Topological **M**ethods in **N**onlinear **A**nalysis
Volume 45, No. 1, 2015, 7–37

MORSE HOMOTOPY AND TOPOLOGICAL CONFORMAL FIELD THEORY

Viktor Fromm

Dedicated to Yuli Rudyak, on the occasion of his 65th birthday

Abstract. By studying spaces of flow graphs in a closed oriented manifold, we equip the Morse complex with the operations of an open topological conformal field theory. This complements previous constructions due to R. Cohen et al., K. Costello, K. Fukaya and M. Kontsevich and is also the Morse theoretic counterpart to a conjectural construction of operations on the chain complex of the Lagrangian Floer homology of the zero section of a cotangent bundle, obtained by studying uncompactified moduli spaces of higher genus pseudoholomorphic curves.

1. Introduction

A flow graph in a manifold M consists of the following data: a graph G, a choice for every edge e of G of a flow Ψ_e on M, and a continuous map $\gamma\colon G \to M$ so that the image of each edge e is a piece of a trajectory of the corresponding flow Ψ_e. Flow graphs can be used to recover invariants of a manifold. The simplest instance of this is the Morse complex, corresponding to the case when G consists of a single edge: by studying the spaces of trajectories of the gradient flow of a Morse function, one constructs a chain complex which computes homology. On the other hand, there are invariants which are not visible in the classical Morse complex but can be recovered using more general flow graphs ([2], [5], [10]).

2010 *Mathematics Subject Classification.* 57R19, 57R56, 81T40, 53D37.
Key words and phrases. Morse complex, topological conformal field theory, flow graph, ribbon graph, moduli space.

In the work of R. Cohen and his collaborators ([2], [5]), flow graphs were used to construct cohomology operations (we remark that our terminology is somewhat different in that we use the term 'flow graph' instead of 'graph flow'). The operations associated to certain special graphs can be identified explicitly and turn out to correspond to invariants known from classical algebraic topology: the cup product, the Steenrod squares, the Stiefel-Whitney classes as well as the Massey products (the latter using a somewhat different approach) can all be encoded in this way. Moreover, the operations satisfy a field-theoretic law: there is a compatibility between gluing together graphs and composing the associated operations.

In this paper we construct, building upon seminal ideas of K. Fukaya ([10]), the conformal version of this theory, obtained by studying flow graphs together with a ribbon structure on the underlying graph. By taking the ribbon structure into account in a suitable way, we define operations that are parametrized by the moduli spaces of Riemann surfaces with boundary. The construction draws upon ideas from Floer homology and Gromov-Witten theory.

Let Σ be a compact connected oriented surface with $m > 0$ boundary components and with $n_+ + n_- \geq 0$ boundary marked points, partitioned into incoming and outgoing points. We assume that $2g - 2 + m > 0$ and denote by $\mathcal{M}_\Sigma$ the space of complex structures on Σ, together with labellings of the marked points by positive real numbers. Let g be a Riemannian metric and f a Morse function on M, whose gradient flow with respect to g is Morse-Smale. Denote by $C^*(f)$ the associated Morse complex, with the grading given by the Morse index and the codifferential given by counts of trajectories of the positive gradient flow. We will use flow graphs to construct cochain maps

$$F_\Sigma^f \colon (C^*(f))^{\otimes n_+} \to C^*(\mathcal{M}_\Sigma) \otimes (C^*(f))^{\otimes n_-}, \tag{1.1}$$

where $C^*(\mathcal{M}_\Sigma)$ is a chain complex computing the cohomology of $\mathcal{M}_\Sigma$ (more precisely, cohomology with coefficients in a certain local system must be used). For example, if Σ is a disk with two incoming and one outgoing marked point, then $\mathcal{M}_\Sigma$ is contractible and the map in cohomology induced by (1.1) is the cup product. To construct the operations (1.1) in the general case, the ribbon graph decomposition of Riemann surfaces ([14], [19]) will be used.

A graph is a one-dimensional CW complex. We refer to the univalent vertices of a graph as the external vertices and to the vertices of valency greater than one as internal. An edge is called external if it is incident to a univalent vertex, and internal otherwise. Consider a graph G together with an embedding $i \colon G \hookrightarrow \Sigma$, so that the univalent vertices of G are mapped to the boundary marked points, the remaining points of G are mapped to the interior of Σ and so that Σ deformation retracts to $i(G)$. Two embeddings are identified if one is obtained from the other by an isotopy which is constant on the univalent vertices. A *ribbon structure* on G

is an equivalence class of embeddings. We write Γ for the ribbon graph and we say that Γ has Σ as its associated oriented surface.

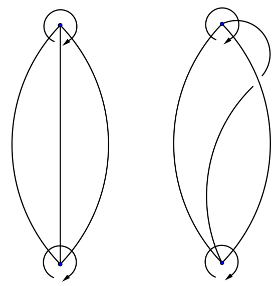

Figure 1. Two ribbon graphs whose associated surfaces are a pair of pants and a torus with a disk removed respectively

A *half-edge* of a graph G is a pair $(v, [\varphi])$, where v is a vertex and $[\varphi]$ is an isotopy class of embeddings $\varphi\colon ([0,1],0) \hookrightarrow (G,v)$. It is a classical observation that a ribbon structure on G can equivalently be defined as a cyclic ordering of the half-edges at every vertex. An *orientation* of an edge is defined as an ordering of the two-element set consisting of the corresponding half-edges. A *metric structure* on G is an assignment to every edge $e \in E(G)$ of a non-negative real number l_e.

Assume that for every marked point on $\partial\Sigma$, a critical point of f is fixed. We write $(\mathbf{p}_+, \mathbf{p}_-)$ for the tuple consisting of the critical points, partitioned into intputs and outputs according to the partition of the corresponding marked points. In Section 2, we associate to every ribbon graph Γ as above a space $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ of flow graphs in M. An element of $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ consists of a metric structure on Γ together with a continuous map $\boldsymbol{\gamma}\colon \Gamma \to M$, so that the restrictions of the map to the edges are pieces of trajectories of the flows of given vector fields $\mathbf{x}$ on M and where incidence conditions corresponding to the critical points $\mathbf{p}_+, \mathbf{p}_-$ are imposed at the external vertices. For generic choice of the vector field datum $\mathbf{x}$, the space $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ is a smooth manifold. Moreover, the partial compactification $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ obtained by allowing internal edges of zero length as well as broken flow lines at the external edges is a manifold with corners. By the ribbon graph decomposition of Riemann surfaces, the space of metric ribbon graphs Γ as above is homeomorphic to $\mathcal{M}_\Sigma$ (see (2.4) below) and thus there is a projection $\pi_\Gamma\colon \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) \to \mathcal{M}_\Sigma$ obtained by forgetting $\boldsymbol{\gamma}$. We show that π_Γ is a proper map. These claims are established in Section 2.2.

We view $Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) = (\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-), \pi_\Gamma)$ as a geometric chain in $\mathcal{M}_\Sigma$. In Section 2.1 we define a chain complex $C^{BM}_*(\mathcal{M}_\Sigma)$ generated by pairs (P, π), where P is a (not necessarily compact) oriented manifold with corners and

$\pi\colon P \to \mathcal{M}_\Sigma$ is a proper continuous map. The complex $C_*^{BM}(\mathcal{M}_\Sigma)$ computes the Borel–Moore homology of $\mathcal{M}_\Sigma$, the latter being isomorphic by rational Poincaré duality to the cohomology.

THEOREM 1.1. *Let*

$$F^f_\Sigma\colon (C^*(f))^{\otimes n_+} \to C_*^{BM}(\mathcal{M}_\Sigma; \det{}^d \otimes or) \otimes (C^*(f))^{\otimes n_-} \tag{1.2}$$

be the linear map defined by

$$\mathbf{p}_+ \mapsto \sum_{\Gamma,\mathbf{p}_-} Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) \otimes \mathbf{p}_-, \tag{1.3}$$

where the sum is over all $\mathbf{p}_- \in (\mathrm{Crit}(f))^{\times n_-}$ and over all the ribbon graphs Γ, so that every internal vertex of Γ is trivalent.

(a) *F^f_Σ is a cochain map.*
(b) *F^f_Σ is independent, up to chain homotopy, of choice of the vector field data, of the Riemannian metric and of the Morse function.*

Here d denotes the dimension of M and det and or are certain local systems on $\mathcal{M}_\Sigma$, defined in Section 3.1. The appearance of local coefficients is due to the fact that the space $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is not canonically oriented, but rather its orientation is determined by fixing linear orderings of the vertices and of the edges of Γ and orientations of the edges. The dependence of the orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ on these choices is described in Section 3.1. Theorem 1.1 is proved in Section 3.2.

We now outline the relationship of Theorem 1.1 to some previous results. As was mentioned above, a construction of operations from spaces of flow graphs was first proposed in [2] (see [5] for a more recent exposition). In this latter approach, one associates to a graph G a cohomology operation, parametrized by the homology of the classifying space of the automorphism group $\mathrm{Aut}(G)$. The main difference of the construction of Theorem 1.1 to these previous ideas is the consideration of ribbon structures. As a consequence, the operations constructed here are parametrized by the moduli spaces of Riemann surfaces instead.

While the operations (1.1) bear some formal resemblence to the structure of Gromov–Witten invariants, there are two pronounced differences: firstly, Σ has non-empty boundary and secondly, the operations are parametrized by the moduli space $\mathcal{M}_\Sigma$ instead of the Deligne–Mumford compactification. The operations (1.1) fit into the framework of an open topological conformal field theory ([21], [13], [7]) – their compatibility with gluing is proved in Section 3.3.

There are two previously known constructions of open TCFTs. The first is an algebraic-combinatorial approach based on an idea of M. Kontsevich in [18] (see also [7], [15]) and produces an open TCFT starting with a (finite-dimensional minimal) cyclic A_∞-algebra – the homotopy associative analogue of a differential graded algebra, equipped with a compatible inner product. The chain complex used in this approach is the so-called ribbon graph complex. The second construction, due to K. Costello, is more analytic and relies on a notion of differential forms on $\mathcal{M}_\Sigma$ ([8]). The starting point here is a so-called Calabi–Yau elliptic space – one of the simplest examples of such an object is the de Rham algebra of a closed oriented manifold. The relationship between these two approaches is well-understood: a cyclic A_∞-algebra can be associated to each Calabi–Yau elliptic space and the open TCFT obtained via the analytic construction on the elliptic space is equivalent to the result of applying the algebraic approach to the associated A_∞-algebra ([8, Section 5]). In a forthcoming paper, we will show that the result of the more geometric construction of Theorem 1.1 is equivalent to what one obtains by applying these previously known approaches.

It is known that for every open TCFT there is an associated universal open-closed, and thus in particular a closed TCFT ([7]). A direct geometric construction of an open-closed TCFT on a manifold, following ideas different from what is presented here, was outlined in [3]. The resulting closed TCFT is expected to be closely related to the BV algebra of loop homology ([13]).

We finish the introduction by sketching the relationship of the construction of Theorem 1.1 to Floer homology and the theory of pseudoholomorphic curves. For a smooth function f on a manifold M, the graph L_{df} of the differential df is an exact Lagrangian submanifold of the total space of the cotangent bundle T^*M. Two graphs L_{df_1} and L_{df_2} intersect transversally if and only if the difference $f_2 - f_1$ is a Morse function. In this case the Lagrangian Floer cohomology of the pair L_{df_1}, L_{df_2} is defined ([9]). The differential of Lagrangian Floer cohomology is constructed by counting the elements of the zero-dimensional components of the moduli spaces of pseudoholomorphic strips in T^*M with L_{df_1} and L_{df_2} as boundary conditions. Extending this idea, we can consider for any oriented surface Σ as in Theorem 1.1 the space of all pseudoholomorphic maps from Σ to T^*M with Lagrangian boundary conditions of the form L_{df}, $f \in \mathcal{C}^\infty(M, \mathbb{R})$ imposed on the components of the complement in $\partial\Sigma$ of the set of marked points. For example, in the case when there are no marked points, we consider the space $\mathcal{M}_\Sigma^{T^*M}$ of all pseudoholomorphic curves in T^*M which map every boundary component of Σ to a submanifold L_{df} for some fixed $f \in \mathcal{C}^\infty(M, \mathbb{R})$. The space $\mathcal{M}_\Sigma^{T^*M}$ is usually non-compact, but if it carries a fundamental class in Borel-Moore homology, then using the fact that the projection $\pi_\Sigma \colon \mathcal{M}_\Sigma^{T^*M} \to \mathcal{M}_\Sigma$ is proper, one would obtain a cocycle in $\mathcal{M}_\Sigma$. More generally, by considering surfaces with marked points on the boundary, we would be led to operations on the chain complex of Lagrangian Floer cohomology, analogous to the ones constructed in Theorem 1.1. If Σ is a disk, this could be made rigorous by the methods developed in [12]. The general case is, to the knowledge of the author, conjectural.

A. Floer showed that for a suitable choice of almost complex structures on T^*M and of a Riemannian metric on M, the chain complex of Lagrangian Floer homology of a pair L_{df_1}, L_{df_2} is isomorphic to the Morse complex of $f_2 - f_1$ ([9]). The main idea is sometimes referred to as an 'adiabatic limit' argument: Floer demonstrated that after multiplying f_1 and f_2 by a sufficiently small number, there is a one-to-one correspondence between isolated pseudoholomorphic strips in T^*M and isolated gradient flow trajectories in M. In [11], this idea was generalized to obtain an identification between spaces of pseudoholomorphic disks with arbitrary number of boundary marked points and spaces of flows of ribbon trees. These results suggest that the operations on the chain complex of Lagrangian Floer homology of the zero section of the cotangent bundle, defined as outlined above from the study of general psudoholomorphic curves in T^*M, should correspond to operations on the Morse complex $C^*(f_2 - f_1)$, obtained from spaces of flows of general ribbon graphs. Theorem 1.1 provides the construction of these latter operations. Throughout this paper, homology and cohomology with real coefficients is used and the coefficient ring is omitted from the notation. Theorem 1.1 continues to hold for coefficients in an arbitrary field of characteristic zero.

The author would like to thank Klaus Mohnke for many fruitful conversations.

2. Ribbon graphs and Morse theory

In this Section we introduce the complex $C_*^{BM}(\mathcal{M}_\Sigma)$ of locally finite geometric chains in $\mathcal{M}_\Sigma$ and use flow graphs in a manifold to define elements of this complex.

2.1. Geometric chains and Borel–Moore homology. The Borel–Moore homology, or homology with closed support, of a topological space X is defined as the homology of the chain complex of locally finite singular chains in X, i.e. of linear combinations $\sum n_\sigma \sigma$ of singular simplices $\sigma\colon \Delta \to X$, so that for every compact subset $K \subset X$ there are only finitely many non-vanishing coefficients n_σ with $\sigma(\Delta) \cap K \neq \emptyset$. Summing up the top-dimensional simplices of a triangulation of a (not necessarily compact) oriented manifold N^d, one defines its fundamental class $[N] \in H_d^{BM}(N)$. The map $x \mapsto [N] \cap x$ yields a Poincaré duality isomorphism

$$H^k(N) \xrightarrow{\sim} H_{d-k}^{BM}(N). \tag{2.1}$$

While Borel–Moore homology is not functorial in general, there is a pushforward

$$f_*\colon H_*^{BM}(X) \to H_*^{BM}(Y)$$

provided that $f\colon X \to Y$ is proper.

If $f\colon X \to Y$ is continuous and on both X and Y there are Poincaré duality isomorphisms as in (2.1), then $f^*\colon H^*(Y) \to H^*(X)$ yields a transfer homomorphism $f_*^{!BM}\colon H_*^{BM}(Y) \to H_{*+d_X-d_Y}^{BM}(X)$ in Borel–Moore homology.

Similarly to the case of singular homology discussed in [16], Borel–Moore homology may be computed as the homology of a complex of geometric chains. To define this complex, we recall that a d-dimensional manifold with corners P is a paracompact Hausdorff space locally modelled on the products $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{d-k}$, $0 \leq k \leq d$. A local boundary component β at a point x of P consists of a choice, for a coordinate neighbourhood U of x, of a connected component of the points of U which lie on the codimension one stratum. With the convention

$$\partial P = \{(x,\beta) : x \in P,\ \beta \text{ a local boundary component of } P \text{ at } x\}, \tag{2.2}$$

the boundary of a manifold with corners is again a manifold with corners and there is a natural map $i_{\partial P}\colon \partial P \to P$.

We denote by $C_*^{BM}(X)$ the vector space of all sums of the form $\sum_P n_P(P, f_P)$, where P is an oriented manifold with corners and $f_P\colon P \to X$ is a proper continuous map. As before, we require that the geometric chain $\sum_P n_P(P, f_P)$ be locally finite: for every $K \subset\subset X$, there are only finitely many summands so that n_P is non-zero and $f_P(P) \cap K \neq \emptyset$. We make the following two identifications. Firstly, if P' is obtained from P by reversing the orientation, then we identify $(P', f_{P'})$ with $-(P, f_P)$. Secondly, if P is the union $P = Q \cup_X R$ of two codimension zero submanifolds with corners Q and R along a (possibly empty) submanifold with corners $X \subset \partial P, \partial Q$, then we identify (P, f_P) with the sum of $(Q, f_P|_Q)$ and $(R, f_P|_R)$. The degree of (P, f_P) is the dimension of P and the boundary is defined by $\partial(P, f_P) = (\partial P, f_P \circ i_{\partial P})$.

$$\mathcal{M}_\Sigma \simeq \bigcup_\Gamma \mathrm{Met}(\Gamma)/\mathrm{Aut}(\Gamma). \tag{2.3}$$

PROPOSITION 2.1. *For any topological space X, the homology of the complex $C_*^{BM}(X)$ is isomorphic to the Borel–Moore homology of X.*

PROOF. Denote by $h_*^{BM}(X)$ the homology of the complex $C_*^{BM}(X)$ and by $H_*^{BM}(X)$ the Borel–Moore homology of X. By definition, the latter is given by equivalence classes of closed locally finite chains $\sum_\sigma n_\sigma \sigma$, where $\sigma\colon \Delta \to X$ is a singular simplex. A map $\psi\colon H_*^{BM}(X) \to h_*^{BM}(X)$ is defined by

$$\Big[\sum_\sigma n_\sigma \sigma\Big] \mapsto \Big[\sum_\sigma n_\sigma(\Delta, \sigma)\Big].$$

Conversely, given a generator (P, f_P) of $C_*^{BM}(X)$, choose a triangulation of P which induces a triangulation of ∂P and so that if two components of ∂P are diffeomorphic, then the induced triangulations coincide. Since f is proper, the sum $\sum\limits_i f_P|_{\Delta_i} =: \sum\limits_i \sigma_i$ over the top-dimensional simplices of the triangulation is a locally finite singular chain in X; it is closed (resp. exact) if (P, f_P) is closed (resp. exact) and its class in $H_*^{BM}(X)$ is independent of the choice of the triangulation. Define $\phi\colon h_*^{BM}(X) \to H_*^{BM}(X)$ by

$$[P, f_P] \mapsto \Big[\sum_i \sigma_i\Big].$$

It is immidiate to check that ϕ and ψ are inverse to each other. □

In Section 3.3, we will use the fact that in two simple special cases the transfer homomorphism $f_*^{!BM}\colon H_*^{BM}(Y) \to H_*^{BM}(X)$ corresponding to a continuous map $f\colon X \to Y$ can be identified as the homomorphism induced by an explicit chain map $f_*^{!CBM}\colon C_*^{BM}(Y) \to C_*^{BM}(X)$. Firstly, in the case of the inclusion map $i\colon X \hookrightarrow Y$ of an open subset $X \subset Y$, the image under $i_*^{!CBM}$ of a generator (P, f_P) of $C_*^{BM}(Y)$ is given by $(f_P^{-1}(X), f_P|_{f_P^{-1}(X)})$. Secondly, if π is the projection $\pi\colon X \to Y$ of a locally trivial fibre bundle, then the image of a generator (P, f_P) of $C_*^{BM}(Y)$ under $\pi_*^{!CBM}$ is given by (f_P^*X, π^*f_P), where $f_P^*X \to P$ is the pullback bundle and π^*f_P the map which makes the following diagram commutative:

$$\begin{array}{ccc} f_P^*X & \xrightarrow{\pi^* f_P} & X \\ \downarrow & & \downarrow \pi \\ P & \xrightarrow[f_P]{} & Y \end{array}$$

We will also consider the complex C_*^{BM} with coefficients in a local system. In this case a generator is a pair (P, f_P) as before, together with a section of the pullback of the local system under f_P. If the local system is graded, then the degree of the chain is given by the sum of the dimension of P and the degree of the section. There is a straightforward analogue of Proposition 2.1 for homology with local coefficients.

As was mentioned above, on any oriented manifold there is a Poincaré duality isomorphism $H_*^{BM}(M) \simeq H^{\dim M - *}(M)$. Over a field of characteristic zero, this is also true for the space $\mathcal{M}_\Sigma$ of complex structures. To explain this fact, we recall that by the ribbon graph decomposition of Riemann surfaces ([14], [19], see also [8]), there is a homeomorphism

$$\mathcal{M}_\Sigma \simeq \bigcup_\Gamma \mathrm{Met}_0(\Gamma)/\sim . \tag{2.4}$$

The union on the right-hand side of (2.4) is over all ribbon graphs Γ whose associated oriented surface is Σ and so that every internal vertex of the graph has valency at least three; the univalent vertices of Γ correspond to the marked points on $\partial\Sigma$. The symbol $\mathrm{Met}_0(\Gamma)$ denotes the space of metric structures on Γ, where we allow edges of length zero, subject to the condition that the sum of the lengths of the edges in any cycle remain positive. The equivalence relation is generated by the following two identifications. Firstly, we identify a metric structure on a ribbon graph Γ with each metric structure obtained as the pullback under a ribbon graph automorphism $\Gamma \to \Gamma$ which fixes each univalent vertex. Secondly, we identify two metric ribbon graphs if one is obtained from the other by collapsing internal edges of zero length.

As a simple example, consider the case of a disk with four marked points on the boundary (for simplicity, the real-valued labellings of the marked points are omitted).

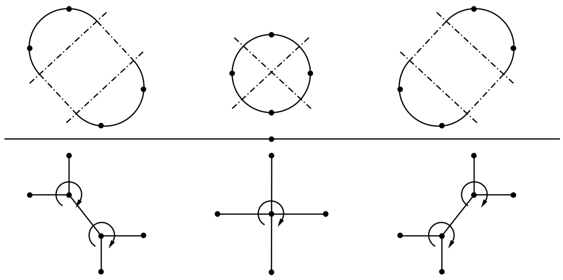

FIGURE 2. The ribbon graph decomposition of the moduli space of holomorphic disks with four marked points on the boundary

The moduli space is a real line which is identified as the result of gluing two copies of $\mathbb{R}_{\geq 0}$ to a point. The homeomorphism (2.4) means that each element of the moduli space is obtained uniquely from the fully symmetric configuration by slicing open along one of the two axes of symmetry which do not contain marked points, and then gluing in a complex strip.

It follows from (2.4) that $\mathcal{M}_\Sigma$ is an orbifold which admits a good cover in the sense of [1, p. 35]. Thus over a field of characteristic zero, there is an isomorphism

$$H^*_c(\mathcal{M}_\Sigma; or) \simeq H_{\dim \mathcal{M}_\Sigma - *}(\mathcal{M}_\Sigma), \tag{2.5}$$

where $H^*_c(\mathcal{M}_\Sigma; or)$ is the cohomology with compact support and coefficients in the orientation sheaf. By the universal coefficient theorem, $H^*_c(\mathcal{M}_\Sigma; or)$ and $H_*(\mathcal{M}_\Sigma)$ are the dual vector spaces to respectively $H^{BM}_*(\mathcal{M}_\Sigma; or)$ and $H^*(\mathcal{M}_\Sigma)$. We conclude the isomorphism

$$H^{BM}_*(\mathcal{M}_\Sigma; or) \simeq H^{\dim \mathcal{M}_\Sigma - *}(\mathcal{M}_\Sigma). \tag{2.6}$$

2.2. Flow graphs in a manifold. We now introduce spaces of flow graphs in a manifold and use them to define elements of $C^{BM}_*(\mathcal{M}_\Sigma)$. The discussion of transversality and the construction of natural compactifications follows a similar line of argument as in the classical setting of Morse theory ([20]).

Let Γ be a ribbon graph which appears on the right-hand side of the ribbon graph decomposition (2.4). Suppose that to every boundary marked point on $\partial\Sigma$ (or, equivalently, to every external edge e of Γ), a critical point p_e of the Morse function f is associated. The partition of the marked points into incoming and outgoing points defines a partition $\mathbf{p} = (\mathbf{p}_+, \mathbf{p}_-)$ of the tuple $\mathbf{p}$ of the critical points. We will associate to this data a Banach manifold $\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$. We first introduce parametrizations of the edges of Γ.

Let e be an external edge of Γ and v be the external vertex incident to e. If v is marked as incoming, then we fix a homeomorphism $\psi_e\colon (-\infty, 0] \xrightarrow{\sim} e - \{v\}$. In the case when v is marked as outgoing, we fix ψ_e as a homeomorphism $\psi_e\colon [0,\infty) \xrightarrow{\sim} e - \{v\}$. In each case we orient e by prescribing ψ_e to be orientation-preserving. To choose parametrizations of the internal edges of Γ, we must take into account the fact that the choice of orientations of these edges is non-canonical.

Definition 2.2. We denote by $\Gamma' \to \Gamma$ the finite cover with fibre given by all choices of orientations of all the internal edges of Γ.

Thus Γ' is a graph whose points are given by pairs consisting of a point of Γ together with a choice of orientations of all the internal edges of Γ. Every edge of Γ' carries a natural orientation and the projection $p_\Gamma\colon \Gamma' \to \Gamma$ preserves the orientations of the external edges. The group T of covering transformations of $\Gamma' \to \Gamma$ is generated by the involutions τ_e given by reversing the orientation of an internal edge e of Γ.

We fix for every internal edge e' of Γ' a continuous map $\psi_{e'}\colon [0,1] \to e'$ whose restriction to $(0,1)$ is a homeomorphism and which induces the natural orientation of e'. If e' is an incoming internal edge of Γ' which is mapped to e under p_Γ, then a map $\psi_{e'}\colon (-\infty,0] \to e'$ is uniquely defined by requiring that the composition of $p_\Gamma \circ \psi_{e'}$ coincide with ψ_e. We define parametrizations $\psi_{e'}\colon [0,\infty) \to e'$ of the outgoing edges of Γ' analogously.

If e' is an incoming external edge of Γ' and p the associated critical point of f, then we denote by $H_{e'}$ the space of all elemenets $\gamma \in W^{1,2}_{\mathrm{loc}}((-\infty,0], M)$ so that there exist $T > 0$ and $\xi \in W^{1,2}([T,\infty), T_pM)$ with $\gamma(-t) = \exp_p(\xi(t))$ for all $t \geq T$. Here $\exp_p$ denotes the exponential map at p, defined in a neighbourhood of the origin of T_pM. In the case when e' is outgoing, we define $H_{e'}$ analogously, but with $(-\infty,0]$ replaced by $[0,\infty)$. For an internal edge e' of Γ', we denote $H_{e'} = W^{1,2}([0,1], M)$.

Definition 2.3. We define $\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ as the space of contunuous maps $\boldsymbol{\gamma}\colon \Gamma' \to M$, so that for every edge e' of Γ', the composition $\gamma_{e'} = \boldsymbol{\gamma} \circ \psi_{e'}$ is an element of $H_{e'}$ and moreover for any pair of edges e of Γ and e' of Γ', where e is an internal edge,

$$\gamma_{\tau_e(e')}(t) = \begin{cases} \gamma_{e'}(t) & \text{if } p_\Gamma(e') \neq e, \\ \gamma_{e'}(1-t) & \text{if } p_\Gamma(e') = e. \end{cases} \tag{2.7}$$

The space $\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ is a Banach manifold. The tangent space $T_{\boldsymbol{\gamma}}\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ at a point $\boldsymbol{\gamma}$ is the closed subspace of $\otimes_{e' \in E(\Gamma')} W^{1,2}(\gamma_{e'}^* TM)$ consisting of all the elements $\mathbf{s} = (s_{e'})_{e' \in E(\Gamma')}$ which define a continuous section of $\boldsymbol{\gamma}^* TM$ and so that for all edges e of Γ and e' of Γ', where e is an internal edge,

$$s_{\tau_e(e')}(t) = \begin{cases} s_{e'}(t) & \text{if } p_\Gamma(e') \neq e, \\ -s_{e'}(1-t) & \text{if } p_\Gamma(e') = e. \end{cases} \tag{2.8}$$

Here $W^{1,2}(\gamma_{e'}^* TM)$ denotes the space of sections of class $W^{1,2}$ and $E(\Gamma')$ is the set of edges of Γ'.

We define a Banach bundle $\mathcal{E} \to \mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ as the pullback of $T\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ under the projection $\mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-) \to \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ to the second factor.

There is a well-defined section

$$\mathbf{S} = (s_{e'})_{e' \in E(\Gamma')} \in L^2(\mathcal{E}) \tag{2.9}$$

determined by the condition that

$$s_{e'}(\boldsymbol{\gamma}) = \frac{d}{dt}\gamma_{e'}(t) \tag{2.10}$$

for every edge e' of Γ'.

We want to consider maps in $\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$, so that the restriction of the map to every edge is a piece of a trajectory of the flow of a given one-parameter family of vector fields. We now introduce the setup for the construction of these vector field data.

We choose for each edge e' of Γ' a one-parameter family of vector fields on M. Formally, consider the vector bundle $E \to \mathrm{Met}_0(\Gamma) \times [0,1] \times M$ given as the pullback of TM under the projection to the last factor. Let X_Γ denote the space of all sections of class $W^{1,2}$ of E.

Definition 2.4. We define $\mathcal{X}_\Gamma$ to be the space of all elements

$$\mathbf{x} = (x_{e'})_{e' \in E(\Gamma')} \in X_\Gamma^{\otimes |E(\Gamma')|} \tag{2.11}$$

which satisfy the following conditions:

(a) For any two edges e of Γ and e' of Γ', where e is an internal edge, and each $t \in [0,1]$,

$$x_{\tau_e(e')}(\,\cdot\,, t, \,\cdot\,) = \begin{cases} x_{e'}(\,\cdot\,, t, \,\cdot\,) & \text{if } p_\Gamma(e') \neq e, \\ -x_{e'}(\,\cdot\,, 1-t, \,\cdot\,) & \text{if } p_\Gamma(e') = e. \end{cases} \tag{2.12}$$

(b) There is a constant $C > 0$, so that for every edge e' of Γ', the estimate

$$\|x_{e'}(\boldsymbol{\ell}, t, \cdot)\|_{W^{1,2}(TM)} < C \tag{2.13}$$

holds true for all $(\boldsymbol{\ell}, t) \in \mathrm{Met}_0(\Gamma) \times [0,1]$.

The first condition will be used to associate to every element of $\mathcal{X}_\Gamma$ a well-defined section of $\mathcal{E}$. The second condition is essential for the construction of compactifications of the spaces of flow graphs.

To every element $\mathbf{x} \in \mathcal{X}_\Gamma$, we associate a section $\mathcal{F}_{\mathbf{x}} = (F_{e'})_{e' \in E(\Gamma')}$ of $\mathcal{E}$ as follows. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $\sigma(t) = 1$ for $|t| \leq 1$ and $\sigma(t) = 0$ for $|t| \geq 2$. If e' is an external edge of Γ', then

$$F_{e'}(\boldsymbol{\ell}, \gamma)(t) = \nabla_g f(\gamma_{e'}(t)) + \sigma(t) x_{e'}(\boldsymbol{\ell}, |t|, \gamma_{e'}(t)). \tag{2.14}$$

If e' is an internal edge of Γ' which is mapped to $e \in E(\Gamma)$ under the projection $\Gamma' \to \Gamma$, then we denote

$$F_{e'}(\boldsymbol{\ell}, \gamma)(t) = l_e x_{e'}(\boldsymbol{\ell}, t, \gamma_{e'}(t)), \tag{2.15}$$

where l_e is the length of e in the metric structure $\boldsymbol{\ell}$.

With this notation in place, we can now define the spaces of flow graphs.

Definition 2.5. For $\mathbf{x} \in \mathcal{X}_\Gamma$, let $\mathbf{S}_{\mathbf{x}} \in L^2(\mathcal{E})$ denote the section given as the difference

$$\mathbf{S}_{\mathbf{x}} = \mathbf{S} - \mathcal{F}_{\mathbf{x}}, \tag{2.16}$$

where $\mathbf{S}$ is defined as in (2.10). We define the space

$$\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) \subset \mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-) \tag{2.17}$$

of flows over Γ subject to the vector field datum $\mathbf{x}$ as the zero locus of $\mathbf{S}_{\mathbf{x}}$.

Definition 2.5 associates to each graph Γ which appears in the ribbon graph decomposition (2.4) of $\mathcal{M}_\Sigma$ a corresponding space $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ of flow graphs in a manifold. This is illustrated by the following simple example.

Example 2.6. Consider the case when Σ is an annulus with two marked points on the same boundary component.

In this example, the moduli space $\mathcal{M}_\Sigma$ is homeomorphic to an open disk (for simplicity, we omit the real-valued labels at the marked points). There are five distinct isomorphism classes of ribbon graphs which appear on the right-hand side of (2.4). Three two-dimensional cells corresponding to the three ribbon graphs with two internal vertices of valency three are glued together along two one-dimensaion cells which correspond to the two graphs with a single internal vertex of valency four. There are no non-trivial automorphisms. The ribbon graph decomposition and the spaces of flow graphs associated to the cells are illustrated in Figure 3. The shaded parts of the graphs indicate the vector field data.

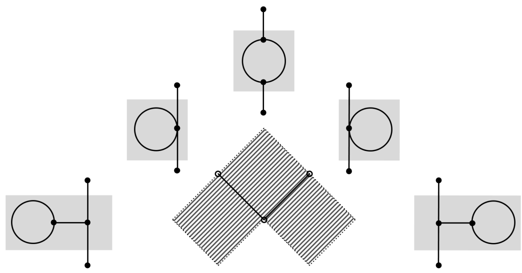

FIGURE 3. Flow graphs corresponding to an annulus with two marked points on the same boundary component

The next proposition summarizes the arguments used to equip the space $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ with the structure of a manifold.

PROPOSITION 2.7.

(a) *For every* $\mathbf{x} \in \mathcal{X}_\Gamma$*, the section*

$$\mathbf{S}_\mathbf{x} \colon \mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-) \to \mathcal{E} \tag{2.18}$$

is Fredholm of index

$$\mathrm{ind}(\mathbf{S}_\mathbf{x}) = |\mathbf{p}_-| - |\mathbf{p}_+| + d\chi(\Sigma) - dn_- + |E(\Gamma)|. \tag{2.19}$$

Here $|\mathbf{p}_+|$ *and* $|\mathbf{p}_-|$ *denote the sum of the Morse indices of the critical points corresponding to the incoming and to the outgoing marked points respectively. The symbol* $|E(\Gamma)|$ *stands for the number of edges of* Γ.

(b) *There is a subset* $\mathcal{X}_{\Gamma,\mathrm{reg}} \subset \mathcal{X}_\Gamma$ *of second category so that for each* $\mathbf{x} \in \mathcal{X}_{\Gamma,\mathrm{reg}}$, $\mathbf{S}_\mathbf{x}$ *is transverse to the zero section of* $\mathcal{E}$.

(c) *Suppose that for every graph* $\widetilde{\Gamma}$ *obtained from* Γ *by collapsing internal edges (where as before no cycle is collapsed), an element* $\mathbf{x}_{\widetilde{\Gamma}} \in \mathcal{X}_{\widetilde{\Gamma},\mathrm{reg}}$ *is fixed. Then there exists a subset of second category of* $\mathcal{X}_\Gamma$*, so that for every element* $\mathbf{x}$ *of that subset, the conclusion of* (b) *holds true and, in addition, the restriction of* $\mathbf{x}$ *to* $\mathrm{Met}_0(\widetilde{\Gamma}) \subset \mathrm{Met}_0(\Gamma)$ *coincides with* $\mathbf{x}_{\widetilde{\Gamma}}$.

PROOF. The arguments are analogous to the classical case of spaces of flow trajectories and we will thus stay brief. Denote by $D\mathbf{S}_x$ the linearization of $\mathbf{S}_x$. For a given $\boldsymbol{\ell} \in \mathrm{Met}(\Gamma)$ and $\mathbf{s} \in T_\gamma\mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$, we can write $D\mathbf{S}_x(0, \mathbf{s})$ in the form

$$(D\mathbf{S}_x(0, \mathbf{s}))_{e'}(t) = \frac{d}{dt} s_{e'}(t) - A(t) s_{e'}(t), \tag{2.20}$$

where $A(t) \in \mathrm{End}(T_{\gamma_{e'}(t)}M)$ are endomorphisms such that if e' is an external edge with $\gamma_{e'}(t) \to p \in \mathrm{Crit}(f)$ for $|t| \to \infty$, then $A(t) \to \mathrm{Hess}_f(p)$. Using the non-degeneracy of the Hessian, one concludes from (2.20) the inequality

$$\|s_{e'}\|_{W^{1,2}} \leq c(\|s_{e'}\|_{L^2} + \|(D\mathbf{S}_x(0,\mathbf{s}))_{e'}\|_{L^2}) \tag{2.21}$$

for a positive constant c. Using the compactness of the embedding

$$W^{1,2}(T_\gamma\mathcal{B}_\Gamma(\mathbf{p}_+,\mathbf{p}_-)) \hookrightarrow L^2(T_\gamma\mathcal{B}_\Gamma(\mathbf{p}_+,\mathbf{p}_-)),$$

it follows from (2.21) that the map $(D\mathbf{S}_x)_2 \colon \mathbf{s} \mapsto D\mathbf{S}_x(0,\mathbf{s})$ has finite-dimensional kernel and closed image. A standard computation using partial integration shows that each element $\mathbf{r} \in L^2(\mathcal{E})$ so that $\langle \mathbf{r}, D\mathbf{S}_x(0,\mathbf{s})\rangle = 0$ for all $\mathbf{s} \in T_\gamma\mathcal{B}_\Gamma(\mathbf{p}_+,\mathbf{p}_-)$ is weakly differentiable and satisfies

$$\frac{d}{dt} r_{e'}(t) + A^T(t) r_{e'}(t) = 0. \tag{2.22}$$

Together with the Sobolev embedding $W^{1,2}_{\mathrm{loc}}(\mathbb{R}, M) \hookrightarrow C^0(\mathbb{R}, M)$ and using uniqueness of solutions of an ordinary differential equation, it follows that the cokernel of $(D\mathbf{S}_x)_2$ is finite-dimensional and thus $(D\mathbf{S}_x)_2$ is Fredholm. Since $\mathrm{Met}(\Gamma)$ is finite-dimensional, we conclude that $D\mathbf{S}_x$ is Fredholm aswell. The index formula (2.19) is straightforward.

To prove the second part of the proposition, we consider the map

$$\mathcal{S}_\mathcal{X} \colon \mathcal{X}_\Gamma \times \mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+,\mathbf{p}_-) \to \mathcal{E}, (\mathbf{x}, \boldsymbol{\ell}, \gamma) \mapsto \mathcal{S}_\mathbf{x}(\boldsymbol{\ell}, \gamma). \tag{2.23}$$

It suffices to show that $\mathcal{S}_\mathcal{X}$ is transverse to the zero section of $\mathcal{E}$. To this end, we must check that if $\mathcal{S}_\mathcal{X}(\mathbf{x}, \boldsymbol{\ell}, \gamma) = 0$, then every element $\mathbf{r} \in L^2(\mathcal{E})$, so that $\langle \mathbf{r}, D\mathcal{S}_\mathcal{X}(\mathbf{y}, \mathbf{m}, \mathbf{s})\rangle = 0$ for all $(\mathbf{y}, \mathbf{m}, \mathbf{s}) \in \mathcal{X}_\Gamma \oplus T_{\boldsymbol{\ell}}\mathrm{Met}(\Gamma) \oplus T_\gamma\mathcal{B}_\Gamma(\mathbf{p}_+,\mathbf{p}_-)$, vanishes identically. By the proof of the first part of the proposition, each component $r_{e'}$, $e' \in E(\Gamma')$, is continuous. Thus it suffices to show that $r_{e'}(t) = 0$ for $t \in (0,1)$. For simplicity, we will only carry this out in the case when e' is an internal edge, the case when e' is an external edge being similar.

From (2.15), we have

$$(D\mathcal{S}_\mathcal{X}(\mathbf{y}, 0, 0))_{e'}(t) = y_{e'}(t, \gamma_{e'}(t)). \tag{2.24}$$

Suppose that $r_{e'}(t_0) \neq 0$, $t_0 \in (0,1)$. Then there exist $0 < \varepsilon < \min(t_0, 1 - t_0)$ and $y_{e'} \in X_\Gamma$, so that

$$\langle r_{e'}(t), y_{e'}(\boldsymbol{\ell}, t, \gamma_{e'}(t))\rangle_g > 0 \quad \text{for } |t - t_0| < \varepsilon \tag{2.25}$$

and

$$y_{e'}(\boldsymbol{\ell}, t, \,\cdot\,) \equiv 0 \quad \text{for } |t - t_0| \geq \varepsilon. \tag{2.26}$$

Using $y_{e'}$, we define $\mathbf{y} \in \mathcal{X}_\Gamma$ as follows. Let e be the edge of Γ corresponding to e' under the projection $\Gamma' \to \Gamma$. If e'' is an edge of Γ' obtained from e' by a covering transformation of $\Gamma' \to \Gamma$ which preserves the orientation of e, then we define $y_{\tau_{e''}(e')}(\boldsymbol{\ell}, t, \cdot\,) = y_{e'}(\boldsymbol{\ell}, t, \cdot\,)$. In the case when e'' is obtained from e' by applying a covering transformation which reverses the orientation of e, we put $y_{\tau_{e'}(e')}(\boldsymbol{\ell}, t, \cdot\,) = -y_{e'}(\boldsymbol{\ell}, 1-t, \cdot\,)$. Finally, define $y_{e''} \equiv 0$ for all other edges e'' of Γ'. Then $\mathbf{y} = (y_{e'})_{e' \in E(\Gamma')}$ is a well-defined element of $\mathcal{X}_\Gamma$. Using (2.8), we compute:

$$\langle \mathbf{r}, D\mathcal{S}(\mathbf{y}, 0, 0)\rangle = 2|E_{\mathrm{int}}(\Gamma)| \int_{t=0}^{\varepsilon} \langle r_{e'}(t), y_{e'}(t, \gamma_{e'}(t))\rangle_g > 0 \tag{2.27}$$

in contradiction to the choice of $\mathbf{r}$. This completes the proof of the second part of the proposition. The proof of the third part is analogous. □

Corollary 2.8.

(a) *For every element* $\mathbf{x} \in \mathcal{X}_{\Gamma,\mathrm{reg}}$*, the space* $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ *is a manifold whose dimension is given by the right-hand side of the index formula* (2.19)*. The space is empty in the case when the right-hand side of* (2.19) *is negative.*

(b) *Let* $\mathcal{X}_\Sigma$ *denote the vector space*

$$\mathcal{X}_\Sigma = \bigoplus_\Gamma \mathcal{X}_\Gamma, \tag{2.28}$$

where the direct sum is over all the ribbon graphs Γ *in the ribbon graph decomposition* (2.4) *of* $\mathcal{M}_\Sigma$*. There is a subset* $\mathcal{X}_{\Sigma,\mathrm{reg}} \subset \mathcal{X}_\Sigma$ *of second category, so that*

$$\mathcal{X}_{\Sigma,\mathrm{reg}} \subset \bigoplus_\Gamma \mathcal{X}_{\Gamma,\mathrm{reg}} \tag{2.29}$$

and, in addition, every element $\mathbf{y} = (\mathbf{x}_\Gamma)_\Gamma \in \mathcal{X}_{\Sigma,\mathrm{reg}}$ *satisfies the following condition: if the graph* $\widetilde{\Gamma}$ *is obtained from* Γ *by collapsing internal edges, then* $\mathbf{x}_{\widetilde{\Gamma}}$ *coincides with the restriction of* $\mathbf{x}_\Gamma$ *to* $\mathrm{Met}_0(\widetilde{\Gamma}) \subset \mathrm{Met}_0(\Gamma)$.

The second part of this Corollary means that the vector field data for different ribbon graphs can be chosen consistently with the attachments of the corresponding cells in the ribbon graph deocmposition.

Proof. The first part of the Corollary follows from the first two parts of Proposition 2.7. To prove the second part, start by associating an arbitrary vector field datum in $\mathcal{X}_{\Gamma,\mathrm{reg}}$ to each graph Γ labelling a lowest-dimenional cell in (2.4) and use the third part of Proposition 2.7 successively to extend over the remaining cells. □

We will consider certain partial compactifications of the spaces $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$. The main idea is as follows. We observe that there are the following three sources of non-compactness of the spaces:

(a) Breaking of the trajectory corresponding to an external edge of the graph into several trajectories connecting critical points.
(b) Convergence to zero of the length l_e of an internal edge e of the graph.
(c) Breaking of a trajectory corresponding to an internal edge of the graph.

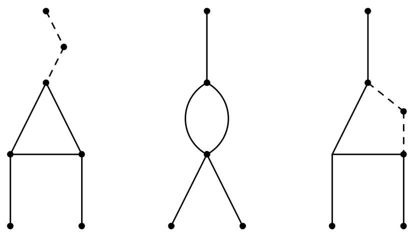

FIGURE 4. Three types of boundary strata of natural compactifications of the spaces of flow graphs

We will consider the partial compactification $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ of $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ obtained by adding the strata of the first and the second (but not the third) type. We now give the formal definition.

Given two critical points p and p' of f, denote by

$$\mathcal{M}(p,p') = \Big\{\gamma\colon \mathbb{R}\to M,\ \dot\gamma(t) = \nabla_g f(\gamma(t)), \lim_{t\to-\infty}\gamma(t) = p,\ \lim_{t\to\infty}\gamma(t) = p'\Big\}\Big/\mathbb{R} \tag{2.30}$$

the space of flow trajectories emanating at p and converging to p'. Denote by

$$\overline{\mathcal{M}}(p,p') = \mathcal{M}(p,p') \cup \bigcup_{\substack{m\geq 1\\ q_1,\dots,q_m}} \mathcal{M}(p,q_1)\times\ldots\times\mathcal{M}(q_m,p') \tag{2.31}$$

the corresponding space of broken trajectories.

Given two n-tuples $\mathbf{p} = (p_1,\dots,p_n)$, $\mathbf{p}' = (p_1',\dots,p_n')$ of critical points, we write

$$\overline{\mathcal{M}}(\mathbf{p},\mathbf{p}') = \overline{\mathcal{M}}(p_1,p_1')\times\ldots\times\overline{\mathcal{M}}(p_n,p_n'). \tag{2.32}$$

DEFINITION 2.9. We define

$$\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) = \bigcup_{\mathbf{q}_+,\mathbf{q}_-,\widetilde\Gamma\prec\Gamma} \overline{\mathcal{M}}(\mathbf{p}_+,\mathbf{q}_+)\times\mathcal{M}_{\widetilde\Gamma,\mathbf{x}}(\mathbf{q}_+,\mathbf{q}_-)\times\overline{\mathcal{M}}(\mathbf{q}_-,\mathbf{p}_-), \tag{2.33}$$

where the union is over all $\mathbf{q}_+\in \mathrm{Crit}(f)^{\times n_+}$, $\mathbf{q}_-\in\mathrm{Crit}(f)^{\times n_-}$ and all ribbon graphs $\widetilde\Gamma$ obtained by collapsing edges of Γ, so that no cycle is collapsed.

We now establish the properties of the spaces $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ which will be used in the proof of Theorem 1.1.

Proposition 2.10.

(a) *For every* $\mathbf{x} \in \mathcal{X}_\Gamma$ *as in the third part of Proposition 2.7, the space* $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ *is a manifold with corners.*

(b) *The boundary* $\partial\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ *is given by the disjoint union*

$$\begin{aligned}\partial\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) = &\Bigg(\coprod_{|\mathbf{q}_+|-|\mathbf{p}_+|=1} \mathcal{M}(\mathbf{p}_+, \mathbf{q}_+) \times \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{q}_+, \mathbf{p}_-)\Bigg) \\ &\coprod\Bigg(\coprod_{|\mathbf{p}_-|-|\mathbf{q}_-|=1} \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{q}_+, \mathbf{p}_-) \times \mathcal{M}(\mathbf{q}_-, \mathbf{p}_-)\Bigg) \\ &\coprod\Bigg(\coprod_{e\in E_{\mathrm{int}}(\Gamma)-L(\Gamma)} \overline{\mathcal{M}}_{\Gamma/e,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)\Bigg),\end{aligned} \tag{2.34}$$

where the last union is over all the internal edges e *of* Γ *which are not loops and where* Γ/e *denotes the ribbon graph obtained from* Γ *by collapsing* e.

(c) *The projection* $\pi_\Gamma \colon \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) \to \mathcal{M}_\Sigma$ *defined by forgetting* γ *is proper.*

Proof. In order to equip the space $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ with the structure of a manifold with corners, we will identify it as a transverse intersection of a manifold and a manifold with corners.

Assume that a connected component Γ'_0 of Γ' is fixed or, equivalently, an orientation of every internal edge of Γ is chosen. For an internal edge e' of Γ'_0, denote by $\Phi^{e'}_{\boldsymbol{\ell},\mathbf{x}}(t, \cdot) \colon M \to M$ the flow of the vector field on the right-hand side of (2.15). We define $M_{e'} = M_{e'}(\boldsymbol{\ell}, \mathbf{x}) \subset M \times M$ as the subspace

$$M_{e'} = \{(q, \Phi^{e'}_{\boldsymbol{\ell},\mathbf{x}}(1, q)) \in M \times M : q \in M\}. \tag{2.35}$$

Thus $M_{e'}$ is a manifold diffeomorphic to M.

Suppose now that e' is an external edge of Γ'_0 with the corresponding critical point $p_{e'}$. Denote by $\Phi^{e'}_{\boldsymbol{\ell},\mathbf{x}}(t, \cdot)$ the flow of the vector field on the right-hand side of (2.14). If e' is marked as incoming, then we define

$$W^u(p_{e'}) = \Big\{q \in M : \lim_{t\to-\infty} \Phi^{e'}_{\boldsymbol{\ell},\mathbf{x}}(t, q) = p_{e'}\Big\}, \tag{2.36}$$

$$\overline{W}^u(p_{e'}) = \bigcup_{p'\in\mathrm{Crit}(f)} \overline{\mathcal{M}}(p_{e'}, p') \times W^u(p') \tag{2.37}$$

and

$$M_{e'} = \{(q, q) : q \in \overline{W}^u(p_{e'})\} \subset M \times M. \tag{2.38}$$

Similarly, if e' is marked as outgoing, then we consider

$$W^s(p_{e'}) = \Big\{q \in M : \lim_{t\to\infty} \Phi^{e'}_{\boldsymbol{\ell},\mathbf{x}}(t, q) = p_{e'}\Big\}, \tag{2.39}$$

$$\overline{W}^s(p_{e'}) = \bigcup_{p' \in \mathrm{Crit}(f)} W^s(p') \times \overline{\mathcal{M}}(p', p_{e'}) \tag{2.40}$$

and

$$M_{e'} = \big\{(q,q) : q \in \overline{W}^s(p_{e'})\big\} \subset M \times M. \tag{2.41}$$

It is well known that $\overline{W}^u(p_{e'})$ and $\overline{W}^s(p_{e'})$ are manifolds with corners, whose boundaries are given by respectively

$$\partial \overline{W}^u(p_{e'}) = \coprod_{\mathrm{ind}_f(p_e) - \mathrm{ind}_f(p') = 1} \mathcal{M}(p_{e'}, p) \times \overline{W}^u(p) \tag{2.42}$$

and

$$\partial \overline{W}^s(p_{e'}) = \coprod_{\mathrm{ind}_f(p') - \mathrm{ind}_f(p_e) = 1} \overline{W}^s(p) \times \mathcal{M}(p, p_{e'}). \tag{2.43}$$

We refer to [22] for a detailed study of trajectory spaces in Morse theory.

We have thus far associated to every edge e' of Γ_0' a manifold with corners $M_{e'}$. As a submanifold of $M \times M$, $M_{e'}$ depends on the choice of $\boldsymbol{\ell} \in \mathrm{Met}_0(\Gamma)$ and of the vector field datum $\mathbf{x}$, however the diffeomorphism type of $M_{e'}$ is independent of these choices. Let us define

$$N = \mathrm{Met}_0(\Gamma) \times (M \times M)^{\times |E(\Gamma_0')|} \tag{2.44}$$

and denote by $L_{\mathbf{x}}$ the subset

$$L_{\mathbf{x}} = \bigcup_{\boldsymbol{\ell} \in \mathrm{Met}_0(\Gamma)} \Bigg(\{\boldsymbol{\ell}\} \times \Bigg(\bigtimes_{e' \in E(\Gamma')} M_{e'}(\boldsymbol{\ell}, \mathbf{x}) \Bigg) \Bigg) \subset N. \tag{2.45}$$

As a product of manifolds with corners, $L_{\mathbf{x}}$ is again a manifold with corners.

Next, we consider the submanifold $L' \subset N$ defined as follows. Recall that each edge of Γ_0' is oriented. We assign to each factor M appearing on the right-hand side of (2.44) a vertex of Γ_0' by the following rule: if $e' \in E(\Gamma_0')$ is incident to the vertices v and v' in this order, then to the two factors of the copy of $M \times M$ corresponding to e' the vertices v and v' respectively are associated. Write each element of N as a $(\boldsymbol{\ell}, \mathbf{q})$, where $\mathbf{q}$ is a tuple whose entries are points of M and denote for each entry q by $v(q)$ the vertex of Γ_0' assigned to the corresponding factor of M. Then $L' \subset N$ is defined as the subset of all those tuples $(\boldsymbol{\ell}, \mathbf{q})$, such that if for two entries q and q' of $\mathbf{q}$, $v(q) = v(q')$ is the same *internal* vertex of Γ_0', then $q = q'$. Thus $L' \subset N$ is a fat diagonal which corresponds to the incidence relations of the internal vertices of the graph. We identify the space $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ as the intersection $L_{\mathbf{x}} \cap L'$: points of $L_{\mathbf{x}}$ are tuples of flow lines associated to the edges of the graph, while intersecting with L' corresponds to imposing the incidence relations that stem from continuity at the internal vertices. It follows from Proposition 2.7 that the intersection $L_{\mathbf{x}} \cap L'$ is transverse. This establishes the first part of the proposition. Formula (2.34) follows from (2.42) and (2.43). The third claim is a consequence of the above discussion of compactifications of the spaces $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$. □

Proposition 2.10 implies that the pair $(\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-),\pi_\Gamma)$, together with a choice of orientation of $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$, defines an element of the chain complex $C_*^{BM}(\mathcal{M}_\Sigma)$ introduced in Section 2.1.

3. The operations

This section contains the discussion of orientations of the spaces of flow graphs as well as the proofs of Theorem 1.1 and of the gluing axiom.

3.1. Orientations of the spaces of flow graphs. The identification given in the proof of Proposition 2.10 of the space $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ as a transverse intersection of submanifolds can be used in the discussion of orientations: to orient $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$, it suffices to fix orientations of the manifolds $L_{\mathbf{x}}$, L' and N. It follows from the definition of these manifolds that their orientations can be determined by choosing orientations of the internal edges of Γ as well as linear orderings of the vertices and of the edges of Γ. Moreover, it is straightforward to determine how the orientations of $L_{\mathbf{x}}$, L' and N, and thus the orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$, change when we reorder vertices and edges or reverse the orientation of an edge. The result of this discussion is summarized in the following proposition.

PROPOSITION 3.1. *Suppose that for every critical point p of f, an orientation of the unstable submanifold $W^u_f(p)$ is fixed.*

(a) *An orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is uniquely defined by choosing orientations of all the internal edges of Γ and linear orderings of all the edges and of the internal vertices of Γ.*

(b) *Reversing the orientation of an internal edge or interchanging two consecutive internal vertices changes the orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ by the factor $(-1)^d$, where d is the dimension of M.*

(c) *Interchanging two consecutive edges e_i, e_j changes the orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ by the factor $(-1)^{k_ik_j}$, where the integers k_i are given by the following rule. If e_i is an internal edge of the graph, then $k_i = d+1$. If e_i is an external edge with the correspoinding critical point p_i, then*

$$k_i = \begin{cases} d - \mathrm{ind}_f(p_i) + 1 & \text{if } e_i \text{ is marked as incoming,} \\ \mathrm{ind}_f(p_i) + 1 & \text{if } e_i \text{ is marked as outgoing.} \end{cases}$$

We can now explain the local systems det and *or* on $\mathcal{M}_\Sigma$ which appear in Theorem 1.1. A local trivialization of det on $\mathrm{Met}(\Gamma)/\mathrm{Aut}(\Gamma) \subset \mathcal{M}_\Sigma$ is defined by a choice of orientations of the internal edges as well as of linear orderings of the vertices, of the internal edges and of those external edges of Γ, which are marked as incoming. Changing the orientation of an internal edge or interchanging two consecutive vertices or edges changes the sign of the trivialization. The fibre of det can be identified with the determinant line of the cohomology $H^*(\Gamma, O_-)$ of Γ relative the vertices corresponding to the outgoing marked points. The local system det is graded by assigning to each section the degree $\chi(\Sigma) - n_-$.

The local system *or* is the orientation sheaf of $\mathcal{M}_\Sigma$. Explicitly, a local trivialization of or on $\mathrm{Met}(\Gamma)/\mathrm{Aut}(\Gamma) \subset \mathcal{M}_\Sigma$ is defined by a choice of orientations of the internal edges and of linear orderings of all the vertices and edges of Γ. Reversing the orientation of an edge or interchanging consecutive vertices or edges reverses the sign of the trivializazion. The fibre of the local system or can be identified with the determinant line of $H^*(\Gamma)$. The grading of the system or is trivial, i.e. every section has degree zero.

LEMMA 3.2. *Suppose that every vertex of the ribbon graph Γ has odd valency. Then the pair $(\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-), \pi_\Gamma)$ defines an element of $C_*^{BM}(\mathcal{M}_\Sigma; \det^{\otimes d} \otimes \mathrm{or})$.*

PROOF. We first note that since every external edge of Γ is marked as either incoming or outgoing and thus has a natural orientation, we could equivalently define the local system det by considering linear orderings of *all* the vertices instead of only the internal ones. Indeed, if e is an external edge with the incident vertices v and v', where v is an internal and v' an external vertex, then we insert v' into a given ordering of the internal vertices as either the predecessor or the successor of v, according to the orientation of e.

We must show that an orientation of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ is the same as a trivialization of the pullback under π_Γ of the local system $\det^{\otimes d} \otimes \mathrm{or}$. Comparing the definitions of det and or with the result of Proposition 3.1, it suffices to check that a trivialization of or is given by a choice an orientation of the vector space spanned by the edges of the graph. This follows from the observation going back to J. Conant and K. Vogtmann ([6, Corollary 1]) that if all vertices of Γ have odd valency, then there is a natural orientation of the vector space spanned by the vertices and the half-edges. □

3.2. Proof of Theorem 1.1. We can now complete the proof of Theorem 1.1. Recall that the top-dimensional cells in the ribbon graph decomposition (2.4) are labelled by the ribbon graphs whose internal vertices have valency three. By Proposition 2.10 and Lemma 3.2, to each such graph Γ and each choice of a vector field datum $\mathbf{x} \in \mathcal{X}_{\Gamma,\mathrm{reg}}$ is associated a geometric chain

$$Z^f_{\Gamma,\mathbf{x}} = (\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-), \pi_\Gamma) \in C_*^{BM}(\mathcal{M}_\Sigma; \det^{\otimes d} \otimes \mathrm{or}). \tag{3.1}$$

Moreover, we may assume that the vector field data for different graphs Γ are chosen as in the second part of Corollary 2.8.

To prove that

$$F^M_\Sigma\colon (C^*(f))^{\otimes n_+} \to C_*^{BM}(\mathcal{M}_\Sigma; \det^{\otimes d} \otimes \mathrm{or}) \otimes (C^*(f))^{\otimes n_-},$$
$$\mathbf{p}_+ \mapsto \sum_{\Gamma, \mathbf{p}_-} Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-) \otimes \mathbf{p}_-$$

is a cochain map, we compute $\Big(\sum_{\Gamma,\mathbf{p}_-} \partial Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)\otimes \mathbf{p}_-\Big)$. By the second part of Proposition 2.10, the latter expression is a sum of terms of two types: the first type corresponds to breaking of trajectories at external edges (see the expressions in the first two lines of (2.34)), while terms of the second type correspond to collapsing an internal edge of Γ (see the expression in the third line of (2.34)). The summands of the first type yield

$$\sum_{\Gamma,\mathbf{p}_-} Z^f_{\Gamma,\mathbf{x}}(d\mathbf{p}_+, \mathbf{p}_-) \otimes \mathbf{p}_- - \sum_{\Gamma,\mathbf{p}_-} (-1)^{\dim \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)} Z^f_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) \otimes d\mathbf{p}_-. \tag{3.2}$$

We must show that the sum of all the terms of the second type is zero. These are of the form $\pm(\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-), \pi_{\widetilde{\Gamma}})$, where $\widetilde{\Gamma}$ is obtained by collapsing a single internal edge in a ribbon graph Γ, all of whose internal vertices have valency three. The sign is determined as follows: $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is oriented as a boundary component of $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ and the trivialization of the pullback to $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ of the local system $\det^{\otimes d} \otimes \mathrm{or}$ is induced by the trivialization of the pullback to $\overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$.

We observe that in the sum of the terms of second type exactly those ribbon graphs $\widetilde{\Gamma}$ appear, where there is a unique internal vertex of valency four and all the remaining internal vertices have valency three. For each such $\widetilde{\Gamma}$, there are exactly two distinct pairs (Γ_1, e_1) and (Γ_2, e_2), so that $\widetilde{\Gamma}$ is obtained from Γ_1 and Γ_2 by collapsing the internal edges $e_1 \in E(\Gamma_1)$ and $e_2 \in E(\Gamma_2)$ respectively: Γ_1 and Γ_2 arise from the two different ways of expanding the four-valent vertex of $\widetilde{\Gamma}$ into two trivalent vertices.

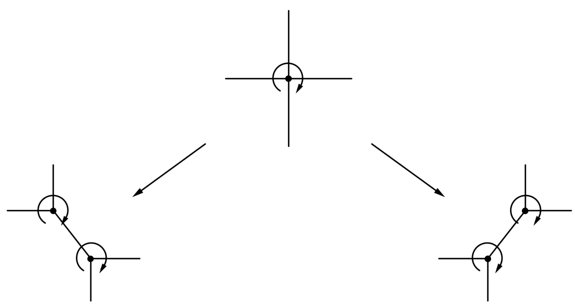

FIGURE 5. The two different ways of expanding a single four-valent vertex of a ribbon graph into two trivalent vertices

To complete the proof of the first part of the Theorem, it suffices to show:

LEMMA 3.3. *The two copies of* $(\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-),\pi_{\widetilde{\Gamma}})$ *corresponding to boundary components of* (Γ_1,e_1) *and of* (Γ_2,e_2) *enter the sum with the opposite sign.*

PROOF. Assume first that d is even. In this case by Proposition 3.1, an orientation of $\overline{\mathcal{M}}_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is the same as an orientation of the vector space $W_{E(\Gamma_1)}$ spanned by the edges of Γ_1. The boundary orientation of $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-) \subset \partial\overline{\mathcal{M}}_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is determined by requiring that the projection

$$W_{E(\Gamma_1)} \simeq \mathbb{R}_{e_1} \oplus W_{E(\widetilde{\Gamma})} \to W_{E(\widetilde{\Gamma})} \tag{3.3}$$

to the second factor be orientation-preserving. Here the symbol $\mathbb{R}_{e_1}$ denotes the direct summand of $\mathbb{R}$ corresponding to the edge e_1. The orientation of $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ as a boundary component of $\overline{\mathcal{M}}_{\Gamma_2,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is defined analogously.

It follows from the definition that a trivialization of the pullback of the local system or to $\overline{\mathcal{M}}_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is given by an orientation of the vector space $W_{E(\Gamma_1)} \oplus W_{V(\Gamma_1)} \oplus W_{H(\Gamma_1)}$, where $W_{V(\Gamma_1)}$ and $W_{H(\Gamma_1)}$ denote the vector spaces generated by the vertices and by the half-edges of Γ_1 respectively. The projection $W_{E(\Gamma_1)} \to W_{E(\widetilde{\Gamma})}$ was described above, while the projection $W_{V(\Gamma_1)} \oplus W_{H(\Gamma_1)} \to W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}$ is given as follows. Denote by h_1 and h_1' the half-edges of e_1, by v_1 and v_1' the corresponding vertices of Γ_1 and by $v \in V(\widetilde{\Gamma})$ the vertex to which e_1 is collapsed. We identify

$$W_{V(\Gamma_1)-\{v',v_1'\}} \simeq W_{V(\widetilde{\Gamma})-\{v\}} \tag{3.4}$$

as well as

$$W_{H(\Gamma_1)-\{h_1,h_1'\}} \simeq W_{H(\widetilde{\Gamma})} \tag{3.5}$$

and consider the map

$$\begin{aligned} W_{V(\Gamma_1)} &\oplus W_{H(\Gamma_1)} \\ &\simeq (\mathbb{R}_{h_1} \oplus \mathbb{R}_{h_1'} \oplus \mathbb{R}_{v_1}) \oplus (\mathbb{R}_{v_1'} \oplus W_{V(\Gamma_1)-\{v_1,v_1'\}} \oplus W_{H(\Gamma_1)-\{h_1,h_1'\}}) \\ &\to \mathbb{R}_v \oplus W_{V(\widetilde{\Gamma})-\{v\}} \oplus W_{H(\widetilde{\Gamma})} \simeq W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}, \end{aligned} \tag{3.6}$$

where the arrow denotes projection to the second factor. A trivialization of or over $\overline{\mathcal{M}}_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ induces a trivialization over $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ by requiring that the projection

$$W_{E(\Gamma_1)} \oplus (W_{V(\Gamma_1)} \oplus W_{H(\Gamma_1)}) \to W_{E(\widetilde{\Gamma})} \oplus (W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}) \tag{3.7}$$

whose components are given by (3.3) and (3.6) be orientation-preserving. In the same way a trivialization over $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$ is induced by a trivialization over $\overline{\mathcal{M}}_{\Gamma_2,\mathbf{x}}(\mathbf{p}_+,\mathbf{p}_-)$.

Recall from the proof of Lemma 3.3 that since Γ_1 and Γ_2 have vertices of odd valency, the ribbon structures of Γ_1 and of Γ_2 define orientations of the vector spaces $W_{V(\Gamma_1)} \oplus W_{H(\Gamma_1)}$ and $W_{V(\Gamma_2)} \oplus W_{H(\Gamma_2)}$ respectively. It is immidiate to check using the explicit description given by (3.6) that the two orientations on $W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}$ induced by the projections $W_{V(\Gamma_1)} \oplus W_{H(\Gamma_1)} \to W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}$ and $W_{V(\Gamma_2)} \oplus W_{H(\Gamma_2)} \to W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}$ are opposite. Thus if the orientations of $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ as boundary of $\overline{\mathcal{M}}_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ and as boundary of $\overline{\mathcal{M}}_{\Gamma_2,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ coincide, i.e. the projection (3.3) and the corresponding projection for Γ_2 are both orientation-preserving, then the orientations of $W_{E(\widetilde{\Gamma})} \oplus W_{V(\widetilde{\Gamma})} \oplus W_{H(\widetilde{\Gamma})}$ induced by the projection (3.7) and by the corresponding projection for Γ_2 are opposite. In this case the two trivializations of or over $\overline{\mathcal{M}}_{\widetilde{\Gamma},\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ differ by a sign. This completes the proof of the Lemma in the case when d is even. The case of odd d follows by the same argument, but with $W_{E(\Gamma)}$ replaced everywhere by $W_{\det} \oplus W_{E(\Gamma)}$, where $W_{\det}$ is the vector space whose orientation corresponds to a trivialization of det (explicitly, $W_{\det} = W_{E(\Gamma)-E_-(\Gamma)} \oplus W_{V(\Gamma)} \oplus W_{H(\Gamma)}$, where $E_-(\Gamma)$ denote the outgoing external edges of Γ). □

We now turn to the proof of the second part of the Theorem. This relies on a flow graph version of the continuation argument, which is classically used to prove the invariance of Morse homology. Given two triples $(f, g, \mathbf{x})$ and $(f', g', \mathbf{x}')$, fix a one-paramater family $(f_t, g_t, \mathbf{x}_t)_{t\in\mathbb{R}}$ so that

$$(f_t, g_t, \mathbf{x}_t) = \begin{cases} (f, g, \mathbf{x}) & \text{if } t \le -1, \\ (f', g', \mathbf{x}') & \text{if } t \ge 1. \end{cases}$$

Recall that the classical continuation principle consists of the following: one observes that for suitable choice of the one-parameter family $(f_t, g_t, \mathbf{x}_t)_{t\in\mathbb{R}}$, each of the spaces

$$\mathcal{N}(p, p') = \Big\{ \gamma\colon \mathbb{R} \to M,\ \dot\gamma(t) = \nabla_{g_t} f_t(\gamma(t)), \lim_{t\to-\infty} \gamma(t) = p,\ \lim_{t\to\infty} \gamma(t) = p' \Big\}, \tag{3.8}$$

where $p \in \mathrm{Crit}(f)$, $p' \in \mathrm{Crit}(f')$ and $\mathrm{ind}_f(p) = \mathrm{ind}_{f'}(p')$, is a compact oriented zero-dimensional manifold. One then shows that the count of the elements yields a quasi-isomorphism $\Psi\colon C^*(f, g) \to C^*(f', g')$:

$$\Psi\colon p \mapsto \sum_{\mathrm{ind}_{f'}(p')=\mathrm{ind}_f(p)} |\mathcal{N}(p, p')| p'. \tag{3.9}$$

Now denote by

$$F^{f'}_\Sigma : (C^*(f', g'))^{\otimes n_+} \to C^{BM}_*(\mathcal{M}_\Sigma; \det{}^d \otimes \mathrm{or}) \otimes (C^*(f', g'))^{\otimes n_-}$$

the map associated by the construction of the first part of the theorem to the triple $(f', g', \mathbf{x}')$. We will construct a chain homotopy Θ between $F_\Sigma^{f'} \circ \Psi^{\otimes n_+}$ and $(\mathrm{Id} \otimes \Psi^{n_-}) \circ F_\Sigma^{f}$. This chain homotopy will be obtained by studying spaces $\overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-)$ which we now introduce.

Recall from Definition 2.5 that $\mathcal{M}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{p}_-)$ is the zero locus of $\mathbf{S} - \mathcal{F}_{\mathbf{x}}$, where $\mathbf{S}$ and $\mathcal{F}_{\mathbf{x}}$ are sections of a Banach bundle $\mathcal{E}$ over $\mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}_-)$ (see Definition 2.3). Given $\mathbf{p}_+ \in \mathrm{Crit}(f)^{\times n_+}$ and $\mathbf{p}'_- \in \mathrm{Crit}(f')^{\times n_-}$, we define $\mathcal{N}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) \subset \mathrm{Met}(\Gamma) \times \mathcal{B}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-)$ as the union

$$\mathcal{N}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) = \bigcup_{T \in \mathbb{R}} \mathcal{N}_{\Gamma,T}(\mathbf{p}_+, \mathbf{p}'_-),$$

where $\mathcal{N}_{\Gamma,T}(\mathbf{p}_+, \mathbf{p}'_-)$ is the zero locus of the section $\mathbf{S} - \mathcal{F}_T$ of $\mathcal{E}$, with $\mathbf{S}$ is defined as in (2.10) and $\mathcal{F}_T$ given as follows. If e' is an external edge of Γ', then

$$F_{e',T}(\boldsymbol{\ell}, \gamma)(t) = \nabla_{g_{t+T}} f_{t+T}(\gamma_{e'}(t)) + \sigma(t) x_{e',t+T}(\boldsymbol{\ell}, |t|, \gamma_{e'}(t)). \tag{3.10}$$

If e' is an internal edge of Γ' which is mapped to $e \in E(\Gamma)$ under the projection $\Gamma' \to \Gamma$, then

$$F_{e',T}(\boldsymbol{\ell}, \gamma)(t) = l_e x_{e',T}(\boldsymbol{\ell}, t, \gamma_{e'}(t)) \tag{3.11}$$

(compare with (2.14) and (2.15) respectively). We denote by $\overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-)$ the partial compactification of $\mathcal{N}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-)$ given by

$$\begin{aligned} \overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) = & \bigcup_{\mathbf{q}_+, \mathbf{q}'_-, \widetilde{\Gamma} \prec \Gamma} \overline{\mathcal{M}}(\mathbf{p}_+, \mathbf{q}_+) \times \mathcal{N}_{\widetilde{\Gamma}}(\mathbf{q}_+, \mathbf{q}'_-) \times \overline{\mathcal{M}}'(\mathbf{q}'_-, \mathbf{p}'_-) \\ & \bigcup \Big(\bigcup_{\mathbf{q}'_+, \mathbf{p}_+} \overline{\mathcal{N}}(\mathbf{p}_+, \mathbf{q}'_+) \times \overline{\mathcal{M}}'_{\Gamma,\mathbf{x}'}(\mathbf{q}'_+, \mathbf{p}'_-) \Big) \\ & \bigcup \Big(\bigcup_{\mathbf{q}_-, \mathbf{q}'_-} \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{q}_-)) \times \overline{\mathcal{N}}(\mathbf{q}_-, \mathbf{p}'_-) \Big), \end{aligned} \tag{3.12}$$

where $\overline{\mathcal{N}}(\mathbf{p}, \mathbf{p}')$ denotes the product

$$\overline{\mathcal{N}}(\mathbf{p}, \mathbf{p}') = \bigcup_{\mathbf{q}, \mathbf{q}'} \overline{\mathcal{M}}(\mathbf{p}, \mathbf{q}) \times \mathcal{N}(\mathbf{q}, \mathbf{q}') \times \overline{\mathcal{M}}'(\mathbf{q}', \mathbf{p}') \tag{3.13}$$

with

$$\mathcal{N}(\mathbf{q}, \mathbf{q}') = \mathcal{N}(q_1, q'_1) \times \ldots \times \mathcal{N}(q_n, q'_n) \tag{3.14}$$

(note that if $|\mathbf{q}| = |\mathbf{q}'|$, then the space $\mathcal{N}(\mathbf{q}, \mathbf{q}')$ is empty unless $\mathrm{ind}_f(q_j) = \mathrm{ind}_{f'}(q'_j)$ for $j = 1, \ldots, n$.)

The meaning of the terms on the right-hand side of (3.12) is as follows. The expression in the first line corresponds to allowing internal edges of length zero as well as breaking of trajectories along the external edges of Γ for fixed $T \in \mathbb{R}$. The terms in the second and third line correspond to partial compactifications at $T \to \infty$ and $T \to -\infty$, respectively.

By a slight abuse of notation, we will again denote by

$$\pi_\Gamma \colon \overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) \to \mathcal{M}_\Sigma$$

the natural projection.

LEMMA 3.4.

(a) *There exists a one-parameter family* $(\mathbf{x}_t)_{t\in\mathbb{R}}$, *so that every pair*

$$Y_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) = (\overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-), \pi_\Gamma) \tag{3.15}$$

defines an element of $C_*^{BM}(\mathcal{M}_\Sigma, \det^{\otimes d} \otimes \mathrm{or})$.

(b) *Define*

$$\begin{aligned} \Theta \colon (C^*(f))^{\otimes n_+} &\to C_*^{BM}(\mathcal{M}_\Sigma; \det^{\otimes d} \otimes \mathrm{or}) \otimes (C^*(f'))^{\otimes n_-}, \\ \mathbf{p}_+ &\mapsto \sum_{\Gamma, \mathbf{p}'_-} Y_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) \otimes \mathbf{p}'_-. \end{aligned} \tag{3.16}$$

Then

$$d \circ \Theta + \Theta \circ d = (\mathrm{Id} \otimes \Psi^{\otimes n_-}) \circ F_\Sigma^f - F_\Sigma^{f'} \circ (\Psi^{\otimes n_+}). \tag{3.17}$$

PROOF. The first part of the Lemma follows by analogous arguments as in the proofs of Propositions 2.7 and 2.10. From (3.12), the boundary of $\overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-)$ may be identified as the disjoint union

$$\begin{aligned} \partial\overline{\mathcal{N}}_\Gamma(\mathbf{p}_+, \mathbf{p}'_-) = &\coprod_{|\mathbf{q}_+|-|\mathbf{p}_+|=1} \mathcal{M}(\mathbf{p}_+, \mathbf{q}_+) \times \overline{\mathcal{N}}_\Gamma(\mathbf{q}_+, \mathbf{p}'_-) \\ &\coprod \Big(\coprod_{|\mathbf{p}'_-|-|\mathbf{q}'_-|=1} \overline{\mathcal{N}}_\Gamma(\mathbf{q}_+, \mathbf{p}'_-) \times \mathcal{M}'(\mathbf{q}'_-, \mathbf{p}'_-) \Big) \\ &\coprod \Big(\coprod_{|\mathbf{q}'_+|=|\mathbf{p}_+|} \mathcal{N}(\mathbf{p}_+, \mathbf{q}'_+) \times \overline{\mathcal{M}}'_{\Gamma,\mathbf{x}'}(\mathbf{q}'_+, \mathbf{p}'_-) \Big) \\ &\coprod \Big(\coprod_{|\mathbf{q}_-|=|\mathbf{p}'_-|} \overline{\mathcal{M}}_{\Gamma,\mathbf{x}}(\mathbf{p}_+, \mathbf{q}_-) \times \mathcal{N}(\mathbf{q}_-, \mathbf{p}'_-) \Big) \\ &\coprod \Big(\coprod_{e\in E_{\mathrm{int}}(\Gamma)-L(\Gamma)} \overline{\mathcal{N}}_{\Gamma/e}(\mathbf{p}_+, \mathbf{p}'_-) \Big). \end{aligned} \tag{3.18}$$

The sum over all Γ of the geometric chains corresponsing to the terms on the left-hand side and in the first two lines of the right-hand side of (3.18) yields the left-hand side of (3.17), while the expressions in the third and fourth lines correspond to the right-hand side of (3.17). Finally, using Lemma 3.3, the sum of the chains corresponding to the terms in the last line of (3.18) vanishes. □

This completes the proof of the second part of Theorem 1.1

3.3. The gluing axiom. The goal of this section is to show that the operations constructed in Theorem 1.1 are compatible with the gluing of surfaces.

Let Σ_1 and Σ_2 be two surfaces as in Theorem 1.1 and assume that the number n_{1-} of the outgoing marked points of Σ_1 coincides with the number n_{2+} of the incoming marked points of Σ_2. Denote this common number by n and denote by Σ the compact orieted surface obtained by attaching Σ_1 to Σ_2 along closed disjoint intervals around the outgoing marked points of Σ_1 respectively around the incoming marked points of Σ_2. Using the homeomorphism (2.4) of the ribbon graph decomposition, there is a map

$$\Xi\colon \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \to \mathcal{M}_{\Sigma} \tag{3.19}$$

defined by first attaching the outgoing edges of a metric ribbon graph corresponding to Σ_1 to the incoming edges of a metric ribbon graph corresponding to Σ_2 and then erasing the bivalent vertices from the resulting graph. Denote by $\pi_1\colon \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \to \mathcal{M}_{\Sigma_1}$ and by $\pi_2\colon \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \to \mathcal{M}_{\Sigma_2}$ the projection to the first respectively to the second factor. The local system *det* is compatible with gluing, i.e. $\Xi^*\det \simeq \pi_1^*\det \oplus \pi_2^*\det$. Thus Ξ induces a map

$$\Xi^*\colon H^*(\mathcal{M}_{\Sigma}; \det^{\otimes d}) \to H^*(\mathcal{M}_{\Sigma_1}; \det^{\otimes d}) \otimes H^*(\mathcal{M}_{\Sigma_2}; \det^{\otimes d}). \tag{3.20}$$

We can express the gluing homomorphism (3.20) using geometric homology as follows (we will for simplicity leave out the local coefficient systems in the notation). Via the Poincaré duality isomorphisms (2.6) on $\mathcal{M}_{\Sigma}$, $\mathcal{M}_{\Sigma_1}$ and $\mathcal{M}_{\Sigma_2}$, one obtains from Ξ^* the corresponding transfer homomorphism

$$\Xi_*^{!BM}\colon H_*^{BM}(\mathcal{M}_{\Sigma}) \to H_{*+n}^{BM}(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}). \tag{3.21}$$

The remarks made in Section 2.1 allow to identify $\Xi_*^{!BM}$ as the homomorphism induced by an explicit chain map

$$\Xi_*^{!CBM}\colon C_*^{BM}(\mathcal{M}_{\Sigma}) \to C_{*+n}^{BM}(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}). \tag{3.22}$$

To this end, we observe that the image $\Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \subset \mathcal{M}_{\Sigma}$ is an open subset: it consists of the equivalence classes of all the metric ribbon graphs Γ which can be obtained by gluing together graphs Γ_1 and Γ_2 from the ribbon graph decompositions of $\mathcal{M}_{\Sigma_1}$ and $\mathcal{M}_{\Sigma_2}$, respectively. There is a homeomorphism

$$\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \simeq \Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \times \mathbb{R}_+^n \tag{3.23}$$

so that Ξ can be identified as the composition of the projection π to the first factor of $\Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \times \mathbb{R}_+^n$ with the inclusion $i\colon \Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \hookrightarrow \mathcal{M}_{\Sigma}$. This yields the following map (3.22): given a genereator (P, f_P) of $C_*^{BM}(\mathcal{M}_{\Sigma})$, we first intersect with $\Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2})$ to obtain

$$(Q, f_Q) = i_*^{!CBM}(P, f_P) \in C_*^{BM}(\Xi(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2})).$$

Then

$$\Xi_*^{!CBM}(P, f_P) = (Q \times \mathbb{R}^n, f_{Q\times\mathbb{R}^n}) = \pi_*^{!CBM}(Q, f_Q)$$

is obtained from (Q, f_Q) by putting a new vertex on each of the edges of Γ which originate from attaching an outgoing edge of Γ_1 to an incoming edge of Γ_2.

The next Proposition establishes the compatibility of the operations constructed in Theorem 1.1 with the gluing maps (3.22).

PROPOSITION 3.5. *The following diagram commutes up to chain homotopy*:

$$\begin{array}{ccc} (C^*(f))^{\otimes n_+} & \xrightarrow{F^f_\Sigma} & C_*^{BM}(\mathcal{M}_\Sigma) \otimes (C^*(f))^{\otimes n} \\ {\scriptstyle F^f_{\Sigma_1}}\downarrow & & \downarrow{\scriptstyle \Xi_*^{!CBM}\otimes \mathrm{Id}} \\ C_*^{BM}(\mathcal{M}_{\Sigma_1}) \otimes (C^*(f))^{\otimes n} & \xrightarrow[\mathrm{Id}\otimes F^f_{\Sigma_2}]{} & C_*^{BM}(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \otimes (C^*(f))^{\otimes n_-} \end{array}$$

PROOF OF PROPOSITION 3.5. Let us constructut a chain homotopy between the compositions $(\Xi_*^{!CBM} \otimes \mathrm{Id}) \circ F^f_\Sigma$ and $(\mathrm{Id} \otimes F^f_{\Sigma_2}) \circ F^f_{\Sigma_1}$.

Denote $\Phi_0 = (\Xi_*^{!CBM} \otimes \mathrm{Id}) \circ F^f_\Sigma$ and $\Phi_n = (\mathrm{Id} \otimes F^f_{\Sigma_2}) \circ F^f_{\Sigma_1}$. We will define cochain maps

$$\Phi_1, \ldots, \Phi_{n-1} \colon (C^*(f))^{\otimes n_+} \to C_*^{BM}(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \otimes (C^*(f))^{\otimes n_-} \tag{3.24}$$

and show that for $k = 0, \ldots, n-1$, there is a chain homotopy between Φ_k and Φ_{k+1}. To this end, denote by $\Sigma(k)$ the surface obtained by gluing Σ_1 to Σ_2 along disjoint closed intervals around the first $n-k$ outgoing marked points on Σ_1, resp. the first $n-k$ incoming marked points on Σ_2. Thus $\Sigma(0) = \Sigma$, while $\Sigma(n)$ is the disjoint union of Σ_1 and Σ_2.

We define $F_k \colon (C^*(f))^{\otimes n_+} \to C_*^{BM}(\mathcal{M}_{\Sigma(k)}) \otimes (C^*(f))^{\otimes n_-}$ by

$$\mathbf{p}_+ \mapsto \sum_{\Gamma(k),\mathbf{q}} Z^f_{\Gamma(k),\mathbf{x}(k)}((\mathbf{p}_+, \mathbf{q}), (\mathbf{q}, \mathbf{p}_-)) \otimes \mathbf{p}_-, \tag{3.25}$$

where the sum is over all the graphs $\Gamma(k)$ in the ribbon graph decomposition (2.4) of $\Sigma(k)$ and over all tuples $\mathbf{q} \in \mathrm{Crit}(f)^{\times k}$. Attaching the last k outgoing edges of $\Gamma(k)$ to the last k incoming edges yields a ribbon graph Γ whose associated surface is Σ. We will assume that the vector field data $\mathbf{x}$ and $\mathbf{x}(k)$ for Σ and $\Sigma(k)$ are chosen so that for each k and all Γ_k, $\mathbf{x}$ and $\mathbf{x}(k)$ coincide on the complement of these edges.

The homomorphism Φ_k is given as

$$\Phi_k = (\Xi_*^{k,!CBM} \otimes \mathrm{Id}) \circ F_k, \tag{3.26}$$

where $\Xi^k \colon \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \to \mathcal{M}_{\Sigma(k)}$ is the gluing map from (3.19).

LEMMA 3.6. *For each $k = 0, \ldots, n-1$, there is a chain homotopy Δ_k between Φ_k and Φ_{k+1}.*

Proof. To simplify terminology, we will refer to a pair consisting of an outgoing edge of Γ_1 and the corresponding incoming edge of Γ_2 as an *attachment pair*.

The above formal definition of the homomorphism Φ_k means the following. The map $\Phi_n = (\mathrm{Id} \otimes F^f_{\Sigma_2}) \circ F^f_{\Sigma_1}$ is given by a count of chains corresponding to pairs of graphs in the ribbon graphs decompositions of $\mathcal{M}_{\Sigma_1}$ and $\mathcal{M}_{\Sigma_2}$, i.e.

$$(\mathrm{Id} \otimes F^f_{\Sigma_2}) \circ F^f_{\Sigma_1} : \mathbf{p}_+ \mapsto \sum_{\Gamma_1,\Gamma_2,\mathbf{q},\mathbf{p}_-} Z^f_{\Gamma_1,\mathbf{x}}(\mathbf{p}_+, \mathbf{q}) \otimes Z^f_{\Gamma_2,\mathbf{x}'}(\mathbf{q}, \mathbf{p}_-) \otimes \mathbf{p}_-, \tag{3.27}$$

where the sum is over ribbon graphs Γ_1 and Γ_2 in the ribbon graph decompositions of Σ_1 and of Σ_2, respectively and over all tuples of critical points $(\mathbf{q}, \mathbf{p}_-) \in (\mathrm{Crit}(f))^{\times(n+n_-)}$. For $0 \leq k < n$, the map Φ_k is obtained by counts of geometric chains as on the right-hand of (3.27), but where instead of a broken trajectory of the gradient flow, to the first $n-k$ attachment pairs, a finite piece of a flow trajectory is associated, namely to these attachment pairs solutions of (2.15) are associated, where the parameter l_e is given as the sum of the lengths of the two edges of the pair. We denote the spaces of such flow graphs, partially compactified as in Definitioin 2.9 by allowing breaking of trajectories along external edges as well as collapsing of internal edges, by $\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+)$ and write

$$\pi_{\Gamma_1,\Gamma_2} : \overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+) \to \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \tag{3.28}$$

for the projection which maps each graph flow to the underlying metric structures on Γ_1 and on Γ_2. Using this notation, Φ_k is given by

$$\Phi_k(\mathbf{p}_+) = \sum_{\Gamma_1,\Gamma_2,\mathbf{q},\mathbf{p}_-} Z^f_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_+, \mathbf{q}, \mathbf{p}_-) \otimes \mathbf{p}_-, \tag{3.29}$$

where Γ_1 and Γ_2 are as in (3.27), $\mathbf{q} \in \mathrm{Crit}(f)^{\times k}$ and

$$Z^f_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_+, \mathbf{q}, \mathbf{p}_-) = (\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+), \pi_{\Gamma_1,\Gamma_2}). \tag{3.30}$$

The construction of a chain homotopy between Φ_k and Φ_{k+1} relies on the study of spaces $\overline{\mathcal{L}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+)$ which we now introduce.

Elements of $\overline{\mathcal{L}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+)$ are pairs consisting of a positive number T together with a flow graph of the same form as in the case of $\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+)$, however the equation (2.15) corresponding to the $(n-k)$-th attachment pair is changed to

$$F_{e'}(\boldsymbol{\ell}, \gamma)(t) = T\nabla_g(f(\gamma_{e'}(t))) + y_{e'}(T, \boldsymbol{\ell}, t, \gamma_{e'}(t)), \tag{3.31}$$

subject to the following:

(a) T is greater or equal to the sum $l_{e_1} + l_{e_2}$ of the lengths of the two edges of the attachment pair.

(b) For $l_{e_1} + l_{e_2} \leq T \leq l_{e_1} + l_{e_2} + 1$,

$$y_{e'}(T, \boldsymbol{\ell}, t, \cdot\,) = x_{e'}(\boldsymbol{\ell}, t, \cdot\,) \quad \text{for all } t \in [0,1]. \tag{3.32}$$

(c) For every $T > l_{e_1} + l_{e_2} + 2$, the vector field $y_{e'}(T, \boldsymbol{\ell}, t, \cdot\,)$ satisfies

$$y_{e'}(T, \boldsymbol{\ell}, t, \cdot\,) = \begin{cases} \sigma(Tt)(\mathbf{x}(k))_{e'_1}(\boldsymbol{\ell}_1, Tt, \cdot\,) & \text{for } 0 \leq t \leq 1/2, \\ \sigma(T(1-t))(\mathbf{x}(k))_{e'_2}(\boldsymbol{\ell}_2, Tt, \cdot\,) & \text{for } 1/2 \leq t \leq 1, \end{cases} \tag{3.33}$$

where e'_1 and e'_2 are the elements of the $(n-k)$th attachment pair and e' the edge obtained by gluing e'_1 and e'_2 and erasing the resulting bivalent vertex. Here $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ are the metric structures on Γ_1 and Γ_2 respectively (we also note that the orientations of e'_1 and e'_2 define an orientation of e'). This space is again partially compactified as in Definition 2.9 by allowing broken trajectories at the edges corresponding to the boundary marked points of $\Sigma(k)$ and collapsing of the remaining edges, but in addition we allow breaking along e' (i.e. we take the union with the spaces $\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k+1}(\mathbf{p}_-, \mathbf{q}', \mathbf{p}_+)$ for $\mathbf{q}' \in (\mathrm{Crit}(f))^{\times(k+1)}$).

Denoting again by $\pi_{\Gamma_1,\Gamma_2} \colon \overline{\mathcal{L}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+) \to \mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}$ the natural projection, it follows as in Proposition 2.7 that

$$X^f_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+) = (\overline{\mathcal{L}}_{\Gamma,\Gamma',k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+), \pi_{\Gamma_1,\Gamma_2})$$

is a well-defined element of $C^{BM}_*(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2})$. We denote

$$\begin{aligned} \Delta_k \colon (C^*(f))^{\otimes n_+} &\to C^{BM}_*(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}) \otimes (C^*(f))^{\otimes n_-}, \\ \mathbf{p}_+ &\mapsto \sum_{\Gamma_1,\Gamma_2,\mathbf{q},\mathbf{p}_-} X^f_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_+, \mathbf{q}, \mathbf{p}_-) \otimes \mathbf{p}_-, \end{aligned} \tag{3.34}$$

where the sum is as in (3.29).

The boundary of $(\overline{\mathcal{L}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+), \pi_{\Gamma_1,\Gamma_2})$ is a disjoint union of components of the following form. Firstly, corresponding to the case $T = l_{e_1} + l_{e_2}$, we have the components of $\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_-, \mathbf{q}, \mathbf{p}_+)$. Secondly, corresponding to the case $T \to \infty$, we have boundary components of the form $\overline{\mathcal{M}}_{\Gamma_1,\Gamma_2,k+1}(\mathbf{p}_-, \mathbf{q}', \mathbf{p}_+)$, where $\mathbf{q}' \in (\mathrm{Crit}(f))^{k+1}$. Thirdly, there are the boundary components corresponding to the breaking of trajectories along the incoming edges of Γ_1 and the outgoing edges of Γ_2. Finally, we have the boundary components corresponding to collapsing an internal edge of Γ_1 or of Γ_2.

Summing up all the boundary components of the first two types yields $\Phi_{k+1} - \Phi_k$, while the sum of the boundary components of the third type and all the expressions $(\partial X^f_{\Gamma_1,\Gamma_2,k}(\mathbf{p}_+, \mathbf{q}, \mathbf{p}_-)) \otimes \mathbf{p}_-$ yields $d \circ \Delta_k + \Delta_k \circ d$. Finally, using Lemma 3.3, the sum of all the components of the last type is zero. □

This completes the proof of Proposition 3.5. □

References

[1] A. Adem, J. Leida and Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge Tracts in Mathematics 171, Cambridge University Pess, Cambridge, 2007.

[2] M. Betz and R.L. Cohen, *Graph moduli spaces and cohomology operations*, Turkish J. Math. **18** (1994), no. 1, 23–41.

[3] A. Blumberg, R. Cohen and C. Teleman, *Open-closed topological field theories, string topology, and Hochschild homology*, Alpine Perspectives on Algebraic Topology, Contemp. Math. 504, Amer. Math. Soc., Providence, RI, 2009, 53–76,

[4] A. Borel and J. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. **7** (1960), 137–159.

[5] R.L. Cohen and P. Norbury, *Morse field theory*, Asian J. Math **16** (2012), no. 4, 661–711.

[6] J. Conant and K. Vogtmann, *On a Theorem of Kontsevich*, Algeb. Geom. Topol. **3** (2003), 1167–1224.

[7] K. Costello, *Topological conformal field theories and Calabi–Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214.

[8] ———, *Topological conformal field theories and gauge theories*, Geom. Topol. **11** (2007), 1539–1579.

[9] A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geometry **30** (1989), no. 1, 207–221.

[10] K. Fukaya, *Morse homotopy and its quantization*, Geometric Topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math. Vol. 2.1, Amer. Math. Soc., Providence, RI, 1997, 409–440.

[11] K. Fukaya and Y.-G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. **1** (1997), no. 1, 96–180.

[12] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Intersection Floer Theory: Anomaly and Obstruction. Part* I, AMS/IP Studies in Advanced Mathematics 46.1, Amer. Math. Soc., Providence, RI; International Press, Somerville, MA, 2009.

[13] E. Getzler, *Batalin–Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285.

[14] H. Harer, *The Cohomology of the Moduli Space of Curves*, Theory of Moduli (Montecatini Terme, 1985), Lecture Notes in Math. 1337, Springer, Berlin, 1988, 138–221.

[15] A. Hamilton, A. Lazarev, *Characteristic classes of A_∞-algebras*, J. Homotopy Relat. Struct. **3** (2008), no. 1, 65–111.

[16] M. Jakob, *An Alternative Approach to Homology*, Une Dégustation Topologique [Topological Morsels]: Homotopy Theory in the Swiss Alps (Arolla, 1999), Contemp. Math., 265, Amer. Math. Soc., Providence, RI, 2000, 87–97.

[17] D. Joyce, *On Manifolds with corners*, preprint (2010), arxiv: 0910.3518.

[18] M. Kontsevich, *Feynman Diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math. 120, Birkhäuser, Basel, 1994, 97–121.

[19] R. Penner, *The Decorated teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), 299–339.

[20] M. Schwarz, *Morse Homology*, Progress in Mathematics 111, Birkhäuser Verlag, Basel, 1993.

[21] G. Segal, *The Definition of conformal field theory*, Topology, Geometry and Quantum Field Theory, London Math. Soc. Lecture Notes Ser. 308, Cambridge Univ. Press, Cambridge, 2004, 421–577.

[22] K. WEHRHEIM, *Smooth structures on Morse trajectory spaces, featuring finite ends and associative gluing*, (2012), Preprint, rxiv:1205.0713.

[23] E. WITTEN, *Chern–Simons gauge theory as a string theory*, The Floer Memorial Volume, Progr. Math. 133, Birkhäuser, Basel, 1995, 637–678.

Manuscript received August 8, 2013

VIKTOR FROMM
Humboldt-Universität Berlin
Institut für Mathematik, Rudower
Chaussee 25
12489 Berlin, GERMANY
E-mail address: frommv@math.hu-berlin.de