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THE MODULE CATEGORY WEIGHT OF COMPACT EXCEPTIONAL LIE GROUPS

Younggi Choi

This is dedicated to Professor Yuli Rudyak

Abstract. We give a lower bound for the Lusternik–Schnirelmann category of compact exceptional Lie groups by computing the module category weight through analyzing several Eilenberg–Moore type spectral sequences.

1. Introduction

The Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of a topological space X is the least integer n such that there exists an open cover $X = U_1 \cup \ldots \cup U_{n+1}$ with each U_i contractible to a point in X. There are other computable homotopy invariants such as cup length, category weight, and module category weight with the relation [5], [14]: $\mathrm{cup}(X;\mathbb{F}_p) \leq \mathrm{wgt}(X;\mathbb{F}_p) \leq \mathrm{Mwgt}(X;\mathbb{F}_p) \leq \mathrm{cat}(X)$.

Toomer introduced the explicit formula for the difference between the cup length and the category weight. Using the formula he calculated the difference $\mathrm{cup}(X;\mathbb{F}_p) - \mathrm{wgt}(X;\mathbb{F}_p)$ of any simply connected compact simple Lie group [16]. In fact, it is precisely F_4, E_6, E_7, E_8 which yield a positive difference.

On the other hand, Iwase and Kono [5] determined $\mathrm{cat}(\mathrm{Spin}(9)) = 8$ by computing the lower bound of the difference between the category weight and

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the module category weight of Spin(9), which is

$$\mathrm{Mwgt}(\mathrm{Spin}(9);\mathbb{F}_2) - \mathrm{wgt}(\mathrm{Spin}(9);\mathbb{F}_2) \geq 2.$$

Here we give a lower bound for the Lusternik–Schnirelmann category of compact exceptional Lie groups by studying the difference between the category weight and the module category weight through several Eilenberg–Moore type spectral sequences.

This paper is organized as follows. In Section 2, we collect some known facts, which will be used in next sections. In Section 3, we compute the module category weight with respect to $\mathbb{F}_2$ coefficients of compact exceptional Lie groups by analyzing several Eilenberg–Moore type spectral sequences. In Section 4, we compute the module category weight with respect to $\mathbb{F}_3$ coefficients of compact exceptional Lie groups by the similar method as the case of $\mathbb{F}_2$ coefficients.

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2. Some known facts

Throughout this paper, the subscript of an element always means the degree of the element, for example, the degree of x_i is i. Let $E(x)$ be the exterior algebra on x and $\mathbb{F}_2[x]$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x which is generated by elements $\gamma_i(x)$ with coproduct

$$\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$$

and the product

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x).$$

We define $\mathrm{cup}(X;\mathbb{F}_p)$, the cup-length with respect to $\mathbb{F}_p$, by the least integer m such that $x_1 \dots x_{m+1} = 0$ for any $m+1$ elements $x_i \in \widetilde{H}^*(X;\mathbb{F}_p)$. Let $P^m(\Omega X)$ be the m th projective space, in the sense of Stasheff [15], such that there is a homotopy equivalence $P^\infty(\Omega X) \simeq X$. Let $e_m\colon P^m(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the inclusion map. Consider $(e_m)^*\colon H^*(X;\mathbb{F}_p) \to H^*(P^m(\Omega X);\mathbb{F}_p)$. Then we can define category weight $\mathrm{wgt}(X;\mathbb{F}_p)$ and module category weight $\mathrm{Mwgt}(X;\mathbb{F}_p)$ as follows [5]:

$$\begin{aligned}\mathrm{wgt}(X;\mathbb{F}_p) =& \min\{m \mid (e_m)^* \text{ is a monomorhism}\},\\ \mathrm{Mwgt}(X;\mathbb{F}_p) =& \min\{m \mid (e_m)^* \text{ is a split monomorphism}\\ &\text{of all Steenrod algebra modules}\}.\end{aligned}$$

Then we have the following relation [5]:

$$\mathrm{cup}(X;\mathbb{F}_p) \leq \mathrm{wgt}(X;\mathbb{F}_p) \leq \mathrm{Mwgt}(X;\mathbb{F}_p) \leq \mathrm{cat}(X).$$

Now we describe the mod p cohomology of the exceptional Lie groups, together with some non-trivial Steenrod operations. We refer [12] for the condensed treatment of these cohomology including Hopf algebra structure and the action of the Steenrod algebra.

THEOREM 2.1. *The* mod 2 *cohomology of the exceptional Lie groups* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$\begin{aligned}
H^*(G_2;\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4)\otimes E(Sq^2x_3),\\
H^*(F_4;\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4)\otimes E(Sq^2x_3, x_{15}, Sq^8x_{15}),\\
H^*(E_6;\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4)\otimes E(Sq^2x_3, Sq^{4,2}x_3, x_{15}, Sq^{8,4,2}x_3, Sq^8x_{15}),\\
H^*(E_7;\mathbb{F}_2) &\cong \mathbb{F}_2[x_3, Sq^2x_3, Sq^{4,2}x_3]/(x_3^4, (Sq^2x_3)^4, (Sq^{4,2}x_3)^4)\\
&\qquad \otimes E(x_{15}, Sq^{8,4,2}x_3, Sq^8x_{15}, Sq^{4,8}x_{15}),\\
H^*(E_8;\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^{16})\otimes \mathbb{F}_2[Sq^2x_3]/((Sq^2x_3)^8)\\
&\qquad \otimes \mathbb{F}_2[Sq^{4,2}x_3, x_{15}]/((Sq^{4,2}x_3)^4, x_{15}^4)\\
&\qquad \otimes E(Sq^{8,4,2}x_3, Sq^8x_{15}, Sq^{4,8}x_{15}, Sq^{2,4,8}x_{15}).
\end{aligned}$$

THEOREM 2.2. *The* mod 3 *cohomology of the exceptional Lie groups* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$\begin{aligned}
H^*(G_2;\mathbb{F}_3) &\cong E(x_3, x_{11}),\\
H^*(F_4;\mathbb{F}_3) &\cong \mathbb{F}_3[\beta\mathcal{P}^1x_3]/((\beta\mathcal{P}^1x_3)^3)\otimes E(x_3, \mathcal{P}^1x_3, x_{11}, \mathcal{P}^1x_{11}),\\
H^*(E_6;\mathbb{F}_3) &\cong \mathbb{F}_3[\beta\mathcal{P}^1x_3]/((\beta\mathcal{P}^1x_3)^3)\otimes E(x_3, \mathcal{P}^1x_3, x_9, x_{11}, \mathcal{P}^1x_{11}, x_{17}),\\
H^*(E_7;\mathbb{F}_3) &\cong \mathbb{F}_3[\beta\mathcal{P}^1x_3]/((\beta\mathcal{P}^1x_3)^3)\\
&\qquad \otimes E(x_3, \mathcal{P}^1x_3, x_{11}, \mathcal{P}^1x_{11}, \mathcal{P}^{3,1}x_3, x_{27}, x_{35}),\\
H^*(E_8;\mathbb{F}_3) &\cong \mathbb{F}_3[\beta\mathcal{P}^1x_3, \beta\mathcal{P}^{3,1}x_3]/((\beta\mathcal{P}^1x_3)^3, (\beta\mathcal{P}^{3,1}x_3)^3)\\
&\qquad \otimes E(x_3, \mathcal{P}^1x_3, x_{15}, \mathcal{P}^{3,1}x_3, \mathcal{P}^3x_{15}, x_{35}, x_{39}, x_{47}).
\end{aligned}$$

3. Module Category Weight with respect to $\mathbb{F}_2$ coefficients

Let $\widetilde{G}$ be the 3-connected cover of G which is the homotopy fibre of the map $G \xrightarrow{\iota} K(Z,3)$ where ι is the fundamental class of $H^3(G;Z)$. Then we have the following fibrations: $CP^\infty \to \widetilde{G} \to G$, $S^1 \to \Omega\widetilde{G} \to \Omega G$. Now we get the following theorem. Some of results can be obtained from the Serre spectral sequence of $\widetilde{G} \to G \to K(Z,3)$ and the Adem relations.

THEOREM 3.1 ([6], [9], [11]). *The* mod 2 *cohomology of the* 3*-connected covers of the exceptional Lie groups* $\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, *and* $\widetilde{E}_8$ *are as follows*:

$$\begin{aligned}
H^*(\widetilde{G}_2;\mathbb{F}_2) &\cong \mathbb{F}_2[x_8]\otimes E(Sq^1x_8, Sq^{2,1}x_8),\\
H^*(\widetilde{F}_4;\mathbb{F}_2) &\cong \mathbb{F}_2[x_8]\otimes E(Sq^1x_8, Sq^{2,1}x_8, Sq^{4,2,1}x_8, Sq^{8,4,2,1}x_8),
\end{aligned}$$

$$H^*(\widetilde{E}_6;\mathbb{F}_2) \cong \mathbb{F}_2[x_{32}] \otimes E(x_9, Sq^2x_9, Sq^{4,2}x_9, Sq^8x_9, x_{23}, Sq^{16,8}x_9),$$
$$H^*(\widetilde{E}_7;\mathbb{F}_2) \cong \mathbb{F}_2[x_{32}] \otimes E(x_{11}, Sq^4x_{11}, Sq^8x_{11}, x_{23}, Sq^{8,8}x_{11}, Sq^1x_{32}, Sq^{16,8}x_{11}),$$
$$H^*(\widetilde{E}_8;\mathbb{F}_2) \cong \mathbb{F}_2[x_{15}]/(x_{15}^4) \otimes \mathbb{F}_2[x_{32}] \otimes E(x_{23}, x_{27}, x_{29}, Sq^1x_{32}, x_{35}, Sq^4x_{35}, Sq^{8,4}x_{35}).$$

To get the module category weight of exceptional Lie groups G, we study the Rothenberg–Steenrod spectral sequence converging to $H^*(G)$ with $E_2 \cong \mathrm{Cotor}_{\mathrm{H}^*(\Omega \mathrm{G};\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2)$. This is a spectral sequence of Hopf algebras but it depends on the coalgebra structure. So we should determine the coalgebra structure of $H^*(\Omega G;\mathbb{F}_2)$. Note that since

$$E_2 \cong \mathrm{Cotor}_{\mathrm{H}^*(\Omega \mathrm{G};\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Ext}_{\mathrm{H}_*(\Omega \mathrm{G};\mathrm{BbbF}_2)}(\mathbb{F}_2,\mathbb{F}_2)$$

(see [2]), we can also use the algebra structure of $H_*(\Omega G;\mathbb{F}_2)$ as in [5].

To get the coalgebra structure of $H^*(\Omega G;\mathbb{F}_2)$, we consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G;\mathbb{F}_2)$ with

$$E_2 \cong \mathrm{Tor}_{H^*(G;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2).$$

Since E_2 concentrates in the even dimensions, the spectral sequence collapses at the E_2-term, i.e. $E_2 = E_\infty$. Then there is no coalgebra extension problem in such a spectral sequence [8]. We refer the reader to [10] for concise treatment of above Eilenberg Moore spectral sequence. So as a coalgebra we have the following

THEOREM 3.2. *The coalgebra structure of the* mod 2 *cohomology of the loop spaces of exceptional Lie groups* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$H^*(\Omega G_2;\mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, b_{10}),$$
$$H^*(\Omega F_4;\mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, b_{10}, a_{14}, a_{22}),$$
$$H^*(\Omega E_6;\mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, a_8, b_{10}, a_{14}, a_{16}, a_{22}),$$
$$H^*(\Omega E_7;\mathbb{F}_2) \cong E(a_2, a_4, a_8) \otimes \Gamma(b_{10}, a_{14}, a_{16}, b_{18}, a_{22}, a_{26}, b_{34}),$$
$$H^*(\Omega E_8;\mathbb{F}_2) \cong E(a_2, a_4, a_8, a_{14}) \otimes \Gamma(a_{16}, a_{22}, a_{26}, a_{28}, b_{34}, b_{38}, b_{46}, b_{58}),$$

especially we have $Sq^4b_{10} = a_{14}$ *and* $Sq^8b_{18} = a_{26}$ *by Theorem* 3.1.

Note that even though there is no coalgebra extension, there are many non-trivial algebra extensions in the above spectral sequence. For example, $a_2^2 = Sq^2a_2 = Sq^2\sigma(x_3) = \sigma(Sq^2x_3) = \sigma(x_5) = a_4$, where σ is the cohomology suspension. Similarly $a_4^2 = a_8$ and $a_8^2 = a_{16}$.

Now we consider the Rothenberg–Steenrod spectral sequence converging to $H^*(G;\mathbb{F}_2)$ with

$$E_2 \cong \mathrm{Cotor}_{\mathrm{H}^*(\Omega \mathrm{G};\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2). \tag{3.1}$$

Then we get the next theorem by the standard Cotor computation of the following monogenic Hopf algebras:

$$\mathrm{Cotor}_{\Gamma(a_{2i})}(\mathbb{F}_2,\mathbb{F}_2)=\mathrm{E}(x_{2i+1}),\qquad \mathrm{Cotor}_{\mathrm{E}(a_{2i})}(\mathbb{F}_2,\mathbb{F}_2)=\mathbb{F}_2[x_{2i+1}].$$

We refer the reader to [13] for detail computation method of this spectral sequence.

THEOREM 3.3. $\mathrm{Cotor}_{H^*(\Omega G;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2)$ *of the exceptional Lie groups* G *for* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$\begin{aligned}
\mathrm{Cotor}_{H^*(\Omega G_2;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]\otimes E(x_5,z_{11}),\\
\mathrm{Cotor}_{H^*(\Omega F_4;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]\otimes E(x_5,z_{11},x_{15},x_{23}),\\
\mathrm{Cotor}_{H^*(\Omega E_6;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \mathbb{F}_2[x_3]\otimes E(x_5,x_9,z_{11},x_{15},x_{17},x_{23}),\\
\mathrm{Cotor}_{H^*(\Omega E_7;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \mathbb{F}_2[x_3,x_5,x_9]\otimes E(z_{11},x_{15},x_{17},z_{19},x_{23},x_{27},z_{35}),\\
\mathrm{Cotor}_{H^*(\Omega E_8;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \mathbb{F}_2[x_3,x_5,x_9,x_{15}]\\
&\qquad \otimes E(x_{17},x_{23},x_{27},x_{29},z_{35},z_{39},z_{47},z_{59}),
\end{aligned}$$

especially we have $Sq^4 z_{11}=x_{15}$ *and* $Sq^8 z_{19}=x_{27}$.

Then from information Theorem 2.1 of $H^*(G;\mathbb{F}_2)$, we can analyze non trivial differentials of the Rothenberg–Steenrod spectral sequence (3.1) converging to $H^*(G;\mathbb{F}_2)$ as follows:

$$\begin{aligned}
d_3(z_{11}) &= x_3^4 &&\text{for } G=G_2,F_4,E_{6,7},\\
d_3(z_{19}) &= x_5^4 &&\text{for } G=E_7,\\
d_3(z_{35}) &= x_9^4 &&\text{for } G=E_7,E_8,\\
d_7(z_{39}) &= x_5^8 &&\text{for } G=E_8,\\
d_{15}(z_{47}) &= x_3^{16} &&\text{for } G=E_8,\\
d_3(z_{59}) &= x_{15}^4 &&\text{for } G=E_8.
\end{aligned}\tag{3.2}$$

Next, as in [4], [5], truncating the above computation with the same differential d_i in (3.2), we can compute the spectral sequence of Stasheff's type converging to $H^*(P^m(\Omega G);\mathbb{F}_2)$. Let $A=H^*(G;\mathbb{F}_2)$ in Theorem 2.1. Then like the result in [5, Proposition 2.1], for low m such as $1\le m\le 3$, we have the following:

$$H^*(P^m(\Omega G);\mathbb{F}_2)=A^{[m]}\oplus\sum_i z_{4i+3}\cdot A^{[m-1]}\oplus S_m,\tag{3.3}$$

$$\begin{cases} i=2 & \text{for } G=G_2,F_4,E_6,\\ i=2,4,8 & \text{for } G=E_7,\\ i=8,9,11,14 & \text{for } G=E_8,\end{cases}$$

as modules where $A^{[m]}, (m \geq 0)$ denotes the quotient module $A/D^{m+1}(A)$ of A by the submodule $D^{m+1}(A) \subseteq A$ generated by all the products of $m+1$ elements in positive dimensions in A, and $z_{4i+3} \cdot A^{[m-1]}$ denotes a submodule corresponding to a submodule in $A \otimes E(z_{4i+3})$, and S_m satisfies $S_m \cdot \widetilde{H}^*(P^m(\Omega G); \mathbb{F}_2) = 0$ and $S_m|_{P^{m-1}(\Omega G)} = 0$. For more detail for S_m, we refer the paper [4]. Now we compute the module category weight using the similar method as in [1], [5].

Theorem 3.4. *The module category weight is as follows*:

$$\mathrm{Mwgt}(G_2; \mathbb{F}_2) \geq 4, \qquad \mathrm{Mwgt}(F_4; \mathbb{F}_2) \geq 8, \qquad \mathrm{Mwgt}(E_6; \mathbb{F}_2) \geq 10,$$
$$\mathrm{Mwgt}(E_7; \mathbb{F}_2) \geq 15, \qquad \mathrm{Mwgt}(E_8; \mathbb{F}_2) \geq 32.$$

Proof. From Theorem 3.3, $Sq^4 z_{11} = x_{15}$ in $H^*(P^1(\Omega G); \mathbb{F}_2)$ for $G = F_4, E_6, E_7$. Then from (3.3), $Sq^4 z_{11} = x_{15}$ modulo S_2 in $H^*(P^2(\Omega G); \mathbb{F}_2)$ for $G = F_4, E_6, E_7$. Since S_2 is even-dimensional [1], [5], the modulo S_2 is trivial so $Sq^4 z_{11} = x_{15}$ in $H^*(P^2(\Omega G); \mathbb{F}_2)$. Thus we have

$$(3.4) \quad \begin{aligned} Sq^4(x_3^3 x_5 z_{11} x_{23}) &= x_3^3 x_5 x_{15} x_{23}, && \text{in } H^*(P^7(\Omega F_4); \mathbb{F}_2), \\ Sq^4(x_3^3 x_5 x_9 z_{11} x_{17} x_{23}) &= x_3^3 x_5 x_9 x_{15} x_{17} x_{23}, && \text{in } H^*(P^9(\Omega E_6); \mathbb{F}_2), \\ Sq^4(x_3^3 x_5^3 x_9^3 z_{11} x_{17} x_{23} x_{27}) &= x_3^3 x_5^3 x_9^3 x_{15} x_{17} x_{23} x_{27}, && \text{in } H^*(P^{14}(\Omega E_7); \mathbb{F}_2). \end{aligned}$$

Note that for $x_j \in A^{[m]}$, $Sq^i x_j = Sq^i((e_m)^* x_j) = (e_m)^*(Sq^i x_j)$. Thus in $H^*(P^m(\Omega G); \mathbb{F}_2)$,

$$(3.5) \quad \begin{aligned} &Sq^4(x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}} z_{11}) \\ &= Sq^4(x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}}) z_{11} + x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}} Sq^4 z_{11} \\ &= (e_m)^*(Sq^4(x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}})) z_{11} + x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}} Sq^4 z_{11}. \end{aligned}$$

Since $x_3^4 = x_5^4 = x_9^4 = 0$, and $x_{\alpha_j}^2 = 0$ for other generators x_{α_j} in $H^*(G; \mathbb{F}_2)$ for $G = F_4, E_6, E_7$, we have

$$Sq^4(x_3^3 x_5 x_{23}) = 0, \quad Sq^4(x_3^3 x_5 x_9 x_{17} x_{23}) = 0, \quad Sq^4(x_3^3 x_5^3 x_9^3 x_{17} x_{23} x_{27}) = 0$$

in $H^*(F_4; \mathbb{F}_2)$, $H^*(E_6; \mathbb{F}_2)$, $H^*(E_7; \mathbb{F}_2)$. So in $H^*(P^m(\Omega G); \mathbb{F}_2)$,

$$Sq^4(x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}} z_{11}) = x_{\alpha_1{}^{s_1}} \dots x_{\alpha_j{}^{s_j}} x_{15}.$$

By the definition in Section 2, $\mathrm{Mwgt}(X; \mathbb{F}_2)$ is the least m such that $(e_m)^*$ is a split monomorphism of all Steenrod algebra modules.

Let $\phi_m \colon H^*(P^m(\Omega G); \mathbb{F}_2) \to H^*(G; \mathbb{F}_2)$ be a epimorphism which preserves all Steenrod actions and $\phi_m \circ (e_m)^* \cong 1_{H^*(G; \mathbb{F}_2)}$. Suppose that there are epimorphisms:

$$\begin{aligned} \phi_7 : H^*(P^7(\Omega F_4); \mathbb{F}_2) &\to H^*(F_4; \mathbb{F}_2), \\ \phi_9 : H^*(P^9(\Omega E_6); \mathbb{F}_2) &\to H^*(E_6; \mathbb{F}_2), \\ \phi_{14} : H^*(P^{14}(\Omega E_7); \mathbb{F}_2) &\to H^*(E_7; \mathbb{F}_2). \end{aligned}$$

Then we have the following diagrams:

$$H^*(P^7(\Omega F_4);\mathbb{F}_2) \xrightarrow{\phi_7} H^*(F_4;\mathbb{F}_2) \tag{3.6}$$

$$\begin{array}{ccc} x_3^3x_5x_{15}x_{23} & \longmapsto & x_3^3x_5x_{15}x_{23} \\ {\scriptstyle Sq^4}\uparrow & & \uparrow{\scriptstyle Sq^4} \\ x_3^3x_5z_{11}x_{23} & \longmapsto & 0 \end{array} \tag{3.7}$$

$$H^*(P^9(\Omega E_6);\mathbb{F}_2) \xrightarrow{\phi_9} H^*(E_6;\mathbb{F}_2) \tag{3.8}$$

$$\begin{array}{ccc} x_3^3x_5x_9x_{15}x_{17}x_{23} & \longmapsto & x_3^3x_5x_9x_{15}x_{17}x_{23} \\ {\scriptstyle Sq^4}\uparrow & & \uparrow{\scriptstyle Sq^4} \\ x_3^3x_5x_9z_{11}x_{17}x_{23} & \longmapsto & 0 \end{array} \tag{3.9}$$

$$H^*(P^{14}(\Omega E_7);\mathbb{F}_2) \xrightarrow{\phi_{14}} H^*(E_7;\mathbb{F}_2) \tag{3.10}$$

$$\begin{array}{ccc} x_3^3x_5^3x_9^3x_{15}x_{17}x_{23}x_{27} & \longmapsto & x_3^3x_5^3x_9^3x_{15}x_{17}x_{23}x_{27} \\ {\scriptstyle Sq^4}\uparrow & & \uparrow{\scriptstyle Sq^4} \\ x_3^3x_5^3x_9^3z_{11}x_{17}x_{23}x_{27} & \longmapsto & 0 \end{array} \tag{3.11}$$

Obviously this is a contradiction. So ϕ_7, ϕ_9, and ϕ_{14} are not epimorphisms. This means that $(e_7)^*$, $(e_9)^*$, and $(e_{14})^*$ can not be split monomorphisms of all Steenrod algebra module. Hence we obtain that

$$\mathrm{Mwgt}(F_4;\mathbb{F}_2) \geq 8, \qquad \mathrm{Mwgt}(\mathrm{E}_6;\mathbb{F}_2) \geq 10, \qquad \mathrm{Mwgt}(E_7;\mathbb{F}_2) \geq 15.$$

Now we consider the category weight. For G_2, $x_3^3x_5 \in H^*(P^4(\Omega G_2);\mathbb{F}_2)$. Hence $(e_4)^*$ is a monomorphism, so $\mathrm{wgt}(G_4;\mathbb{F}_2) = 4$. In the same way, $\mathrm{wgt}(G;\mathbb{F}_2)$ is 6 for $G = F_4$, 8 for $G = E_6$, 13 for $G = E_7$, 32 for $G = E_8$. In fact the category weight is the same as the Toomer's invariant, the filtration length, that is, $\mathrm{wgt}(G;\mathbb{F}_2) = f_2(G)$ in [16].

For the case of G_2 and E_8, by dimensional reason, any generator of type x_- can not be of the form $Sq^i(z_-)$ for any i and for any generator of type z. So we can not apply the method in (3.6)–(3.11). Hence we do not obtain any positive difference between the category weight and the module category weight. Hence we have

$$\mathrm{Mwgt}(G_2;\mathbb{F}_2) \geq \mathrm{wgt}(G_2;\mathbb{F}_2) = 4, \qquad \mathrm{Mwgt}(E_8;\mathbb{F}_2) \geq wgt(E_8;\mathbb{F}_2) = 32. \quad \square$$

Summarizing above results, we have:

X	wgt$(X;\mathbb{F}_2)$	Mwgt$(X;\mathbb{F}_2)$	cat(X)
G_2	4	$\geq\ 4$	4
F_4	6	$\geq\ 8$	?
E_6	8	≥ 10	?
E_7	13	≥ 15	?
E_8	32	≥ 32	?

4. Module Category Weight with respect to $\mathbb{F}_3$ coefficients

Now we turn to the case of $\mathbb{F}_3$ coefficients.

Theorem 4.1 ([3], [7], [9], [11]). *The* mod 3 *cohomology of the* 3*-connected covers of the exceptional Lie groups $\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$ are as follows*:

$$\begin{aligned}
H^*(\widetilde{G}_2;\mathbb{F}_3) &\cong \mathbb{F}_3[y_6]\otimes E(x_{11},\beta y_6),\\
H^*(\widetilde{F}_4;\mathbb{F}_3) &\cong \mathbb{F}_3[y_{18}]\otimes E(x_{11},\mathcal{P}^1x_{11},\beta y_{18},\mathcal{P}^1\beta y_{18}),\\
H^*(\widetilde{E}_6;\mathbb{F}_3) &\cong \mathbb{F}_3[y_{18}]\otimes E(x_9,x_{11},\mathcal{P}^1x_{11},x_{17},\beta y_{18},\mathcal{P}^1\beta y_{18}),\\
H^*(\widetilde{E}_7;\mathbb{F}_3) &\cong \mathbb{F}_3[y_{54}]\otimes E(x_{11},\mathcal{P}^1x_{11},x_{19},\mathcal{P}^1x_{19},\mathcal{P}^1\mathcal{P}^1x_{19},x_{35},\beta y_{54}),\\
H^*(\widetilde{E}_8;\mathbb{F}_3) &\cong \mathbb{F}_3[y_{54}]\otimes E(x_{15},z_{23},\mathcal{P}^1z_{23},x_{35},x_{39},x_{47},\beta y_{54},y_{59}).
\end{aligned}$$

Note that from the following morphisms of fibrations

$$\begin{array}{ccccc}
\widetilde{E}_7 & \longrightarrow & E_7 & \longrightarrow & K(Z,3)\\
{\scriptstyle i}\downarrow & & \downarrow & & \downarrow\\
\widetilde{E}_8 & \longrightarrow & E_8 & \longrightarrow & K(Z,3)
\end{array}$$

we can choose generators x_{19} in $H^*(\widetilde{E}_7;\mathbb{F}_3)$ such that $i^*(z_{23}) = \mathcal{P}^1x_{19}$ and $i^*(\mathcal{P}^1z_{23}) = \mathcal{P}^1\mathcal{P}^1x_{19}$ [12, VII, Theorem 5.8]. Since $\mathcal{P}^1\mathcal{P}^1 = 2\mathcal{P}^2$ by the Adem relation, we can also choose generators x_{19}', $\mathcal{P}^1x_{19}'$, $\mathcal{P}^2x_{19}'$ in $H^*(\widetilde{E}_7;\mathbb{F}_3)$.

To get the coalgebra structure of $H^*(\Omega G;\mathbb{F}_3)$, we consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G;\mathbb{F}_3)$ with

$$E_2 \cong \mathrm{Tor}_{H^*(G;\mathbb{F}_3)}(\mathbb{F}_3,\mathbb{F}_3).$$

For an odd prime p, $\beta\mathcal{P}^1x_3$ and $\beta\mathcal{P}^3\mathcal{P}^1x_3$ are even-dimensional for $x_3 \in H^3(G;\mathbb{F}_p)$. Since the cohomology of the loop space of a compact simple Lie group is concentrated on even degrees, we have the following non-trivial differentials in E_2:

$$\begin{aligned}
d_2(\gamma_3(\sigma(x_3))) &= \sigma(\beta\mathcal{P}^1x_3) && \text{for } G = F_4, E_6, E_7, E_8,\\
d_2(\gamma_3(\sigma(\mathcal{P}^1x_3))) &= \sigma(\beta\mathcal{P}^3\mathcal{P}^1x_3) && \text{for } G = E_8,
\end{aligned}$$

where σ is the cohomology suspension. Now the E_3 term is even-dimensional, so that $E_3 \cong E_\infty$. Here we put $\sigma(x_3) = a_2$, $\sigma(\mathcal{P}^1 x_3) = a_6$. Then we get the following

Theorem 4.2. *The coalgebra structure of the* mod 3 *cohomology of the loop spaces of exceptional Lie groups* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$\begin{aligned}
&H^*(\Omega G_2;\mathbb{F}_3) \cong \Gamma(a_2, a_{10}),\\
&H^*(\Omega F_4;\mathbb{F}_3) \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_{10}, a_{14}, b_{22}),\\
&H^*(\Omega E_6;\mathbb{F}_3) \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_8, a_{10}, a_{14}, a_{16}, b_{22}),\\
&H^*(\Omega E_7;\mathbb{F}_3) \cong| \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_{10}, a_{14}, a_{18}, b_{22}, a_{26}, a_{34}),\\
&H^*(\Omega E_8;\mathbb{F}_3) \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \mathbb{F}_3[a_6]/(a_6^3) \otimes \Gamma(a_{14}, a_{18}, b_{22}, a_{26}, a_{34}, a_{38}, a_{46}, b_{58}),
\end{aligned}$$

especially we have $\mathcal{P}^1 b_{22} = a_{26}$ *by Theorem* 4.1.

Consider the Rothenberg–Steenrod spectral sequence converging to $H^*(G;\mathbb{F}_3)$ with

$$E_2 \cong \mathrm{Cotor}_{\mathrm{H^*(\Omega G;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3). \tag{4.1}$$

Then we obtain the next theorem by the standard Cotor computation of following monogenic Hopf algebras:

$$\begin{aligned}
\mathrm{Cotor}_{\Gamma(\mathrm{a_{2i}})}(\mathbb{F}_3, \mathbb{F}_3) &= E(x_{2i+1}),\\
\mathrm{Cotor}_{\mathbb{F}_3[\mathrm{a_{2i}}]/(\mathrm{a_{2i}^{3^n}})}(\mathbb{F}_3, \mathbb{F}_3) &= E(x_{2i+1}) \otimes \mathbb{F}_3[x_{(2i)\cdot 3^n+2}].
\end{aligned}$$

Theorem 4.3. $\mathrm{Cotor}_{\mathrm{H^*(\Omega G;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3)$ *of the exceptional Lie groups* G *for* G_2, F_4, E_6, E_7, *and* E_8 *are as follows*:

$$\begin{aligned}
&\mathrm{Cotor}_{\mathrm{H^*(\Omega G_2;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3) \cong E(x_3, x_{11}),\\
&\mathrm{Cotor}_{\mathrm{H^*(\Omega F_4;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}, z_{23}),\\
&\mathrm{Cotor}_{\mathrm{H^*(\Omega E_6;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3]\\
&\qquad\qquad\otimes E(\mathcal{P}^1 x_3, x_9, x_{11}, \mathcal{P}^1 x_{11}, x_{17}, z_{23}),\\
&\mathrm{Cotor}_{\mathrm{H^*(\Omega E_7;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3]\\
&\qquad\qquad\otimes E(\mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}, x_{19}, z_{23}, x_{27}, x_{35}),\\
&\mathrm{Cotor}_{\mathrm{H^*(\Omega E_8;\mathbb{F}_3)}}(\mathbb{F}_3, \mathbb{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^3\mathcal{P}^1 x_3]\\
&\qquad\qquad\otimes E(x_{15}, x_{19}, z_{23}, x_{27}, x_{35}, x_{39}, x_{47}, z_{59}),
\end{aligned}$$

especially we have $\mathcal{P}^1 z_{23} = x_{27}$.

Then from information Theorem 2.2 of $H^*(G;\mathbb{F}_3)$, we can analyze non trivial differentials of the Rothenberg–Steenrod spectral sequence (4.1) converging to

$H^*(G;\mathbb{F}_3)$ as follows:

$$(4.2)\qquad \begin{aligned} d_3(z_{23}) &= (\beta\mathcal{P}^1x_3)^3, &&\text{for } G = F_4, E_6, E_7, E_8\\ d_3(z_{59}) &= (\beta\mathcal{P}^3\mathcal{P}^1x_3)^3, &&\text{for } G = E_8. \end{aligned}$$

Let $A = H^*(G;\mathbb{F}_3)$ in Theorem 2.2. Then like the result in (3.3), for low m such as $1 \le m \le 3$, we have the following:

$$(4.3)\quad H^*(P^m(\Omega G);\mathbb{F}_3) = A^{[m]} \oplus \sum_i z_{4i+3}\cdot A^{[m-1]} \oplus S_m,$$

$$\begin{cases} i = 5 & \text{for } G = F_4, E_6, E_7,\\ i = 5, 14 & \text{for } G = E_8, \end{cases}$$

as modules. Now we compute the module category weight using the same method in Theorem 3.4.

Theorem 4.4. *The module category weight is as follows*:

$$\mathrm{Mwgt}(G_2;\mathbb{F}_3) \ge 2,\qquad \mathrm{Mwgt}(F_4;\mathbb{F}_3) \ge 8,\qquad \mathrm{Mwgt}(E_6;\mathbb{F}_3) \ge 10,$$
$$\mathrm{Mwgt}(E_7;\mathbb{F}_3) \ge 13,\qquad \mathrm{Mwgt}(E_8;\mathbb{F}_3) \ge 18.$$

Proof. From Theorem 4.2, we get $\mathcal{P}^1z_{23} = x_{27}$ in $H^*(P^1(\Omega G);\mathbb{F}_3)$ for $G = E_7, E_8$. Then $\mathcal{P}^1z_{23} = x_{27}$ modulo S_2 in $H^*(P^2(\Omega G);\mathbb{F}_3)$ for $G = E_7, E_8$ from (4.3). Since S_2 is even-dimensional [1], [5], the modulo S_2 is trivial and $\mathcal{P}^1z_{23} = x_{27}$ in $H^*(P^2(\Omega G);\mathbb{F}_3)$. Thus by the similar reason (3.5) as the case of $\mathbb{F}_2$ coefficients, we have

$$\mathcal{P}^1((\beta\mathcal{P}^1x_3)^2x_3x_7x_{11}x_{15}x_{19}z_{23}x_{35}) = (\beta\mathcal{P}^1x_3)^2x_3x_7x_{11}x_{15}x_{19}x_{27}x_{35},$$

$$\begin{aligned}\mathcal{P}^1((\beta\mathcal{P}^1x_3)^2&(\beta\mathcal{P}^3\mathcal{P}^1x_3)^2x_3x_7x_{15}x_{19}z_{23}x_{35}x_{39}x_{47})\\ &= (\beta\mathcal{P}^1x_3)^2(\beta\mathcal{P}^3\mathcal{P}^1x_3)^2x_3x_7x_{15}x_{19}x_{27}, x_{35}x_{39}x_{47},\end{aligned}$$

in $H^*(P^{12}(\Omega E_7);\mathbb{F}_3)$ and $H^*(P^{17}(\Omega E_8);\mathbb{F}_3)$. Note that the filtration lengths of $\beta\mathcal{P}^1x_3$ and $\beta\mathcal{P}^3\mathcal{P}^1x_3$ are both 2 by the result in [16].

Let $\phi_m\colon H^*(P^m(\Omega G);\mathbb{F}_p) \to H^*(G;\mathbb{F}_p)$ be an epimorphism which preserves all Steenrod actions and $\phi_m \circ (e_m)^* \cong 1_{H^*(G;\mathbb{F}_p)}$. Suppose that there are epimorphisms

$$\begin{aligned}\phi_{12}\colon H^*(P^{12}(\Omega E_7);\mathbb{F}_3) &\to H^*(E_7;\mathbb{F}_3),\\ \phi_{17}\colon H^*(P^{17}(\Omega E_8);\mathbb{F}_3) &\to H^*(E_8;\mathbb{F}_3).\end{aligned}$$

Then we have the following diagrams:

$$(4.4)\qquad H^*(P^{12}(\Omega E_7);\mathbb{F}_3) \xrightarrow{\phi_{12}} H^*(E_7;\mathbb{F}_3)$$

$$
\begin{array}{ccc}
(\beta\mathcal{P}^1x_3)^2x_3x_7x_{11}x_{15}x_{19}x_{27}x_{35} & \longmapsto & (\beta\mathcal{P}^1x_3)^2x_3x_7x_{11}x_{15}x_{19}x_{27}x_{35} \\
\Big\uparrow{\scriptstyle \mathcal{P}^1} & & \Big\uparrow{\scriptstyle \mathcal{P}^1} \\
x_{11}x_{15}x_{19}z_{23}x_{35} & \longmapsto & 0
\end{array}
\tag{4.5}
$$

$$
H^*(P^{17}(\Omega E_{78});\mathbb{F}_3) \xrightarrow{\phi_{17}} H^*(E_8;\mathbb{F}_3)
\tag{4.6}
$$

$$
\begin{array}{ccc}
(\beta\mathcal{P}^1x_3)^2(\beta\mathcal{P}^3\mathcal{P}^1x_3)^2X_1 & \longmapsto & (\beta\mathcal{P}^1x_3)^2(\beta\mathcal{P}^3\mathcal{P}^1x_3)^2X_1 \\
\Big\uparrow{\scriptstyle \mathcal{P}^1} & & \Big\uparrow{\scriptstyle \mathcal{P}^1} \\
\mathcal{P}^1((\beta\mathcal{P}^1x_3)^2(\beta\mathcal{P}^3\mathcal{P}^1x_3)^2X_2 & \longmapsto & 0
\end{array}
\tag{4.7}
$$

where $X_1 = x_3x_7x_{15}x_{19}x_{27}x_{35}x_{39}x_{47}$, $X_2 = x_3x_7x_{15}x_{19}z_{23}x_{35}x_{39}x_{47}$.

Obviously this is a contradiction. So ϕ_{12} and ϕ_{17} are not epimorphisms. This means that $(e_{12})^*$, and $(e_{17})^*$ can not be split monomorphisms of all Steenrod algebra module. Hence we obtain that

$$\text{Mwgt}(E_7;\mathbb{F}_3) \geq 13, \qquad \text{Mwgt}(E_8;\mathbb{F}_3) \geq 18.$$

Now we consider the category weight. For G_2, $x_3x_5 \in H^*(P^2(\Omega G_2);\mathbb{F}_3)$, so $(e_2)^*$ is a monomorphism, so $\text{wgt}(G_2;\mathbb{F}_3) = 2$. For F_4, $(\beta\mathcal{P}^1x_3)^2x_3x_7x_{11}x_{15} \in H^*(P^8(\Omega F_4);\mathbb{F}_3)$, so $(e_8)^*$ is a monomorphism, so $\text{wgt}(F_4;\mathbb{F}_3) = 8$. By the same way, $\text{wgt}(E_6;\mathbb{F}_3) = 10$, $\text{wgt}(E_7;\mathbb{F}_3) = 11$ and $\text{wgt}(E_8;\mathbb{F}_3) = 16$. Here the category weight is the same as the the filtration length in [16], that is, $\text{wgt}(G;\mathbb{F}_3) = f_3(G)$.

For the case of G_2, F_4, and E_6, by dimensional reason, any generator of type x_- can not be of the form $\mathcal{P}^i(z_-)$ or $\beta\mathcal{P}^i(z_-)$ for any i and for any generator of type z. So we can not apply the method in (4.4)–(4.7). Hence we do not obtain any positive difference between the category weight and the module category weight. Hence we have

$$\text{Mwgt}(G_2;\mathbb{F}_3) \geq \text{wgt}(G_2;\mathbb{F}_3) = 2, \qquad \text{Mwgt}(F_4;\mathbb{F}_3) \geq \text{wgt}(F_4;\mathbb{F}_3) = 8,$$
$$\text{Mwgt}(E_6;\mathbb{F}_3) \geq \text{wgt}(E_6;\mathbb{F}_3) = 10.$$ □

REMARK 4.5. Combined with Toomer's result in [16], we have the following conclusion:

G	$\text{wgt}(G;\mathbb{F}_3)-\text{cup}(G;\mathbb{F}_3)$	$\text{Mwgt}(G;\mathbb{F}_2)-\text{wgt}(G;\mathbb{F}_2)$	$\text{Mwgt}(G;\mathbb{F}_3)-\text{wgt}(G;\mathbb{F}_3)$
G_2	0	≥ 0	≥ 0
F_4	2	≥ 2	≥ 0
E_6	2	≥ 2	≥ 0
E_7	2	≥ 2	≥ 2
E_8	4	≥ 0	≥ 2

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Younggi Choi
Department of Mathematics Education
Seoul National University
Seoul 151-748, KOREA

E-mail address: yochoi@snu.ac.kr