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A GENERAL CLASS OF IMPULSIVE EVOLUTION EQUATIONS

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Abstract. One of the novelty of this paper is the study of a general class of impulsive differential equations, which is more reasonable to show dynamics of evolution processes in Pharmacotherapy. This fact reduces many difficulties in applying analysis methods and techniques in Bielecki's normed Banach spaces and thus makes the study of existence and uniqueness theorems interesting. The other novelties of this paper are new concepts of Ulam's type stability and Ulam–Hyers–Rassias stability results on compact and unbounded intervals.

1. Introduction

The dynamic of evolution processes in the real world is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. In general, these short-term perturbations are usually regarded as having acted instantaneously or appearing in the form of instantaneous impulses involving the corresponding differential equations. Many authors were devoted to study mild solutions to impulsive evolution equations with instantaneous impulses of the

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form

$$
\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J' := J \setminus \{t_1, \dots, t_m\},\ J := [0, T], \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), & k = 1, \dots, m, \end{cases} \tag{1.1}
$$

where the linear unbounded operator A: $D(A) \subseteq X \to X$ is the generator of a C_0-semigroup (analytic or compact) $\{T(t),\ t \geq 0\}$ on a Banach space X with a norm $\|\cdot\|$, $f\colon J \times X \to X$ and $I_k\colon X \to X$ and fixed impulsive time t_k satisfy $0 = t_0 < t_1 < \dots, t_m < t_{m+1} = T$, the symbols $x(t_k^+) := \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$ and $x(t_k^-) := \lim_{\varepsilon \to 0^-} x(t_k + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Note that I_k in (1.1) is a sequences of instantaneously impulse operators and has been developed in physics, population dynamics, biotechnology, and so forth. For more details on differential equations with instantaneous impulses, one can see for instance the monographs [8], [10], [33], the works on not time variable impulses problem [4], [5], [12], [14]–[16], [26], [34], [36], [37] and time variable impulses problem [1]–[3], [17]–[19] and the references therein.

However, the action of instantaneous impulses seems do not describe some certain dynamics of evolution processes in Pharmacotherapy. Taking into consideration the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Thus, we do not expect to use the model (1.1) to describe such process. In fact, the above situation is fallen in a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. To this end, Hernández and O'Regan [21] introduce a new class of impulsive evolution equations (with not instantaneous impulses) of the form

$$
\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, \dots, m, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \dots, m, \\ x(0) = x_0 \in X, \end{cases} \tag{1.2}
$$

where A and f are the same as (1.1) and the fixed points s_i and t_i satisfy $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \dots < s_{m-1} < t_m < s_m < t_{m+1} = T$, and $g_i\colon [t_i, s_i] \times X \to X$ is continuous for all $i = 1, \dots, m$. Here g_i is regarded as continuous action process.

The concepts of mild solutions and classical solutions are introduced by Hernández and O'Regan [21] (see Definitions 2.1 and 2.2). Meanwhile, existence and uniqueness results of (1.2) are presented by using the theory of strongly continuous semigroup and compact semigroup via fixed point theorems (see Theorems 2.1 and 2.2, [21] and Theorems 2.1 and 2.2, [29]). Next, Pierri et al. [29] continue the work and development in [21] and study the existence and uniqueness of mild solutions to semilinear impulsive differential equations with not instantaneous impulses in the fractional power space by virtue of the theory of analytic semigroup.

Next, we have a remark on the conditions in (1.2):

$$x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \tag{1.3}$$

where $g_i \in C([t_i, s_i] \times X, X)$ and there are positive constants L_{g_i}, $i = 1, \dots, m$ such that

$$\|g_i(t, u_1) - g_i(t, u_2)\| \le L_{g_i} \|u_1 - u_2\| \quad \text{for each } t \in [t_i, s_i] \text{ and all } u_1, u_2 \in X.$$

It follows from Theorems 2.1 and 2.2 in [21], [29], that a necessary condition $\max\{L_{g_i} : i = 1, \dots, m\} < 1$ is considered. Then Banach fixed point theorem gives a unique $z_i \in C([t_i, s_i], X)$ so that $z = g_i(t, z)$ if and only if $z = z_i(t)$. So (1.3) is equivalent to

$$x(t) = z_i(t), \quad t \in (t_i, s_i],\ i = 1, \dots, m, \tag{1.4}$$

which does not depend on the state $x(\,\cdot\,)$. Thus it is necessary to modify (1.3) and we recommend to consider the conditions

$$x(t) = g_i(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, \dots, m. \tag{1.5}$$

Comparing with (1.3), the new conditions (1.5) is a better and reasonable generalization of sudden impulses to not instantaneous ones.

Motivated by the above remark, we study impulsive evolution equations of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}),\ i = 0, \dots, m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1, \dots, m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i],\ i = 1, \dots, m. \end{cases} \tag{1.6}$$

Note that we consider in (1.6) that $x \in C((t_k, t_{k+1}], X)$, $k = 0, \dots, m$, and there exist $x(t_k^-)$, $x(t_k^+)$, $k = 1, \dots, m$ with $x(t_k^-) = x(t_k)$. Next this model is more suitable to show dynamics of evolution processes in Pharmacotherapy: the first equation denotes the health status of a patient; the second equation denotes the doctor takes some actions to test medicine for the patient practicably; the third equation denotes the testing medicine is valid for this patient and then begin to deal with the effect of patient for some time.

The rest of this paper is organized as follows. In Section 2, we give an existence and uniqueness result of (1.6) with $x(0) = x_0 \in X$. In Section 3, we introduce new four types of Ulam's type stability for differential equations with not instantaneous impulses (see Definitions 3.1–3.4). Ulam problem [35] has been attracted by many researchers, one can refer to the monographs of Cădariu [11], Hyers [22], [23], Jung [24], Rassias [30] and other mathematicians. For more recent contribution on such important fields, one can see the papers [6], [7], [13], [20], [25], [27], [31], [32], [37] and reference therein. We mainly present the generalized Ulam–Hyers–Rassias stability results for the equation (1.6) on a compact interval.

Finally in Section 4, we extend our study to

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}),\ i \in \mathbb{M}, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i \in \mathbb{M}, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i],\ i \in \mathbb{M}, \end{cases} \tag{1.7}$$

where $t \in \mathbb{R}_+ := [0, \infty)$, the fixed points s_i and t_i satisfy $t_i < s_i < t_{i+1}$ and either $\mathbb{M} = \{1, \ldots, m\}$ or $\mathbb{M} = \mathbb{N}$ and then with $\lim\limits_{i\to\infty} t_i = \infty$. We also set $t_{m+1} = \infty$ for $\mathbb{M} = \{1, \ldots, m\}$. Some extensions of Ulam–Hyers–Rassias stability for the case with infinite impulses are given. The existence and uniqueness result is also presented for this case.

2. An existence and uniqueness result

Set $J := [0, T]$. Throughout this paper, we need the Banach space

$$\begin{aligned} PC(J, X) := \{x \colon J \to X : x \in C((t_k, t_{k+1}], X),\ k = 0, \ldots, m \\ \text{and there exist } x(t_k^-) \text{ and } x(t_k^+),\ k = 1, \ldots, m \text{ with } x(t_k^-) = x(t_k)\} \end{aligned}$$

endowed either with the Chebyshev PC-norm $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$ or with the Bielecki PCB-norm $\|x\|_{PCB} := \sup\{|x(t)|e^{-\Omega t} : t \in J\}$ for some $\Omega \in \mathbb{R}$.

We recall the following concepts of mild solutions of semilinear evolution equations with not instantaneous impulses.

Definition 2.1 (see Definition 2.1, [21]). A function $x \in PC(J, X)$ is called a mild solution of the problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}),\ i = 0, \ldots, m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1, \ldots, m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i, s_i],\ i = 1, \ldots, m, \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

if $x(0) = x_0$ and

$$\begin{aligned} x(t) &= g_i(t, x(t_i^-)), && t \in (t_i, s_i],\ i = 1, \ldots, m; \\ x(t_i^+) &= g_i(t_i, x(t_i^-)), && i = 1, \ldots, m, \\ x(t) &= T(t)x_0 + \int_0^t T(t-s) f(s, x(s))\, ds, && t \in [0, t_1]; \\ x(t) &= T(t - s_i) g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t-s) f(s, x(s))\, ds, \\ & && t \in [s_i, t_{i+1}],\ i = 1, \ldots, m. \end{aligned}$$

Note that we consider $x \in C((t_i, t_{i+1}], X)$, $i = 0, \dots, m$.

Cornering the existence and uniqueness of solutions to the problem (2.1), Hernández and O'Regan [21] initial obtain a interesting result under strong conditions via PC-norm. Here, we give another result under weak conditions via PCB-norm.

We introduce the following conditions:

- (H$_0$) $A\colon D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{T(t),\ t \geq 0\}$ on a Banach space X with a norm $\|\cdot\|$. Then $\|T(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ [28].
- (H$_1$) $f \in C(J \times X, X)$.
- (H$_2$) There exists a positive constant L_f such that

$$\|f(t,u_1) - f(t,u_2)\| \leq L_f \|u_1 - u_2\| \quad \text{for each } t \in J \text{ and all } u_1, u_2 \in X.$$

- (H$_3$) $g_i \in C([t_i, s_i] \times X, X)$ and there are positive constants L_{g_i}, $i = 1, \dots, m$ such that

$$\|g_i(t,u_1) - g_i(t,u_2)\| \leq L_{g_i} \|u_1 - u_2\| \quad \text{for each } t \in [t_i, s_i] \text{ and all } u_1, u_2 \in X.$$

Theorem 2.2. *Assume that* (H$_0$)–(H$_3$) *are satisfied. Then the equation* (2.1) *has a unique mild solution $x \in PC(J,X)$.*

Proof. Consider a mapping $F\colon PC(J,X) \to PC(J,X)$ defined by

$$\begin{aligned}
(Fx)(0) &= x_0;\\
(Fx)(t) &= g_i\bigg(t, T(t_i - s_{i-1}) g_{i-1}(s_{i-1}, x(t_{i-1}^-)) + \int_{s_{i-1}}^{t_i} T(t_i - s) f(s, x(s))\, ds\bigg),\\
&\qquad\qquad t \in (t_i, s_i],\ i = 1, \cdots, m;\\
(Fx)(t) &= T(t)x_0 + \int_0^t T(t-s) f(s, x(s))\, ds, \qquad t \in [0, t_1];\\
(Fx)(t) &= T(t - s_i) g_i(s_i, x(t_i^-)) + \int_{s_i}^t T(t-s) f(s, x(s))\, ds,\\
&\qquad\qquad t \in (s_i, t_{i+1}],\ i = 1, \dots, m,
\end{aligned}$$

where we set $g_0(t,x) := x_0$ and so $L_{g_0} = 0$. Obviously, F is well defined due to (H$_1$).

Supposing $\Omega > \omega$, for any $x, y \in PC(J,X)$ and $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\|\\
&\quad \leq Me^{\omega(t-s_i)} L_{g_i} \|x(t_i^-) - y(t_i^-)\| + ML_f \int_{s_i}^t e^{\omega(t-s)} \|x(s) - y(s)\|\, ds\\
&\quad \leq Me^{\omega(t-s_i) + \Omega t_i} L_{g_i} \|x - y\|_{PCB} + ML_f \int_{s_i}^t e^{\omega(t-s) + \Omega s}\, ds \|x - y\|_{PCB}
\end{aligned}$$

$$\leq M\bigg(e^{\omega(t-s_i)+\Omega t_i}L_{g_i}+\frac{e^{\Omega t}L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB},$$

which implies that

$$\begin{aligned}e^{-\Omega t}\|(Fx)(t)-(Fy)(t)\| &\leq M\bigg(e^{\omega(t-s_i)+\Omega(t_i-t)}L_{g_i}+\frac{L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB}\\ &\leq M\bigg(e^{\Omega(t_i-s_i)}L_{g_i}+\frac{L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB},\end{aligned}$$

for $t\in(s_i,t_{i+1}]$. Proceeding as above for $t\in[0,t_1]$, we obtain that

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\| &\leq ML_f\int_0^t e^{\omega(t-s)}\|x(s)-y(s)\|\,ds\\ &\leq ML_f\int_0^t e^{\omega(t-s)+\Omega s}\,ds\|x-y\|_{PCB}\leq\frac{ML_fe^{\Omega t}}{\Omega-\omega}\|x-y\|_{PCB},\end{aligned}$$

which implies that

$$e^{-\Omega t}\|(Fx)(t)-(Fy)(t)\|\leq\frac{ML_f}{\Omega-\omega}\|x-y\|_{PCB},\quad t\in[0,t_1].$$

Using the above estimates, similarly for $t\in(t_i,s_i]$, $i=1,\dots,m$, we derive

$$\begin{aligned}\|(Fx)(t)-(Fy)(t)\| &\leq ML_{g_i}\bigg(e^{\omega(t_i-s_{i-1})}L_{g_{i-1}}\|x(t_{i-1}^-)-y(t_{i-1}^-)\|\\ &\qquad+L_f\int_{s_{i-1}}^{t_i}e^{\omega(t_i-s)}\|x(s)-y(s)\|\,ds\bigg)\\ &\leq ML_{g_i}\bigg(e^{\omega(t_i-s_{i-1})+\Omega t_{i-1}}L_{g_{i-1}}+\frac{e^{\Omega t_i}L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB},\end{aligned}$$

which implies that

$$\begin{aligned}&e^{-\Omega t}\|(Fx)(t)-(Fy)(t)\|\\ &\qquad\leq ML_{g_i}\bigg(e^{\omega(t_i-s_{i-1})+\Omega(t_{i-1}-t_i)}L_{g_{i-1}}+\frac{L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB}\\ &\qquad\leq ML_{g_i}\bigg(e^{\Omega(t_{i-1}-s_{i-1})}L_{g_{i-1}}+\frac{L_f}{\Omega-\omega}\bigg)\|x-y\|_{PCB},\end{aligned}$$

for $t\in(t_i,s_i]$, $i=1,\dots,m$. Summarizing the above estimates, we have that

$$\|Fx-Fy\|_{PCB}\leq L_F\|x-y\|_{PCB}$$

for

$$\begin{aligned}L_F:=M\times\max_{1\leq i\leq m}\bigg\{\frac{L_f}{\Omega-\omega},L_{g_i}\bigg(e^{\Omega(t_{i-1}-s_{i-1})}L_{g_{i-1}}+\frac{L_f}{\Omega-\omega}\bigg),\\ e^{\Omega(t_i-s_i)}L_{g_i}+\frac{L_f}{\Omega-\omega}\bigg\}.\end{aligned}$$

Obviously, one can choose a sufficient large $\Omega>\omega$ such that $L_F<1$, and so F is a contraction mapping. Then, one can derive the results immediately. □

3. Concepts and results of Ulam's type stability on a compact interval

In this section, we introduce Ulam's type stability concepts for the equation (1.6).

Let $\varepsilon > 0$, $\psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}_+)$ be nondecreasing. We consider the following inequalities

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon, & t \in (s_i, t_{i+1}),\ i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \varepsilon, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon, & t \in (t_i, s_i],\ i = 1, \dots, m, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varphi(t), & t \in (s_i, t_{i+1}),\ i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \psi, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \psi, & t \in (t_i, s_i],\ i = 1, \dots, m, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon\varphi(t), & t \in (s_i, t_{i+1}),\ i = 0, \dots, m, \\ \|y(t_i^+) - g_i(t_i, y(t_i^-))\| \leq \varepsilon\psi, & i = 1, \dots, m, \\ \|y(t) - g_i(t, y(t_i^-))\| \leq \varepsilon\psi, & t \in (t_i, s_i],\ i = 1, \dots, m. \end{cases} \tag{3.3}$$

Now we set the vector space

$$Z := PC(J, X) \bigcap_{i=0}^{m} C^1((s_i, t_{i+1}), X) \bigcap_{i=0}^{m} C((s_i, t_{i+1}), D(A))$$

The following concepts are inspired by Wang et al. [37].

DEFINITION 3.1. The equation (1.6) is Ulam–Hyers stable if there exists a real number $c_{f,m,g} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in Z$ of the inequality (3.1), there exists a mild solution $x \in PC(J, X)$ of the equation (1.6) with

$$\|y(t) - x(t)\| \leq c_{f,m,g}\varepsilon, \quad t \in J. \tag{3.4}$$

DEFINITION 3.2. The equation (1.6) is generalized Ulam–Hyers stable if there exists $\theta_{f,m,g} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f,m,g}(0) = 0$ such that for each solution $y \in Z$ of the inequality (3.1), there exists a mild solution $x \in PC(J, X)$ of the equation (1.6) with

$$\|y(t) - x(t)\| \leq \theta_{f,m,g}(\varepsilon), \quad t \in J. \tag{3.5}$$

DEFINITION 3.3. The equation (1.6) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,g,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in Z$ of the inequality (3.3), there exists a mild solution $x \in PC(J,X)$ of the equation (1.6) with

$$\|y(t)-x(t)\| \le c_{f,m,g,\varphi}\varepsilon(\varphi(t)+\psi), \quad t \in J. \tag{3.6}$$

Definition 3.4. The equation (1.6) is generalized Ulam–Hyers–Rassias stable with respect to (φ,ψ) if there exists $c_{f,m,g,\varphi} > 0$ such that for each solution $y \in Z$ of the inequality (3.2) there exists a mild solution $x \in PC(J,X)$ of the equation (1.6) with

$$\|y(t)-x(t)\| \le c_{f,m,g,\varphi}(\varphi(t)+\psi), \quad t \in J. \tag{3.7}$$

Remark 3.5. It is clear that: (a) Definition 3.1 ⇒ Definition 3.2; (b) Definition 3.3 ⇒ Definition 3.4; (c) Definition 3.3 for $\varphi(t)=\psi=1$ ⇒ Definition 3.1.

Remark 3.6. A function $y \in Z$ is a solution of the inequality (3.3) if and only if there is $G \in \bigcap_{i=0}^{m} C^1((s_i,t_{i+1}),X) \bigcap_{i=0}^{m} C((s_i,t_{i+1}),D(A))$ and $g \in \bigcap_{i=1}^{m} C([t_i,s_i],X)$ (which depend on y) such that:

(a) $\|G(t)\| \le \varepsilon\varphi(t)$, $t \in \bigcup_{i=0}^{m}(s_i,t_{i+1})$ and $\|g(t)\| \le \varepsilon\psi$, $t \in \bigcup_{i=0}^{m}[t_i,s_i]$;
(b) $y'(t) = Ay(t) + f(t,y(t)) + G(t)$, $t \in (s_i,t_{i+1})$, $i=0,\ldots,m$;
(c) $y(t) = g_i(t,y(t_i^-)) + g(t)$, $t \in (t_i,s_i]$, $i=1,\ldots,m$;
(d) $y(t_i^+) = g_i(t_i,y(t_i^-)) + g(t_i)$, $i=1,\ldots,m$.

One can have similar remarks for the inequalities (3.1) and (3.2).

Remark 3.7. If $y \in Z$ is a solution of the inequality (3.3) then y is a solution of the following integral inequality

$$\begin{cases} \|y(t)-g_i(t,y(t_i^-))\| \le \varepsilon\psi, \qquad t\in(t_i,s_i],\ i=1,\ldots,m; \\ \left\| y(t) - T(t)y(0) - \displaystyle\int_0^t T(t-s)(s,y(s))\,ds \right\| \\ \qquad \le \varepsilon M \displaystyle\int_0^t e^{\omega(t-s)}\varphi(s)\,ds, \quad t\in[0,t_1]; \\ \|y(t_i^+) - g_i(t_i,y(t_i^-))\| \le \varepsilon\psi, \quad i=1,\ldots,m; \\ \left\| y(t) - T(t-s_i)g_i(s_i,y(t_i^-)) - \displaystyle\int_{s_i}^t T(t-s)f(s,y(s))ds \right\| \\ \qquad \le \varepsilon M e^{\omega(t-s_i)}\psi + \varepsilon M \displaystyle\int_{s_i}^t e^{\omega(t-s)}\varphi(s)\,ds, \\ \qquad\qquad t\in[s_i,t_{i+1}],\ i=1,\ldots,m. \end{cases} \tag{3.8}$$

In fact, by Remark 3.6 we get

$$\begin{cases} y'(t) = Ay(t) + f(t,y(t)) + G(t), & t\in(s_i,t_{i+1}),\ i=1,\ldots,m; \\ y(t_i^+) = g_i(t_i,y(t_i^-)) + g(t_i), & i=1,\ldots,m; \\ y(t) = g_i(t,y(t_i^-)) + g(t), & t\in(t_i,s_i], i=1,\ldots,m. \end{cases} \tag{3.9}$$

Clearly [28, p. 105], the solution $y \in Z$ of the equation (3.9) is given by

$$\begin{aligned}
y(t) &= g_i(t, y(t_i^-)) + g(t), && t \in (t_i, s_i],\ i = 1, \ldots, m; \\
y(t) &= T(t)y(0) + \int_0^t T(t-s)(f(s, y(s)) + G(s))\, ds, && t \in [0, t_1]; \\
y(t) &= T(t-s_i)(g_i(s_i, y(t_i^-)) + g(t_i)) \\
&\quad + \int_{s_i}^t T(t-s)(f(s, y(s)) + G(s))\, ds, && t \in [s_i, t_{i+1}],\ i = 1, \ldots, m.
\end{aligned}$$

For $t \in [s_i, t_{i+1}]$, $i = 0, \ldots, m$, we get

$$\begin{aligned}
\bigg\| y(t) - T(t-s_i) g_i(s_i, y(t_i^-)) - \int_{s_i}^t T(t-s) f(s, y(s))\, ds \bigg\| \\
\le M e^{\omega(t-s_i)} \|g(t_i)\| + \int_{s_i}^t e^{\omega(t-s)} \|G(s)\|\, ds \\
\le \varepsilon M e^{\omega(t-s_i)} \psi + \varepsilon M \int_{s_i}^t e^{\omega(t-s)} \varphi(s)\, ds.
\end{aligned}$$

Proceeding as above, we derive that

$$\|y(t) - g_i(t, y(t_i^-))\| \le \|g(t)\| \le \varepsilon\psi, \quad t \in (t_j, s_j],\ j = 1, \ldots, m;$$

$$\begin{aligned}
\bigg\| y(t) - T(t)y(0) - \int_0^t T(t-s) f(s, y(s))\, ds \bigg\| \le M \int_0^t e^{\omega(t-s)} \|G(s)\|\, ds \\
\le \varepsilon M \int_0^t e^{\omega(t-s)} \varphi(s)\, ds, \quad t \in [0, t_1].
\end{aligned}$$

Similarly, one can give similar remarks for the solutions of the inequalities (3.2) and (3.1).

To discuss stability, we need the following additional assumption:

(H_4) Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $c_\varphi > 0$ such that

$$\int_0^t \varphi(s)\, ds \le c_\varphi \varphi(t), \quad \text{for each } t \in J.$$

We need an impulsive Gronwall inequality which was given by Bainov and Simeonov (see Theorem 16.4, [9]).

LEMMA 3.8. *Let* $\mathbb{M}_0 := \mathbb{M} \cup \{0\}$ *and the following inequality holds*

$$u(t) \le a(t) + \int_0^t b(s)u(s)\, ds + \sum_{0 < t_k < t} \beta_k u(t_k^-), \quad t \ge 0, \tag{3.10}$$

where u, $a, b \in PC(\mathbb{R}_+, \mathbb{R}_+)$, a *is nondecreasing and* $b(t) > 0$, $\beta_k > 0$, $k \in \mathbb{M}$. *Then, for* $t \in \mathbb{R}_+$, *the following inequality is valid*:

$$u(t) \le a(t)(1+\beta)^k \exp\bigg(\int_0^t b(s)\, ds\bigg), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0, \tag{3.11}$$

where $\beta = \sup_{k\in\mathbb{M}}\{\beta_k\}$.

Now we are ready to state our main results in this section.

Theorem 3.9. *Assume that* (H_1)–(H_4) *are satisfied. Then the equation* (1.6) *is Ulam–Hyers–Rassias stable with respect to* (φ,ψ).

Proof. Let $y \in PC(J,D(A)) \cap C^1((s_i,t_{i+1}],X)$ be a solution of the inequality (3.3). Denote by x the unique mild solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t)), & t \in (s_i,t_{i+1}),\ i = 0,\dots,m, \\ x(t_i^+) = g_i(t_i, x(t_i^-)), & i = 1,\dots,m, \\ x(t) = g_i(t, x(t_i^-)), & t \in (t_i,s_i],\ i = 1,\dots,m, \\ x(0) = y(0). \end{cases} \tag{3.12}$$

Then we get

$$x(t) = \begin{cases} g_i(t,x(t_j^-)), & t \in (t_j,s_j],\ j = 1,\dots,m, \\ T(t)y(0) + \displaystyle\int_0^t T(t-s)f(s,x(s))\,ds, & t \in [0,t_1], \\ T(t-s_i)g_i(s_i,x(t_i^-)) + \displaystyle\int_{s_i}^t T(t-s)f(s,x(s))\,ds, & \\ & t \in (s_i,t_{i+1}],\ i = 1,\dots,m. \end{cases}$$

Keep in mind of (3.8), for each $t \in (s_i,t_{i+1}]$, $i = 1,\dots,m$, we have

$$\begin{aligned} &\bigg\| y(t) - T(t-s_i)g_i(s_i,y(t_i^-)) - \int_{s_i}^t T(t-s)f(s,y(s))\,ds \bigg\| \\ &\qquad \le \varepsilon M e^{|\omega|T}\bigg(\psi + \int_0^t \varphi(s)\,ds\bigg) \le \varepsilon M e^{|\omega|T}(\psi + c_\varphi\varphi(t)), \end{aligned}$$

and for $t \in (t_j,s_j]$, $j = 1,\dots,m$, we have

$$\|y(t) - g_i(t,y(t_i))\| \le \varepsilon\psi,$$

and for $t \in [0,t_1]$, we have

$$\bigg\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s,y(s))\,ds \bigg\| \le \varepsilon M e^{|\omega|T} c_\varphi \varphi(t).$$

Hence, for each $t \in (s_i,t_{i+1}]$, $i = 1,\dots,m$, we get

$$\begin{aligned} \|y(t) - x(t)\| \le{}& \bigg\| y(t) - T(t-s_i)g_i(s_i,y(t_i^-)) - \int_{s_i}^t T(t-s)f(s,y(s))\,ds \bigg\| \\ &+ M e^{|\omega|T}\|g_i(s_i,y(t_i^-)) - g_i(s_i,x(t_i^-))\| \\ &+ M e^{|\omega|T}\int_{s_i}^t \|f(s,y(s)) - f(s,x(s))\|\,ds \end{aligned}$$

$$\leq Me^{|\omega|T}\bigg(\varepsilon(1+c_\varphi)[\psi+\varphi(t)]+L_{g_i}\|y(t_i^-)-x(t_i^-)\| + L_f\int_{s_i}^t\|y(s)-x(s)\|\,ds\bigg)$$

$$\leq Me^{|\omega|T}\bigg(\varepsilon(1+c_\varphi)[\psi+\varphi(t)]+L_f\int_0^t\|y(s)-x(s)\|\,ds + \sum_{j=1}^{i}L_{g_j}\|y(t_i^-)-x(t_i^-)\|\bigg).$$

Further, for $t\in(t_j,s_j],\ j=1,\ldots,m$, we have

$$\begin{aligned}\|y(t)-x(t)\| &\leq \|y(t)-g_i(t,y(t_i^-))\|+\|g_i(t,y(t_i^-))-g_i(t,x(t_i^-))\|\\ &\leq \varepsilon\psi+\sum_{j=1}^{i}L_{g_j}\|y(t_j^-)-x(t_j^-)\|\\ &\leq Me^{|\omega|T}\bigg(\varepsilon(1+c_\varphi)[\psi+\varphi(t)]\\ &\quad + L_f\int_0^t\|y(s)-x(s)\|\,ds+\sum_{j=1}^{i}L_{g_j}\|y(t_i^-)-x(t_i^-)\|\bigg).\end{aligned}$$

Next, for $t\in[0,t_1]$, we have

$$\begin{aligned}\|y(t)-x(t)\| &\leq Me^{|\omega|T}\bigg(\varepsilon c_\varphi\varphi(t)+L_f\int_0^t\|y(s)-x(s)\|\,ds\bigg)\\ &\leq Me^{|\omega|T}\bigg(\varepsilon(1+c_\varphi)[\psi+\varphi(t)]+L_f\int_0^t\|y(s)-x(s)\|\,ds\bigg).\end{aligned}$$

Consequently, for $t\in(t_i,t_{i+1}]$, using Lemma 3.8, we derive that

$$\begin{aligned}\|y(t)-x(t)\| &\leq Me^{|\omega|T}(1+c_\varphi)(1+Me^{|\omega|T}L_g)^i e^{Me^{|\omega|T}L_f t}\varepsilon(\psi+\varphi(t))\\ &\leq Me^{|\omega|T}(1+c_\varphi)(1+Me^{|\omega|T}L_g)^m e^{Me^{|\omega|T}L_f T}\varepsilon(\psi+\varphi(t))\\ &:= c_{f,m,g,\varphi}\varepsilon(\psi+\varphi(t)),\end{aligned}$$

for $L_g:=\sup_{i\in\mathbb{M}}L_{g_i}$, which implies that the equation (1.6) is Ulam–Hyers–Rassias stable with respect to (φ,ψ). The proof is completed. □

Just repeating the procedure in Theorem 3.9, we establish without proof the following results.

Remark 3.10. Under the assumptions of Theorem 3.9, we consider the equation (1.6) and the inequality (3.2). One can repeat the same process to verify that the equation (1.6) is generalized Ulam–Hyers–Rassias stable with respect to (φ,ψ).

Remark 3.11. Under the assumptions of Theorem 3.9, we consider the equation (1.6) and the inequality (3.1). One can repeat the same process to verify that the equation (1.6) is Ulam-Hyers stable.

To end this section, we give an example to illustrate our abstract results above.

Example 3.12. We consider one-dimensional diffusion processes with not instantaneous changes of states. The example can explain either evolution of the temperature of the rod or the chemical concentration of the substance.

Consider the following impulsive partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,y) = \dfrac{\partial^2}{\partial y^2}x(t,y), & y \in (0,1),\ t \in [0,1) \cup (2,3], \\ \dfrac{\partial}{\partial y}x(t,0) = \dfrac{\partial}{\partial y}x(t,1) = 0, & t \in [0,1) \cup (2,3], \\ x(t,y) = \lambda x(1^-,y), & \lambda \in (-1,1),\ t \in (1,2],\ y \in (0,1). \end{cases} \tag{3.13}$$

Hence $J = [0,3]$, $0 = t_0 = s_0$, $t_1 = 1$, $s_1 = 2$ and $T = 3$. Let $X = L^2(0,1)$. Define

$$Ax = \frac{\partial^2}{\partial y^2}x \quad \text{for } x \in D(A)$$

with

$$D(A) = \left\{ x \in X : \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X,\ x(0) = x(1) = 0 \right\}.$$

Then, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X. Moreover, $\|T(t)\| \leq 1$ for all $t \geq 0$. So $M = 1$ and $\omega = 0$.

Denote $x(t)(y) = x(t,y)$, $f(t,x)(y) = 0$ and $g_1(t,x)(y) = \lambda x(y)$, then the problem (3.13) can be abstracted into

$$\begin{cases} x'(t) = Ax(t), & t \in (0,1) \cup (2,3), \\ x(t) = \lambda x(1^-), & t \in (1,2]. \end{cases} \tag{3.14}$$

Clearly, (H_0)–(H_3) are satisfied. Then the equation (3.14) with a given initial value has a unique solution $x \in PC(J,X)$. Moreover, we put $\varphi(t) = e^t$ and $\psi = 1$. Then (H_4) holds if $c_\varphi = 1$. Thus, by Theorem 3.9, the equation (3.13) is Ulam–Hyers–Rassias stable with respect to $(e^t, 1)$ on $[0,3]$ and $c_{f,\psi,g,\varphi} = 2(1 + |\lambda|)$.

4. Extension stability results on the unbounded interval

In this section we consider the case $J = \mathbb{R}_+$. Then one can replace $i = 1, \ldots, m$ by $i \in \mathbb{N}$ and $i = 0, \ldots, m$ by $i \in \{0\} \cup \mathbb{N}$ in the inequalities (3.1)–(3.3) and (3.4)–(3.7), respectively. Then, one can rewrite four parallel stability definitions like Definitions 3.1–3.4 when we take

$$Z := PC(J,X) \bigcap_{i\in\mathbb{M}_0} C^1((s_i,t_{i+1}),X) \bigcap_{i\in\mathbb{M}_0} C((s_i,t_{i+1}),D(A))$$

We state the following assumptions:

(H_0') C_0-semigroup $\{T(t), t\ge 0\}$ is exponentially stable, that is, $\omega<0$ in (H_0).

(H_1') $f\in C([0,\infty)\times X, X)$.

(H_2') There exists positive constant $L_f\in C(\mathbb{R}_+,(0,\infty))$ such that

$$\|f(t,u_1)-f(t,u_2)\|\le L_f\|u_1-u_2\|$$

for each $t\in[0,\infty)$ and all $u_1,u_2\in X$.

(H_3') $g_i\in C([t_i,s_i]\times X,X)$ and there are positive constants L_{g_i}, $i\in\mathbb{N}$ such that

$$\|g_i(t,u_1)-g_i(t,u_2)\|\le L_{g_i}\|u_1-u_2\|$$

for each $t\in[t_i,s_i]$ and all $u_1,u_2\in X$.

(H_4') There exists $c_\varphi\ge 1$ such that

$$\int_{s_i}^t e^{\omega(s_i-s)}\varphi(s)\,ds\le c_\varphi\varphi(t)$$

for any $t\in[s_i,t_{i+1}]$ and $i\in\mathbb{M}_0$.

First we have the following extension of Theorem 2.2.

Theorem 4.1. *Assume that* (H_0), (H_1')–(H_3') *are satisfied, then the equation* (1.7) *has a unique mild solution* $x\in PC([0,\infty),X)$ *with* $x(0)=x_0\in X$.

Proof. Take $T\in\mathbb{R}_+' := \mathbb{R}_+\setminus\mathbb{M}$. Then by Theorem 2.2 the equation (1.7) has a unique mild solution $x\in PC([0,T]),X)$ with $x(0)=x_0\in X$. Taking $T\to\infty$, the proof is finished. □

Now we pass to the stability problem.

Theorem 4.2. *Assume that* (H_0')–(H_4') *are satisfied. Then the equation* (1.7) *is Ulam–Hyers–Rassias stable with respect to* (φ,ψ) *provided that*

$$L_g := \sup_{i\in\mathbb{M}} L_{g_i} < \infty, \qquad \alpha := \inf_{i\in\mathbb{M}}(t_{i+1}-s_i)>0,$$
$$\omega+ML_f<0, \qquad \beta := ML_g e^{(\omega+ML_f)\alpha}<1.$$

Proof. Let $y\in Z$ be a solution of the inequality (3.3). Denote by x the unique mild solution of

$$\begin{cases} x'(t)=Ax(t)+f(t,x(t)), & t\in(s_i,t_{i+1}),\ i\in\mathbb{M}_0,\\ x(t_i^+)=g_i(t_i,x(t_i^-)), & i\in\mathbb{M},\\ x(t)=g_i(t,x(t_i^-)), & t\in(t_i,s_i],\ i\in\mathbb{M},\\ x(0)=y(0). \end{cases} \tag{4.1}$$

Then we get

$$x(t)=\begin{cases} g_i(t,x(t_i^-)), & t\in(t_i,s_i],\ i\in\mathbb{M},\\ T(t)x_0+\displaystyle\int_0^t T(t-s)f(s,x(s))ds, & t\in[0,t_1],\\ T(t-s_i)g_i(s_i,x(t_i^-))+\displaystyle\int_{s_i}^t T(t-s)f(s,x(s))ds, & t\in(s_i,t_{i+1}],\ i\in\mathbb{M}.\end{cases}$$

Here we note that we mean $[a,\infty]=[a,\infty)$ to avoid confusion. By (3.8), for each $t\in(s_i,t_{i+1}]$, $i\in\mathbb{M}$, we have

$$\begin{aligned}\bigg\|y(t)-T(t-s_i)g_i(s_i,y(t_i^-))-\int_{s_i}^t T(t-s)f(s,y(s))\,ds\bigg\|\\ \le \varepsilon Me^{\omega(t-s_i)}\psi+\varepsilon M\int_{s_i}^t e^{\omega(t-s)}\varphi(s)\,ds\le \varepsilon Me^{\omega(t-s_i)}c_\varphi(\psi+\varphi(t)),\end{aligned}$$

and for $t\in(t_i,s_i]$, $i\in\mathbb{M}$, we have $\|y(t)-g_i(t,y(t_i^-))\|\le\varepsilon\psi$, and for $t\in[0,t_1]$, we have

$$\begin{aligned}\bigg\|y(t)-T(t)y(0)-\int_0^t T(t-s)f(s,y(s))ds\bigg\|&\le\varepsilon M\int_0^t e^{\omega(t-s)}\varphi(s)\,ds\\ &\le\varepsilon Mc_\varphi e^{\omega t}\varphi(t).\end{aligned}$$

Hence, for each $t\in(s_i,t_{i+1}]$, $i\in\mathbb{M}$, we get

$$\begin{aligned}\|y(t)-x(t)\|\le&\bigg\|y(t)-T(t-s_i)g_i(s_i,y(t_i^-))-\int_{s_i}^t T(t-s)f(s,y(s))\,ds\bigg\|\\ &+Me^{\omega(t-s_i)}\|g_i(s_i,y(t_i^-))-g_i(s_i,x(t_i^-))\|\\ &+M\int_{s_i}^t e^{\omega(t-s)}\|f(s,y(s))-f(s,x(s))\|\,ds\\ \le&\,\varepsilon Mc_\varphi e^{\omega(t-s_i)}(\psi+\varphi(t))+Me^{\omega(t-s_i)}L_g\|y(t_i^-)-x(t_i^-)\|\\ &+ML_f\int_{s_i}^t e^{\omega(t-s)}\|y(s)-x(s)\|\,ds,\end{aligned}$$

which implies for $\overline{y}(t):=e^{-\omega t}y(t)$ and $\overline{x}(t):=e^{-\omega t}x(t)$ that

$$\begin{aligned}\|\overline{y}(t)-\overline{x}(t)\|\le&\,\varepsilon Mc_\varphi e^{-\omega s_i}(\psi+\varphi(t))\\ &+Me^{\omega(t_i-s_i)}L_g\|\overline{y}(t_i^-)-\overline{x}(t_i^-)\|+ML_f\int_{s_i}^t\|\overline{y}(s)-\overline{x}(s)\|\,ds\end{aligned}$$

for $t\in[s_i,t_{i+1}]$, $i\in\mathbb{M}$. By the Gronwall inequality (note $\varphi(t)$ is nondecreasing), we derive

$$\|\overline{y}(t)-\overline{x}(t)\|\le(\varepsilon Mc_\varphi e^{-\omega s_i}(\psi+\varphi(t))+Me^{\omega(t_i-s_i)}L_g\|\overline{y}(t_i^-)-\overline{x}(t_i^-)\|)e^{ML_f(t-s_i)},$$

which gives back

$$\|y(t)-x(t)\|\le Me^{(\omega+ML_f)(t-s_i)}(\varepsilon c_\varphi(\psi+\varphi(t))+L_g\|y(t_i^-)-x(t_i^-)\|)\tag{4.2}$$

for $t \in [s_i, t_{i+1}]$, $i \in \mathbb{M}$, and in particular

$$\begin{aligned}(4.3)\quad \|y(t_{i+1}^-) - x(t_{i+1}^-)\| &\le Me^{(\omega+ML_f)(t_{i+1}-s_i)}(\varepsilon c_\varphi(\psi + \varphi(t_{i+1}^-)) + L_g\|y(t_i^-) - x(t_i^-)\|) \\ &\le Mc_\varphi\varepsilon(\psi + \varphi(t_{i+1}^-)) + \beta\|y(t_i^-) - x(t_i^-)\|\end{aligned}$$

for $i \in \mathbb{M}$ when $t_{i+1} < \infty$. Further, for $t \in (t_i, s_i]$, $i \in \mathbb{M}$, we have

$$\begin{aligned}(4.4)\quad \|y(t) - x(t)\| &\le \|y(t) - g_i(t, y(t_i^-))\| + \|g_i(t, y(t_i^-)) - g_i(t, x(t_i^-))\| \\ &\le \varepsilon\psi + L_g\|y(t_i^-) - x(t_i^-)\|.\end{aligned}$$

Moreover, for $t \in [0, t_1]$, we have

$$\begin{aligned}\|y(t) - x(t)\| &\le \varepsilon M\int_0^t e^{\omega(t-s)}\varphi(s)\,ds + M\int_0^t e^{\omega(t-s)}L_f\|y(s) - x(s)\|\,ds \\ &\le \varepsilon Mc_\varphi e^{\omega t}\varphi(t) + ML_f\int_0^t e^{\omega(t-s)}\|y(s) - x(s)\|\,ds\end{aligned}$$

which yields like above that

$$(4.5)\quad \|y(t) - x(t)\| \le \varepsilon Mc_\varphi e^{(\omega+ML_f)t}\varphi(t) \le \varepsilon Mc_\varphi\varphi(t)$$

for $t \in [0, t_1]$, and in particular

$$(4.6)\quad \|y(t_1^-) - x(t_1^-)\| \le \varepsilon Mc_\varphi\varphi(t_1^-).$$

Solving (4.3) and using (4.6) we derive

$$\begin{aligned}(4.7)\quad \|y(t_i^-) - x(t_i^-)\| &\le M\varepsilon c_\varphi\sum_{j=2}^{i}(\psi + \varphi(t_j^-))\beta^{i-j} + \beta^{i-1}\|y(t_1^-) - x(t_1^-)\| \\ &\le \frac{M\varepsilon c_\varphi(\psi + \varphi(t_i^-))}{1-\beta} + \varepsilon Mc_\varphi\varphi(t_i^-),\end{aligned}$$

since $\varphi(\cdot)$ is nondecreasing.

Now let $t \ge 0$. Then either $t \in [0, t_1]$ and (4.5) gives

$$(4.8)\quad \|y(t) - x(t)\| \le \varepsilon Mc_\varphi\varphi(t),$$

or $t \in (t_i, t_{i+1}]$ for some $i \in \mathbb{M}_0$. Then either $t \in (t_i, s_i]$ and then (4.4) and (4.7) give

$$\begin{aligned}(4.9)\quad \|y(t) - x(t)\| &\le \varepsilon\psi + L_g\left(\frac{M\varepsilon c_\varphi(\psi + \varphi(t_i^-))}{1-\beta} + \varepsilon Mc_\varphi\varphi(t_i^-)\right) \\ &\le \varepsilon\left(\left(1 + \frac{ML_gc_\varphi}{1-\beta}\right)\psi + \left(\frac{ML_gc_\varphi}{1-\beta} + Mc_\varphi\right)\varphi(t)\right),\end{aligned}$$

or $t \in (s_i, t_{i+1}]$ and then (4.2) and (4.7) give

$$\begin{aligned}\|y(t)-x(t)\| &\le Me^{(\omega+ML_f)(t-s_i)}\bigg(\varepsilon c_\varphi(\psi+\varphi(t)) \\ &\quad + L_g\bigg(\frac{M\varepsilon c_\varphi(\psi+\varphi(t_i^-))}{1-\beta}+\varepsilon M c_\varphi \varphi(t_i^-)\bigg)\bigg) \\ &\le M\varepsilon c_\varphi\bigg(\bigg(1+\frac{L_g}{1-\beta}\bigg)\psi+\bigg(2+\frac{L_g}{1-\beta}\bigg)\varphi(t)\bigg).\end{aligned} \tag{4.10}$$

Using (4.8), (4.9) and (4.10) we have

$$\|y(t)-x(t)\| \le c_{f,g,\varphi}\varepsilon(\psi+\varphi(t))$$

for any $t \ge 0$, where $c_{f,g,\varphi} := Mc_\varphi(2+L_g/(1-\beta))$. Summarizing, we see that the equation (1.7) is Ulam–Hyers–Rassias stable with respect to (φ,ψ). □

Remark 4.3. If $\gamma := \sup_{i\in\mathbb{M}_0}(t_{i+1}-s_i) < \infty$, then assumption ($\mathrm{H}_4'$) holds for any $\varphi(t) = ce^{\omega' t}$, $c>0$ and $\omega' > \omega$. Indeed, we compute

$$\int_{s_i}^t e^{\omega(s_i-s)}\varphi(s)\,ds = c\int_{s_i}^t e^{\omega(s_i-s)+\omega' s}\,ds \le c\frac{e^{\omega(s_i-t)+\omega' t}}{\omega'-\omega} \le \frac{e^{-\omega\gamma}}{\omega'-\omega}\varphi(t)$$

for any $t \in [s_i, t_{i+1}]$ and $i \in \mathbb{M}_0$, so $c_\varphi = e^{-\omega\gamma}/(\omega'-\omega)$. In particular, the constant function $\varphi(t) = \varphi$ can be also used with $\omega' = 0$.

Finally, we give an example to illustrate our above results.

Example 4.4. Consider

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,y) = (\Delta_y - 3I)x(t,y), & y\in\Omega, \\ & t\in(2i+1,2(i+1)],\ i\in\{0\}\cup\mathbb{N}, \\ \dfrac{\partial}{\partial y}x(t,y) = 0, & y\in\partial\Omega,\ t\ge 0, \\ x(t,y) = \sin i\cdot x(2i^-,y), & y\in\Omega,\ (2i,2i+1],\ i\in\mathbb{N}, \end{cases} \tag{4.11}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, Δ_y is the Laplace operator in $\mathbb{R}^2$, and $\partial\Omega \in C^2$. Note now $s_i = 2i+1$ and $t_i = 2i$, $i \in \{0\}\cup\mathbb{N}$. Here we consider infinitely many impulses on infinite time interval $\mathbb{R}_+$.

Let $X = L_2(\Omega)$, $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$. Define $Ax = (\Delta_y - 3I)x$, $x \in D(A)$. By Theorem 2.5 of [28], A is just the infinitesimal generator of a contraction C_0-semigroup in $L_2(\Omega)$, that is, $\|T(t)\| \le e^{-3t}$ for $t \ge 0$, so $m = 1$ and $\omega = -3 < 0$.

Denote $x(\,\cdot\,)(y) = x(\,\cdot\,,y)$, $f(\,\cdot\,,x)(y) = 0$ and $g_i(\,\cdot\,,x)(y) = (\sin i)\cdot x(y)$, then the problem (4.11) can be abstracted into

$$\begin{cases} x'(t) = Ax(t),\ t\in(2i+1,2(i+1)), & i\in\{0\}\cup\mathbb{N}, \\ x(2i^+) = x(2i^-), & i\in\mathbb{N}, \\ x(t) = (\sin i)x(2i^-), & t\in(2i,2i+1],\ i\in\mathbb{N}. \end{cases} \tag{4.12}$$

Clearly, (H_0')–(H_3') are satisfied, and $L_g = 1$, $\alpha = \gamma = 1$, $L_f = 0$, $\omega + ML_f = -3$, $\omega = e^{-3} < 1$ and by Remark 4.3, we can take $\varphi(t) = e^{\omega' t}$, $\omega' > -3$ and $\psi = 1$. Then (H_4') holds for $c_\varphi = e^3/(\omega' + 3)$. Thus, applying Theorem 4.2, the equation (4.11) is Ulam–Hyers–Rassias stable with respect to $(e^{\omega' t}, 1)$ on $\mathbb{R}_+$ with $\omega' > -3$ and

$$c_{f,\mathbb{N},g,\varphi} = \frac{e^3}{\omega' + 3}\left(2 + \frac{1}{1 - e^{-3}}\right).$$

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