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POSITIVE SOLUTIONS TO p-LAPLACE REACTION-DIFFUSION SYSTEMS WITH NONPOSITIVE RIGHT-HAND SIDE

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Abstract. The aim of the paper is to show the existence of positive solutions to the elliptic system of partial differential equations involving the p-Laplace operator

$$\begin{cases} -\Delta_p u_i(x) = f_i(u_1(x), u_2(x), \ldots, u_m(x)), & x \in \Omega,\ 1 \le i \le m, \\ u_i(x) \ge 0, & x \in \Omega,\ 1 \le i \le m, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We consider the case of nonpositive right-hand side f_i, $i = 1, \ldots, m$. The sufficient conditions entails spectral bounds of the matrices associated with $f = (f_1, \ldots, f_m)$. We employ the degree theory from [5] for tangent perturbations of maximal monotone operators in Banach spaces.

1. Introduction

In the recent paper [5] the following nonlinear boundary value problem was discussed:

$$\begin{cases} -\Delta_p u(x) = f(x, u(x)), & x \in \Omega, \\ u(x) \ge 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$, for $p \geq 2$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is called the p-Laplace operator or p-Laplacian and $f\colon \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function, which is *not* necessarily positive. The authors, exploiting the idea originated in [4], introduced the topological degree Deg_M for mappings that are tangent to the set of constraints M and they applied it to obtain sufficient conditions under which the problem (1.1) possesses at least one weak solution:

Theorem 1.1. *Suppose that the following conditions are satisfied*:

(a) *there is* $C > 0$ *such that* $|f(x,s)| \leq C(1+s^{p-1})$ *for all* $s \geq 0$ *and alomost all* $x \in \Omega$,

(b) $\displaystyle\lim_{s\to 0+} \frac{f(x,s)}{s^{p-1}} = \rho_0(x)$ *and* $\displaystyle\lim_{s\to\infty} \frac{f(x,s)}{s^{p-1}} = \rho_\infty(x)$ *uniformly with respect to* $x \in \Omega$,

where $\rho_0, \rho_\infty \in L^\infty(\Omega)$. *If the principal eigenvalue* $\lambda_{1,p}$ *of* p*-Laplacian lies between* ρ_0 *and* ρ_∞, *i.e. either* $\rho_0 < \lambda_{1,p} < \rho_\infty$ *or* $\rho_\infty < \lambda_{1,p} < \rho_0$ *almost everywhere, then the problem* (1.1) *admits at least one nontrivial weak solution* $u \in W_0^{1,p}(\Omega)$.

The question we are concerned with is whether or not the results obtained in [5] can be generalised to the case of the system of equations of the type (1.1). In this paper we are focused on an autonomous case. Let us consider the system

$$
\begin{cases}
-\Delta_p u(x) = f(u(x)), & x \in \Omega,\\
u_i(x) \geq 0, & x \in \Omega,\ 1 \leq i \leq m,\\
u(x) = 0, & x \in \partial\Omega,
\end{cases}
\tag{1.2}
$$

where $u = (u_1,\ldots,u_m)$, $\Delta_p u = (\Delta_p u_1,\ldots,\Delta_p u_m)$ and $f\colon [0,+\infty)^m \to \mathbb{R}^m$ is a continuous function. The problem was investigated in [14] for $p = 2$ and a multivalued right-hand side. The existence of a positive solutions to systems involving p-Laplacian has been investigated by means of a topological approach, for example [2], [12], [13], [15], [19], as well as of a variational approach, for example [1], [18]. In all these papers, the right-hand side of the quasilinear elliptic system is assumed to be nonnegative. It seems this assumption is present in most of the articles related to the subject. One of the goals of our paper is to drop this assumption in favour of a weaker one.

Let us denote by θ the real function $\theta\colon \mathbb{R} \ni s \mapsto |s|^{p-2}s$ and let $\Theta = \theta \times \ldots \times \theta$ be the Cartesian product of m copies of θ.

The following assumption reflects the assumption (b) of Theorem 1.1 from the one-dimensional case:

$$
f(u) = D_0\Theta(u) + o_0(u), \quad f(u) = D_\infty\Theta(u) + o_\infty(u), \quad u \in \mathbb{R}^m_+
\tag{1.3}
$$

where $\mathbb{R}^m_+ = [0,\infty)^m$, $D_0, D_\infty \colon \mathbb{R}^m \to \mathbb{R}^m$ are linear mappings and

$$\lim_{u\to 0} \frac{o_0(u)}{|u|^{p-1}} = 0, \qquad \lim_{|u|\to\infty} \frac{o_\infty(u)}{|u|^{p-1}} = 0. \tag{1.4}$$

We shall call the functions D_0 and D_∞ *linearisations* of f at 0 and at infinity, respectively. Note that D_0 and D_∞ can be identified with matrices.

Moreover, we shall use the following assumption: f is tangent to $\mathbb{R}^m_+$, i.e.

$$f(s) \in T_{\mathbb{R}^m_+}(s) \quad \text{for } s \in \mathbb{R}^m_+. \tag{1.5}$$

Here, we applied Clarke's tangent cone (see [3]), which for a closed convex subset M of a Banach space E is equal to

$$T_M(x) = \overline{\bigcup_{h>0} h\cdot(M-x)}, \quad x \in M. \tag{1.6}$$

Because

$$T_{\mathbb{R}^m_+}(u) = \{v \in \mathbb{R}^m \mid v_i \geq 0 \text{ if } u_i = 0,\ i = 1,\dots,m\}, \quad u \in \mathbb{R}^m_+, \tag{1.7}$$

we can restate the assumption (1.5) in the following manner:

$$f_i(s) \geq 0 \quad \text{if } s_i = 0 \text{ for all } s \in \mathbb{R}^m_+, \tag{1.8}$$

where $f_i(s)$ and s_i are the i-th components of the vectors $f(s)$ and s, respectively. In the case of $m = 1$ the condition (1.8) is equivalent to the inequality $f(0) \geq 0$ and therefore it is implied by (1.3).

The reaction-diffusion system (1.2) describes the equilibrium state of a distribution of substances (functions u_i) under the influence of chemical reactions (which are described by the right-hand side) and the process of diffusion (which is described by the left-hand side). In this physical interpretation, the most commonly assumed condition $f_i \geq 0$ means that each substance $u_1,\dots,u_m$ is produced during the reaction. This is not a realistic assumption, because some chemicals (substrates) vanish in favour of the other (products). Our assumption (1.8) is natural, physically justified and can be interpreted as follows: the amount of each chemical cannot decrease if it equals zero.

The paper is organized as follows. Section 2 is devoted to recalling the degree theory defined in [5] in the form adequate to deal with the p-Laplace operator. Section 3 is intended to transform the initial problem (1.2) into an abstract one, to apply the topological degree Deg_M introduced in [5] and thereby to prove the main theorem of the paper (see Theorem 3.8). Section 4 describes the families $\mathcal{D}_0$ and $\mathcal{D}_1$ arising in the statement of Theorem 3.8.

Notation. If E is a normed space, then by $\|\cdot\|_E$ (or $\|\cdot\|$, for short) we denote its norm. If $B\subset E$, then ∂B and $\overline{B}$ stand for the boundary of B and the closure of B respectively. If $M=\overline{M}\subset E$ and $B\subset M$, then by $\partial_M B$ we denote the relative boundary (with respect to M) of B.

If $x_0\in E$ and $r>0$, then $B_M(x_0,r):=\{x\in M\mid \|x-x_0\|<r\}$.

If E^* is a dual space of E (space of all continuous linear functionals), then $\langle\cdot\,,\cdot\rangle=\langle\cdot\,,\cdot\rangle_E\colon E^*\times E\to\mathbb{R}$ denotes the duality operator $\langle p,u\rangle:=p(u)$, $p\in E^*$, $u\in E$.

We put $\langle\cdot\,,\cdot\rangle_p=\langle\cdot\,,\cdot\rangle_{L^p(\Omega)}$ and $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega)}$. Treating $L^p(\Omega)^*$ as $L^q(\Omega)$ for $p^{-1}+q^{-1}=1$, $p\in(1,\infty)$, we can write $\langle f,g\rangle_p=\int_\Omega f\cdot g$.

For $x\in\mathbb{R}^N$, $N\ge 1$, $|x|$ denotes the Euclidean norm of x and $x\cdot y$ is the Euclidean scalar product of $x,y\in\mathbb{R}^N$.

2. Degree theory for tangent mappings

In order to demonstrate the existence of nontrivial solutions to the problem (1.2) we shall exploit the coincidence degree function Deg_M constructed in [5]. For the convenience of the reader, we recall its domain and properties in the form that is useful for our considerations. To do this, let X and Y be reflexive Banach spaces with a dense and compact embedding $i\colon Y\to X$, i.e. the mapping i is linear and compact with its range $i(Y)$ dense in X. Suppose that a closed convex cone $M\subset X$ and functionals $\mathbf{a}\colon Y\to\mathbb{R}$ and $\mathbf{n}\colon X\to\mathbb{R}$ satisfy the following conditions:

(a1) $\mathbf{a}$ and $\mathbf{n}$ are coercive C^1 functionals, i.e. counterimages $\mathbf{a}^{-1}((-\infty,m))$, $\mathbf{n}^{-1}((-\infty,m))$ are bounded for all $m\in\mathbb{R}$.

(a2) there exists a continuous function $\kappa\colon[0,+\infty)\to[0,+\infty)$ such that $\kappa^{-1}(\{0\})=\{0\}$, $\lim\limits_{s\to+\infty}\kappa(s)=+\infty$ and

$$\langle D\mathbf{a}(u_1)-D\mathbf{a}(u_2),u_1-u_2\rangle_Y\ge\kappa(\|u_1-u_2\|_Y)\|u_1-u_2\|_Y \text{ for all } u_1,u_2\in Y,$$
$$\langle D\mathbf{n}(u_1)-D\mathbf{n}(u_2),u_1-u_2\rangle_X\ge\kappa(\|u_1-u_2\|_X)\|u_1-u_2\|_X \text{ for all } u_1,u_2\in X;$$

(a3) for any $u\in X$ there exist $u^+,u^-\in M$ such that $u=u^+-u^-$ and $\mathbf{n}(u^+)\le\mathbf{n}(u)$; if $u\in i(Y)$, then $u^+,u^-\in i(Y)$ and $\mathbf{a}(i^{-1}u^+)\le\mathbf{a}(i^{-1}u)$;

(a4) $\mathbf{n}$ is bounded on bounded sets and monotone with respect to M, i.e. $\mathbf{n}(u+v)\ge\mathbf{n}(u)$ for any $u,v\in M$.

Note that the coercivity from (a1) is a consequence of the assumption (a2).

Let $\mathcal{A}\colon Y\to Y^*$ and $N\colon X\to X^*$ be defined by $\mathcal{A}:=D\mathbf{a}$ and $N:=D\mathbf{n}$. Define $A\colon D(A)\to X^*$ by

$$D(A):=i(\mathcal{A}^{-1}(i^*(X^*)))\quad\text{and}\quad Au:=(i^*)^{-1}(\mathcal{A}i^{-1}u),\quad\text{for } u\in D(A). \tag{2.1}$$

The operation of obtaining A by restricting $\mathcal{A}$ to $D(A)$ is a generalisation of the analogical one that is usually considered in the case of a Gelfand triple $Y \subset X \subset Y^*$ where X is a Hilbert space.

Let us also consider a continuous mapping $F\colon M \to X^*$ *tangent* to M, i.e.

$$F(u) \in T_{M^*}(N(u)) \quad \text{for } u \in M, \tag{2.2}$$

where $M^* = N(M)$. To calculate the cone $T_{M^*}(N(u))$ tangent to M^* in $N(u)$ one can employ (1.6), since M^* is a closed convex cone (see [5]).

THEOREM 2.1. *Suppose that there exist a functional* $\mathbf{n}$ *and a cone* M *as above. Then there exists a coincidence degree* Deg_M*, which to any* A*,* F *as above and for any open bounded subset* $U \subset M$ *with* $Au \neq F(u)$ *for* $u \in \partial_M U \cap D(A)$*, it attains an integer number* $\mathrm{Deg}_M(A, F, U)$*, such that the following properties are satisfied*:

(a) (Existence) *if* $\mathrm{Deg}_M(A, F, U) \neq 0$*, then there exists* $u \in U \cap D(A)$ *such that* $Au = F(u)$;
(b) (Additivity) *if* U_1, U_2 *are open disjoint subsets of an open* $U \subset X$ *and* $Au \neq F(u)$ *for* $u \in \overline{U} \setminus (U_1 \cup U_2)$*, then*
$$\mathrm{Deg}_M(A, F, U) = \mathrm{Deg}_M(A, F, U_1) + \mathrm{Deg}_M(A, F, U_2);$$
(c) (Homotopy invariance) *if* $H\colon M \times [0,1] \to X^*$ *is a continuous and bounded mapping* (*maps bounded subsets of the domain into the bounded subsets of* X^*) *such that*
$$H(u,t) \in T_{M^*}(N(u)) \quad \textit{for all } u \in M,\ t \in [0,1],$$
and $Au \neq H(u,t)$ *for all* $u \in \partial_M U \cap D(A)$ *and* $t \in [0,1]$*, then*
$$\mathrm{Deg}_M(A, H(\cdot, 0), U) = \mathrm{Deg}_M(A, H(\,\cdot\,, 1), U).$$
(d) (Normalisation) $\mathrm{Deg}_M(A, 0, U) = 1$ *if* $0 \in U$.

We highlight that this is a special case of a more general theory from [5].

PROOF. Because (a)–(c) follow directly from [5, Theorem 2.5], it suffices to prove (d). By the definition of the degree Deg_M (see [5]), it follows that $\mathrm{Deg}_M(A, 0, U) = \mathrm{ind}_M(\Phi_\alpha, U)$, where ind_M stands for the fixed point index for compact mappings of absolute neighbourhood retracts due to Granas (see [10] or [11] for details) and

$$\Phi_\alpha(u) = (N + \alpha A)^{-1}(r(N(u) + \alpha \cdot 0)) = (N + \alpha A)^{-1}(N(u))$$

for some retraction $r\colon X^* \to M^*$ and a small number $\alpha > 0$.

Consider the homotopy $H(u,t) = t \cdot \Phi_\alpha(u)$, for $u \in \overline{U}$, $t \in [0,1]$. We shall prove that there are no solutions $u \neq 0$ to the equation $H(u,t) = u$ for $t \in [0,1]$. Suppose on the contrary that there is $u \neq 0$ and $0 \leq t \leq 1$ such that $H(u,t) = u$.

Then $t > 0$ and $N(u) = (N + \alpha A)(t^{-1}u)$. If $t = 1$, then $A(u) = 0$ and thereby $u = 0$ and we arrive at a contradiction. If $t < 1$ then

$$-\alpha(t^{-1} - 1)\langle Au, u\rangle = \langle (N + \alpha A)(t^{-1}u) - (N + \alpha A)(u), t^{-1}u - u\rangle. \tag{2.3}$$

By the assumption (a2) we obtain that for $u \neq 0$ the left-hand side of (2.3) is negative and the right-hand side is positive. The contradiction shows that the homotopy H is fixed-point free on the boundary $\partial_M U$ of U and therefore

$$\mathrm{Deg}_M(A, 0, U) = \mathrm{ind}_M(H(\,\cdot\,, 1), U) = \mathrm{ind}_M(H(\,\cdot\,, 0), U) = \mathrm{ind}_M(0, U) = 1. \qquad \square$$

The major advantage of using the degree Deg_M lies in the possibility of considering mappings F with values being *outside* the domain M. This enables us to relax the standard nonnegativity condition of f in (1.2) and instead of it, to consider the assumption (1.8).

3. Abstract setting for the system

As it was mentioned in the introduction, we seek *weak solutions* $u = (u_1, \ldots, u_m)$ of the system (1.2), i.e. mappings $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ such that $u_i(x) \geq 0$ for almost all $x \in \Omega$, $1 \leq i \leq m$ and

$$\int_\Omega |\nabla u_i|^{p-2} \nabla u_i(x) \cdot \nabla \eta(x)\, dx = \int_\Omega f_i(u(x))\eta(x)\, dx \tag{3.1}$$

for all $\eta \in W_0^{1,p}(\Omega)$, $1 \leq i \leq m$.

For one equation (i.e. when $m = 1$) and for $f(x) = \lambda\theta(x) := \lambda|x|^{p-2}x$ we arrive at the eigenvalue problem for p-Laplacian with the Dirichlet Boundary Condition:

$$-\Delta_p u = \lambda|u|^{p-2}u, \quad u \geq 0, \quad u|\partial\Omega \equiv 0. \tag{3.2}$$

It is well known that the problem (3.2) admits nonzero solutions only if λ is the principal eigenvalue $\lambda_{1,p}$ of $-\Delta_p$, which is given by the Rayleigh formula

$$\lambda_{1,p} := \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p\, dx}{\displaystyle\int_\Omega |u(x)|^p\, dx}. \tag{3.3}$$

Then u is strictly positive in Ω, it belongs to $L^\infty(\Omega)$ and is uniquely determined up to a positive multiplier (see [16], [17]).

Let $X_0 = L^p(\Omega, \mathbb{R})$, $X = L^p(\Omega, \mathbb{R}^m) = X_0^m$, $Y_0 = W_0^{1,p}(\Omega)$, $Y = Y_0^m$, where the spaces X and Y are equipped with the following norms:

$$\|u\|_X = \bigg(\sum_{i=1}^m \|u_i\|_{X_0}^p\bigg)^{1/p}, \quad u \in X, \qquad \|u\|_Y = \bigg(\sum_{i=1}^m \|u_i\|_{Y_0}^p\bigg)^{1/p}, \quad u \in Y.$$

By the Rellich–Kondrachov Compactness Theorem, the natural embedding $i\colon Y \to X$ is compact and dense. Consider the cone of nonnegative vector functions

$$M = \{u = (u_1, \dots, u_m) \in X \mid u_1, \dots, u_m \geq 0 \text{ a.e. on } \Omega\}.$$

Let $\mathbf{a}\colon Y \to \mathbb{R}$, $\mathbf{n}\colon X \to \mathbb{R}$ be defined by

$$\mathbf{a}(u) = \frac{1}{p}\sum_{i=1}^{m}\int_\Omega |\nabla u_i|^p, \qquad \mathbf{n}(u) = \frac{1}{p}\sum_{i=1}^{m}\int_\Omega |u_i|^p.$$

PROPOSITION 3.1. *The assumptions* (a1)–(a4) *from the previous section are satisfied. Moreover, if we put $N = D\mathbf{n}$ and $\mathcal{A} = D\mathbf{a}$, then*:

$$\begin{aligned}\langle \mathcal{A}u, v\rangle &= \langle D\mathbf{a}(u), v\rangle = \sum_{i=1}^{m}\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla v_i, \quad u, v \in Y,\\ \langle Nu, v\rangle &= \langle D\mathbf{n}(u), v\rangle = \sum_{i=1}^{m}\int_\Omega |u_i|^{p-2}u_i v_i, \qquad u, v \in X.\end{aligned} \tag{3.4}$$

PROOF. Note that the functionals $\mathbf{a}$ and $\mathbf{n}$ are Gateaux differentiable and the formulae (3.4) are satisfied. Since these Gateaux derivatives are continuous, $\mathbf{a}$ and $\mathbf{n}$ are Frêchet differentiable. The coercivity follows from the relations

$$\begin{aligned}\mathbf{a}(u) &= \frac{1}{p}\sum_{i=1}^{m}\|u_i\|_{Y_0}^p = \frac{1}{p}\|u\|_Y^p, \quad u \in Y,\\ \mathbf{n}(u) &= \frac{1}{p}\sum_{i=1}^{m}\|u_i\|_{X_0}^p = \frac{1}{p}\|u\|_X^p, \quad u \in X.\end{aligned}$$

The condition (a2) is a consequence of the estimates

$$\begin{aligned}\langle D\mathbf{a}(u) - D\mathbf{a}(v), u - v\rangle &\geq 2^{2-p}\|u - v\|_Y^p, \quad u, v \in Y,\\ \langle D\mathbf{n}(u) - D\mathbf{n}(v), u - v\rangle &\geq 2^{2-p}\|u - v\|_X^p, \quad u, v \in X,\end{aligned}$$

which are implied by the inequality

$$(|x|^{p-2}x - |y|^{p-2}y)\cdot(x - y) \geq 2^{2-p}|x - y|^p, \quad x, y \in \mathbb{R}^M,\ M \geq 1. \tag{3.5}$$

In order to prove (a3), for a given function $u = (u_i)_{i=1}^m \in X$ let us define $u^+, u^- \in X$ such that $(u^\pm)_i = (u_i)^\pm := \max\{\pm u_i, 0\}$. Then $u = u^+ - u^-$ and

$$\mathbf{n}(u^+) = \frac{1}{p}\sum_{i=1}^{m}\int_\Omega |{u_i}^+|^p \leq \frac{1}{p}\sum_{i=1}^{m}\int_\Omega |u_i|^p = \mathbf{n}(u).$$

If $u \in Y$, then $u_i \in W_0^{1,p}(\Omega)$ for $i \in \{1, \dots, m\}$ and, by [9, Lemma 7.6], ${u_i}^+ \in W_0^{1,p}(\Omega)$ with $\nabla {u_i}^+(x) = 0$ if $u_i(x) \leq 0$ and $\nabla {u_i}^+(x) = \nabla u_i(x)$ if $u_i > 0$. Therefore $\mathbf{a}(u^+) \leq \mathbf{a}(u)$.

Finally, (a4) follows directly from the estimate $|u_i|^p \leq |u_i + v_i|^p$ for $u, v \in M$. $\square$

By the symbol N_f we denote the Nemytskiĭ operator generated by the right-hand side f in the equation (1.2), i.e. $N_f(u)(x) = f(u(x))$. By the assumption (1.3) we have the growth condition:

$$\text{there is } c > 0 \text{ such that } |f(s)| \le c(1 + |s|^{p-1}) \text{ for all } s \in \mathbb{R}^m_+, \tag{3.6}$$

which implies that $N_f\colon M \to L^q(\Omega, \mathbb{R}^m)$ is continuous. Using the Riesz representation isomorphism between X^* and $L^q(\Omega, \mathbb{R}^m)$, $1/p + 1/q = 1$, we can identify N_f with the operator

$$F\colon X \to X^*, \quad \langle F(u), v\rangle = \sum_{i=1}^m \int_\Omega f_i(u(x))v_i(x)\,dx, \quad u, v \in X \tag{3.7}$$

and N with N_Θ.

Using the identifications $X^* \approx (X_0^*)^m$, $Y^* \approx (Y_0^*)^m$, the relations (3.4) and (3.7), we can restate the system (3.1) as follows: $\langle \mathcal{A}u, v\rangle = \langle F(u), v\rangle$, $u, v \in Y$. Hence, the functional $\mathcal{A}u \in Y^*$ is in fact an element of $X^* \subset Y^*$. From the definition (2.1) of $D(A)$ and A we see that $u \in D(A)$ and that (3.1) is equivalent to the equation $Au = F(u)$. Having in mind the constraints $u_i \ge 0$, we arrive at the abstract formulation of the initial system:

$$\begin{cases} Au = F(u), \\ u \in M. \end{cases} \tag{3.8}$$

In order to apply the degree from Section 2 to investigate the existence of the abstract operator equation (3.8), we need to ensure the condition (2.2), i.e. the tangency of the operator N_f. This is done by the following theorem:

Theorem 3.2. *Assume that $f\colon \mathbb{R}^m_+ \to \mathbb{R}^m$ is a continuous function with the growth condition* (3.6). *Then the following conditions are equivalent:*

(a) *the condition* (1.8)*;*
(b) *the tangency of f to $\mathbb{R}^m_+$, i.e. $f(s) \in T_{\mathbb{R}^m_+}(s)$ for $s \in \mathbb{R}^m_+$;*
(c) *the tangency of $F = N_f$ to M, i.e. the condition* (2.2).

In order to prove it, we need to establish the description of the tangent cone in L^q spaces.

Lemma 3.3. *Let $1 \le q < \infty$ and put*

$$M_q = \{u \in L^q(\Omega, \mathbb{R}^m) \mid u(x) \ge 0 \textit{ for a.a. } x \in \Omega\}. \tag{3.9}$$

Then, for $u \in M_q$,

$$T_{M_q}(u) = \{v \in L^q(\Omega, \mathbb{R}^m) \mid v(x) \in T_{\mathbb{R}^m_+}(u(x)) \textit{ for a.a. } x \in \Omega\}. \tag{3.10}$$

Proof. Fix $u \in M_q$. Temporarily, denote the right-hand side of (3.10) by T. Let $m \in M_q$ and $h > 0$. Then $h(m - u) \in T$ by (1.7) and therefore $h(M_q - u) \subset T$. This and closedness of T implies that $T_{M_q}(u) \subset T$.

Now, let $v \in T$. For $n \in \mathbb{N}$ let us put

$$m^n = (v/n + u)^+ = ((v_1/n + u_1)^+, \dots, (v_m/n + u_m)^+)$$

and $v^n = n(m^n - u)$. Clearly, $m^n \in M_q$ and $v^n \in T_{M_q}(u)$. Since $v_i(x) \geq 0$ if $u_i(x) = 0$, $m^n(x) = v(x)/n + u(x)$ for sufficiently large n and for almost all $x \in \Omega$. For such n and x, $v^n(x) = v(x)$. This shows that $v^n \to v$ almost everywhere. Since also $|v_i^n| \leq |v_i|$, Lebesgue's Dominated Convergence Theorem shows that $v^n \to v$ in $L^q(\Omega, \mathbb{R}^m)$. Therefore $v \in T_{M_q}(u)$. □

PROOF OF THEOREM 3.2. The equivalence of (a) and (b) follows from (1.7).

Observe that using the Riesz representation isomorphism ρ between $X^* = (L^p(\Omega, \mathbb{R}^m))^*$ and $L^q(\Omega, \mathbb{R}^m)$, $1/p + 1/q = 1$, we can treat $M^* = N(M)$ as the cone $M_q \subset L^q(\Omega, \mathbb{R}^m)$ defined by (3.9). Because of the linearity and continuity of ρ, the tangency condition $F(u) \in T_{M^*}(N(u))$ can be rewritten in the form

$$N_f(u) \in T_{M_q}(N_\Theta(u)).$$

By (3.10), this is equivalent to the statement

$$f(u(x)) \in T_{\mathbb{R}^m_+}(u^{p-1}(x)) = T_{\mathbb{R}^m_+}(u(x)) \quad \text{for a.a. } x \in \Omega. \tag{3.11}$$

Now it is clear that (c) is implied by (b). Conversely, consider any $s \in \mathbb{R}^m_+$. From (3.11) applied to $u(x) \equiv s$, we obtain (b). □

Thus, all the assumptions of the degree theory for tangent mappings are satisfied.

PROPOSITION 3.4. *The tangency* (1.8) *of f to $\mathbb{R}^m_+$ implies the tangency of the mappings $D_0 \circ \Theta$, $D_\infty \circ \Theta$ to $\mathbb{R}^m_+$ and therefore the tangency of the linearisations D_0, D_∞ to $\mathbb{R}^m_+$. This is equivalent to the quasinonnegativity of the matrices D_0, D_∞.*

Let us recall that a matrix $D = (d_{ij})_{i,j=1}^m$ is *quasinonnegative*, if $d_{ij} \geq 0$ for $i \neq j$.

PROOF. Fix $u \in \mathbb{R}^m_+$. By the tangency of f to $\mathbb{R}^m_+$ we have that

$$t^{p-1} D_0 \Theta(u) + o_0(tu) = D_0 \Theta(tu) + o_0(tu) = f(tu) \in T_{\mathbb{R}^m_+}(tu) = T_{\mathbb{R}^m_+}(u),$$

for $t > 0$. Therefore $D_0 \Theta(u) + t^{-(p-1)} o_0(tu) \in T_{\mathbb{R}^m_+}(u)$. Letting $t \to 0$ and using the closedness of $T_{\mathbb{R}^m_+}(u)$, we obtain $D_0 \Theta(u) \in T_{\mathbb{R}^m_+}(u)$, $u \in \mathbb{R}^m_+$. Now, putting $u := \Theta^{-1}(v)$, we obtain $D_0(v) \in T_{\mathbb{R}^m_+}(\Theta^{-1}(v)) = T_{\mathbb{R}^m_+}(v)$, $v \in \mathbb{R}^m_+$.

Similarly we can prove the tangency of $D_\infty \circ \Theta$ and D_∞.

Now we shall prove that the tangency of D to $\mathbb{R}^m_+$ is equivalent to the quasinonnegativity of D. Let us treat D as a matrix $(d_{ij})_{i,j=1}^m$. Assume that $(D(u))_i \geq 0$ if $u_i = 0$, for $u \in \mathbb{R}^m_+$ and $i = 1, \dots, m$ and use it for $u = e^j$, i.e. for the j-th vector from the canonical basis of $\mathbb{R}^m$. Since $(D(e^j))_i = d_{ij}$ and $e_i^j = 0$ if $i \neq j$, we obtain that $d_{ij} \geq 0$ if $i \neq j$. Thus, D is quasinonnegative.

Now, assume that D is quasinonnegative. Let $u \in \mathbb{R}_+^m$ and let $i \in \{1, \ldots, m\}$ be such that $u_i = 0$. Then

$$(Du)_i = \sum_{j=1}^{m} d_{ij} u_j = \sum_{j \neq i} d_{ij} u_j \geq 0.$$

This proves the tangency of D to $\mathbb{R}_+^m$. □

In what follows, we shall be using the following classes of matrices:

$$\begin{aligned}
\mathcal{S} &:= \{D \mid D \text{ is a quasinonnegative matrix}\},\\
\mathcal{D} &:= \{D \in \mathcal{S} \mid (A - N_{D\circ\Theta})^{-1}(\{0\}) \cap M = \{0\}\},\\
\mathcal{D}_0' &:= \{D \in \mathcal{S} \mid \exists\,(\tau_0 \in M^*)\ (A - N_{D\circ\Theta})^{-1}(\{\tau_0\}) \cap M = \emptyset\},\\
\mathcal{D}_0 &:= \mathcal{D} \cap \mathcal{D}_0',\\
\mathcal{D}_1 &:= \{\mathcal{D} \in \mathcal{S} \mid \forall (t \in [0,1])\ tD \in \mathcal{D}\}.
\end{aligned}$$

The importance of the families $\mathcal{D}_0$ and $\mathcal{D}_1$ is presented in Proposition 3.7. The description of them is postponed until the next section.

Note that $D \in \mathcal{D}$ if and only if the system

$$\begin{cases} Au = N_{D\circ\Theta}(u), \\ u \in M, \end{cases} \quad\Leftrightarrow\quad \begin{cases} -\Delta_p u(x) = D\Theta(u(x)), & x \in \Omega, \\ u_i(x) \geq 0, & x \in \Omega,\ 1 \leq i \leq m, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.12}$$

possesses no weak solutions $u \neq 0$.

Lemma 3.5. *The homotopies $H_1, H_2 \colon M \times [0,1] \to X$,*

$$H_1(u,t) = \begin{cases} \dfrac{1}{t^{p-1}} N_f(tu), & t > 0, \\ N_{D_0\Theta}(u), & t = 0, \end{cases} \qquad H_2(u,t) = \begin{cases} t^{p-1} N_f\left(\dfrac{u}{t}\right), & t > 0, \\ N_{D_\infty\Theta}(u), & t = 0, \end{cases}$$

are continuous, bounded on bounded sets and $H_i(u,t) \in T_{M^*}(N(u))$, $i = 1, 2$.

Proof. Note that

$$H_1(u,t)(x) = D_0\Theta(u(x)) + \alpha(tu(x)) \cdot |u(x)|^{p-1}, \quad \text{for } u \in M,\ t > 0, \tag{3.13}$$

where $\alpha \colon \mathbb{R}_+^m \to \mathbb{R}^m$ is a continuous function such that $\alpha(0) = 0$ and $o_0(u) = \alpha(u)|u|^{p-1}$. From (1.4) and (3.6) we obtain the existence of $K > 0$ such that $|\alpha(u)| \leq K$ for all $u \in M$. This implies that H_1 is bounded on bounded subsets of $M \times [0,1]$.

To prove the continuity of H_1 we only need to show that if $M \ni u_n \to u \in M$ and $(0,1] \ni t_n \to 0$, then $H_1(u_{n_k}, t_{n_k}) \to H_1(u,0)$ for some $\mathbb{N} \ni n_k \to \infty$. We can find a subsequence of u_n (still denoted by u_n) such that u_n converges almost everywhere to u and $|u_n(x)| \le w(x)$ for some $w \in L^p(\Omega)$ and almost all $x \in \Omega$. Observe that, by (3.13), $H_1(u_n, t_n) = N_{D_0\Theta}(u_n) + \alpha_n$, where

$$\alpha_n(x) = \alpha(t_n u_n(x))|u_n(x)|^{p-1} \to 0 \quad \text{for a.a } x \in \Omega. \tag{3.14}$$

Moreover, from (3.14) we have

$$|\alpha_n(x)| \le K|u_n(x)|^{p-1} \le Kw^{p-1}(x)$$

for almost all $x \in \Omega$, which shows that the sequence α_n is dominated in $L^q(\Omega, \mathbb{R}^n)$. From Lebesgue's Dominated Convergence Theorem we have $\alpha_n \to 0$ in $L^q(\Omega, \mathbb{R}^m)$ and consequently, $H_1(u_n, t_n) \to H_1(u, 0)$. In a similar fashion we can demonstrate the continuity and the boundedness of H_2.

The last part of the conclusion follows from Theorem 3.2, Proposition 3.4 and the fact that $T_{M^*}(t\tau) = T_{M^*}(\tau)$ for $\tau \in M^*$ and $t > 0$. □

The proposition that follows is the generalization of [5, Theorem 4.10].

Proposition 3.6. *Let the assumptions* (1.3) *and* (1.8) *be satisfied.*

(a) *Assume that $D_0 \in \mathcal{D}$. Then there exists $\delta > 0$ such that $A(u) \neq F(u)$ for all $0 \neq u \in D(A) \cap B_M(0, \delta)$ and*

$$\mathrm{Deg}_M(A, N_f, B_M(0, \delta)) = \mathrm{Deg}_M(A, N_{D_0 \circ \Theta}, B_M(0, \delta)).$$

(b) *Assume that $D_\infty \in \mathcal{D}$. Then there exists $R > 0$ such that $A(u) \neq F(u)$ for all $u \notin D(A) \cap B_M(0, \delta)$ and*

$$\mathrm{Deg}_M(A, N_f, B_M(0, R)) = \mathrm{Deg}_M(A, N_{D_\infty \circ \Theta}, B_M(0, R)).$$

Proof. (a) Consider the homotopy H_1 from Lemma 3.5. We shall show that there exists $\delta > 0$ such that

$$A(u) \neq H_1(u, t), \quad \text{for all } 0 \neq u \in D(A) \cap B_M(0, \delta) \text{ and } t \in [0, 1].$$

If not, there exists a sequence of points $0 \neq u_n \in M \cap D(A)$, $t_n \in [0, 1]$ such that $A(u_n) = H_1(u_n, t_n)$ and $u_n \to 0$. By the assumption $D_0 \in \mathcal{D}$ we know that the equation $Au = N_{D_0\Theta}(u)$ has no solutions in $M \setminus \{0\}$. Therefore, $t_n > 0$ and then

$$A(w_n) = H_1(w_n, s_n), \quad \text{where } t_n u_n = s_n w_n,\ 0 < s_n \to 0,\ \|w_n\| = 1. \tag{3.15}$$

Here, we used the relation $A(tu) = t^{p-1}A(u)$ for all $t > 0$ and $u \in Y$.

To deal with the discontinuity of A, we rewrite (3.15) in the following way:

$$w_n = (N + A)^{-1}(v_n) \quad \text{with } v_n := N(w_n) + H_1(w_n, s_n). \tag{3.16}$$

It follows from [5, Proposition 3.1] that since (a1)–(a2) are satisfied and Y is compactly embedded in X, the operator $(N + A)^{-1}$ is completely continuous, i.e. it is continuous and maps bounded sets into relatively compact ones. As v_n is bounded, we can assume that w_n is convergent in X to some w with $\|w\| = 1$.

From the continuity of H_1 and $(N+A)^{-1}$ we conclude from (3.16) that

$$A(w) = H_1(w,0) = N_{D_0\Theta}(w).$$

But this is contradicted by the assumption $D_0 \in D$. Therefore we are allowed to use the homotopy invariance of the degree (Theorem 2.1(c)), which proves the assertion.

(b) The proof is analogous. $\square$

Proposition 3.6 provides a broad class of functions f for which the degree of a pair (A, N_f) is well defined and describes its value by the degree of pairs $(A, N_{D_0\circ\Theta})$ or $(A, N_{D_\infty\circ\Theta})$. The question arises, how to calculate such degree.

Proposition 3.7. *If $D \in \mathcal{D}_i$ for $i \in \{0,1\}$, then, for every $r > 0$,*

$$\mathrm{Deg}_M(A, N_{D\circ\Theta}, B_M(0,r)) = i.$$

Proof. We shall proceed as in the proof of Theorem 2.6 in [5]. Firstly, let us observe, that $D \in \mathcal{D}$, which assures us the equation $Au = N_{D\circ\Theta}$ admits no nontrivial solutions in M. Therefore the degree $d := \mathrm{Deg}_M(A, N_{D\circ\Theta}, B_M(0,r))$ is well defined and is independent of $r > 0$, the last being a consequence of the additivity property of the degree.

Suppose that $D \in \mathcal{D}_1$. Consider the homotopy $H(u,t) = tN_{D\circ\Theta} = N_{t\cdot D\circ\Theta}$, $u \in M$, $t \in [0,1]$. The assumption $D \in D_1$ implies $tD \in \mathcal{D}$ and that there are no nontrivial solutions of the equation $A(u) = H(u,t)$, $t \in [0,1]$. The homotopy invariance yields

$$\mathrm{Deg}_M(A, N_{D\circ\Theta}, B_M(0,r)) = \mathrm{Deg}_M(A, 0, B_M(0,r)) = 1,$$

where the latter equality follows from Theorem 2.1(iv).

Suppose now that $D \in \mathcal{D}_0$. Define $H(u,t) = N_{D\circ\Theta} + t\tau_0$, where $\tau_0 \in M^*$ is from the definition of the family $\mathcal{D}_0'$. Homogeneity of A and Θ, together with the assumption $D \in \mathcal{D}_0'$, shows that $A(u) = H(u,t)$ is satisfied only for $u = 0$, $t = 0$. As a consequence, the operator $N_{D\circ\Theta}$ is homotopic to a solution-free operator $N_{D\circ\Theta} + \tau_0$. From the homotopy and the existence property we deduce that $\mathrm{Deg}_M(A, N_{D\circ\Theta}, B_M(0,r)) = 0$.

We are now ready to state the main existence theorem.

Theorem 3.8. *Let the assumptions* (1.3) *and* (1.8) *be satisfied. Assume that $D_0 \in \mathcal{D}_0$ and $D_\infty \in \mathcal{D}_1$ or $D_0 \in \mathcal{D}_1$ and $D_\infty \in \mathcal{D}_0$. Then there exists at least one nonzero solution to the reaction-diffusion system* (1.2).

Proof. Proposition 3.6 yields there are two radii $0 < \delta < R$ such that

$$\mathrm{Deg}_M(A, N_f, B_M(0,\delta)) = \mathrm{Deg}_M(A, N_{D_0\circ\Theta}, B_M(0,\delta)),$$
$$\mathrm{Deg}_M(A, N_f, B_M(0,R)) = \mathrm{Deg}_M(A, N_{D_\infty\circ\Theta}, B_M(0,R)).$$

By Proposition 3.7 we obtain that

$$\mathrm{Deg}_M(A, N_f, B_M(0,\delta)) \neq \mathrm{Deg}_M(A, N_f, B_M(0,R)).$$

The additivity property of the degree gives $\mathrm{Deg}_M(A, N_f, B_M(0,R)\setminus B_M(0,\delta)) \neq 0$ and consequently, by the existence property, there exists at least one solution $u \in B_M(0,R)\setminus B_M(0,\delta)$ of (3.8), which is the nontrivial solution of the system (1.2). □

As we see, the degree theory for tangent mappings established in [5] can be successfully applied to investigating the solutions of (1.2). Now we shall focus on describing the families $\mathcal{D}_0$ and $\mathcal{D}_1$, thereby making Theorem 3.8 applicable.

4. Description of the families $\mathcal{D}_0$ and $\mathcal{D}_1$

In what follows we shall be using the following subset of a spectrum of a given matrix D:

$$\sigma_\oplus(D) := \{\lambda \in \mathbb{R} \mid \text{there exists } 0 \neq u \in \mathbb{R}^m_+ \text{ such that } Du = \lambda u\}.$$

We can now restate the famous Perron–Frobenius Theorem in the following manner:

LEMMA 4.1 (Perron–Frobenius [8]). *If D is a nonnegative square matrix, then $r(D) \in \sigma_\oplus(D)$, where $r(D)$ stands for the spectral radius of D. In particular,*

$$s(D) := \max \mathrm{Re}(\sigma(D)) = \max \sigma_\oplus(D) \tag{4.1}$$

for nonnegative matrices D.

The value $s(D)$ is called the spectral bound of D.

PROPOSITION 4.2. *If D is a quasinonnegative matrix, then the set $\sigma_\oplus(D)$ is nonempty and the relation* (4.1) *holds.*

PROOF. By the definition of quasinonnegativity, the matrix $D + \alpha I$ is nonnegative for some $\alpha > 0$, where I stands for the identity matrix. From Lemma 4.1 we have

$$s(D) + \alpha = s(D + \alpha I) = \max \sigma_\oplus(D + \alpha I) = \max \sigma_\oplus(D) + \alpha. \qquad \square$$

The following characterisation of the families $\mathcal{D}_0$, $\mathcal{D}_1$ in the case of $p = 2$ is provided by [14]:

$$\begin{aligned} \mathcal{D} &= \{D \in \mathcal{S} \mid \lambda_{1,p} \notin \sigma_\oplus(D)\}, \\ \mathcal{D}_0 &= \{D \in \mathcal{D} \mid s(D) > \lambda_{1,p}\}, \\ \mathcal{D}_1 &= \{D \in \mathcal{D} \mid s(D) < \lambda_{1,p}\}. \end{aligned} \tag{4.2}$$

If $m = 1$, the same characterisation was presented in the proof of Theorem 4.4 in [5] for $p > 2$.

Proposition 4.3. *If $D \in \mathcal{D}$, then $\lambda_{1,p} \notin \sigma_\oplus(D)$.*

We do not know whether the converse holds (compare with Theorem 4.12).

Proof. Suppose that $\lambda_{1,p} \in \sigma_\oplus(D)$. Let $v \in \mathbb{R}^m_+$ be a nonnegative nonzero eigenvector of the matrix D corresponding with the eigenvalue $\lambda_{1,p}$ and let $v' = \Theta^{-1}(v)$. Denote the positive eigenfunction corresponding with the first eigenvalue of one-dimensional p-Laplacian by ω. Then $u := v'\omega \in M$ satisfies the equation (3.12) and consequently $D \notin \mathcal{D}$. $\square$

The equality (4.2) shows that in the case $p = 2$ the relation between values $s(D)$ and $\lambda_{1,p}$ is crucial. To investigate the relation in the case $p > 2$ we will be using the following property of quasinonnegative matrices:

Fact 4.4. *Let D be a quasinonnegative matrix. Let $i_1, \ldots, i_k$ be any indices from $\{1, \ldots, m\}$ and $\widehat{D} = (d_{i_j,i_l})_{1\le j,l\le k}$ be the square matrix of dimension k. Then $s(D) \geq s(\widehat{D})$.*

Proof. Since $s(D + \alpha I) = s(D) + \alpha$, $s(\widehat{D} + \alpha I) = s(\widehat{D}) + \alpha$ and $d_{ij} \geq 0$ for $i \neq j$, it suffices to prove the assertion for D nonnegative. From Lemma 4.1 we have that $s(D) = r(D)$ where $r(D)$ is a spectral radius of D. Consider the square matrix $\widetilde{D}$ of dimension m, obtained by replacing coordinates d_{jl} of D for $(j, l) \notin \{i_1, \ldots, i_k\} \times \{i_1, \ldots, i_k\}$ by zeros. Clearly, $r(\widehat{D}) = r(\widetilde{D})$.

As the matrix D is nonnegative, the coordinates of $\widetilde{D}$ does not exceed that of D. The same holds for the matrices $(\widetilde{D})^h$ and D^h for any natural h. Thus $\|D^h\| \geq \|\widetilde{D}^h\|$ and finally

$$s(D) = r(D) = \inf\left\{ \sqrt[h]{\|D^h\|} \;\middle|\; h = 1, 2, \ldots \right\} \geq r(\widetilde{D}) = r(\widehat{D}) = s(\widehat{D}). \qquad \square$$

Corollary 4.5. *If a quasinonnegative matrix D satisfies $s(D) < 0$ then its reciprocal has all coordinates nonpositive.*

Proof. The proof is by induction on the dimension m of D.

If $m = 1$, the assertion trivially holds true.

Let $m > 1$. Suppose that the assertion is satisfied for all matrices of dimension $m - 1$. Let D be of dimension m and let E be the matrix obtained from D by removing the last column and the last row. Therefore $s(E) \leq s(D) < 0$ by Fact 4.4 and the matrix E^{-1} has all coordinates nonpositive. An easy computation shows that multiplying D by the matrix

$$C_1 := \begin{pmatrix} I & 0 \\ \mathbf{d} & 1 \end{pmatrix} \cdot \begin{pmatrix} -E^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

with $\mathbf{d} = (d_{m1}, \ldots, d_{m,m-1})$ we obtain

$$C_1 D = \begin{pmatrix} -I & \mathbf{e} \\ 0 & \alpha \end{pmatrix} \tag{4.4}$$

for some column vector $\mathbf{e} = (e_1, \ldots, e_{m-1})^T$ with nonnegative coordinates and some real $\alpha \in \mathbb{R}$.

This yields $1 \cdot (-1)^{m-1} (\det E)^{-1} \cdot \det D = (-1)^{m-1} \cdot \alpha$. Since $s(E), s(D) < 0$, it follows that $-\operatorname{sgn} \det E = \operatorname{sgn} \det D = (-1)^m$ and finally $\alpha < 0$.

Multiplying (4.4) by

$$C_2 := \begin{pmatrix} I & \mathbf{e}/-\alpha \\ 0 & 1/-\alpha \end{pmatrix} \tag{4.5}$$

we conclude that $D^{-1} = -C_2 \cdot C_1$. This shows that D^{-1} has nonpositive coordinates and proves the assertion. □

THEOREM 4.6. *If D is a quasinonnegative matrix with $s(D) < \lambda_{1,p}$, then $D \in \mathcal{D}_1$.*

PROOF. Since $s(tD) = ts(D)$ for $t \in [0,1]$, we shall have established the lemma, if we prove that $D \in \mathcal{D}$. To this end, we suppose that there exists a nontrivial solution of the system (3.12).

Notice, that we can assume that $\|u_i\|_p \neq 0$ for $i = 1, \ldots, m$. Indeed, if $u_i \equiv 0$ for some i, then we can remove the i-th equation from (3.12) and the matrix $(d_{jk})_{j,k \neq i}$ of the new system has negative spectral bound, which follows from Fact 4.4.

Evaluating the i-th equation on the function u_i (treating it as the equality of functionals) one can show that

$$\begin{aligned} \lambda_{1,p} \|u_i\|_p^p &\leq \langle -\Delta_p u_i, u_i \rangle_{W_0^{1,p}(\Omega)} \\ &= \sum_{j=1}^m d_{ij} \langle \theta u_j, u_i \rangle_p \leq \sum_{j=1}^m d_{ij} \|u_j\|_p^{p-1} \|u_i\|_p, \end{aligned} \tag{4.6}$$

the first estimate being a consequence of the Rayleigh formula (3.3), whereas the second one – of the Hölder inequality.

By (4.6) we obtain $(D - \lambda_{1,p} I)\xi \geq 0$, where we set $\xi := (\|u_i\|_p^{p-1})_{i=1}^m$. Since $s(D - \lambda_{1,p} I) < 0$, the above result implies that the matrix $-(D - \lambda_{1,p} I)^{-1}$ is of nonnegative coordinates. Then

$$-\xi = (-(D - \lambda_{1,p} I)^{-1}) \cdot (D - \lambda_{1,p} I)\xi \geq 0.$$

This proves that $\xi = 0$, i.e. all the functions u_i are equal to zero. A contradiction. □

It is worth mentioning about the result from [7] concerning the special case of the system of two equations (i.e. $m = 2$) but without the assumption of quasinonnegativity of the matrix. Proposition 4.2(b) in [7] states that if D is of dimension 2, possesses a negative eigenvalue and $\lambda_{1,p} \notin \sigma(D)$, then (3.12) does not have any nontrivial solutions.

The description of the family $\mathcal{D}_0$ will involve the value

$$m(D) := \max\{d_{ii} \mid i = 1, \dots, m\}.$$

The relation between $s(D)$ and $m(D)$ is established in the following proposition, being an immediate consequence of Fact 4.4:

Corollary 4.7. *The inequality $s(D) \geq m(D)$ holds for all quasinonnegative matrices D, and the equality holds for triangular matrices, among others.*

Proof. The first part of the assertion follows from the inequality

$$s(D) \geq s((d_{jl})_{j,l=i}) = d_{ii}, \quad i = 1, \dots, m.$$

The second one is trivial. $\square$

We shall need the following two facts.

Lemma 4.8 (see [6, Theorem 1]). *If a function $h \in L^\infty(\Omega)$ is nonnegative and nonzero, then the equation*

$$-\Delta_p u = \lambda_{1,p}|u|^{p-2}u + h$$

possesses no weak solutions $u \in W_0^{1,p}(\Omega)$.

Lemma 4.9. *If $u = (u_1, \dots, u_m) \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ is a weak solution of the system*

$$\begin{cases} -\Delta_p u_1 = \displaystyle\sum_{j=1}^m d_{1j}|u_j|^{p-2}u_j + h_1, \\ -\Delta_p u_2 = \displaystyle\sum_{j=1}^m d_{2j}|u_j|^{p-2}u_j + h_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\Delta_p u_m = \displaystyle\sum_{j=1}^m d_{mj}|u_j|^{p-2}u_j + h_m, \end{cases} \tag{4.7}$$

where $d_{ij}, h_i \in L^\infty(\Omega)$, $i, j = 1, \dots, m$, then $u_i \in L^\infty(\Omega)$.

Lemma 4.9 is a far generalisation of [5, Lemma 4.7].

Proof. We adapt the arguments from [17]. Let $u := \max\{|u_1|, \dots, |u_m|\} \in W_0^{1,p}(\Omega)$ and let $k > 1$ be real. Set

$$I_i^\pm := \int_{\{\pm u_i \geq k\}} |\nabla u_i|^p.$$

We can assume that $I_i^+ \geq I_i^-$, because otherwise we can simultaneously change u_i, h_i and d_{ij}, d_{ji}, $i \neq j$ in (4.7) into the opposite ones. Therefore,

$$I_i := \int_{\{|u_i| \geq k\}} |\nabla u_i|^p = I_i^+ + I_i^- \leq 2I_i^+. \tag{4.8}$$

Using a test function $\max\{u_i - k, 0\}$ in the i-th equation in (4.7) we obtain

$$\begin{aligned} I_i^+ &= \sum_{j=1}^m \int_{\{u_i \ge k\}} a_{ij}|u_j|^{p-2}u_j(u_i - k) + \int_{\{u_i \ge k\}} h_i(u_i - k) \\ &\le \sum_{j=1}^m \|a_{ij}\|_\infty \int_{\{u_i \ge k\}} |u_j|^{p-1}(|u_i| - k) + \|h_i\|_\infty \int_{\{u_i \ge k\}} (|u_i| - k) \\ &\le C \int_{\Omega_k} u^{p-1}(u - k) + C \int_{\Omega_k} (u - k), \end{aligned}$$

where $\Omega_k = \{x \in \Omega \mid u(x) \ge k\}$ and $C > 0$ is independent on k. The convexity of the function $s \mapsto |s|^{p-1}$, the choice of k and (4.8) then yields

$$I_i \le C \int_{\Omega_k} (u-k)^p + C(k^{p-1}+1) \int_{\Omega_k} (u-k) \le C \int_{\Omega_k} (u-k)^p + Ck^{p-1} \int_{\Omega_k} (u-k)$$

(the constant C has changed). Moreover, it is not difficult to verify that

$$\sum_{i=1}^m I_i \ge \int_{\Omega_k} |\nabla(u - k)|^p \ge \frac{1}{C|\Omega_k|^s} \int_{\Omega_k} (u - k)^p, \quad \text{for some } s > 0,$$

the last inequality being a consequence of Hölder and Sobolev inequality (see [5, Lemma 4.6]). Hence

$$(1 - C|\Omega_k|^s) \int_{\Omega_k} (u - k)^p \le C|\Omega_k|^s k^{p-1} \int_{\Omega_k} (u - k).$$

But $|\Omega_k| \to 0$ as $k \to \infty$. Therefore

$$\int_{\Omega_k} (u - k)^p \le C|\Omega_k|^s k^{p-1} \int_{\Omega_k} (u - k) \tag{4.9}$$

for $k \ge k_0$ where k_0 is sufficiently large. Applying Hölder inequality

$$\left(\int_{\Omega_k} (u - k) \right)^p \le |\Omega_k|^{p-1} \int_{\Omega_k} (u - k)^p$$

to (4.9) we obtain

$$\int_{\Omega_k} (u - k) \le C \cdot |\Omega_k|^{1+\alpha} k \quad \text{for some } C, \alpha > 0 \text{ and } k \ge k_0. \tag{4.10}$$

Let us introduce the function $f\colon (0,\infty) \to [0,\infty)$ defined by

$$f(k) := \int_{\Omega_k} (u - k) = \int_k^\infty |\Omega_t| \, dt$$

(the second equality follows from the Tonelli–Fubini theorem). Evidently, f is nonnegative, nondecreasing, absolutely continuous and $f'(k) = -|\Omega_k|$ almost everywhere.

Suppose $f(k) \ne 0$ for $k \ge k_0$. Then the inequality (4.10) can be expressed in the following manner

$$k^{-\varepsilon} \le -Cf(k)^{-\varepsilon} f'(k), \quad C > 0, \ \varepsilon \in (0, 1).$$

Integrating over the interval $[k_0, k]$ for any $k > k_0$ we conclude that

$$k^{1-\varepsilon} \leq k_0^{1-\varepsilon} + Cf(k_0)^{1-\varepsilon},$$

which fails to be true for k large enough. This shows that $f(k) = 0$ for some k and that $u \leq k$ almost everywhere. This is equivalent to the assertion. $\square$

Lemma 4.10. *Let $u = (u_1, \ldots, u_m) \in M$ be a solution to the system* (3.12) *and let $1 \leq i \leq m$.*

(a) *If $d_{ii} \geq \lambda_{1,p}$, then*

$$d_{ik} = 0 \quad \text{or} \quad u_k = 0 \quad \text{for all } k \neq i, \tag{4.11}$$

u_i is a nonnegative eigenfunction of p-Laplacian corresponding with the first eigenvalue and $D \in \mathcal{D}_0'$.

(b) *If $d_{ii} > \lambda_{1,p}$, then $u_i = 0$.*

(c) *If $u_i = 0$, then the condition* (4.11) *is satisfied.*

Proof. (a)–(b). Let $d_{ii} \geq \lambda_{1,p}$. Analysing the i-th equality of the system (3.12) we obtain that

$$-\Delta_p(u_i) = \sum_{k=1}^{m} d_{ik}\theta(u_k) = \lambda_{1,p}\theta(u_i) + h \tag{4.12}$$

where

$$L^q(\Omega) \ni h := \sum_{k \neq i} d_{ik}\theta(u_k) + (d_{ii} - \lambda_{1,p})\theta(u_i)$$

is a nonnegative function. From Lemmata 4.8 and 4.9 we conclude that (4.12) has no solutions provided $h \neq 0$. Hence $h = 0$. By this and the fact that all u_i's are nonnegative, the following is satisfied:

- u_i is the eigenfunction of p-Laplacian corresponding with the first eigenvalue $\lambda_{1,p}$;
- for every number $k \neq i$ we have $d_{ik}\theta(u_k) = 0$, i.e. either $d_{ik} = 0$ or $u_k = 0$;
- $(d_{ii} - \lambda_{1,p})\theta(u_i) = 0$, i.e. either $d_{ii} = \lambda_{1,p}$ or $u_i = 0$.

If $D \notin \mathcal{D}_0'$, then for a fixed nonnegative nonzero function $v \in L^p(\Omega)$ there would exist $w = (w_1, \ldots, w_m) \in M$ such that

$$Aw = N_{D\Theta}(w) + N_{\Theta}(ve^i),$$

where e^i is the i-th vector from the canonical basis of $\mathbb{R}^m$. As before, we can obtain the equation (4.12), where the function

$$h := \sum_{k \neq i} d_{ik}\theta(u_k) + (d_{ii} - \lambda_{1,p})\theta(u_i) + \theta(v)$$

is nonnegative and nonzero. From Lemmata 4.8 and 4.9 it follows that the equation (4.12) possesses no solution, contrary to the assumption.

(c) If $u_i = 0$, then, as before, from the system (3.12) it follows that

$$0 = \sum_{k \neq i} d_{ik} \theta(u_k).$$

Since $d_{ik} \geq 0$ and $u_k \geq 0$, we deduce that (4.11) is satisfied. □

COROLLARY 4.11. *If D is a quasinonnegative matrix with $m(D) \geq \lambda_{1,p}$, then $D \in \mathcal{D}_0'$.*

The analogous description of the family $\mathcal{D}_i$ to (4.2) in the nonlinear case is true for triangular matrices.

THEOREM 4.12. *Suppose that D is a quasinonnegative triangular matrix. Then the following equivalence hold:*

(a) *$D \in \mathcal{D}$ if and only if $\lambda_{1,p} \notin \sigma_{\oplus}(D)$,*
(b) *$D \in \mathcal{D}_0'$ if and only if $s(D) \geq \lambda_{1,p}$,*
(c) *$D \in \mathcal{D}_1$ if and only if $s(D) < \lambda_{1,p}$.*

PROOF. We give the proof for an upper triangular matrix D (the other case can be proved similarly).

On account of Proposition 4.3, to prove the first equivalence (i), it suffices to show that if $D \notin \mathcal{D}$ then $\lambda_{1,p} \in \sigma_{\oplus}(D)$. Assume that there exists a nonzero function $u \in M$ satisfying the system (3.12). Let l be the greatest natural number for which $u_l \neq 0$. Then the l-th equation of the system (3.12) is equivalent to:

$$-\Delta_p u_l = \sum_{i=l}^{m} d_{li} \theta(u_i) = d_{ll} \theta(u_l).$$

Therefore $d_{ll} = \lambda_{1,p}$ and u_l is a positive eigenfunction of p-Laplacian. Hence $\lambda_{1,p} \in \sigma(D)$.

We shall prove that $\lambda_{1,p} \in \sigma_{\oplus}(D)$, which is equivalent to the existence of nonnegative numbers $x_1, \ldots, x_m$, not all of which being equal to zero, satisfying the equation:

$$(D - \lambda_{1,p} I) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It is equivalent to the system of equations:

$$\sum_{j=i+1}^{m} d_{ij} x_j = (\lambda_{1,p} - d_{ii}) x_i, \quad i = 1, \ldots, m. \tag{4.13}$$

For this purpose, let k be the smallest natural number, for which $d_{kk} = \lambda_{1,p}$ and $u_k \neq 0$ (the existence of such k follows from $\lambda_{1,p} \in \sigma(D)$).

Consider the numbers $x_1, \ldots, x_m$ defined recursively as follows:

- $x_k = 1$, $x_{k+1} = x_{k+2} = \ldots = x_m = 0$;

- if $i < k$ then

$$x_i = \begin{cases} \dfrac{1}{\lambda_{1,p} - d_{ii}} \displaystyle\sum_{j=i+1}^{m} d_{ij} x_j & u_i \neq 0, \\ 0 & u_i = 0. \end{cases} \tag{4.14}$$

These numbers are well defined and nonnegative, because if $u_i \neq 0$ then from Lemma 4.10(b) we conclude that $d_{ii} \leq \lambda_{1,p}$ and the choice of k implies that $d_{ii} \neq \lambda_{1,p}$ for $i < k$.

It remains to prove that all the equalities in (4.13) hold. Clearly, they do for $i \geq k$ ($x_k = 0$ for $i > k$ and $d_{kk} = \lambda_{1,p}$). Moreover, they do also for such $i < k$ that $u_i \neq 0$.

Let $i < k$ and $u_i = 0$. We have to show that

$$\sum_{j=i+1}^{k} d_{ij} x_j = 0.$$

Let $i < j \leq k$. If $u_j = 0$, then $x_j = 0$ from (4.14). If $u_j \neq 0$, then Lemma 4.10(c) shows that $d_{ij} = 0$, by $u_i = 0$. Hence $d_{ij} x_j = 0$ for $i < j \leq k$.

The conclusion (c) is a consequence of the following sequence of implications:

$$\begin{aligned} D \in \mathcal{D}_1 &\Leftrightarrow \forall (0 \leq t \leq 1)\ tD \in \mathcal{D} \Leftrightarrow \forall (0 < t \leq 1)\ \lambda_{1,p} \notin \sigma_{\oplus}(tD) \\ &\Leftrightarrow \forall (0 < t \leq 1)\ \frac{\lambda_{1,p}}{t} \notin \sigma_{\oplus}(D) \Leftrightarrow [\lambda_{1,p}, \infty) \cap \sigma_{\oplus}(D) = \emptyset \Leftrightarrow s(D) < \lambda_{1,p}. \end{aligned}$$

To prove (b), it is convenient to apply Corollaries 4.11 and 4.7. They imply that if $s(D) = m(D) \geq \lambda_{1,p}$, then $D \in \mathcal{D}_0'$. The converse follows from the third part of the conclusion. Namely, if $s(D) < \lambda_{1,p}$, then $D \in \mathcal{D}_1$. Assume, contrary to our claim, that $D \in \mathcal{D}_0'$. Then $D \in \mathcal{D}_0 \cap \mathcal{D}_1 = \emptyset$ by Proposition 3.7, which is impossible. □

The second kind of matrices considered in the paper, other than triangular matrices, is the family of irreducible matrices.

Definition 4.13. The matrix D is *irreducible* if there does not exist such a permutation matrix P that the product PDP^{-1} is of the form

$$\begin{pmatrix} D' & Q \\ 0 & D'' \end{pmatrix},$$

where D', D'' are square matrices. Equivalently, the matrix is irreducible if there is impossible to divide the set of indices $1, \ldots, m$ into two nonempty disjoint sets $i_1, \ldots, i_l$ and $j_1, \ldots, j_k$ such that $d_{i_\alpha j_\beta} = 0$, $0 \leq \alpha \leq l$, $0 \leq \beta \leq k$.

Fact 4.14. *Let $D \in \mathcal{S}$ be irreducible and let $u \in L^p(\Omega, \mathbb{R}^m)$, $u \neq 0$, satisfies the system* (3.12). *Then $u_i \neq 0$ for every $i = 1, \ldots, m$.*

PROOF. Let $u \in M$ be as above. From Lemma 4.10(c) we deduce that $d_{ij} = 0$ for such i, j that $u_i = 0$, $u_j \neq 0$. By irreducibility of D we find out that either all the functions u_i are zero or all of them are nonzero. The assumption $u \neq 0$ excludes the first possibility. □

THEOREM 4.15. *Let D be a quasinonnegative irreducible matrix of dimension greater than* 1. *If $m(D) \geq \lambda_{1,p}$, then $D \in \mathcal{D}_0$.*

It is worth noting that if D is irreducible or triangular, then assuming that $d_{ii} \geq \lambda_1$ *for all* $1 \leq i \leq m$ in order to obtain $D \in \mathcal{D}'_0$ is not necessary. This corresponds to the fact, that it is sufficient to assume that *at least* one component of the nonlinearity at zero or at infinity is above the principal eigenvalue to use Theorem 4.12 or 4.15. Example 4.19 demonstrates this observation. The similar condition is made in [15, Lemma 2.8].

PROOF. Suppose that $m(D) \geq \lambda_{1,p}$ and that $D \notin \mathcal{D}_0$. Corollary 4.11 implies that $D \notin \mathcal{D}$. Therefore there exists a nonzero function $u \in X$ that satisfies the system (3.12). From Fact 4.14 we conclude that $u_1, \ldots, u_n \neq 0$. From Lemma 4.10(b) it follows that $m(D) \leq \lambda_{1,p}$, and consequently, $m(D) = \lambda_{1,p}$. There exists an index i such that $d_{ii} = \lambda_{1,p}$. By Lemma 4.10(a), for $j \neq i$ we have $d_{ij} = 0$, which contradicts the irreducibility of D. □

Evidently, if D is of dimension 1, then the assertion of Theorem 4.15 holds with strict inequality.

REMARK 4.16. It is now clear that in the case of irreducible matrix D of dimension at least 2, the following relations hold:

(a) if $s(D) < \lambda_{1,p}$ then $D \in \mathcal{D}_1$,
(b) if $m(D) \geq \lambda_{1,p}$ then $D \in \mathcal{D}_0$.

It is unknown how to verify the relations $D \in D_0$, $D \in D_1$ when $\lambda_{1,p} \in (m(D), s(D)]$. The author makes a hypothesis that the characterisation (4.2) holds for all quasinonnegative matrices and $p \geq 2$.

If the matrix is neither triangular nor irreducible, one can apply the following observation:

PROPOSITION 4.17. *Consider a quasinonnegative matrix D. If P is a permutation matrix, then $D \in D_i$ if and only if $PDP^{-1} \in \mathcal{D}_i$. Moreover, if D is of the form $\begin{pmatrix} D_1 & Q \\ 0 & D_2 \end{pmatrix}$, where D_1, D_2 are quasinonnegative and Q is with nonnegative coefficients, then*

(a) *if $D_1, D_2 \in \mathcal{D}$ then $D \in \mathcal{D}$,*
(b) *if $D_1, D_2 \in \mathcal{D}_1$ then $D \in \mathcal{D}_1$,*
(c) *if $D_2 \in D'_0$ then $D \in \mathcal{D}'_0$.*

In particular, if $D_1 \in \mathcal{D}$ and $D_2 \in \mathcal{D}_0$, then $D \in \mathcal{D}_0$.

PROOF. The proof is straightforward and it is only based on the definitions of $\mathcal{D}$, $\mathcal{D}_0'$, $\mathcal{D}_1$. $\square$

EXAMPLE 4.18. This example is meant to demonstrate how to exploit Theorems 4.6, 4.12, 4.15 and Proposition 4.17 to examine if the given matrix D belongs to the family $\mathcal{D}_0$ or $\mathcal{D}_1$.

Consider the domain $\Omega \subset \mathbb{R}^N$ and $p \geq 2$ such that $\lambda_{1,p} = 4$. Using Theorem 4.6 we can verify that the following two matrices:

$$D_1 = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & -2 & 1 & 0 \\ 2 & 0 & -3 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

belong to the family $\mathcal{D}_1$, since $s(D_1) \approx 3.4 < \lambda_{1,p}$ and $s(D_2) \approx 3.7 < \lambda_{1,p}$.

Now we shall demonstrate that the matrices

$$D_3 = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & 0 & 5 & 3 & 3 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} 3 & 0 & 1 & 2 & 1 \\ 1 & 5 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 \\ 1 & 0 & -3 & 0 & 0 \\ 1 & 3 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

belong to $\mathcal{D}_0$. The matrix D_3 is upper triangular with $s(D) = m(D) = 5 > \lambda_{1,p}$. Therefore, from Theorem 4.12 we obtain $D_3 \in \mathcal{D}_0'$. Note that $\lambda_{1,p} \notin \sigma_{\oplus}(D_3)$, though $\lambda_{1,p} \in \sigma(D_3)$. Therefore, $D_3 \in \mathcal{D}$ and finally $D_3 \in \mathcal{D}_0$.

The matrix D_4 has the form $\left(\begin{smallmatrix} D' & Q \\ 0 & D'' \end{smallmatrix}\right)$, where $D', D'' \in \mathcal{D}_0$. Indeed, D' is lower triangular with $s(D') = 5$ and $\lambda_{1,p} \notin \sigma(D')$. Additionally, D'' is irreducible with $m(D) \geq \lambda_{1,p}$. As a result, Proposition 4.17 proves that $D_4 \in \mathcal{D}_0$.

To prove that $D_5 \in \mathcal{D}_0$ it is convenient to change the order of columns and rows of D_5. By Proposition 4.17 we obtain that $D_5 \in \mathcal{D}_0$ if and only if $E = PD_5P^{-1} \in \mathcal{D}_0$, where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 2 & 2 & 3 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}.$$

Because E is of the form $\begin{pmatrix} D' & Q \\ 0 & D'' \end{pmatrix}$, where $D' \in \mathcal{D}_1$ and $D'' \in \mathcal{D}_0$, we obtain that $E \in D_0$.

EXAMPLE 4.19. Set $p = 3$ and let $\Omega \subset \mathbb{R}^N$ be such that $\lambda_{1,p} \leq 4$. Define

$$f(x,y) = \bigg(y^2 - \arctan x^2, \frac{4xy(x^2+y^2)}{1+xy}\bigg).$$

Observe that, although f is not positive, it is tangent to $\mathbb{R}^2_+$, because $f_1(0,y) = y^2 \geq 0$ and $f_2(x,0) = 0 \geq 0$.

Direct calculations yield $D_0 = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $D_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 4 & 4 \end{smallmatrix}\right)$. Since $s(D_0) = 0 < \lambda_{1,p}$, we have $D_0 \in \mathcal{D}_1$, by Theorem 4.6. Since D_∞ is irreducible and $m(D_\infty) = 4$, we have $D_\infty \in \mathcal{D}_0$, by Theorem 4.15. Therefore, Theorem 3.8 implies the existence of nontrivial nonnegative solution to the system

$$-\Delta_p u_1(x) = f_1(u_1(x), u_2(x)), \qquad -\Delta_p u_2(x) = f_2(u_1(x), u_2(x))$$

with Dirichlet boundary condition.

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