

**EXISTENCE OF SOLUTIONS FOR A KIRCHHOFF TYPE
FRACTIONAL DIFFERENTIAL EQUATIONS
VIA MINIMAL PRINCIPLE AND MORSE THEORY**

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ABSTRACT. In this paper by using the minimal principle and Morse theory, we prove the existence of solutions to the following Kirchhoff type fractional differential equation:

$$\begin{cases} M \left(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) \\ \quad \cdot ({}_tD_\infty^\alpha ({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t)) = f(t, u(t)), \quad t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where $\alpha \in (1/2, 1)$, ${}_tD_\infty^\alpha$ and ${}_{-\infty}D_t^\alpha$ are the right and left inverse operators of the corresponding Liouville–Weyl fractional integrals of order α respectively, H^α is the classical fractional Sobolev Space, $u \in \mathbb{R}$, $b: \mathbb{R} \rightarrow \mathbb{R}$, $\inf_{t \in \mathbb{R}} b(t) > 0$, $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ Carathéodory function and $M: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function that satisfy some suitable conditions.

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1. Introduction

The aim of this paper is to establish the existence of nontrivial solutions for the following Kirchhoff type fractional differential problem

$$(1.1) \quad \begin{cases} M\left(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt\right) \\ \quad \cdot ({}_tD_\infty^\alpha ({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t)) = f(t, u(t)), \quad t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where $\alpha \in (1/2, 1)$, ${}_tD_\infty^\alpha$ and ${}_{-\infty}D_t^\alpha$ are the right and left inverse operators of the corresponding Liouville–Weyl fractional integrals of order α respectively, H^α is the classical fractional Sobolev Space, $u \in \mathbb{R}$, $b: \mathbb{R} \rightarrow \mathbb{R}$, $\inf_{t \in \mathbb{R}} b(t) > 0$, $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ Carathéodory function and $M: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function that satisfy some suitable conditions.

Fractional differential equations have been receiving great interest recently. This is due to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., [1], [8], [10], [15], [16], [18], [21] and the references therein.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the Several years back, the critical point theory has become to a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [14], Rabinowitz [17] and the references listed therein.

Recently Jiao and Zhou [9], have studied the following fractional boundary value problem

$$\begin{cases} {}_tD_T^\alpha ({}_0D_t^\alpha u(t)) = \nabla F(t, u(t)), \quad t \in [0, T], \\ u(0) = u(T) = 0. \end{cases}$$

They proved the existence of solutions to this problem by using critical point theory.

In [20], by using the Mountain Pass Theorem, Torres investigates the existence of solutions for the fractional Hamiltonian systems

$$(1.2) \quad \begin{cases} ({}_tD_\infty^\alpha ({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t)) = \nabla W(t, u(t)), \quad t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}). \end{cases}$$

In a later work, by using the genus properties in the critical theory, Zhang and Yuan [22] obtained the infinitely many solutions for the problem (1.2).

The problem (1.1) is related to the stationary analogue of the equation

$$u_{xx} + M \left(\int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) ({}_tD_\infty^\alpha ({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t)) = f(t, x),$$

proposed by Kirchhoff [11] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string during the vibration. The reader is referred to [2]–[4], [7], [11], [19] and the references therein for previous work on this subject. In particular, these papers discuss the historical development of the problem as well as describe situations that can be realistically modeled by (1.1) with a nonconstant M .

Inspired by the above articles, in this paper, we would like to investigate the existence of solutions for problem (1.1). The technical tools are the minimal principle and Morse theory.

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed. Sections 3 and 4 are devoted to our results on existence of one solution by the minimal principle and existence of two nontrivial solutions by Morse theory, respectively.

2. Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. For the convenience of the reader, we also present here the necessary definitions.

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \rightarrow \mathbb{R}$ be a functional. First, we recall the definition of the Palais-Smale condition which plays an important role in our paper.

DEFINITION 2.1. We say that φ satisfies the Palais-Smale condition if any sequence $(u_n) \in X$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence.

Also, for the convenience of the reader, we also present here the necessary definitions of fractional calculus theory. We refer the reader to [10].

DEFINITION 2.2. The left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis \mathbb{R} are defined by

$$(2.1) \quad {}_{-\infty}I_x^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - \xi)^{\alpha-1} \phi(\xi) d\xi,$$

$$(2.2) \quad {}_x I_\infty^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi - x)^{\alpha-1} \phi(\xi) d\xi,$$

respectively, where $x \in \mathbb{R}$. The left and right Liouville–Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis \mathbb{R} are defined by

$$(2.3) \quad {}_{-\infty} D_x^\alpha \phi(x) = \frac{d}{dx} {}_{-\infty} I_x^{1-\alpha} \phi(x),$$

$$(2.4) \quad {}_x D_\infty^\alpha \phi(x) = -\frac{d}{dx} {}_x I_\infty^{1-\alpha} \phi(x).$$

respectively, where $x \in \mathbb{R}$. The definitions (2.3) and (2.4) may be written in an alternative form:

$$(2.5) \quad {}_{-\infty} D_x^\alpha \phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\phi(x) - \phi(x-\xi)}{\xi^{\alpha+1}} d\xi,$$

$$(2.6) \quad {}_x D_\infty^\alpha \phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\phi(x) - \phi(x+\xi)}{\xi^{\alpha+1}} d\xi.$$

Also, we define the Fourier transform $\mathcal{F}(u)(\xi)$ of $u(x)$ is defined by

$$\mathcal{F}(u)(\xi) = \int_{-\infty}^\infty e^{-ix \cdot \xi} u(x) dx.$$

Let us recall that, for any $\alpha > 0$, the semi-norm (see [20])

$$|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty} D_x^\alpha u\|_{L^2},$$

and norm

$$(2.7) \quad \|u\|_{I_{-\infty}^\alpha} = (\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2)^{1/2},$$

and let the space $I_{-\infty}^\alpha(\mathbb{R})$ denote the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I_{-\infty}^\alpha}$.

Next, for $0 < \alpha < 1$, we give the relationship between classical fractional Sobolev space $H^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$, where $H^\alpha(\mathbb{R})$ is defined by

$$H^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha}.$$

with the norm

$$(2.8) \quad \|u\|_\alpha = (\|u\|_{L^2}^2 + |u|_\alpha^2)^{1/2},$$

and semi-norm

$$|u|_\alpha = \|\xi|^\alpha \mathcal{F}(u)\|_{L^2}.$$

We note the spaces $H^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equal and have equivalent norms (see [20]). Therefore, we define $H^\alpha(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \xi|^\alpha \mathcal{F}(u) \in L^2(\mathbb{R})\}$.

Before starting our results, we need the following assumptions:

(L) $b: \mathbb{R} \rightarrow (0, +\infty)$ is continuous and $b(t) \rightarrow +\infty$ as $|t| \rightarrow \infty$.

Let

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R}) \mid \int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt < \infty \right\}.$$

The space X^α is a reflexive and separable Hilbert space with the inner product

$$(2.9) \quad \langle u, v \rangle_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + b(t)u(t)v(t)) dt,$$

and the corresponding norm $\|u\|_{X^\alpha}^2 = \langle u, u \rangle_{X^\alpha}$.

Similar to proofs of Lemma 2.1, Lemma 2.2 and Theorem 2.1 in [20], we can get the following lemmas.

LEMMA 2.3. *Suppose $b(t)$ satisfies (L). Then the space X^α is continuously embedded in $H^\alpha(\mathbb{R})$.*

LEMMA 2.4. *Suppose that (L) holds. Then the imbedding of X^α in $L^2(\mathbb{R})$ is continuous and compact.*

LEMMA 2.5. *Let $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_\alpha$ such that*

$$(2.10) \quad \sup_{x \in \mathbb{R}} |u(x)| \leq C \|u\|_\alpha,$$

Also by Lemma 2.5, there is a constant $C_\alpha > 0$ such that

$$(2.11) \quad \|u\|_\infty \leq C_\alpha \|u\|_{X^\alpha}.$$

REMARK 2.6. From Lemma 2.5, we know that if $u \in H^\alpha(\mathbb{R})$ with $1/2 < \alpha < 1$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty)$, because

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_\infty^{q-2} \|u\|_{L^2(\mathbb{R})}^2.$$

REMARK 2.7. From Remark 2.6 and Lemma 2.4, it is easy to verify that the imbedding of X^α in $L^q(\mathbb{R})$ is also compact for $q \in (2, \infty)$. Hence, for all $2 \leq q < \infty$, the imbedding of X^α in $L^q(\mathbb{R})$ is continuous and compact and then this with Lemma 2.5, for all $q \in [2, \infty)$, there exists $C_q > 0$ such that

$$\|u\|_{L^q(\mathbb{R})} \leq C_q \|u\|_{X^\alpha}.$$

First, we consider the eigenvalue problem

$$(2.12) \quad \begin{cases} {}_tD_\infty^\alpha ({}_{-\infty}D_t^\alpha u(t)) + b(t)u(t) = \lambda u, & t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}). \end{cases}$$

We mean by a weak solution of systems (2.12), any $u \in X^\alpha$ such that

$$(2.13) \quad \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + b(t)u(t)v(t)) dt = \lambda \int_{\mathbb{R}} u(t)v(t) dt,$$

for every $v \in X^\alpha$.

THEOREM 2.8. *Suppose that (L) holds. Then each eigenvalue of (2.12) is real and if we repeat each eigenvalue according to its multiplicity, we have $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. λ_1 can be characterized as*

$$(2.14) \quad \lambda_1 = \inf_{u \in X^\alpha \setminus \{0\}} \frac{\int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt}{\int_{\mathbb{R}} |u(t)|^2 dt}.$$

Furthermore, there exists an orthogonal basis $\{w_k\}_{k=1}^\infty$ of X^α , where $w_k \in X^\alpha$ is an eigenfunction corresponding to λ_k for $k = 1, 2, \dots$

PROOF. First we know that X^α is a Hilbert space with the inner product (2.9). Next, we will transform (2.13) into a problem about symmetric compact operator. Since the imbedding of X^α in $L^2(\mathbb{R})$ is compact, then there exist a constant C_L such that

$$(2.15) \quad \|u\|_{L^2(\mathbb{R})} \leq C_L \|u\|_{X^\alpha}.$$

From Hölder inequality and (2.15), for given $u \in L^2(\mathbb{R})$ and any $v \in X^\alpha$,

$$\left| \int_{\mathbb{R}} u(t)v(t) dt \right| \leq \|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} \leq C_L \|u\|_{L^2(\mathbb{R})} \|v\|_{X^\alpha}.$$

In view of the Riesz theorem, there exists a unique $\omega_0 \in X^\alpha$ such that

$$\int_{\mathbb{R}} u(t)v(t) dt = \langle \omega_0, v \rangle_{X^\alpha}, \quad \text{for all } v \in X^\alpha.$$

We now define the operator $K: L^2(\mathbb{R}) \rightarrow X^\alpha$ as $Ku = \omega_0$, so

$$\|Ku\|_{X^\alpha} \leq C_L \|u\|_{L^2(\mathbb{R})},$$

and K is a bounded linear operator from $L^2(\mathbb{R})$ to X^α . Let $S: X^\alpha \rightarrow L^2(\mathbb{R})$ be an imbedding operator, by Lemma 2.4, S is compact. Hence (2.13) is equivalent to

$$\langle u, v \rangle_{X^\alpha} = \langle \lambda \omega_0, v \rangle_{X^\alpha} = \langle \lambda KSu, v \rangle_{X^\alpha}, \quad \text{for all } v \in X^\alpha.$$

Therefore, $(I - \lambda KS)u = 0$.

Since X^α is separable and KS is symmetric and compact, by Riesz–Schauder theory, we know that all eigenvalue $\{\lambda_k\}$ of KS are positive real numbers and there are corresponding eigenfunctions which make up an orthogonal basis of X^α and (2.14) holds. □

Let $V = \text{span}\{e_1\}$ be the one-dimensional eigenspace associated with λ_1 , where $e_1 > 0$ in \mathbb{R} and $\|e_1\|_{X^\alpha} = 1$. Taking one subspace $Y \subset X^\alpha$ completing V such that $X^\alpha = V \oplus Y$, there exists $\bar{\lambda} > \lambda_1$ such that

$$\int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \geq \bar{\lambda} \int_{\mathbb{R}} |u(t)|^2 dt, \quad \text{for } u \in Y.$$

3. Existence solution by the minimal principle

The functional $J: X^\alpha \rightarrow \mathbb{R}$ corresponding to problem (1.1) is defined by

$$(3.1) \quad J(u) = \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) - \int_{\mathbb{R}} F(t, u(t)) dt$$

$$= \frac{1}{2} \overline{M} (\|u\|_{X^\alpha}^2) - \int_{\mathbb{R}} F(t, u(t)) dt,$$

where $\overline{M}(s) = \int_0^s M(t) dt$ and $F(x, t) = \int_0^t f(x, s) ds$ and

$$(3.2) \quad \Phi(u) = \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right).$$

In this section, we assume that $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function with subcritical growth with respect to t , that is:

- (F1) $|f(t, x)| \leq h(t)|x|^{q-1}$ hold for all $t \in \mathbb{R}, x \in \mathbb{R}$, where $2 \leq q < \infty$ and $h: \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function such that $h \in L^\infty(\mathbb{R})$; and $M: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following condition
- (H0) $M(t) \geq mt^{\alpha_0-1}$ for all $t \in \mathbb{R}^+$, where $m > 0, \alpha_0 > 1$ and $q < 2\alpha_0$;

Now, we can state our main result in this section.

THEOREM 3.1. *Let L satisfies assumption (L). Then under assumptions (H0) and (F1) the problem (1.1) has at least one weak solution in X^α .*

PROOF. Let (u_n) be a sequence that converges weakly to u in X_0 , so by Remark 2.7, we have

$$(3.3) \quad \begin{cases} u_n \rightharpoonup u & \text{weakly in } X^\alpha, \\ u_n \rightarrow u & \text{strongly in } L^q(\mathbb{R}), \quad 2 \leq q < \infty. \end{cases}$$

Therefore, by the weak lower semicontinuous of the norm, one can get

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u_n(t)|^2 + b(t)|u_n(t)|^2) dt \geq \int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt.$$

Combining this with the continuity and monotonicity of the function \overline{M} , we have

$$(3.4) \quad \begin{aligned} \liminf_{n \rightarrow \infty} \Phi(u_n) &= \liminf_{n \rightarrow \infty} \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u_n(t)|^2 + b(t)|u_n(t)|^2) dt \right) \\ &\geq \frac{1}{2} \overline{M} \left(\liminf_{n \rightarrow \infty} \int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u_n(t)|^2 + b(t)|u_n(t)|^2) dt \right) \\ &\geq \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) = \Phi(u). \end{aligned}$$

By (F1), Remark 2.7 and the Hölder inequality, we get

$$(3.5) \quad \begin{aligned} &\int_{\mathbb{R}} (F(t, u_n(t)) - F(t, u(t))) dt \\ &\leq \int_{\mathbb{R}} f(t, u + \theta_n(u_n - u))|u_n - u| dt \end{aligned}$$

$$\begin{aligned} &\leq \int_{\mathbb{R}} h(t)|u + \theta_n(u_n - u)|^{q-1}|u_n(t) - u(t)| dx \\ &\leq \|h\|_{L^\infty(\mathbb{R})} \|u + \theta_n(u_n - u)\|_{L^q}^{q-1} \|u_n - u\|_{L^q}, \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$, where $0 \leq \theta_n(t) \leq 1$, for all $t \in \mathbb{R}$. From (3.4) and (3.5), the functional J is weakly lower semicontinuous in X^α .

On the other hand, by assumptions (H0), (F1) and Remark 2.7, we have

$$\begin{aligned} (3.6) \quad J(u) &= \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) - \int_{\mathbb{R}} F(t, u(t)) dt \\ &\geq \frac{m}{2\alpha_0} \|u\|_{X^\alpha}^{2\alpha_0} - \int_{\mathbb{R}} h(t)|u(t)|^q dt \\ &\geq \frac{m}{2\alpha_0} \|u\|_{X^\alpha}^{2\alpha_0} - \|h\|_{L^\infty(\mathbb{R})} \|u\|_{L^q}^q \geq \frac{m}{2\alpha_0} \|u\|_{X^\alpha}^{2\alpha_0} - \|h\|_{L^\infty(\mathbb{R})} C_q^q \|u\|_{X^\alpha}^q. \end{aligned}$$

Since $q < 2\alpha_0$, it follows from (3.6), that the functional J is coercive. Thus, using the minimal principle, we deduce that the functional J has at least one weak solution and therefore the problem (1.1) has at least one weak solution. \square

4. Existence solution by Morse theory

In this section, we assume that $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with the following conditions:

(M0) there exists a constant $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;

(M1) there exists a constant $m_1 > 0$ such that $M(t) \leq m_1$ for all $t \geq 0$.

Also, we make the following assumptions:

(F2) there exist $r > 0$, $\tilde{\lambda} \in (\lambda_1, \bar{\lambda})$ such that $m_1 \lambda_1 < m_0 \tilde{\lambda}$, and $|u| \leq r$ implies

$$m_1 \lambda_1 |u|^2 \leq 2F(t, u) \leq m_0 \tilde{\lambda} |u|^2;$$

(F3) $2F(t, x)/|x|^2 < m_0(\lambda_1 - \mu)$ For any $\mu > 0$ and for all $(t, x) \in (\mathbb{R}, \mathbb{R})$;

(F4) $\lim_{|x| \rightarrow \infty} (f(t, x)x - 2F(t, x)) = +\infty$.

LEMMA 4.1. *Assume that (L) and (M0) hold. Then Φ' is of type (S_+) , i.e. if $u_n \rightharpoonup u$ in X and*

$$\overline{\lim}_{n \rightarrow \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = 0,$$

then $u_n \rightarrow u$ in X^α .

PROOF. For all $u, v \in X^\alpha$, we have

$$\begin{aligned} \langle \Phi'(u) - \Phi'(v), u - v \rangle &= M \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha (u - v)(t)|^2 + b(t)|(u - v)(t)|^2) dt \right) \\ &\quad \times \int_{\mathbb{R}} (|_{-\infty} D_t^\alpha (u - v)(t)|^2 + b(t)|(u - v)(t)|^2) dt \geq m_0 \|u - v\|_{X^\alpha}^2. \end{aligned}$$

So, the operator Φ' is strongly monotone, then possesses the property of type (S_+) . \square

LEMMA 4.2. Assume that $\alpha \in (1/2, 1]$ and $b(t)$ satisfies assumption (L). Under assumptions (M0) and (F1), any bounded sequence $\{u_n\}$ in X^α such that $J'(u_n) \rightarrow 0$ in $(X^\alpha)^*$ as $n \rightarrow \infty$ has a convergent subsequence.

PROOF. Since (u_n) is bounded in X^α and X^α is a reflexive Banach space and so by passing to a subsequence (for simplicity denoted again by $\{u_n\}$) if necessary, by Remark 2.7, we may assume that

$$(4.1) \quad \begin{cases} u_n \rightharpoonup u & \text{weakly in } X^\alpha, \\ u_n \rightarrow u & \text{strongly in } L^q(\mathbb{R}^n) \quad (2 \leq q < \infty). \end{cases}$$

Therefore

$$\langle J'(u_n), u_n - u \rangle \rightarrow 0, \quad \int_{\mathbb{R}} f(t, u_n(t))(u_n(t) - u(t)) dt \rightarrow 0,$$

so, we get

$$\varepsilon_n \|u_n - u\| \geq \langle J'(u_n), u_n - u \rangle = \langle \Phi'(u_n), u_n - u \rangle - \int_{\mathbb{R}} f(t, u_n(t))(u_n(t) - u(t)) dt$$

with $\varepsilon_n \rightarrow 0$. Thus $\limsup_{n \rightarrow \infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0$. By (4.1), it is easy to get $\lim_{n \rightarrow \infty} \langle \Phi'(u), u_n - u \rangle = 0$. Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \\ \leq \limsup_{n \rightarrow \infty} \langle \Phi'(u_n), u_n - u \rangle - \liminf_{n \rightarrow \infty} \langle \Phi'(u), u_n - u \rangle \leq 0. \end{aligned}$$

Since Φ' is of type (S₊) (see Lemma 4.1), so we obtain $u_n \rightarrow u$ as $n \rightarrow \infty$ in X^α . □

LEMMA 4.3. Assume that $\alpha \in (1/2, 1]$ and $b(t)$ satisfies assumption (L). Under assumptions (M0), (F1) and (F3), the functional J is coercive in X^α .

PROOF. By (F3) one can get for small enough $\mu > 0$,

$$F(t, x) \leq \frac{m_0}{2}(\lambda_1 - \mu)|x|^2, \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}.$$

Thus, by definition of λ_1 , for $u \in X^\alpha$

$$\begin{aligned} J(u) &= \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty}D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) - \int_{\mathbb{R}} F(t, u(t)) dt \\ &\geq \frac{m_0}{2} \|u\|_{X^\alpha}^2 - \frac{m_0}{2} (\lambda_1 - \mu) \|u\|_{L^2}^2 \\ &\geq \frac{m_0}{2} \left(1 - \frac{\lambda_1 - \mu}{\lambda_1} \right) \|u\|_{X^\alpha}^2 \rightarrow +\infty \quad \text{as } \|u\|_{X^\alpha} \rightarrow \infty. \end{aligned}$$

Therefore, we have that J is coercive in X^α . □

By Lemmas 4.2 and 4.3, it follows that:

LEMMA 4.4. *Under assumptions (M0), (F1) and (F3) (or substitute (F3) for (F4)), the functional J satisfies the Palais–Smale condition.*

Now, we recall the definition of local linking to proceed with our proof.

DEFINITION 4.5. We say that a functional φ has a local linking to the decomposition of the space $X = V \oplus Y$ near the origin 0 if and only if there is a small ball B_ρ with the center at 0 and small radius $\rho > 0$ such that

$$\begin{aligned}\varphi(v_1) &> \varphi(0), \quad \text{for } v_1 \in B_\rho \cap Y \setminus \{0\}, \\ \varphi(v_2) &\leq \varphi(0), \quad \text{for } v_2 \in B_\rho \cap V.\end{aligned}$$

LEMMA 4.6. *Assume that $\alpha \in (1/2, 1]$ and $b(t)$ satisfies assumption (L), then under assumptions (M0), (M1), (F1) and (F2), the functional J has a local linking at the origin with respect to $X^\alpha = V \oplus Y$, where V and Y are functional subspaces of X^α in the Section 1.*

PROOF. First, we take $u \in V$; since V is finite dimensional, by (2.11), we can see that $\|u\|_{X^\alpha} \leq \rho$ implies $|u| \leq r := \rho C_\alpha$, for all $t \in \mathbb{R}$ and $\rho > 0$ small enough. Thus by (F2), for $\|u\|_{X^\alpha} \leq \rho$, we get

$$\begin{aligned}J(u) &= \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) - \int_{\mathbb{R}} F(t, u(t)) dt \\ &\leq \frac{m_1}{2} \lambda_1 \int_{\mathbb{R}} |u(t)|^2 dt - \int_{\mathbb{R}} F(t, u(t)) dt \\ &= \int_{|u| \leq r} \left(\frac{m_1}{2} \lambda_1 |u(t)|^2 - F(t, u(t)) \right) dt \leq 0.\end{aligned}$$

On the other hand, we take $u \in Y$; from (F1), (F2) and the definition of $\tilde{\lambda}$, one gets

$$\begin{aligned}J(u) &= \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt \right) - \int_{\mathbb{R}} F(t, u(t)) dt \\ &\geq \frac{m_0}{2} \left(\int_{\mathbb{R}} (|_{-\infty} D_t^\alpha u(t)|^2 + b(t)|u(t)|^2) dt - \tilde{\lambda} \int_{\mathbb{R}} |u(t)|^2 dt \right) \\ &\quad - \int_{\{|u| \leq r\}} \left(F(t, u(t)) - \frac{\tilde{\lambda} m_0}{2} |u(t)|^2 \right) dt \\ &\quad - \int_{\{|u| > r\}} \left(F(t, u(t)) - \frac{\tilde{\lambda} m_0}{2} |u(t)|^2 \right) dt \\ &\geq \frac{m_0}{2} \left(1 - \frac{\tilde{\lambda}}{\lambda} \right) \|u\|_{X^\alpha}^2 - C_3 \int_{\{|u| > r\}} |u(t)|^s dt \\ &\geq \frac{m_0}{2} \left(1 - \frac{\tilde{\lambda}}{\lambda} \right) \|u\|_{X^\alpha}^2 - C_3 C_s^s \|u\|_{X^\alpha}^s,\end{aligned}$$

($2 < s < \infty$). So we can derive that when $u \in X^\alpha$ and $0 < \|u\|_{X^\alpha} \leq \rho$ and $\rho > 0$ small, $J(u) > 0$, which completes the proof. \square

REMARK 4.7. From the proof of Lemma 4.4, we can get a stronger result: there exists a $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$, B_ρ satisfies all the conditions required by the definition of local linking. From this point of view, we can conclude that $0 \in X^\alpha$ is the unique critical point of J in a ball that is small enough.

For an isolated critical point $u \in \mathcal{K}$ of a C^1 functional $f: E \rightarrow \mathbb{R}$, we define a the critical group of f at u as follows:

$$C_q(f, u) = H_q(f_c \cap B_\rho, f_c \cap B_\rho \setminus \{u\}), \quad q = 0, 1, \dots,$$

in which $c = f(u)$, $f_c = \{v \in E : f(v) \leq c\}$, $\rho > 0$ is small and $H_q(\cdot, \cdot)$ the q -th singular relative homology group with integer coefficients.

We say that u is an homological nontrivial critical point of f if at least one of its critical groups is nontrivial.

DEFINITION 4.8 ([5], [6]). We say f satisfies the deformation condition at level $c \in \mathbb{R}$ if, for any $\delta > 0$ and any neighbourhood B of c , there are $\varepsilon > 0$ and a continuous deformation $\eta: E \times [0, 1] \rightarrow E$ such that

- (a) $\eta(u, t) = u$ for either $t = 0$ or $u \notin f^{-1}[c - \delta, c + \delta]$,
- (b) $f(\eta(u, t))$ is nonincreasing in t for any $u \in E$,
- (c) $\eta(f_{c+\varepsilon} \setminus B) \subset f_{c-\varepsilon}$.

We note here that the usual Palais–Smale condition can imply the deformation condition.

Let f satisfy the deformation condition and let $\#\mathcal{K} < \infty$. Take $a < \inf f_c$. By the Morse theory, the information of all critical point of f are contained in the Morse inequality

$$(4.2) \quad \sum_{u \in \mathcal{K}} P(t, u) = P(t, \infty) + (1 + t)Q(t),$$

where

$$(4.3) \quad P(t, u) = \sum_{q=0}^{\infty} \dim C_q(f, u)t^q, \quad P(t, \infty) = \sum_{q=0}^{\infty} \dim H_q(E, f_a)t^q,$$

are the Poincaré polynomials of f at $u \in \mathcal{K}$ and at infinity and $Q(t)$ is a formal series with nonnegative integer coefficients.

Now, we can give a result from Morse theory as follows.

LEMMA 4.9 ([12, Theorem 2.1]). *Let E be a Banach space and $f: E \rightarrow \mathbb{R}$ a C^1 functional satisfying the Palais–Smale condition. Suppose that E has a decomposition $E = W \oplus Z$, where W is a finite dimensional subspace, say $\dim W = m < \infty$. Suppose that there exists a small ball B_ρ with its center at the origin 0 and small radius $\rho > 0$ such that*

$$f(z) > f(0), \quad \text{for } z \in B_\rho \cap Z \setminus \{0\};$$

$$f(w) \leq f(0), \quad \text{for } w \in B_\rho \cap W.$$

If $0 \in E$ is the unique critical point of f in B_ρ , then

$$C_m(f, 0) = H_m(f_c \cap B_\rho, f_c \cap B_\rho \setminus \{0\}) \neq 0.$$

Here, since $\dim W = 1 < \infty$, by Lemma 4.6, Remark 4.7 and Lemma 4.9, we have the following lemma:

LEMMA 4.10. *Assume that $\alpha \in (1/2, 1]$ and $b(t)$ satisfies assumption (L), then under assumptions (M0), (M1), (F1) and (F2), 0 is a critical point of J and $C_1(J, 0) \neq 0$.*

PROPOSITION 4.11 ([12]). *Assume that J has a critical point $u = 0$ with $J(0) = 0$. If J has a local linking at 0 with respect to $X = V \oplus W$, $k = \dim V < \infty$, i.e. there exists $\rho > 0$ small such that*

$$\begin{aligned} J(0) &\leq 0, & u \in V, \quad \|u\| \leq \rho, \\ J(u) &> 0, & u \in W, \quad 0 < \|u\| \leq \rho, \end{aligned}$$

Then $C_k(J, 0) \not\cong 0$; that is, 0 is an homological nontrivial critical point of J .

Now, we can state our main result.

THEOREM 4.12. *Let $\alpha \in (1/2, 1]$ and $b(t)$ satisfies assumption (L). Assume (M0), (M1), (F1)–(F3) hold, then the problem (1.1) has at least two nontrivial weak solutions in X^α .*

PROOF. By Lemmas 4.3, 4.4 and Definition 4.8, J is coercive and satisfies the Palais–Smale condition and deformation condition. Thus J is bounded below. By Lemmas 4.6, 4.10 and Proposition 4.11, the trivial solution $0 \in X^\alpha$ is a homologically nontrivial critical point of J but not a minimizer. Since J is bounded from below and satisfies the deformation property, it follows that J attains its minimum at some $u^* \in X^\alpha$ and $C_q(J, u^*) \cong \delta_{q,0}\mathbb{Z}$. Set $a < \inf J(X^\alpha)$, then $H_q(X^\alpha, J_a) \cong \delta_{q,0}\mathbb{Z}$. Now, we may assume that u^* is the only minimizer of J . Assume that J has only two critical points 0 and u^* , i.e. $K = \{0, u^*\}$. Then the Morse inequality (4.2) reads as

$$(4.4) \quad P(t, 0) + P(t, u^*) = P(t, \infty) + (1 + t)Q(t).$$

It follows that $P(t, 0) = (1 + t)Q(t)$. Because 0 is not a minimizer of J and is homologically nontrivial, we see that $C_0(J, 0) \cong 0$ and $P(t, 0) \neq 0$. So by Lemma 4.10, we have $H_1(J_{c_0}, J_{c_0} \setminus \{0\}) = C_1(J, 0) \neq 0$ where $c_0 = J(0)$. By the deformations $X^\alpha \simeq J_{c_0}$ and $J_{c_0} \setminus \{0\} \simeq \{u^*\}$, one can get

$$H_1(J_{c_0}) \cong H_1(X^\alpha), \quad H_1(J_{c_0} \setminus \{0\}) \cong H_0(J_{c_0} \setminus \{0\}) \cong 0.$$

Now from the exact sequence

$$\cdots \rightarrow H_1(J_{c_0} \setminus \{0\}) \rightarrow H_1(J_{c_0}) \rightarrow H_1(J_{c_0})(J_{c_0}, J_{c_0} \setminus \{0\}) \rightarrow H_0(J_{c_0} \setminus \{0\}) \rightarrow \cdots,$$

we see easily that $H_1(J_{c_0})(J_{c_0}, J_{c_0} \setminus \{0\}) = C_1(J, 0) \cong 0$. This is contradiction from Lemma 4.10. Then J at least three critical points. Also, by Lemma 4.10, 0 is a trivial critical point of J . Therefore, taking into account that the critical points of the functional J are exactly the weak solutions of systems (1.1), we have the conclusion. \square

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