v\|_q^s \, d\tau \right)^{1/s} < \infty \right\} \tag{3.1}$$

with norm $\|v\|_{B_T^{q,s}} = \|v\|_{B_T^{q,s}(\Omega)}$. This normed space, which has been introduced in [6]–[8], is important for our results. Equipped with the norm $v \mapsto \|v\|_{B_T^{q,s}} + \|v\|_{L^2_\sigma}$ it is a Banach space.

In (3.1) $A = A_q$ denotes the Stokes operator, and $S(\tau) = e^{-\tau A}$, $0 \le \tau < \infty$ the exponentially decaying analytic semi-group generated by $-A$. Using the fractional powers A^α we will exploit for $v \in L^2_\sigma(\Omega)$ the estimates

$$\|A^{-\alpha} v\|_q \le C\|v\|_2, \tag{3.2}$$

$$\|S(\tau)v\|_q \le C\tau^{-\alpha} e^{-\delta\tau} \|A^{-\alpha} v\|_q \le C\tau^{-\alpha} e^{-\delta\tau}\|v\|_2, \tag{3.3}$$

with $0 < \alpha < 3/4$, $2\alpha + 3/q = 3/2$, $\delta = \delta(\Omega, q) > 0$, $C = C(\Omega, q, \alpha, \delta) > 0$, see [6, (1.14)]. By (3.3) we conclude that the function $\tau \mapsto \|S(\tau)v\|_q^s$ is well defined on $(0,T)$. Therefore, $v \in B_T^{q,s}(\Omega) \subseteq L^2_\sigma(\Omega)$ simply means that this function is Lebesgue integrable on $(0,T)$.

Moreover, let $W^{1,2}_{0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Then we define the Banach space

$$
\begin{aligned}
(3.4)\quad X := \big\{ & v \in L^2_{\mathrm{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega)); \\
& (A^{-1/2}v)_t, A^{1/2}v \in L^{s/2}(0,T; L^{q/2}(\Omega)),\ A^{-1/2}v|_{t=0} = 0 \big\}
\end{aligned}
$$

equipped with the norm

$$
\|v\|_X := \|(A^{-1/2}v)_t\|_{q/2,s/2,T} + \|A^{1/2}v\|_{q/2,s/2,T}. \tag{3.5}
$$

Additionally to the data given in Assumptions 1.1, in the following propositions we need a vector field

$$
f_0 \in L^{s/2}(0,T; L^{q/2}(\Omega)) \quad \text{with } q,\ s \text{ as in Assumption 1.1(a).} \tag{3.6}
$$

Note that $f = \operatorname{div} F$, f_0 in Assumption 1.1(a), (3.6), respectively, satisfy

$$
F \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)), \quad f_0 \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)). \tag{3.7}
$$

Next we consider several well known results on the linear nonstationary Stokes system in $[0,T) \times \Omega$ given by

$$
v_t - \Delta v + \nabla h = f + f_0, \quad \operatorname{div} v = 0, \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0 \tag{3.8}
$$

with associated pressure ∇h.

Proposition 3.1. *Let $f = \operatorname{div} F$, f_0, v_0 be as in Assumptions* 1.1 *and* (3.6), *and let $E_{f,f_0,v_0} := v \in L^2_{\mathrm{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega))$ be a weak solution of the system* (3.8) *in the usual sense, defined by the relation*

$$
-\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} = \langle v_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} + \langle f_0, w\rangle_{\Omega,T} \tag{3.9}
$$

for each $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$. Then we obtain the following properties:

(a) *The function*

$$
v = E_{f,f_0,v_0} \colon [0,T) \to L^2_\sigma(\Omega) \tag{3.10}
$$

is strongly continuous, after redefinition on a null set of $[0,T)$, and it holds the energy equality

$$
\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau = \frac{1}{2}\|v_0\|_2^2 + \langle f_0, v\rangle_{\Omega,t} - \langle F, \nabla v\rangle_{\Omega,t} \tag{3.11}
$$

for each $t \in [0,T)$. Further, v is uniquely determined by f, f_0, v_0.

(b) *Let* $S(t) := e^{-tA}$, $t \in [0,T)$. *Then* $v = E_{f,f_0,v_0}$ *has the representation*

$$v(t) = S(t)v_0 + \int_0^t S(t-\tau)Pf_0(\tau)\,d\tau + A^{1/2}\int_0^t S(t-\tau)A^{-1/2}P\operatorname{div}F(\tau)\,d\tau \tag{3.12}$$

for each $t \in [0,T)$, *and it holds* $E_{f,f_0,v_0} = S(\,\cdot\,)v_0 + E_{f,f_0,0}$.

(c) *If* $v_0 = 0$, *then* $\widetilde{v} = E_{f,f_0,0} \in X$, *i.e.*

$$(A^{-1/2}\widetilde{v})_t,\ A^{1/2}\widetilde{v} \in L^{s/2}(0,T;L^{q/2}(\Omega)), \quad A^{-1/2}\widetilde{v}|_{t=0} = 0, \tag{3.13}$$

and

$$(A^{-1/2}\widetilde{v})_t + A^{1/2}\widetilde{v} = A^{-1/2}P\operatorname{div}F + A^{-1/2}Pf_0, \quad t \in [0,T), \tag{3.14}$$

$$\|\widetilde{v}\|_X \le C(\|F\|_{q/2,s/2,T} + \|f_0\|_{q/2,s/2,T}) \tag{3.15}$$

with some constant $C = C(\Omega,q) > 0$ *independent of* T.

(d) *Conversely, let* $\widetilde{v} \in L^2_{\rm loc}([0,T);W^{1,2}_{0,\sigma}(\Omega))$ *satisfy the properties* (3.13)–(3.15). *Then* $\widetilde{v} = E_{f,f_0,0}$ *is a weak solution of the system* (3.8) *with* $v_0 = 0$, *and* $E_{f,f_0,v_0} = E_{f,f_0,0} + Sv_0$ *is a weak solution of* (3.8) *with the given data* f, f_0, v_0.

(e) *There holds*

$$\|E_{f,f_0,v_0}\|_{q,s,T} \le \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,f_0,0}\|_{q,s,T} < \infty, \tag{3.16}$$

and there exists some constant $C = C(\Omega,q) > 0$ *independent of* T *such that*

$$\|E_{f,f_0,0}\|_{q,s,T} \le C\big(\|(A^{-1/2}E_{f,f_0,0})_t\|_{q/2,s/2,T} + \|A^{1/2}E_{f,f_0,0}\|_{q/2,s/2,T}\big). \tag{3.17}$$

Proof. (a) Using (3.7) and (3.8) we obtain for $v = E_{f,f_0,v_0}$ and $0 < T' < T$ the strong continuity (3.10), the energy equality (3.11), and the estimate

$$\frac{1}{2}\|v\|^2_{2,\infty,T'} + \|\nabla v\|^2_{2,2,T'} \le 2\|v_0\|_2^2 + 8\|f_0\|^2_{2,1,T'} + 4\|F\|^2_{2,2,T'}, \tag{3.18}$$

see [13, IV. Theorem 2.3.1], and v is uniquely determined by f, f_0, v_0.

(b) The representation (3.12) follows from [13, IV. Theorem 2.4.1]. Note that $A^{-1/2}P\operatorname{div}$, defined by $\langle A^{-1/2}P\operatorname{div}F,\varphi\rangle = \langle -F,\nabla A^{-1/2}\varphi\rangle$ for $\varphi \in C^\infty_{0,\sigma}(\Omega)$, is a bounded operator satisfying

$$\|A^{-1/2}P\operatorname{div}F(t)\|_{q/2} \le C\|F(t)\|_{q/2} \quad \text{for a.a. } t \in [0,T) \tag{3.19}$$

with $C = C(\Omega,q) > 0$; see [3, Examples 3), (2.14)].

(c) Applying $A^{-1/2}$ to (3.12) when $v_0 = 0$, we obtain for $\widetilde{v} := E_{f,f_0,0}$ that $A^{-1/2}\widetilde{v}$ is a weak solution of the system (3.14) in $[0,T) \times \Omega$. By the maximal regularity estimate, see, e.g. [6, (2.7)], we obtain the estimate (3.15) and it holds (3.13).

(d) Let $v \in L^2_{\mathrm{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega))$ satisfy (3.13)–(3.15). Testing (3.14) with $A^{1/2}w$, $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$, we obtain the relation (3.9) for $\widetilde{v} = E_{f,f_0,0}$ where $v_0 = 0$. Here we need the properties

$$\frac{d}{dt}\langle v, w\rangle_\Omega = \frac{d}{dt}\,\langle A^{-1/2}v, A^{1/2}w\rangle_\Omega \in L^1(0,T),$$

and $v_0 = 0$, $w(T) = 0$ yielding

$$-\langle v, w_t\rangle_{\Omega,T} = -\int_0^T \langle A^{-1/2}v, (A^{1/2}w)_t\rangle_\Omega\,dt = \int_0^T \langle (A^{-1/2}v)_t, A^{1/2}w\rangle_\Omega\,dt.$$

Next we use that $E_{0,0,v_0} = Sv_0$ satisfies (3.9) for $f = 0$, $f_0 = 0$ as weak solution. This implies, together with (3.9) for $\widetilde{v} = E_{f,f_0,0}$, that $E_{f,f_0,v_0} = E_{0,0,v_0} + E_{f,f_0,0}$ solves (3.9) and is a weak solution of (3.8).

(e) Setting $\widetilde{v} = E_{f,f_0,0}$ when $v_0 = 0$, we obtain using (3.12), (3.14) the representation

$$\begin{aligned} A^{-1/2}\widetilde{v}(t) &= \int_0^t S(t-\tau)A^{-1/2}Pf_0\,d\tau + \int_0^t S(t-\tau)A^{-1/2}P\operatorname{div}F\,d\tau \\ &= \int_0^t S(t-\tau)\big((A^{-1/2}\widetilde{v})_t + A^{1/2}\widetilde{v}\big)\,d\tau. \end{aligned} \tag{3.20}$$

By the fractional Sobolev embedding estimate $\|w\|_q \le c\|A^{3/(2q)}w\|_{q/2}$ and the Hardy–Littlewood inequality we obtain (3.17) from (3.20), cf. [6, (2.24)].

Next we obtain for $E_{f,f_0,v_0} = E_{f,f_0,0} + Sv_0$ that

$$\|E_{f,f_0,v_0}\|_{q,s,T} \le \|Sv_0\|_{q,s,T} + \|E_{f,f_0,0}\|_{q,s,T} = \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,f_0,0}\|_{q,s,T}$$

which proves (3.16). This completes the proof of Proposition 3.1. $\square$

4. Proof of the main results

4.1. Proof of Theorem 2.1. Let $f = \operatorname{div}F, v_0, k, g, E = E_{k,g}$, $0 < T \le \infty$ be given as in Theorem 2.1. We will need several steps for the proof.

Preliminaries. Writing a solution u of (1.1) in the form $u = v + E_{k,g}$ we are looking for a solution v of (1.10). To work in the space X, see (3.4), we have to turn to $\widehat{v} = v - E_{f,0,v_0}$ which is a solution of the equation

$$\widehat{v}_t - \Delta\widehat{v} + \nabla h = k(\widehat{v} + \widehat{E}) - \operatorname{div}((\widehat{v} + \widehat{E})(\widehat{v} + \widehat{E})), \quad \widehat{E} := E_{f,0,v_0} + E_{k,g}, \tag{4.1}$$

together with $\operatorname{div}\widehat{v} = 0$, $\widehat{v}(0) = 0$ and $\widehat{v}|_{\partial\Omega} = 0$. Here $k(\widehat{v} + \widehat{E})$ plays the role of f_0 in Proposition 3.1. To reformulate the fixed point problem (4.1) we define

$$\begin{aligned} \widehat{F}(\widehat{v}) &:= -(\widehat{v} + \widehat{E})(\widehat{v} + \widehat{E}), \\ \widehat{f}(\widehat{v}) &:= \operatorname{div}\widehat{F}(\widehat{v}), \\ \widehat{f}_0(\widehat{v}) &:= k(\widehat{v} + \widehat{E}). \end{aligned} \tag{4.2}$$

By Hölder's inequality and (3.17) we obtain that $\widehat{E}, E_{f,0,v_0}, E = E_{k,g} \in L^s(0,T;L^q(\Omega))$ as well as $\widehat{v}, v \in L^s(0,T;L^q(\Omega))$, and that

$$\begin{aligned} \|\widehat{F}(\widehat{v})\|_{q/2,s/2,T} &\le \|\widehat{v}+\widehat{E}\|^2_{q,s,T} \le \big(\|\widehat{v}\|_{q,s,T}+\|\widehat{E}\|_{q,s,T}\big)^2 < \infty, \\ \|\widehat{f}_0(\widehat{v})\|_{q/2,s/2,T} &\le \|k\|_{q,s,T}\big(\|\widehat{v}\|_{q,s,T}+\|\widehat{E}\|_{q,s,T}\big) < \infty. \end{aligned} \tag{4.3}$$

Correspondingly, we set

$$\begin{aligned} F(v) &:= -(v+E)(v+E) = -(\widehat{v}+\widehat{E})(\widehat{v}+\widehat{E}), \\ f(v) &:= \operatorname{div} F(v) = \operatorname{div}\widehat{F}(\widehat{v}) = \widehat{f}(\widehat{v}), \\ f_0(v) &:= k(v+E) = k(\widehat{v}+\widehat{E}), \end{aligned} \tag{4.4}$$

and get estimates of $F(v)$, $f_0(v)$ in $L^{s/2}(0,T;L^{q/2}(\Omega))$ as those for $\widehat{F}(\widehat{v})$, $\widehat{f}_0(\widehat{v})$ in (4.3).

Next we mention an estimate for $b(T)$ as given in (2.1). We obtain, using (3.16), (3.17), (3.15) with $f_0 = 0$ and (1.6), the estimate

$$\begin{aligned} \|\widehat{E}\|_{q,s,T} &\le \|E_{f,0,v_0}\|_{q,s,T} + \|E\|_{q,s,T} \\ &\le \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,0,0}\|_{q,s,T} + \|E\|_{q,s,T} \\ &\le \|v_0\|_{B^{q,s}_T(\Omega)} + C\big(\|F\|_{q/2,s/2,T} + \|k\|_{q,s,T} + \|g\|_{-1/q,q,s,T}\big) \end{aligned}$$

with constant $C = C(\Omega,q) > 0$. We may assume that $C \ge 1$. Hence

$$\|\widehat{E}\|_{q,s,T} \le Cb(T), \qquad \|k\|_{q,s,T} \le b(T) \tag{4.5}$$

with $C = C(\Omega,q) \ge 1$ independent of T.

The nonlinear operator $\mathcal{F}$. Let $\widehat{v} \in X$ and let $\mathcal{F}(\widehat{v}) := w$ be the solution of the system

$$(A^{-1/2}w)_t + A^{1/2}w = A^{-1/2}P\operatorname{div}\widehat{F}(\widehat{v}) + A^{-1/2}P\widehat{f}_0(\widehat{v}), \quad w \in X, \tag{4.6}$$

as in (3.14).

Using, step by step, (3.17), (3.15) and (4.3), we obtain that

$$\begin{aligned} \|\mathcal{F}(\widehat{v})\|_{q,s,T} &= \|w\|_{q,s,T} \\ &\le C_1(\|(A^{-1/2}w)_t\|_{q/2,s/2,T} + \|A^{1/2}w\|_{q/2,s/2,T}) \\ &\le C_2(\|\widehat{F}(\widehat{v})\|_{q/2,s/2,T} + \|\widehat{f}_0(\widehat{v})\|_{q/2,s/2,T}) \\ &\le C_3((\|\widehat{v}\|_{q,s,T}+\|\widehat{E}\|_{q,s,T})^2 + \|k\|_{q,s,T}(\|\widehat{v}\|_{q,s,T}+\|\widehat{E}\|_{q,s,T})). \end{aligned} \tag{4.7}$$

Moreover, by (3.17) for $\|\widehat{v}\|_{q,s,T}$ and (4.5) for $\|\widehat{E}\|_{q,s,T}$ and $\|k\|_{q,s,T}$, we get that

$$\|\mathcal{F}(\widehat{v})\|_{q,s,T} \le C_4(\|\widehat{v}\|_X + b(T))^2 \tag{4.8}$$

with constants $C_1,\dots,C_4 > 0$ depending on Ω, q. Consequently,

$$\begin{aligned} \|\mathcal{F}(\widehat{v})\|_{q,s,T} \le C_1\|\mathcal{F}(\widehat{v})\|_X &\le C_4(\|\widehat{v}\|_X + b(T))^2, \\ \|\mathcal{F}(\widehat{v})\|_X &\le a(\|\widehat{v}\|_X + b)^2, \quad a = C_4/C_1,\ b = b(T) \end{aligned} \tag{4.9}$$

with constants $C_1, C_2 > 0$ depending on Ω, q.

Next we estimate the expression $\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})$ with $\widehat{v}, \widetilde{v} \in X$, using the representation formula (3.12) with v, f_0, F replaced by $\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})$ and $\widehat{f}_0(\widehat{v}) - \widehat{f}_0(\widetilde{v})$, $\widehat{F}(\widehat{v}) - \widehat{F}(\widetilde{v})$, respectively. We obtain that

$$\begin{aligned}
&(\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v}))(t)\\
&= \int_0^t A^{1/2} S(t-\tau) A^{-1/2} P[\operatorname{div}(\widehat{F}(\widehat{v}) - \widehat{F}(\widetilde{v})) + \widehat{f}_0(\widehat{v}) - \widehat{f}_0(\widetilde{v})]\, d\tau\\
&= \int_0^t A^{1/2} S(t-\tau) A^{-1/2} P[\operatorname{div}((\widetilde{v} + \widehat{E})(\widetilde{v} - \widehat{v}) + (\widetilde{v} - \widehat{v})(\widehat{v} + \widehat{E})) + k(\widehat{v} - \widetilde{v})]\, d\tau.
\end{aligned}$$

Then we apply the same arguments as in (4.7)–(4.9) to get for $\widehat{v}, \widetilde{v} \in X$ the estimate

$$\begin{aligned}
\|\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})\|_X &\le C_1(\|\widehat{v}\|_X + b + \|\widetilde{v}\|_X + b + b)\|\widehat{v} - \widetilde{v}\|_X)\\
&\le C_2(\|\widehat{v}\|_X + b + \|\widetilde{v}\|_X + b)\|\widehat{v} - \widetilde{v}\|_X
\end{aligned}$$

with $C_1, C_2 > 0$ depending on Ω, q. We may assume that $C_2 = a$ with a as in (4.9). Thus we obtain that

$$\|\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})\|_X \le a(\|\widehat{v}\|_X + b + \|\widetilde{v}\|_X + b)\|\widehat{v} - \widetilde{v}\|_X. \tag{4.10}$$

The fixed point problem $\widehat{v} = \mathcal{F}(\widehat{v})$. Here we use similar arguments as in the existence proof of very weak solutions, see [3], [6], [7].

Let $\widehat{v}, \widetilde{v} \in X$. Then $\mathcal{F}(\widehat{v})$, $\mathcal{F}(\widetilde{v})$ satisfy the estimates (4.9), (4.10), i.e. with $a = a(\Omega, q) > 0$ and $b = b(T)$

$$\|\mathcal{F}(\widehat{v})\|_X \le a(\|\widehat{v}\|_X + b)^2, \tag{4.11}$$

$$\|\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})\|_X \le a\big(\|\widetilde{v}\|_X + b + \|\widehat{v}\|_X + b\big)\|\widehat{v} - \widetilde{v}\|_X. \tag{4.12}$$

For the given data $f = \operatorname{div} F$, v_0, k, g as in Assumptions 1.1 and with $b(T)$ defined in (2.1) we suppose the smallness condition

$$4ab = 4ab(T) = 4a(\|v_0\|_{B_T^{q,s}} + \|F\|_{q/2,s/2,T} + \|k\|_{q,s,T} + \|g\|_{-1/q,q,s,T}) < 1. \tag{4.13}$$

Using $4ab < 1$ we choose

$$0 < y_1 := 2b(1 + \sqrt{1 - 4ab})^{-1} < 2b, \quad y_1 = a y_1^2 + b > b$$

and $B := \{v \in X;\ \|v\|_X \le y_1 - b\}$. Then, if $\widehat{v} \in B$, we obtain from the estimate

$$\|\mathcal{F}(\widehat{v})\|_X \le a(\|\widehat{v}\|_X + b)^2 \le a y_1^2 = y_1 - b$$

that $\mathcal{F}(B) \subseteq B$. Further we use (4.12) and obtain with $\widehat{v}, \widetilde{v} \in B$ that

$$\begin{aligned}
\|\mathcal{F}(\widehat{v}) - \mathcal{F}(\widetilde{v})\|_X &\le a(\|\widetilde{v}\|_X + b + \|\widehat{v}\|_X + b)\|\widehat{v} - \widetilde{v}\|_X\\
&\le 2a y_1 \|\widehat{v} - \widetilde{v}\|_X \le 4ab\|\widehat{v} - \widetilde{v}\|_X.
\end{aligned}$$

Thus $\mathcal{F}\colon B \to B$ is a strict contraction, and Banach's fixed point principle yields a unique $\widehat{v} \in B$ satisfying $\widehat{v} = \mathcal{F}(\widehat{v})$.

Uniqueness in X. Suppose that $\widehat{v}_1, \widehat{v}_2 \in X$ are fixed points of $\mathcal{F}$. Then we conclude from (4.10) with $\|\cdot\|_X$ replaced by $\|\cdot\|_{q;s,T}$ that

$$\begin{aligned}(4.14)\qquad \|\widehat{v}_1 - \widehat{v}_2\|_{q,s,T} &= \|\mathcal{F}(\widehat{v}_1) - \mathcal{F}(\widehat{v}_2)\|_{q,s,T} \\ &\le a\big(\|\widehat{v}_1\|_{q,s,T} + b(T) + \|\widehat{v}_2\|_{q,s,T} + b(T)\big)\|\widehat{v}_1 - \widehat{v}_2\|_{q,s,T}\end{aligned}$$

with $b = b(T)$ as in (4.5), and with $a = a(\Omega, q) > 0$ as in (4.9).

Consider any subinterval $[0, T')$, $0 < T' < T$. Then we obtain the same estimate (4.14) with $\|\cdot\|_{q,s,T}$, $b(T)$ replaced by $\|\cdot\|_{q,s,T'}$, $b(T')$, and choose $0 < T' < T$ such that

$$a(\|\widehat{v}_1\|_{q,s,T'} + b(T') + \|\widehat{v}_2\|_{q,s,T'} + b(T')) \le \frac{1}{2};$$

thus we conclude that $\|\widehat{v}_1 - \widehat{v}_2\|_{q,s,T'}/2 \le 0$, $\widehat{v}_1 = \widehat{v}_2$. Finally, we continue this argument with $[0, T')$ replaced by $[T', 2T')$ with the same constant a, and so on. This yields $\widehat{v}_1 = \widehat{v}_2$ in $[0, T)$.

The condition $\mathcal{F}(\widehat{v}) = \widehat{v}$, $\widehat{v} \in X$. Assume that $\widehat{v} \in X$ satisfies the condition $\mathcal{F}(\widehat{v}) = \widehat{v}$. Then we show that

$$(4.15)\qquad v := \widehat{v} + E_{f,0,v_0} \text{ is a strong solution of (1.9)}$$

as in Definition 1.2(b). Obviously, by (3.17) $\widehat{v} \in X \subset L^s(0, T; L^q(\Omega))$, and by (3.16) $E_{f,0,v_0} \in L^s(0, T; L^q(\Omega))$; hence

$$(4.16)\qquad v \in L^s(0, T; L^q(\Omega)).$$

Then we use Proposition 3.1(d), and conclude that $\widehat{v}$ is a weak solution of the system (3.8) with f, f_0 replaced by $\widehat{f}(\widehat{v})$, $\widehat{f}_0(\widehat{v})$, and with $\widehat{v}(0) = 0$.

Now we consider $\widehat{E} = E_{f,0,v_0} + E$, $E = E_{k,g}$, as in (4.2) and use the relation (3.9) for $(v =) E_{f,0,v_0}$ with $f = \operatorname{div} F$, $f_0 = 0$ as well as for the weak solution $\widehat{v} = \mathcal{F}(\widehat{v})$ of (3.9). We conclude that

$$(4.17)\qquad v = \widehat{v} + E_{f,0,v_0} \text{ satisfies (1.10) with } E = E_{k,g}.$$

Thus it holds (3.9) with F, f_0 replaced by $F(v)$, $f_0(v)$, see (4.4). Using Proposition 3.1(a) we conclude that $v = \widehat{v} + E_{f,0,v_0}$ satisfies the properties (a)–(d) in Definition 1.2 of the perturbed system (1.9) and, consequently, $u = v + E_{k,g}$ is a strong solution of the general system (1.1).

Finally, in view of (4.13) we may choose $\varepsilon^* = \varepsilon^*(\Omega, q) = (8a)^{-1}$. This completes the proof of Theorem 2.1. □

4.2. Proof of Corollary 2.2. Let f, v_0, k, g, q, s, $[0, T)$, $E = E_{k,g}$, $b = b(T)$ and $\varepsilon^* = \varepsilon^*(\Omega, q) > 0$ be given as in Corollary 2.2.

Since $b(T) \to 0$ as $T \to 0$, we find some $T^* \in (0, T]$ satisfying $b(T^*) \le \varepsilon^*$. By Theorem 2.1 we get the uniquely determined solutions $v, u = v + E$. □

4.3. Proof of Theorem 2.3. Assume that the given data $f = \operatorname{div} F$, v_0, k, g additionally satisfy the regularity conditions (2.3). From [3, Corollary 5] we obtain for the solution $E = E_{k,g}$ of the Stokes system (1.3), in addition to (1.6), that

$$E \in L^s(0,T;W^{2,q}(\Omega)), \quad E_t \in L^s(0,T;L^q(\Omega)). \tag{4.18}$$

Then we have to prove the regularity properties (2.4) for the solution v of (1.9), written in the form

$$\begin{gathered} v_t - \Delta v + (v+E)\cdot\nabla v + \nabla p^* = f^* := \operatorname{div} F - (v+E)\cdot\nabla E, \\ v|_{t=0} = v_0, \quad v|_{\partial\Omega} = 0, \quad \operatorname{div} v = 0. \end{gathered} \tag{4.19}$$

For a moment let $v := \widehat{v}$ be the corresponding solution for the well known case $k = 0$, $g = 0$, $E = 0$. In this case the regularity properties (2.4) for $v = \widehat{v}$ have been shown in [13, V. Theorem 1.8.1, p. 298], where the critical expression, now written in the form $\widehat{v}\cdot\nabla\widehat{v}$, has been treated using the Yosida operators $J_k = (I + A^{1/2}/k)^{-1}$, $k \in \mathbb{N}$, and $\widehat{v}\cdot\nabla\widehat{v}_k$ with $\widehat{v}_k = J_k\widehat{v}$, and using $\widehat{v} \in L^s(0,T;L^q(\Omega))$.

We can reduce our regularity problem for (4.19) to this known case. Since $v + E \in L^s(0,T;L^q(\Omega))$, we use the approximation $(v+E)\cdot\nabla v_k$, $k \in \mathbb{N}$, for the critical term $(v+E)\cdot\nabla v$. Furthermore, since by (4.18) $E, \nabla E \in L^s(0,T;L^q(\Omega))$ and $v \in L^\infty_{\mathrm{loc}}([0,T);L^2(\Omega))$ and since $q \geq 4$, $s \geq 4$, we obtain that

$$v\cdot\nabla E, E\cdot\nabla E \in L^2_{\mathrm{loc}}([0,T);L^2(\Omega)). \tag{4.20}$$

Thus we obtain from (4.20), (2.3) that $f^* \in L^2_{\mathrm{loc}}([0,T);L^2(\Omega))$. Then we get for (4.19) - as in [13, V. Theorem 1.8.1] - the properties (2.4), (2.6), (2.7). This completes the proof of Theorem 2.3. □

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