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MEASURABLE PATTERNS, NECKLACES AND SETS INDISCERNIBLE BY MEASURE

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Abstract. In some recent papers the classical 'splitting necklace theorem' is linked in an interesting way with a geometric 'pattern avoidance problem', see Alon et al. (Proc. Amer. Math. Soc., 2009), Grytczuk and Lubawski (arXiv:1209.1809 [math.CO]), and Lasoń (arXiv:1304.5390v1 [math.CO]). Following these authors we explore the topological constraints on the existence of a (relaxed) measurable coloring of $\mathbb{R}^d$ such that any two distinct, non-degenerate cubes (parallelepipeds) are measure discernible. For example, motivated by a conjecture of Lasoń, we show that for every collection $\mu_1, \ldots, \mu_{2d-1}$ of $2d-1$ continuous, signed locally finite measures on $\mathbb{R}^d$, there exist two nontrivial axis-aligned d-dimensional cuboids (rectangular parallelepipeds) C_1 and C_2 such that $\mu_i(C_1) = \mu_i(C_2)$ for each $i \in \{1, \ldots, 2d-1\}$. We also show by examples that the bound $2d-1$ cannot be improved in general. These results are steps in the direction of studying general topological obstructions for the existence of non-repetitive colorings of measurable spaces.

1. Introduction

The following definition explains in what sense two objects (measurable sets) can be *measure discernible* (or *indiscernible*).

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Definition 1.1. Let $(X, \mathcal{B}, \mu_1, \dots, \mu_d)$ be a measure space with a collection $\mu = (\mu_1, \dots, \mu_d)$ of measures. We say that two measurable sets $A, B \in \mathcal{B}$ are μ-indiscernible or *measure indiscernible* if $\mu(A) = \mu(B)$ as vectors in $\mathbb{R}^d$ or, equivalently, if

$$\mu_j(A) = \mu_j(B) \quad \text{for each } j = 1, \dots, d. \tag{1.1}$$

In the opposite case, i.e. if at least one of equalities in (1.1) does not hold, the sets A and B are measure discernible.

There are many interesting combinatorial geometric results which claim the existence of measure indiscernible partitions of the ambient space X. The classical 'Ham Sandwich Theorem' is a result of this type. Indeed, if $X = \mathbb{R}^d$ then it claims the existence of two μ-indiscernible half-spaces which have a common boundary hyperplane. Much more recent is the result of Hubard and Aronov [9], Karasev [11], and Soberón [19], who showed that for a given collection of d continuous measures $\mu_1, \dots, \mu_d$, defined on $\mathbb{R}^d$, and an integer $k \geq 2$, there exists a partition of $\mathbb{R}^d$ into k convex sets which are μ-indiscernible.

Some questions (and results) about indiscernible partitions are better known as problems about *fair division*, *consensus partitions*, *envy free divisions*, or simply as *equipartitions* of measures, [16], [13], [15], [21], [22]. One of the best known results of this type is the *'splitting necklace theorem'* of Alon [1, 2] which says that each necklace with $k \cdot a_i$ beads of color $i = 1, \dots, n$ can be fairly divided between k *thieves* by at most $n(k-1)$ cuts. Alon deduced this result from the fact that such a division is possible also in the case of a continuous necklace $[0, 1]$ where beads of given color are interpreted as measurable sets $A_i \subset [0, 1]$ (or more generally as continuous measures μ_i).

Some 'pattern avoidance problems' [5] also appear to be directly related to questions about measure indiscernible sets, however until recently [3], [8], [12] these areas seem to have had completely independent development. For illustration, Erdös [6] asked whether there is a 4-coloring of the integers such that each two adjacent intervals are (in our terminology) 'color discernible', meaning that they remain different even after some permutation of their elements. Continuous (measure theoretic) analogues of these questions were formulated and studied in [3], [8] and [12].

The paper [3] establishes an interesting link between the pattern avoidance problem of Erdös and the splitting necklaces problem and focuses on the question whether the number of cuts can be reduced for some subinterval of a line measurably colored by a prescribed number of colors. For example they showed that there exists a measurable 4-coloring of the real line such that two adjacent intervals are always color discernible.

Papers [8] and [12] continued this research, connecting the higher dimensional pattern avoidance problem with the higher dimensional extensions of the splitting necklace theorem [14]. In particular the results and conjectures of Lasoń [12] are our immediate motivation for exploring these and other aspects of measurable colorings of Euclidean spaces.

1.1. Our paper. Our objective is to identify and explore the topological constraints for the existence of 'non-repetitive' or 'pattern avoiding' colorings (measures) of $\mathbb{R}^d$ which are not necessarily measurable partitions (Definition 2.1). For a given family $\mathcal{F}$ of measurable sets in $\mathbb{R}^d$ we introduce the 'pattern-avoiding number' $\nu(\mathcal{F})$ and the 'relaxed pattern-avoiding number' $\nu_{\mathrm{rel}}(\mathcal{F})$ (Definition 2.3) which detect the critical number of colors when color repetitions in $\mathcal{F}$ are always present.

Some results and conjectures of Lasoń [12] (see Section 2.1 for an outline) are naturally interpreted as results about the invariant $\nu(\mathcal{F})$. Our focus is on the closely related invariant $\nu_{\mathrm{rel}}(\mathcal{F})$ which is easier to handle so we are able to provide much more precise information, including some exact calculations.

Following [3], [8], [12] we put some emphasis on the class $\mathcal{C}_d$ of d-cubes and the class $\mathcal{P}_d$ od d-cuboids (rectangular parallelepipeds) in $\mathbb{R}^d$. Our first exact evaluation (Theorem 2.8, Examples 2.9, 2.10, and 2.11) shows that

$$\nu_{\mathrm{rel}}(\mathcal{C}_d) = d + 1 \quad \text{and} \quad \nu_{\mathrm{rel}}(\mathcal{P}_d) = 2d.$$

The class $\mathcal{P}_d$ is somewhat special in the sense that it is invariant with respect to a very large group of auto-homeomorphisms of $\mathbb{R}^d$ (Remark 4.2). As a consequence one can calculate the generalized ν-invariant (in the sense of Remark 2.4) for the class $\mathcal{P}_d$ not only for signed measures but for some other classes including the positive and probability measures on $\mathbb{R}^d$.

In other directions we show that Theorem 2.8 admits several extensions of different nature. In Theorem 3.3 we prove, by using more powerful topological tools, that one can often guarantee the existence of an arbitrarily large finite family of measure indiscernible cubes (cuboids). The same method yields an even stronger result involving families obtained by more general Lie group actions (Theorem 4.1).

In the special case of the Lie group G_{DL}, generated by positive, axis-aligned dilatations and translations in $\mathbb{R}^d$, we calculate (Theorem 5.1) the ν_{rel}-invariant of the associated families of centrally symmetric convex bodies.

We study also other aspects of Theorem 2.8 and show for example (Section 3.1) that in same instances of the problem one can guarantee the existence of *disjoint* measure indiscernible cubes.

2. Non-repetitive colorings of $\mathbb{R}^d$

If not specified otherwise, all measures are signed, locally finite Borel measures on $\mathbb{R}^d$ which are absolutely continuous with respect to the Lebesgue measure dm.

Definition 2.1. A measurable k-coloring of $\mathbb{R}^d$ is a partition $\mathbb{R}^d = A_1 \cup \ldots \cup A_k$ of the ambient d-space into k-measurable sets. A *relaxed* k-coloring of $\mathbb{R}^d$ is a collection $\mu = (\mu_1, \ldots, \mu_k)$ of continuous measures, $d\mu_i = f_i\, dm$, where $f_i(x)$ is the 'intensity' of color i at x, and (more importantly) $\mu_i(A)$ is the total amount of color i used for coloring of the measurable set A.

The following definition explains in what sense a (relaxed) measurable k-coloring of $\mathbb{R}^d$ may be *non-repetitive* (pattern avoiding). The definition is formulated in the language of measures (relaxed colorings) but we tacitly use it also for strict colorings (partitions).

Definition 2.2. Let $\mathcal{F}$ be a family of Lebesgue measurable sets in $\mathbb{R}^d$, such as the family of all axis-aligned cubes $\mathcal{C}_d$ or the family $\mathcal{P}_d$ of all axis-aligned cuboids (d-parallelepipeds). We say that a (relaxed) k-coloring $\mu = (\mu_1, \ldots, \mu_k)$ of $\mathbb{R}^d$ is $\mathcal{F}$-non-repetitive (cube non-repetitive, cuboid non-repetitive) if each two distinct elements $A, B \in \mathcal{F}$ are μ-discernible (Definition 1.1) i.e. if $\mu_i(A) \neq \mu_i(B)$ for at least one of the indices $i \in [k]$.

Definition 2.3. Given a family $\mathcal{F}$ of Lebesgue measurable sets in $\mathbb{R}^d$ we define the corresponding (measure) 'pattern-avoiding number' of $\mathcal{F}$ as the number,

$$\nu(\mathcal{F}) = \mathrm{Inf}\{k \in \mathbb{N} \mid \exists\, \mathcal{F}\text{-non-repetitive } k\text{-coloring of } \mathbb{R}^d\}. \tag{2.1}$$

Similarly, if we allow relaxed colorings we have the corresponding 'relaxed pattern avoiding number' $\nu_{\mathrm{rel}}(\mathcal{F})$ defined as the minimum (infimum) of all k such that there exists a relaxed $\mathcal{F}$-non-repetitive k-coloring of $\mathbb{R}^d$.

Remark 2.4. Perhaps a more systematic approach would involve more general invariants $\nu(\mathcal{F}, \mathcal{C})$ where, aside from the family $\mathcal{F}$ of measurable sets, one also specifies in advance the family $\mathcal{C}$ of admissible colorings (measures). Here we deal mainly with 'relaxed colorings' and the corresponding invariant $\nu_{\mathrm{rel}}(\mathcal{F})$, where $\mathcal{C}$ is the class of all signed, continuous, locally finite Borel measures. The invariant $\nu(\mathcal{F})$ is recovered if $\mathcal{C}$ is the family of all colorings with disjoint measurable set. Other cases of interest would include positive (probabilistic) measures, measures satisfying a condition on their support, etc.

2.1. Some known results about $\nu(\mathcal{F})$. Continuing the research from [3] and [8], and in particular improving over some bounds established in [8], Lasoń in [12] described a method of constructing measurable k-colorings of $\mathbb{R}^d$ which are cube (or cuboid) non-repetitive.

Theorem 2.5 [12, Theorem 3.6]. *For every $d \geq 1$ there exists a measurable $(2d+3)$-coloring of $\mathbb{R}^d$ such that no two nontrivial axis-aligned d-dimensional cubes have the same measure of each color. In other words there exists a measurable $(2d+3)$-coloring (partition) of $\mathbb{R}^d$ which is $\mathcal{C}_d$-non-repetitive.*

Theorem 2.6 [12, Theorem 3.8]. *There exists a measurable $(4d+1)$-coloring (partition) of $\mathbb{R}^d$ which is $\mathcal{P}_d$-non-repetitive. In other words there exists a measurable $(4d+1)$-coloring of $\mathbb{R}^d$ such that no two nontrivial axis-aligned d-dimensional cuboids have the same measure of each color.*

It is natural to ask whether the bounds $2d+3$ and $4d+1$ in Theorems 2.5 and 2.6 are the best possible so Lasoń formulated also the following conjecture.

Conjecture 2.7 [12, Conjecture 3.7]. For every measurable $(2d+2)$-coloring of $\mathbb{R}^d$ there exist two non-degenerate axis-aligned d-dimensional cubes which have the same measure of each color. In other words each $(2d+2)$-coloring of $\mathbb{R}^d$ is 'pattern-repetitive' in the sense that there always exist two distinct cubes that are measure indiscernible.

2.2. Repetitive relaxed colorings of $\mathbb{R}^d$. Here we address the question of the existence of cube (cuboid) non-repetitive patterns in the class of *relaxed* measurable colorings of $\mathbb{R}^d$ (Definition 2.1). In other words we consider exactly the same questions addresses by Theorems 2.5 and 2.6 and Conjecture 2.7 but we allow more general colorings provided by continuous measures which do not necessarily correspond to measurable partitions.

Aside from proving the counterparts of Theorems 2.5 and 2.6 we also provide examples showing that the bounds are the best possible in this case.

Theorem 2.8. *For every collection $\mu_1, \ldots, \mu_d$ of d continuous, signed locally finite measures on $\mathbb{R}^d$, there are two nontrivial axis-aligned d-dimensional cubes C_1 and C_2 such that $\mu_i(C_1) = \mu_i(C_2)$ for all $i = 1, \ldots, d$.*

For every collection $\mu_1, \ldots, \mu_{2d-1}$ of $2d-1$ continuous, signed locally finite measures on $\mathbb{R}^d$, there are two nontrivial axis-aligned d-dimensional cuboids (rectangular parallelepiped) C_1 and C_2 such that $\mu_i(C_1) = \mu_i(C_2)$ for all $i = 1, \ldots, 2d-1$.

Proof. Each nontrivial axis-aligned cube in $\mathbb{R}^d$ is uniquely determined by its vertex $a = (a_1, \ldots, a_d)$ with smallest coordinates and with the length l of its edge. So, the space of all such cubes is homeomorphic to $\mathbb{R}^d \times (0, \infty)$.

The configuration space of all pairs of distinct, axis-aligned cubes in $\mathbb{R}^d$ can be described as $(\mathbb{R}^d \times (0, \infty))^2 \setminus \Delta$, where Δ is the diagonal in the product space. This space is obviously $\mathbb{Z}/2$-equivariantly homotopy equivalent to the sphere S^d. (The antipodal action on the configuration space is the obvious one.)

Let us consider the $\mathbb{Z}/2$-equivariant mapping $F\colon (\mathbb{R}^d \times (0,\infty))^2 \setminus \Delta \to \mathbb{R}^d$ given by

$$F((a,l_1),(c,l_2)) = (\mu_1(a,l_1) - \mu_1(c,l_2), \ldots, \mu_d(a,l_1) - \mu_d(c,l_2)).$$

If there are no measure indiscernible cubes, this mapping would miss the origin. This would lead to an antipodal map from S^d to S^{d-1}, which is a contradiction establishing the 'cube case' of the theorem.

Each nontrivial axis-aligned cuboid in $\mathbb{R}^d$ is uniquely determined by its vertex $a = (a_1, \ldots, a_d)$ with the smallest coordinates and with its vertex $b = (b_1, \ldots, b_d)$ with the biggest coordinates. Here $a_i < b_i$ for each $i = 1, \ldots, d$. So, for every i, (a_i, b_i) always belongs to the open half-plane P (above the line $y = x$). Therefore, the space of all such cuboids is homeomorphic to $\mathcal{P}_d \approx (\mathbb{R}^2)^d$.

As a consequence, the configuration space of all pairs of two distinct nontrivial axis-aligned cuboids in $\mathbb{R}^d$ can be described as $(\mathcal{P}_d)^2 \setminus \Delta$, where Δ is the diagonal in the product space. This space is obviously $\mathbb{Z}/2$-equivariantly homotopy equivalent to the sphere S^{2d-1}.

Let us consider the $\mathbb{Z}/2$-equivariant mapping $G\colon (\mathcal{P}_d)^2 \setminus \Delta \to \mathbb{R}^{2d-1}$ given by

$$F((a,b),(c,d)) = (\mu_1(a,b) - \mu_1(c,d), \ldots, \mu_{2d-1}(a,b) - \mu_{2d-1}(c,d)). \tag{2.2}$$

If there are no pairs of measure indiscernible cuboids, the map described by (2.2) would miss the origin. This would imply the existence of an antipodal map from S^{2d-1} to S^{2d-2}, which leads to the desired contradiction. □

We complete this section by providing examples showing that the estimates obtained in Theorem 2.8 are the best possible. We describe the densities of continuous *signed* measures on $\mathbb{R}^d$ which restricted on $I^d = (0,1)^d$ yield (after normalizing) probability measures on the open d-cube I^d. In light of Remark 4.2 these measures can be pulled back to $\mathbb{R}^d$ to yield desired probability measures on $\mathbb{R}^d$.

Example 2.9. For illustration we initially treat the case $d = 1$, and give two measures on the real line $\mathbb{R}$ such that no two distinct intervals in $\mathbb{R}$ contain the same amount of both measures. Notice that in the case $d = 1$ both parts of Theorem 2.8 reduce to the same statement.

The measures μ_1 and μ_2 are described by their density functions $\varphi_1(x) = 1-x$ and $\varphi_2(x) = x$. Any two intervals containing the same amount of both measures would be of the same length, since the density of the measure $\mu_1+\mu_2$ is constant. But, if they are different, one of them would be more "on the left" and that one would contain greater amount of the measure μ_1 and smaller amount of the measure μ_2 than the other interval.

Notice that this example is also related to the conjecture 3.4 in [12]. Namely, it is conjectured there that for any partition of $\mathbb{R}^d$ in k measurable sets, there is an axis-aligned cube which has a fair q-splitting using at most $k(q-1)-d-1$ axis-aligned hyperplane cuts. This example shows that in the case of measures (and not partitions) for $k=q=2$ and $d=1$, one cut is not enough, while $k(q-1)-d-1=0$.

EXAMPLE 2.10. Let us now take care of the general case. We construct first a collection of $d+1$ measures on $\mathbb{R}^d$ such that no two distinct cubes contain the same amount of every measure. Let these measures be given by their density functions $\varphi_1,\dots,\varphi_{d+1}\colon \mathbb{R}^d\to\mathbb{R}$,

$$\begin{aligned}
\varphi_1(x_1,\dots,x_d) &= x_1x_2\cdot\ldots\cdot x_d,\\
\varphi_2(x_1,\dots,x_d) &= (1-x_1)x_2\cdot\ldots\cdot x_d,\\
\varphi_3(x_1,\dots,x_d) &= (1-x_2)x_3\cdot\ldots\cdot x_d,\\
\varphi_4(x_1,\dots,x_d) &= (1-x_3)x_4\cdot\ldots\cdot x_d,\dots,\varphi_{d+1}(x_1,\dots,x_d)=1-x_d.
\end{aligned}$$

It is easy to verify that the length of the edges of two cubes containing the same amount of every measure have to be equal, and than that the coordinates of the vertex with the smallest coordinates should be also equal for these two cubes. (We first notice that for the coordinate x_d, then x_{d-1} etc.)

EXAMPLE 2.11. Let us describe $2d$ measures on $\mathbb{R}^d$ by their density functions:

$$\begin{aligned}
\varphi_0(x) &= 1,\\
\varphi_i(x) &= x_i \quad \text{for } i\in\{1,\dots,d\},\\
\varphi_{d+i}(x) &= x_i^3 \quad \text{for } i\in\{1,\dots,d-1\}.
\end{aligned}$$

Let us denote by $[a,b]$ and $[c,d]$ two cuboids containing the same amount of every measure. Here $a=(a_1,\dots,a_d)$ and $c=(c_1,\dots,c_d)$ are vertices of these two cuboids with smallest coordinates and b and d the vertices with biggest coordinates.

The requirement that these cuboids contain the same amount of every measure provide us with the following equalities. Measure μ_0 gives us the equality

$$\prod_{i=1}^{d}(b_i-a_i)=\prod_{i=1}^{d}(d_i-c_i).$$

Measures μ_j for $j\in\{1,\dots,d\}$ give us the equalities

$$(a_j+b_j)\prod_{i=1}^{d}(b_i-a_i)=(c_j+d_j)\prod_{i=1}^{d}(d_i-c_i).$$

Together with the first equality these give us the equalities $a_j+b_j=c_j+d_j$, for all $j\in\{1,\dots,d\}$. Measures μ_{d+j} for $j\in\{1,\dots,d-1\}$ give us the equalities

$$(a_j^2+b_j^2)(a_j+b_j)\prod_{i=1}^{d}(b_i-a_i)=(c_j^2+d_j^2)(c_j+d_j)\prod_{i=1}^{d}(d_i-c_i).$$

Together with previous equalities these give us

$$a_j^2 + b_j^2 = c_j^2 + d_j^2 \quad \text{for } j \in \{1, \dots, d-1\}.$$

Furthermore, these equalities together with the equalities $a_j + b_j = c_j + d_j$ give us directly $a_j = c_j$ and $b_j = d_j$ for $j \in \{1, \dots, d-1\}$. Then, from the first equality we have $b_d - a_d = d_d - c_d$, and from another one $a_d + b_d = c_d + d_d$. This gives us $a_d = c_d$ and $b_d = d_d$, and so these two cuboids coincide. This means that two distinct cuboids could not contain the same amount of every of the described $2d$ measures.

Corollary 2.12. *If $\mathcal{C}_d$ and $\mathcal{P}_d$ are the families of all cubes (respectively cuboids) in the d-dimensional space $\mathbb{R}^d$ then,*

$$\nu_{\mathrm{rel}}(\mathcal{C}_d) = d + 1 \quad \textit{and} \quad \nu_{\mathrm{rel}}(\mathcal{P}_d) = 2d. \tag{2.3}$$

Let us compare the results obtained in this section with the results of [12] (see also Section 2.1), dealing (instead of measures) with the partitions of $\mathbb{R}^d$ in disjoint measurable subsets. Lasoń proved that there exists a partition of $\mathbb{R}^d$ in $2d + 3$ disjoint measurable sets such that no two distinct nontrivial axis-aligned cubes contain the same amount of every of these sets. Also, it is proved that there exists a partition of $\mathbb{R}^d$ in $4d + 1$ disjoint measurable sets such that no two distinct nontrivial axis-aligned cuboids contain the same amount of every of these sets. It is conjectured (Conjecture 2.7 in Section 2.1) that these estimates are the best possible.

We work with measures and prove the corresponding results with $2d+3$ being replaced by $d + 1$, and with $4d + 1$ being replaced by $2d$, and show that these results are the best possible in this case.

3. A generalization

Definition 3.1. For each topological space X, the associated *configuration space* $F(X, n)$ of all n-tuples of labelled points in X is the space,

$$F(X, n) := \{x \in X^n \mid x_i \neq x_j \text{ for each } i \neq j\}.$$

The obvious action of the symmetric group S_n on X^n, restricts to a free action on the associated configuration space $F(X, n)$.

As shown by examples in the previous section, Theorem 2.8 is optimal as far as the number of measures is concerned. However, we show that it can be considerably improved in a different direction. Indeed, it turns out that instead of two cubes (cuboids) we can prove the existence of a finite family of cubes (cuboids) of any size which are μ-indiscernible in the sense of Definition 1.1.

More explicitly, we extend the results from the previous section to the case of n cubes (cuboids) in $\mathbb{R}^d$ where $n = p^k$ is a power of a prime p. Instead of the

Borsuk–Ulam theorem, used in the proof of Theorem 2.8, we apply the following result, see [20], [10], [7], [11], [9], [4].

Theorem 3.2. *Suppose that* $n = p^k$ *is a power of a prime* $p \geq 2$ *and let* $m \geq 2$. *Let* W_n *be the* $(n-1)$*-dimensional, real representation of the symmetric group* S_n *which arises as the orthogonal complement of the diagonal in the permutation representation* $\mathbb{R}^n$. *Then each equivariant map* $\Phi\colon F(\mathbb{R}^m, n) \to W_n^{\oplus(m-1)}$ *must have a zero.*

Theorem 3.3. *For each collection* $\mu_1, \dots, \mu_d$ *of* d *continuous, signed locally finite measures on* $\mathbb{R}^d$ *and any natural number* n*, there exists a collection of* n *pairwise distinct nontrivial, axis-aligned* d*-dimensional cubes* $C_1, \dots, C_n$ *which are* μ*-indiscernible in the sense that* $\mu_i(C_j) = \mu_i(C_k)$ *for all* $i = 1, \dots, d$ *and* $j, k \in \{1, \dots, n\}$. *For each collection* $\mu_1, \dots, \mu_{2d-1}$ *of* $2d-1$ *continuous signed locally finite measures on* $\mathbb{R}^d$*, there exists a collection of* n *pairwise distinct nontrivial, axis-aligned* d*-dimensional cuboids* (*rectangular parallelepipeds*) $C_1, \dots, C_n$ *which are* μ*-indiscernible in the sense that* $\mu_i(C_j) = \mu_i(C_k)$ *for all* $i = 1, \dots, 2d-1$ *and* $j, k \in \{1, \dots, n\}$.

Proof. We outline the proof of the second statement. Without loss of generality we can assume that $n = p^k$ is a power of a prime number. As already observed in the proof of Theorem 2.8, the variety $\mathcal{P}_d$ of all cuboids in $\mathbb{R}^d$ is homeomorphic to $\mathbb{R}^{2d}$. Given a collection $C = (C_1, \dots, C_n) \in F(\mathcal{P}_d, n)$ of pairwise distinct cuboids and a measure μ_i let $\mu_i(C) := (\mu_i(C_1), \dots, \mu_i(C_n)) \in \mathbb{R}^n$. Obviously the cuboids $\{C_j\}_{j=1}^n$ are μ_i-indiscernible if and only if $\pi(\mu_i(C)) = 0$ where $\pi\colon \mathbb{R}^n \to W_n$ is the natural projection.

Let $\phi_i\colon F(\mathcal{P}_d, n) \to W_n$ be the map defined by $\phi_i(C) = \pi(\mu_i(C))$. Let $\Phi\colon F(\mathcal{P}_d, n) \to (W_n)^{\oplus(2d-1)}$ be the associated map where $\Phi(C) = (\phi_1(C), \dots, \phi_{2d-1}(C))$. Since $\mathcal{P}_d \cong \mathbb{R}^{2d}$ it follows from Theorem 3.2 that for some $C \in F(\mathcal{P}_d, n)$, $\Phi(C) = 0$ which completes the proof of the theorem. □

3.1. The case of pairwise disjoint cubes and cuboids. A natural question is whether one can strengthen Theorems 2.8 and 3.3 by claiming the existence of *pairwise disjoint* cuboids (cubes) which are μ-indiscernible.

Proposition 3.4. *The configuration space of all ordered collections of* $n \geq 2$ *pairwise disjoint cuboids in* $\mathbb{R}^d$ *is* S_n*-equivariantly homotopy equivalent to the configuration space* $F(\mathbb{R}^d, n)$. *Moreover, the configuration space of all ordered collections of* $n \geq 2$ *pairwise disjoint cubes in* $\mathbb{R}^d$ *of the same size is also* S_n*-equivariantly homotopy equivalent to the configuration space* $F(\mathbb{R}^d, n)$.

Proof. As shown by May [17, Theorem 4.8], the configuration space of all ordered collections of n pairwise disjoint, axis-aligned cuboids in $\mathbb{R}^d$ is S_n-equivariantly homotopy equivalent to the configuration space $F(\mathbb{R}^d, n)$. Actually

May puts more emphasis in his proof on axis-aligned cuboids with disjoint interiors but the argument in [17] can be easily modified to cover the case of disjoint cuboids as well.

A similar result holds for disjoint cubes. Indeed, for a given axis-aligned cuboid C let $\widehat{C}$ be the $\subseteq$-maximal cube in the set of all axis-aligned cubes $D \subset C$ which have the same barycenter as C. Then the map which sends an ordered collection $(C_1, \ldots, C_n)$ of n pairwise disjoint cuboids to the corresponding collections $(\widehat{C}_1, \ldots, \widehat{C}_n)$ of cubes is easily shown to be a deformation retraction. Finally, if all cubes $\widehat{C_i}$ are shrank to the cubes of the same size we obtain a deformation retraction that establishes the second part of the proposition. $\square$

Proposition 3.4 shows that we cannot improve the 'cuboid case' of Theorem 3.3 (by an argument based on Theorem 3.2) to pairwise disjoint cuboids unless we drastically reduce the number of measures. However, the situation with cubes is different. The following result shows that one can always find two or more pairwise disjoint, measure indiscernible cubes of the *same size* if one allows not more than $(d-1)$ colors (i.e. one less than in the 'cube case' of Theorem 3.3). The proof relies on Proposition 3.4 and follows closely the proof of Theorem 3.3 so the details are omitted.

Proposition 3.5. *For each collection $\mu_1, \ldots, \mu_{d-1}$ of $(d-1)$ continuous, signed locally finite measures on $\mathbb{R}^d$ and any natural number n, there exists a collection of n pairwise disjoint, axis-aligned d-dimensional cubes $C_1, \ldots, C_n$ of the* same size *which are μ-indiscernible in the sense that $\mu_i(C_j) = \mu_i(C_k)$ for all $i = 1, \ldots, d$ and $j, k \in \{1, \ldots, n\}$.*

Conjecture 3.6. The result from Proposition 3.5 is the best possible in the following stronger sense. There exists a collection of d continuous, signed locally finite measures on $\mathbb{R}^d$ such that not only pairs of disjoint cubes but the pairs of disjoint *cuboids* are also measure discernible.

4. Towards general non-repetitive colorings

Here we show that there is nothing special about cubes and cuboids and that theorems from the previous sections hold also for balls, ellipsoids, and even more generally for measurable sets of suitable form. Moreover, there is nothing special about choosing a preferred position for the selected geometric shape (say axis-aligned or similar). Perhaps the most natural framework for this problem is given by the following result which involves arbitrary Lie group actions on $\mathbb{R}^d$.

Theorem 4.1. *Let Q be a polytope in $\mathbb{R}^d$ (more generally a convex body or just a measurable set). Let G be a Lie group acting on $\mathbb{R}^d$. Let $\mathcal{F} = O_Q = \{g(Q) \mid g \in G\}$ be the set (orbit) of all images of Q with respect to actions of elements from G. Assume that O_Q is a smooth manifold (possibly with singularities) of* *geometric dimension* ν*. Then for each* relaxed measurable coloring *of* $\mathbb{R}^d$ *with* $\nu - 1$ *colors* (*Definition* 2.1) *and each integer* $n \geq 2$ *there exist a collection of* n *distinct elements in* $\mathcal{F}$ *which are pairwise measure indiscernible* (*in the sense of Definition* 1.1)*. In particular,* $\nu_{\mathrm{rel}}(\mathcal{F}) \geq \nu$*.*

Proof. By passing to a larger number if necessary we can assume that $n = p^k$ is power of a prime. By assumption O_Q is a manifold of dimension ν so there is a subset $U \subset O_Q$ homeomorphic to $\mathbb{R}^\nu$. The test map for the existence of a collection of n measure indiscernible sets in U is $\phi_i \colon F(U, n) \to W_n^{\oplus(\nu-1)}$ which by Theorem 3.2 must have a zero. □

4.1. Remarks and examples. Typically the orbit space O_Q that appears in Theorem 4.1 is homeomorphic to a homogeneous manifold G/H where $H = \{g \in G \mid g(Q) = Q\}$. For example it is well-known that the group of all isometries of a convex body K in $\mathbb{R}^d$ is a Lie subgroup H of the group $G = \mathrm{Isom}(\mathbb{R}^d)$ of all isometries of the ambient space. In this case Theorem 4.1 establishes a connection between the dimension of the isometry (symmetry) group H of K and the ν_{rel}-invariant of the associated family $\mathcal{F}_K$ of isometric copies of K in $\mathbb{R}^d$,

$$d(d+1)/2 < \dim(H) + \nu_{\mathrm{rel}}(\mathcal{F}_K). \tag{4.1}$$

If G is the Lie group of all maps $f : \mathbb{R}^d \to \mathbb{R}^d$ where $f(x) = Ax + b$ for some diagonal matrix A with positive entries and $b \in \mathbb{R}^d$ we obtain a generalization of Theorem 2.8 to the case of G-orbits of convex bodies, including for example the case of axis-aligned ellipsoids.

Remark 4.2. Perhaps as a justification of treating separately the case of cuboids (Theorems 2.8 and 3.3) from the general case (Theorem 4.1), here we argue that after all there is something special about the family $\mathcal{P}_d$.

Suppose that $f_i \colon \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, d)$ is a family of homeomorphisms. If $f = \prod f_i$ is the associated auto-homeomorphism of $\mathbb{R}^d$ then f obviously sends cuboids to cuboids. Similarly if $f_i \colon \mathbb{R} \to (0,1)$ are homeomorphisms, the associated product homeomorphism $f \colon \mathbb{R}^d \to (0,1)^d$ sends bijectively the cuboids from $\mathbb{R}^d$ to cuboids from $(0,1)^d$. By restricting on $(0,1)^d$ the functions described in the Example 2.11, normalizing and pulling back to $\mathbb{R}^d$ by f, we can easily construct $2d$ probability measures on $\mathbb{R}^d$ which distinguish cuboids one from another. As a consequence we can prove that $\nu(\mathcal{P}_d, \mathcal{P}) = 2d$ (see Remark 2.4) where $\mathcal{P}$ is the family of probability measures on $\mathbb{R}^d$. This is a result that does not obviously hold for other classes $\mathcal{F}$ covered by Theorem 4.1.

5. Some exact values for $\nu_{\mathrm{rel}}(\mathcal{F})$

In this section we show that the invariant $\nu_{\mathrm{rel}}(\mathcal{F})$ can be evaluated for many other classes of convex bodies (measurable sets) in $\mathbb{R}^d$. In particular we show that the same bounds that we determined in the case of cuboids (Theorem 2.8), also hold in the case of ellipsoids and other centrally symmetric, axis-aligned bodies in $\mathbb{R}^d$.

We begin with some preliminary definitions. Let $\chi\colon \mathbb{R}^d \to \mathbb{R}$ be a non-negative, bounded, measurable function with bounded support in $\mathbb{R}^d$. Our main example of such a function is the indicator function χ_K of a convex body $K \subset \mathbb{R}^d$.

We put some emphasis on the case of centrally symmetric convex bodies. Motivated by that we say that $\chi\colon \mathbb{R}^d \to \mathbb{R}$ is an even function, relative $a \in \mathbb{R}^d$, if

$$\chi(a+x) = \chi(a-x) \quad \text{for each } x \in \mathbb{R}^d. \tag{5.1}$$

Let G_{DT} be the group of all transformations of $\mathbb{R}^d$ generated by translations and positive axis-aligned dilatations. More explicitly, $L \in G_{DT}$ if there exists a diagonal matrix $A = \operatorname{diag}\{C_1, \ldots, C_d\}$ with positive entries, and a vector $b \in \mathbb{R}^d$ such that $L(x) = A(x) + b$ for each $x \in \mathbb{R}^d$.

It is not difficult to check that the geometric dimension of the family $\mathcal{F}_K = \{L(K) \mid L \in G_{DT}\}$ is equal to $2d$. In light of Theorem 4.1 we know that $\nu_{\mathrm{rel}}(\mathcal{F}_K) \geq 2d$. The following proposition establishes the opposite inequality in the case of centrally symmetric convex bodies.

Theorem 5.1. *Let K be a centrally symmetric convex body in $\mathbb{R}^d$. Let*

$$\mathcal{F}_K = \{L(K) \mid L \in G_{DT}\}$$

be the associated family of all convex bodies obtained from K by successive positive, axis-aligned dilatations and translations. Then $\nu_{\mathrm{rel}}(\mathcal{F}_K) \leq 2d$. More explicitly, the required relaxed $(2d)$-coloring of $\mathbb{R}^d$ is provided by the measures $d\mu_i = \phi_i dm$ with the following density functions,

$$\phi_0 = 1, \quad \phi_i(x) = x_i \quad (i = 1, \ldots, d), \quad \phi_{d+i} = x_i^2 \quad (i = 1, \ldots, d-1). \tag{5.2}$$

Before we commence the proof of Theorem 5.1 we establish the following lemma.

Lemma 5.2. *Suppose that $\alpha\colon \mathbb{R} \to \mathbb{R}$ is a non-negative, integrable function such that for some $r \in \mathbb{R}$, $\alpha(r-x) = \alpha(r+x)$ for each $x \in \mathbb{R}$. Assume that $\int_{\mathbb{R}} \alpha > 0$. Suppose that $\mathfrak{u}_1(x) = a_1 x + b_1$ and $\mathfrak{u}_2(x) = a_2 x + b_2$ are two increasing linear functions $(a_1, a_2 > 0)$ such that*

$$\begin{aligned} \int_{\mathbb{R}} \mathfrak{u}_1(x)\alpha(x)\,dx &= \int_{\mathbb{R}} \mathfrak{u}_2(x)\alpha(x)\,dx, \\ \int_{\mathbb{R}} \mathfrak{u}_1^2(x)\alpha(x)\,dx &= \int_{\mathbb{R}} \mathfrak{u}_2^2(x)\alpha(x)\,dx. \end{aligned} \tag{5.3}$$

Then $(a_1, b_1) = (a_2, b_2)$, i.e. $\mathfrak{u}_1(x) = \mathfrak{u}_2(x)$ for each $x \in \mathbb{R}$.

Proof. Without loss of generality we may assume that $r = 0$ which means that α is an even function. Since $\mathfrak{u}_i(x) + \mathfrak{u}_i(-x) = 2b_i$, it easily follows from the first equality in (5.3) that $b_1 = b_2$.

For contradiction let us assume that $a_2 > a_1 > 0$. It follows that $\mathfrak{u}_2^2(x) + \mathfrak{u}_2^2(-x) > \mathfrak{u}_1^2(x) + \mathfrak{u}_1^2(-x)$ for each $x \neq 0$, which is essentially a consequence of the convexity of the function $x \mapsto |x|^2$. However this is in contradiction with the second equality in (5.3) since α is non-negative and by a change of variables,

$$\int_{\mathbb{R}} [\mathfrak{u}_1^2(x) + \mathfrak{u}_1^2(-x)]\alpha(x)\,dx = \int_{\mathbb{R}} [\mathfrak{u}_2^2(x) + \mathfrak{u}_2^2(-x)]\alpha(x)\,dx. \tag{5.4}$$ □

Proof of Theorem 5.1. For a given $\chi\colon \mathbb{R}^d \to \mathbb{R}$ let $\mathcal{F}_\chi = \{\chi \circ L \mid L \in G_{DT}\}$. For example if $\chi = \chi_K$ is the indicator function of a convex body K then $\mathcal{F}_{\chi_K}$ is the set of indicator functions of elements from $\mathcal{F}_K$.

We prove a slightly more general statement by showing that if $\chi\colon \mathbb{R}^d \to \mathbb{R}$ is an even, non-negative, integrable function with non-zero integral, then each two elements $\chi_1, \chi_2 \in \mathcal{F}_\chi$ are measure discernible in the sense that

$$\int_{\mathbb{R}^d} \phi_i \chi_1 = \int_{\mathbb{R}^d} \phi_i \chi_2 \quad \text{for each } i \in \{0, 1, \ldots, 2d-1\} \quad \Rightarrow \quad \chi_1 = \chi_2. \tag{5.5}$$

Suppose that $\chi_i(x) = \chi(L_i^{-1})(x)$ $(i = 1, 2)$ where $L_i(x) = A_i(x) + b_i$ and

$A_1 = \mathrm{diag}(c_1', \ldots, c_d')$, $A_2 = \mathrm{diag}(c_1'', \ldots, c_d'')$, $b_1 = (b_1', \ldots, b_d')$, $b_2 = (b_1'', \ldots, b_d'')$.

The assumption from (5.5) on the functions χ_1 and χ_2 is by a change of variables equivalent to,

$$\prod_{i=1}^{d} c_i' \int_{\mathbb{R}^d} \phi_i(L_1(x))\chi(x)\,dx = \prod_{i=1}^{d} c_i'' \int_{\mathbb{R}^d} \phi_i(L_2(x))\chi(x)\,dx. \tag{5.6}$$

Since $\phi_0 = 1$, and by assumption $\int \chi \neq 0$, we deduce from (5.6) that

$$\prod_{i=1}^{d} c_i' = \prod_{i=1}^{d} c_i''.$$

The functions $\phi_j(x) = x_j$ and $\phi_{d+j}(x) = x_j^2$ depend only on the variable x_j. If $i \in \{j, d+j\}$ then the equality (5.6) (after cancelling out the products) can be rewritten as follows:

$$\int_{\mathbb{R}} \phi_i(L_1(x))\widehat{\chi}(x)\,dx = \int_{\mathbb{R}} \phi_i(L_2(x))\widehat{\chi}(x)\,dx \tag{5.7}$$

where $\widehat{\chi}$ is the density of the measure on $\mathbb{R}$ obtained as the pushforward of the measure $\chi\,dx$ (defined on $\mathbb{R}^d$) with respect to the projection on the x_j-axis.

Since $\phi_j(L_1(x)) = c_j' x_j + b_j'$ and $\phi_{d+j}(L_1(x)) = (c_j' x_j + b_j')^2$ the equalities (5.7) provide exactly the input needed for the application of Lemma 5.2. As a consequence we have the equality $(c_j', b_j') = (c_j'', b_j'')$ for each $j = 1, \ldots, d-1$.

From here and the equality of products we observe that $c'_d = c''_d$. By choosing the remaining unused function $\phi_d(x) = x_d$, and by one more application of (5.7), we deduce that $b'_d = b''_d$ which completes the proof of the proposition. □

Corollary 5.3. *Suppose that K is a centrally symmetric convex body in $\mathbb{R}^d$. If $\mathcal{F}_K = \{L(K) \mid L \in G_{DT}\}$ is the associated family of all convex bodies obtained by successive dilatations (positive, axis-aligned) and translations, then*

$$\nu_{\mathrm{rel}}(\mathcal{F}_K) = 2d.$$

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