Topological **M**ethods in **N**onlinear **A**nalysis
Volume 45, No. 2, 2015, 385–397

MULTIPLE NONSEMITRIVIAL SOLUTIONS FOR A CLASS OF DEGENERATE QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. We prove the existence of multiple nonnegative nonsemitrivial solutions for a degenerate quasilinear elliptic system. Our technical approach is based on variational methods.

1. Introduction

In this paper, we prove the multiplicity of solutions for the following degenerate quasilinear elliptic system, defined on Ω,

$$(1.1_\lambda)\qquad \begin{aligned} -\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v \\ &\quad + \frac{\mu(x)}{(\alpha+1)(\delta+1)}|u|^{\gamma-1}|v|^{\delta+1}u, \\ -\nabla(\nu_2(x)|\nabla v|^{q-2}\nabla v) &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u \\ &\quad + \frac{\mu(x)}{(\beta+1)(\gamma+1)}|u|^{\gamma+1}|v|^{\delta-1}v, \end{aligned}$$

$$(1.2_\lambda)\qquad u|_{\partial\Omega} = v|_{\partial\Omega} = 0,$$

2010 *Mathematics Subject Classification.* 35J70, 35P30, 35B32, 35J50.
Key words and phrases. Degenerate equations, principal eigenvalue, multiple solutions, bifurcation, Ekeland's variational principle.

where Ω is a bounded and connected subset of $\mathbb{R}^N$, $N \geq 2$. The degeneracy of this system is considered in the sense that the measurable, nonnegative diffusion coefficients ν_1, ν_2 are allowed to vanish in Ω, (as well as at the boundary $\partial\Omega$) and/or to blow up in $\overline{\Omega}$.

Throughout this paper, we assume the following hypotheses:

($\mathcal{H}$) $N > p > 1$, $N > q > 1$, $\alpha \geq 0$ and $\beta \geq 0$ satisfying $(\alpha+1)/p + (\beta+1)/q = 1$, $\gamma \geq 0$, $\delta \geq 0$ and $p < \gamma + 1$ or $q < \delta + 1$ satisfying $(\gamma+1)/p^* + (\delta+1)/q^* < 1$.

The quantities p^* and q^* are defined in the next section.

($\mathcal{H}_1$) The exponents α, β, γ and δ satisfy also the general condition

$$\frac{1}{(\alpha+1)(\delta+1)} + \frac{1}{(\beta+1)(\gamma+1)} < 1.$$

We introduce the function space

$(\mathcal{N})_p$ which consists of nonnegative weighted functions $\nu\colon \Omega \subset \mathbb{R}^N \to \mathbb{R}$ such that ν vanishes and/or tends to infinity at finite points at most, $\nu \in L^1_{\rm loc}(\Omega)$, $\nu^{-1/(p-1)} \in L^1_{\rm loc}(\Omega)$ and $\nu^{-s} \in L^1(\Omega)$, for some $p > 1$, $s > \max\{N/p, 1/(p-1)\}$ satisfying $ps \leq N(s+1)$.

Then for the weight functions ν_1, ν_2 we assume the following hypothesis:

($\mathcal{N}$) There exist functions μ_1 satisfying condition $(\mathcal{N})_p$, for some s_p, and μ_2 satisfying condition $(\mathcal{N})_q$, for some s_q, such that

$$\frac{\mu_1(x)}{c_1} \leq \nu_1(x) \leq c_1\,\mu_1(x) \quad \text{and} \quad \frac{\mu_2(x)}{c_2} \leq \nu_2(x) \leq c_2\,\mu_2(x), \tag{1.3}$$

almost everywhere in Ω, for some constants $c_1 > 1$ and $c_2 > 1$.

Furthermore, we suppose that the coefficient functions $a(x)$, $d(x)$, $b(x)$ and $\mu(x)$ satisfy the following conditions:

(Υ_1) a is a smooth function, at least $C^{0,\zeta}_{\rm loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $a \in L^{p^*/(p^*-p)}(\Omega)$ and either there exists $\Omega_a^+ \subset \Omega$ of positive Lebesgue measure, i.e. $|\Omega_a^+| > 0$ such that $a(x) > 0$, for all $x \in \Omega_a^+$, neither $a(x) \equiv 0$, in Ω.

(Υ_2) d is a smooth function, at least $C^{0,\zeta}_{\rm loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $d \in L^{q^*/(q^*-q)}(\Omega)$ and either there exists $\Omega_d^+ \subset \Omega$ of positive Lebesgue measure, i.e. $|\Omega_d^+| > 0$ such that $d(x) > 0$, for all $x \in \Omega_d^+$, neither $d(x) \equiv 0$ in Ω.

(Υ_3) $b(x) \geq 0$ almost everywhere in Ω, $b \not\equiv 0$ and $b \in L^{\omega_1}(\Omega) \cap L^\infty(\Omega)$, where $\omega_1 = [1 - (\alpha+1)/p^* - (\beta+1)/q^*]^{-1}$.

(Υ_4) μ is sign changing (i.e. $\mu^+ \not\equiv 0$, $\mu^- \not\equiv 0$) and $\mu \in L^{\omega_2}(\Omega) \cap L^\infty(\Omega)$, where $\omega_2 = [1 - (\gamma+1)/p^* - (\delta+1)/q^*]^{-1}$.

In addition the function $\mu(x)$ satisfies the following key condition:

(Υ_5) $\displaystyle\int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx < 0,$

where (u_1, v_1) is the positive normalized eigenfunction of the unperturbed system:

$$\begin{aligned} -\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v, \quad x \in \Omega, \\ -\nabla(\nu_2(x)|\nabla v|^{q-2}\nabla v) &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u, \quad x \in \Omega, \end{aligned} \tag{1.4$_\lambda$}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{1.5$_\lambda$}$$

corresponding to the positive principal eigenvalue λ_1.

As it was proved in [13] (see also Section 2, Theorem 2.4), system (1.4$_\lambda$) is in fact an eigenvalue problem which admits a positive principal eigenvalue λ_1 and the corresponding normalized eigenfunctions (u_1, v_1) are positive, up to singular and/or degenerate points, componentwise. Moreover, up to the singular/degenerate points of ν_1 and ν_2, they are also bounded and sufficiently smooth.

REMARK 1.1. An example of the weighted function $\mu(x)$ which satisfies both conditions (Υ_4) and (Υ_5) may be the following; Let $\mu(x)$ be a smooth function in $\overline{\Omega}$, which is zero at a neighbourhood of the singular/degenerate points and $\mu(x)$ satisfies

$$\int_\Omega \mu^-(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx > \int_\Omega \mu^+(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx, \tag{1.6}$$

for $\mu(x) = \mu^+(x) - \mu^-(x)$, i.e. μ^+ and μ^- are the positive and the negative part of μ, respectively. More precisely, let z_i, $i = 1, \ldots, n$, be the finite singular and/or degenerate points. Assume for some small enough ε, the spheres $B_\varepsilon(z_i)$ centered at z_i. Since z_i are finite we can find $\widetilde{\Omega} \subset \Omega$, such that

$$\widetilde{\Omega} \cap \overline{\bigcup_i B_\varepsilon(z_i)} = \emptyset.$$

Note that $\widetilde{\Omega}$ may be chosen such that, both u_i and v_1 are uniformly bounded from above and uniformly bounded away from zero.

We define now μ to be continuous in $\overline{\Omega}$, such that $\mu(x) = 0$, for $x \in \bigcup_i \overline{B_\varepsilon(z_i)}$, μ is positive in $\Omega \setminus \overline{\widetilde{\Omega} \cup \bigcup_i B_\varepsilon(z_i)}$, with $\mu(x) < \delta$, in $\Omega \setminus \overline{\widetilde{\Omega} \cup \bigcup_i B_\varepsilon(z_i)}$, δ small enough and μ is negative in $\widetilde{\Omega}$, with sufficiently large L^∞ norm, such that

$$\int_{\widetilde{\Omega}} |\mu(x)||u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx > \int_{\Omega\setminus\widetilde{\Omega}} \mu^+(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx,$$

i.e. (1.6) holds.

An example of the physical motivation of the assumptions $(\mathcal{N})$, $(\mathcal{N})_p$ may be found in [4, p. 79]. These assumptions are related to the modelling of reaction diffusion processes in composite materials occupying a bounded domain Ω, which at some points they behave as perfect insulators. When at some points the medium is perfectly insulating, it is natural to assume that $\nu_1(x)$ and/or $\nu_2(x)$ vanish in $\overline{\Omega}$. For more information we refer to [13] and the references therein.

Multiplicity results for semilinear and quasilinear elliptic systems have received a great deal of interest in recent years; see, for instance, the papers [1]–[3], [5], [6], [8]–[10], [12] and the references therein.

We note that the procedure here is based on the arguments developed in [6] and [8]. Following along the same lines as in [8], we will prove multiplicity of nonsemitrivial solutions for the system (1.1_λ)–(1.2_λ).

2. The eigenvalue problem (1.4_λ)–(1.5_λ)

Let $\nu(x)$ be a nonnegative weight function in Ω which satisfies condition $(\mathcal{N})_p$. We consider the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\nu)$ to be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu)} := \bigg(\int_\Omega \nu(x)|\nabla u|^p\bigg)^{1/p}.$$

The space $\mathcal{D}_0^{1,p}(\Omega,\nu)$ is a reflexive Banach space. For a discussion about the space setting we refer to [7] and the references therein. Let

$$p_s^* := \frac{Nps}{N(s+1)-ps}.$$

Lemma 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ and the weight ν satisfies $(\mathcal{N})_p$. Then the following embeddings hold*:

(a) $\mathcal{D}_0^{1,p}(\Omega,\nu) \hookrightarrow L^{p_s^*}(\Omega)$ *continuously for* $1 < p_s^* < N$,
(b) $\mathcal{D}_0^{1,p}(\Omega,\nu) \hookrightarrow L^r(\Omega)$ *compactly for any* $r \in [1, p_s^*)$.

The space setting for our problem is the product space

$$Z := \mathcal{D}_0^{1,p}(\Omega,\nu_1) \times \mathcal{D}_0^{1,q}(\Omega,\nu_2)$$

equipped with the norm

$$\|z\|_Z := \|u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \|v\|_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}, \quad z=(u,v)\in Z.$$

Observe that inequalities (1.3) in condition $(\mathcal{N})$ implies that the functional spaces $\mathcal{D}_0^{1,p}(\Omega,\nu_1) \times \mathcal{D}_0^{1,q}(\Omega,\nu_2)$ and $\mathcal{D}_0^{1,p}(\Omega,\mu_1) \times \mathcal{D}_0^{1,q}(\Omega,\mu_2)$ are equivalent.

In the sequel we denote by p^* and q^* the quantities $p_{s_p}^*$ and $p_{s_q}^*$, respectively, where s_p and s_q are induced by condition $(\mathcal{N})$. Also, we use $\|\cdot\|_{1,p}$ and $\|\cdot\|_{1,q}$ for the norms $\|\cdot\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)}$ and $\|\cdot\|_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}$, respectively.

We introduce the functionals $J, D, B, M\colon Z \to \mathbb{R}$ as follows:

$$\begin{aligned} J(u,v) &:= \frac{\alpha+1}{p}\int_\Omega \nu_1(x)|\nabla u|^p\,dx + \frac{\beta+1}{q}\int_\Omega \nu_2(x)|\nabla v|^q\,dx, \\ D(u,v) &:= \frac{\alpha+1}{p}\int_\Omega a(x)|u|^p\,dx + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q\,dx, \\ B(u,v) &:= \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx, \\ M(u,v) &:= \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,dx. \end{aligned}$$

It is a standard procedure (see [8], [12]) to prove the following properties of these functionals.

LEMMA 2.2. *The functionals $J, D, B,$ and M are well defined. Moreover, J is continuous and D, B and M are compact.*

Next, we introduce the functionals $A_\lambda, I_\lambda\colon Z \to \mathbb{R}$ in the following way:

$$\begin{aligned} A_\lambda(u,v) &:= J(u,v) - \lambda D(u,v) - \lambda B(u,v), \\ I_\lambda(u,v) &:= A_\lambda(u,v) - \frac{1}{(\gamma+1)(\delta+1)}M(u,v). \end{aligned}$$

The functionals A_λ and I_λ are well defined, and they are weakly lower semicontinuous. Clearly, $I_\lambda \in C^1(Z,\mathbb{R})$.

DEFINITION 2.3. We say that (u,v) is a *weak solution* of the system (1.1_λ)–(1.2_λ) if (u,v) is a critical point of the functional I_λ, i.e.

$$\begin{aligned} \int_\Omega \nu_1(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx &= \lambda\int_\Omega a(x)|u|^{p-2}u\phi\,dx + \lambda\int_\Omega b(x)|u|^{\alpha-1}|v|^{\beta+1}u\phi\,dx \\ &\quad + \frac{1}{(\alpha+1)(\delta+1)}\int_\Omega \mu(x)|u|^{\gamma-1}|v|^{\delta+1}u\phi\,dx, \\ \int_\Omega \nu_2(x)|\nabla v|^{q-2}\nabla v\cdot\nabla\psi\,dx &= \lambda\int_\Omega d(x)|v|^{q-2}v\psi\,dx + \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta-1}v\psi\,dx, \\ &\quad + \frac{1}{(\beta+1)(\gamma+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta-1}v\psi\,dx \end{aligned}$$

for any $(\phi,\psi) \in Z$.

By a *semitrivial solution*, we mean any weak solution $(u,v) \in Z$ of the form $(u,0)$ or $(0,v)$. Otherwise, the solution is called *nonsemitrivial*.

Concerning the eigenvalue problem (1.4_λ)–(1.5_λ) we have the following result, which was proved in [13].

THEOREM 2.4 ([13, Theorem 1.1]). *The system (1.4_λ)–(1.5_λ) admits a positive principal eigenvalue λ_1, given by*

$$\lambda_1 = \inf_{D(u,v)+B(u,v)=1} J(u,v). \tag{2.1}$$

The associated normalized eigenfunction (u_1, v_1) *belongs to* Z *and each component is nonnegative. In addition,*

(a) *the set of all eigenfunctions corresponding to the principal eigenvalue* λ_1 *forms a one-dimensional manifold* $E_1 \subset Z$*, which is defined by*
$$E_1 = \{(c_1 u_1, c_1^{p/q} v_1) : c_1 \in \mathbb{R}\}.$$
(b) λ_1 *is the only eigenvalue of* (1.4_λ)–(1.5_λ) *to which corresponds a componentwise nonnegative eigenfunction.*

(c) λ_1 *is isolated in the following sense: there exists* $\eta > 0$*, such that the interval* $(0, \lambda_1 + \eta)$ *does not contain any other eigenvalue than* λ_1*.*

Based on the properties of λ_1, the authors in [11], proved certain bifurcation results:

Definition 2.5. Let $E = \mathbb{R} \times Z$ be equipped with the norm
$$\|(\lambda, u, v)\|_E = (|\lambda|^2 + \|(u, v)\|_Z^2)^{1/2}, \quad (\lambda, u, v) \in E. \tag{2.2}$$
We say that the set
$$C = \{(\lambda, u, v) \in E : (\lambda, u, v) \text{ solves } (1.1_\lambda),\ (u, v) \neq (0, 0)\}$$
is a continuum of nontrivial solutions of (1.1_λ), if it is a connected set in E with respect to the topology induced by the norm (2.2). We say $\lambda_0 \in \mathbb{R}$ is a bifurcation point of the system (1.1_λ) (in the sense of Rabinowitz), if there is a continuum of nontrivial solutions C of (1.1_λ) such that $(\lambda_0, 0, 0) \in \overline{C}$ and C is either unbounded in E or there is an eigenvalue $\widehat{\lambda} \neq \lambda_0$, such that $(\widehat{\lambda}, 0, 0) \in \overline{C}$.

More precisely, from [11, Theorem 4.6 and Proposition 4.7] we have that

Theorem 2.6. *The principal eigenvalue* $\lambda_1 > 0$ *of the unperturbed problem* (1.4_λ)–(1.5_λ) *is a bifurcation point* (*in the sense of Rabinowitz*) *of the perturbed system* (1.1_λ)*. Moreover, there exists an* $\eta > 0$ *small enough, such that for each* $(\lambda, u, v) \in C \cap B_\eta(\lambda_1, 0)$*, we have* $u(x) \geq 0$ *and* $v(x) \geq 0$*, almost everywhere in* Ω*.*

Based now on the properties of the scalar eigenvalue problem, we may prove the following properties of the solutions of (1.1_λ)–(1.2_λ).

Lemma 2.7. *Let* λ *be close enough to* λ_1*. Every nontrivial solution* (u, v) *of* (1.1_λ)–(1.2_λ) *is nonsemitrivial.*

Proof. First consider the following eigenvalue problems:
$$-\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u, \quad x \in \Omega, \tag{2.3_λ}$$
$$u|_{\partial\Omega} = 0, \tag{2.4_λ}$$

and

$$(2.5_\lambda) \qquad -\nabla(\nu_2(x)|\nabla v|^{q-2}\nabla v) = \lambda d(x)|v|^{q-2}v, \quad x \in \Omega,$$

$$(2.6_\lambda) \qquad v|_{\partial\Omega} = 0.$$

It is known (see [7]) that the problem (2.3_λ)–(2.4_λ) ((2.5_λ)–(2.6_λ), resp.) has a positive principal eigenvalue $\lambda^{'}(\lambda^{''}$, resp.), which is characterized variationally by

$$\lambda^{'} = \inf_{u\in\mathcal{D}_0^{1,p}(\Omega,\nu_1)\setminus\{0\}} \frac{\int_\Omega \nu_1(x)|\nabla u|^p\,dx}{\int_\Omega a(x)|u|^p\,dx}$$

$$\left(\lambda^{''} = \inf_{v\in\mathcal{D}_0^{1,q}(\Omega,\nu_2)\setminus\{0\}} \frac{\int_\Omega \nu_2(x)|\nabla v|^q\,dx}{\int_\Omega d(x)|v|^q\,dx}, \text{ resp.}\right).$$

This eigenvalues is simple and isolated and it is the only one having a positive eigenfunction $\phi^{'}(\phi^{''}$, resp.). Now, observe that the nonzero component of any semitrivial solution of the system (1.1_λ)–(1.2_λ) corresponds to an eigenfunction of (2.3_λ)–(2.4_λ) or (2.5_λ)–(2.6_λ). So it suffices to prove that $\lambda_1 < \min\{\lambda^{'},\lambda^{''}\}$. Suppose not. Then the system $(2.4_{\lambda'})$–$(2.5_{\lambda'})$ ($(2.4_{\lambda''})$–$(2.5_{\lambda''})$, resp.) would have a solution $(\phi^{'},0)$ ($(0,\phi^{''})$, resp.). From the variational characterization (2.1) of the eigenvalue λ_1 this is a contradiction, and so the proof is complete.□

3. Main results

First, we introduce some notations. Let Λ_λ be the Nehari manifold associated with (1.1_λ)–(1.2_λ); i.e.

$$\Lambda_\lambda := \{(u,v) \in Z : \langle I'_\lambda(u,v),(u,v)\rangle = 0\}.$$

It is clear that Λ_λ is closed in Z and all critical points of I_λ must lie on Λ_λ. So, $(u,v) \in \Lambda_\lambda$ if and only if

$$\int_\Omega \nu_1(x)|\nabla u|^p\,dx - \lambda\int_\Omega a(x)|u|^p\,dx - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx = \frac{1}{(\alpha+1)(\delta+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,dx,$$

$$\int_\Omega \nu_2(x)|\nabla v|^q\,dx - \lambda\int_\Omega d(x)|v|^q\,dx - \lambda\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx = \frac{1}{(\beta+1)(\gamma+1)}\int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,dx.$$

Now, we define the following disjoint subsets of Λ_λ:

$$\Lambda_\lambda^+ := \bigg\{(u,v) \in \Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,dx < 0\bigg\},$$

$$\Lambda_\lambda^0 := \bigg\{(u,v) \in \Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,dx = 0\bigg\},$$

$$\Lambda_\lambda^- := \bigg\{(u,v) \in \Lambda_\lambda : \int_\Omega \mu(x)|u|^{\lambda+1}|v|^{\delta+1}\,dx > 0\bigg\}.$$

Note that the condition (Υ_5) implies that $(u_1,v_1) \notin \Lambda_\lambda^-$.

Lemma 3.1. *The solution branch C bends to the right of λ_1 at $(\lambda_1,0,0)$; i.e. there exists $\rho > 0$, such that $(\lambda,u,v) \in C$ and $\|u\|_{1,p} + \|v\|_{1,q} < \rho$, implies $\lambda > \lambda_1$.*

Proof. Suppose not. Then, there exists a sequence $(\lambda_n,u_n,v_n) \in C$, such that $(u_n,v_n) \to 0$ in Z, $\lambda_n \le \lambda_1$, $\lambda_n \to \lambda_1$ and

$$\int_\Omega \nu_1(x)|\nabla u_n|^p\,dx - \lambda_n\int_\Omega a(x)|u_n|^p\,dx - \lambda_n\int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx = \frac{1}{(\alpha+1)(\delta+1)}\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx, \tag{3.1}$$

$$\int_\Omega \nu_2(x)|\nabla v_n|^q\,dx - \lambda_n\int_\Omega d(x)|v_n|^q\,dx - \lambda_n\int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx = \frac{1}{(\beta+1)(\gamma+1)}\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx. \tag{3.2}$$

We introduce the sequences $\widetilde{u}_n$ and $\widetilde{v}_n$ in the following way:

$$\widetilde{u}_n = \frac{u_n}{(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q)^{1/p}} \quad\text{and}\quad \widetilde{v}_n = \frac{v_n}{(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q)^{1/q}} \tag{3.3}$$

which are bounded sequences. Indeed, we have

$$\|\widetilde{u}_n\|_{1,p}^p + \|\widetilde{v}_n\|_{1,q}^q = 1 \quad \text{for every } n \in \mathbb{N}.$$

Thus, we may assume $(\widetilde{u}_n,\widetilde{v}_n) \rightharpoonup (\widetilde{u}_0,\widetilde{v}_0)$ in Z. Using $(\alpha+1)/p + (\beta+1)/q = 1$ in the condition $(\mathcal{H})$, we have

$$\int_\Omega b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,dx = \frac{\displaystyle\int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q}.$$

Moreover, the range of exponents and Lemma 2.1 implies

$$\frac{\displaystyle\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q} \le \frac{\|\mu\|_{\omega_2}\|u_n\|_{p^*}^{\gamma+1}\|v_n\|_{q^*}^{\delta+1}}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q} \to 0,$$

as $(u_n, v_n) \to 0$ in Z. Using (3.1) and (3.2), we obtain

$$\int_\Omega (\nu_1(x)|\nabla \widetilde{u}_n|^p - \lambda_n a(x)|\widetilde{u}_n|^p - \lambda_n b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1})\,dx \to 0,$$

$$\int_\Omega (\nu_2(x)|\nabla \widetilde{v}_n|^q - \lambda_n d(x)|\widetilde{v}_n|^q - \lambda_n b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1})\,dx \to 0,$$

as $n \to \infty$. Moreover, the compactness of the operators D and B implies that

$$\lambda_n \int_\Omega a(x)|\widetilde{u}_n|^p\,dx \to \lambda_1 \int_\Omega a(x)|\widetilde{u}_0|^p\,dx,$$

$$\lambda_n \int_\Omega d(x)|\widetilde{v}_n|^q\,dx \to \lambda_1 \int_\Omega d(x)|\widetilde{v}_0|^q\,dx,$$

$$\lambda_n \int_\Omega b(x)|\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\,dx \to \lambda_1 \int_\Omega b(x)|\widetilde{u}_0|^{\alpha+1}|\widetilde{v}_0|^{\beta+1}\,dx,$$

as $n \to \infty$. Hence, $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}_0, \widetilde{v}_0) \neq (0,0)$, since $\|(\widetilde{u}_n, \widetilde{v}_n)\|_Z = 1$, for every $n \in \mathbb{N}$. Also, $(\widetilde{u}_0, \widetilde{v}_0)$ is a solution of (1.4_{λ_1})–(1.5_{λ_1}). By Theorem 2.4(a), λ_1 is simple. Thus, $\widetilde{u}_0 = k^p u_1$ and $\widetilde{v}_0 = k^q v_1$, for some positive constant k. Multiplying equations (3.1) and (3.2) by $(\alpha+1)/p$ and $(\beta+1)/q$, respectively, adding the resulting equations, and using condition $(\mathcal{H})$, we deduce that

$$A_{\lambda_n}(u_n, v_n) = c_1 \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx, \quad \text{for any } n \in \mathbb{N}, \tag{3.4}$$

where $c_1 = 1/(p(\delta+1)) + 1/(q(\gamma+1))$. From the variational characterization (2.1) of the eigenvalue λ_1, equation (3.4), and condition (Υ_5) we conclude that

$$0 \le \lim_{n\to\infty} c_1 \int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,dx = c_2 \int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx < 0,$$

for some $c_2 = c_2(c_1, k) > 0$, which is a contradiction, and so the proof is complete. □

COROLLARY 3.2. *Suppose that $(\lambda, u, v) \in C$, such that (λ, u, v) is close enough to $(\lambda_1, 0, 0)$; then $(u, v) \in \Lambda_\lambda^+$.*

PROOF. Let $(\lambda_n, u_n, v_n) \in C$, such that $(u_n, v_n) \to (0,0)$ in Z and $\lambda_n \to \lambda_1$. Then, using the same arguments as in Lemma 3.1 we may prove that

$$\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx < 0, \quad \text{for } n \text{ large enough};$$

i.e. $(u_n, v_n) \in \Lambda_\lambda^+$, when n is large enough. □

LEMMA 3.3. *There exists $\lambda^0 > \lambda_1$, such that for every $\lambda \in (\lambda_1, \lambda^0)$ the set Λ_λ^- is closed in Z.*

Proof. First note that $\Lambda_\lambda^- \neq \emptyset$, since $\mu^+ \not\equiv 0$. We have to prove that for any $(u_n, v_n) \in \Lambda_\lambda^-$ such that $(u_n, v_n) \to (u, v)$ in Z, we have $(u, v) \in \Lambda_\lambda^-$, when $\lambda \in (\lambda_1, \lambda^0)$. Due to the compactness of the operator M, this will be the case if

$$\int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,dx \to \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,dx > 0.$$

Assume that such a λ^0 does not exist. Then, there exists a sequence (λ_n, u_n, v_n), with $(u_n, v_n) \in \Lambda_\lambda^-$, such that

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx \to 0.$$

Since (u_n, v_n) is a solution for the system (1.1_{λ_n})–(1.2_{λ_n}), we have that

$$\int_\Omega (\nu_1(x)|\nabla u_n|^p - \lambda_n a(x)|u_n|^p - \lambda_n b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1})\,dx \to 0,$$

$$\int_\Omega (\nu_2(x)|\nabla \widetilde{v}_n|^q - \lambda_n d(x)|v_n|^q - \lambda_n b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1})\,dx \to 0,$$

as $n \to \infty$. Similar to Lemma 3.1, we may prove that the sequences $\{\widetilde{u}_n\}$ and $\{\widetilde{v}_n\}$ converge strongly to some $(\widetilde{u}_0, \widetilde{v}_0)$, and we have $\widetilde{u}_0 = k^p u_1$ and $\widetilde{v}_0 = k^q v_1$, for some positive constant k. The compactness of the operator M implies that

$$0 \leq \lim_{n\to\infty} c_3 \int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,dx = c_4 \int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx < 0,$$

for some positive constants c_3 and c_4, which is a contradiction. Thus, Λ_λ^- is closed in Z. □

Lemma 3.4. *The functional I_λ satisfies the* (PS) *condition on Λ_λ^-, whenever λ is close enough to λ_1.*

Proof. Let the sequence $(u_n, v_n) \in \Lambda_\lambda^-$ be such that $I_\lambda(u_n, v_n) \leq c$ and $I'_\lambda(u_n, v_n) \to 0$, as $n \to \infty$. We first prove that (u_n, v_n) is a bounded sequence. The quantity $M(u_n, v_n)$ is bounded, for all $n \in \mathbb{N}$, since

$$I_\lambda(u_n, v_n) - \left\langle I'_\lambda(u_n, v_n), \left(\frac{u_n}{p}, \frac{v_n}{q}\right)\right\rangle = \left[\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\right] M(u_n, v_n).$$

Therefore, $A_\lambda(u_n, v_n)$ must be bounded, too. Next, we claim that there exists a positive constant σ, such that

$$\frac{A_\lambda(u_n, v_n)}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q} \geq \sigma > 0, \quad \text{for every } n \in \mathbb{N},$$

which would imply the boundedness of (u_n, v_n) in Z. Suppose not. Then, there exists a sequence (λ_n, u_n, v_n), with $(u_n, v_n) \in \Lambda_\lambda^-$, such that $\lambda_n \to \lambda_1$ and

$$\frac{A_{\lambda_n}(u_n, v_n)}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q} = A_{\lambda_n}(\widetilde{u}_n, \widetilde{v}_n) \to 0,$$

where $(\widetilde{u}_n, \widetilde{v}_n)$ are the sequences introduced by (3.3). The boundedness of $(\widetilde{u}_n, \widetilde{v}_n)$ implies that

$$(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}_0, \widetilde{v}_0)$$

in Z, for some $(\widetilde{u}_0, \widetilde{v}_0) \in Z$. From the variational characterization (2.1) of λ_1 and By the weak lower semicontinuity of A_λ we have

$$0 \le A_{\lambda_1}(\widetilde{u}_0, \widetilde{v}_0) \le \liminf_{n\to\infty} A_{\lambda_n}(\widetilde{u}_n, \widetilde{v}_n) = 0. \tag{3.5}$$

We claim that $(\widetilde{u}_0, \widetilde{v}_0) \ne (0,0)$. Otherwise, from the compactness of the functionals D and B we have

$$\lim_{n\to\infty} D(\widetilde{u}_n, \widetilde{v}_n) = \lim_{n\to\infty} B(\widetilde{u}_n, \widetilde{v}_n) = 0.$$

Hence, from (3.5) we conclude that $(\widetilde{u}_n, \widetilde{v}_n) \to (0,0)$ in Z, which contradicts the fact that $\|(\widetilde{u}_n, \widetilde{v}_n)\|_Z = 1$, for every $n \in \mathbb{N}$.

Now, from (3.5) we must have that $\widetilde{u}_0 = k^p u_1$ and $\widetilde{v}_0 = k^q v_1$, for some positive constant k. Then from (Υ_5) we have

$$0 < \int_\Omega \mu(x)|\widetilde{u}_n|^{\gamma+1}|\widetilde{v}_n|^{\delta+1}\,dx \to c_5 \int_\Omega \mu(x)|u_1|^{\gamma+1}|v_1|^{\delta+1}\,dx < 0,$$

which is a contradiction. Thus (u_n, v_n) is a bounded sequence. Using the compactness of the functionals D, B, and M and following the procedure from [12, Lemma 2.3] we obtain that (u_n, v_n) has a convergent subsequence, and so the proof is complete. □

Our main result is the following theorem.

THEOREM 3.5. *There exists $\lambda^* > \lambda_1$, such that the system (1.1_λ)–(1.2_λ) has two nonnegative nonsemitrivial solutions, for every $\lambda \in (\lambda_1, \lambda^*)$.*

PROOF. By Theorem 2.6 and Corollary 3.2, there exist a nonsemitrivial solution for the system (1.1_λ)–(1.2_λ), which belongs in Λ_λ^+. We prove the existence of a solution, which belongs in Λ_λ^-. Consider the set Λ_λ^- equipped with the metric $d(\widetilde{z}_1, \widetilde{z}_2) = \|\widetilde{z}_1 - \widetilde{z}_2\|_Z$, for every $\widetilde{z}_1$ and $\widetilde{z}_2$ in Λ_λ^-. It is clear from Lemma 3.3, that for λ^* close to λ_1, the set Λ_λ^- becomes a complete metric space. Using condition $(\mathcal{H})$, we observe that

$$\begin{aligned} A_\lambda(u,v) = \frac{\alpha+1}{p}\int_\Omega (\nu_1(x)|\nabla u|^p - \lambda a(x)|u|^p - \lambda b(x)|u|^{\alpha+1}|v|^{\beta+1})\,dx \\ + \frac{\beta+1}{q}\int_\Omega (\nu_2(x)|\nabla v|^q - \lambda d(x)|v|^q - \lambda b(x)|u|^{\alpha+1}|v|^{\beta+1})\,dx. \end{aligned}$$

Hence, for every $(u,v) \in \Lambda_\lambda$, using $(\alpha+1)/p + (\beta+1)/q = 1$, we have

$$I(u,v) = \left[\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\right] \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}\,dx.$$

Since $p < \gamma+1$ or $q < \delta+1$, we conclude that $I_\lambda(u,v) > 0$ whenever $(u,v) \in \Lambda_\lambda^-$ and I_λ is bounded below in Λ_λ^-, i.e.

$$\inf_{(u,v)\in\Lambda_\lambda^-} I_\lambda(u,v) \geq 0.$$

On the other hand, the functional I_λ satisfies the (PS) condition in Λ_λ^- (by Lemma 3.4). Thus, Ekeland's variational principle implies the existence of a solution for the system (1.1_λ)–(1.2_λ). This solution is nonnegative, since $I_\lambda(|u|,|v|) = I_\lambda(u,v)$, and in addition, Lemma 2.7 implies that it is also nonsemitrivial. □

Acknowledgements. The authors would like to thank the unknown referee for his/her valuable comments and suggestions.

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Manuscript received November 28, 2012

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