Topological Methods in Nonlinear Analysis
Volume 44, No. 2, 2014, 539–553

INFINITELY MANY SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATION WITH CONCAVE AND CONVEX TERMS

Leran Xia — Minbo Yang — Fukun Zhao

Abstract. In this paper, we are concerned with the following quasilinear elliptic equation with concave and convex terms

$$(\mathrm{P}) \qquad -\Delta u - \frac{1}{2}\, u\Delta(|u|^2) = \alpha|u|^{p-2}u + \beta|u|^{q-2}u, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < 2$, $4 < q \leq 22^*$. The existence of infinitely many solutions is obtained by the perturbation methods

1. Introduction

In the present paper, we are concerned with the following quasilinear elliptic equation with concave and convex terms

$$(\mathrm{P}) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}\, u\Delta(|u|^2) = \alpha|u|^{p-2}u + \beta|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

2010 *Mathematics Subject Classification.* 35B05, 35B45.

Key words and phrases. Concave and convex terms, fountain theorem, perturbation methods.

The first author was supported by the key programm of Yunnan Normal University, China.

The second author was supported by NSFC (11101374 and 11271331).

The corresponding author was supported partly by NSFC(11061040 and 11361078), Key Project of Chinese Ministry of Education(No212162), NSFY of Yunnan Province (2011CI020) and China Scholarship Council, China.

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\alpha, \beta \in \mathbb{R}$ are parameters, $1 < p < 2$, $4 < q \le 22^*$, $2^* = 2N/(N-2)$ if $N \ge 3$ and $2^* = \infty$ if $N = 1, 2$. Such a problem is referred to as the so-called modified Schrödinger equation (see [3], [24] and [19]). Our motivation comes from the works about the semilinear case (see, for example, [2] and [6])

$$\begin{cases} -\Delta u = \alpha |u|^{p-2} u + \beta |u|^{q-2} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < 2 < q < 2^*$. In [2] Ambrosetti, Brezis and Cerami showed that, for $\alpha > 0$ small and $\beta > 0$, (1.1) has infinitely many solutions with negative energy and infinitely many solutions with positive energy in $H_0^1(\Omega)$. In [6] Bartsch and Willem dropped the restriction on α via the fountain theorem and its dual version. More precisely, they show that for $\beta > 0$ and $\alpha \in \mathbb{R}$ (1.1) has infinitely many solutions with energy going to infinity, and for $\alpha > 0$ and $\beta \in \mathbb{R}$ (1.1) has infinitely many solutions with negative energy going to zero.

In the case $q = 2^*$, the existence of a positive solution was obtained by Brezis and Nirenberg in [9]. See [17] for the multiplicity result. A natural problem is whether the same conclusions hold true or not for the quasilinear problem (P)?

In this paper, we will give positive answer for the subcritical case $q < 22^*$ and the critical case $q = 22^*$. The main idea of our arguments comes from the works concerning the fountain theorem and its dual version (see [4], [5] and [6]).

Problems similar to (P) were considered recently in some papers. The minimization methods was used in [19], [24]. The main tool in [3], [20] is the Nehari method. A change of variables argument was involved in [11], [21]. With this change of variables the quasilinear problem is transformed to a semilinear problem and various existing methods for semilinear problems can be adopted and modified to treat the resulting equation such as done recently in [1], [11]–[14], [18], [23], [25] and the references therein. In particular, in [13], the authors obtained the existence of a positive solution of a similar problem on $\mathbb{R}^N$.

The weak form of (P) is

$$\int_\Omega [(1+u^2)\nabla u \nabla \phi + u|\nabla u|^2 \phi - \alpha |u|^{p-2} u\phi - \beta |u|^{q-2} u\phi] \, dx = 0, \tag{1.2}$$

for all $\phi \in C_0^\infty(\Omega)$, which is formally the variational formulation of the following functional

$$I_0(u) = \frac{1}{2} \int_\Omega (1+u^2)|\nabla u|^2 \, dx - \frac{\alpha}{p} \int_\Omega |u|^p \, dx - \frac{\beta}{q} \int_\Omega |u|^q \, dx. \tag{1.3}$$

We may define the derivative of I_0 at u in the direction of $\phi \in C_0^\infty(\Omega)$ as follows

$$\langle I_0'(u), \phi \rangle = \int_\Omega [(1+u^2)\nabla u \nabla \phi + u|\nabla u|^2 \phi] \, dx \tag{1.4}$$

$$- \alpha \int_\Omega |u|^{p-2} u\phi \, dx - \beta \int_\Omega |u|^{q-2} u\phi \, dx.$$

We call u a critical point of I_0 if $u \in W_0^{1,2}(\Omega)$, $\int_\Omega u^2 |\nabla u|^2 dx < \infty$ and $\langle I_0'(u), \phi\rangle = 0$ for all $\phi \in C_0^\infty(\Omega)$. That is, u is a weak solution of (P).

The main difficulty in our problems is that there is no suitable space on which the functional I_0 enjoys both smoothness and compactness, so the standard critical point theory can not be applied directly. To overcome this difficulty, we use a perturbation method developed recently in [22]. The main idea is, to find a family of C^1-functionals I_μ with compactness on a suitable work space, adding a perturbation term to the original functional I_0. So, we seek for a sequence $\{u_{\mu,n}\}$ of critical points of I_μ for $\mu > 0$ small via the arguments in fountain theorem (see [4], [5], [6] and [26]) and establish suitable estimates for the critical points as $\mu \to 0$, and hence we may pass to the limit to get a sequence of solutions of the original problem. More precisely, we consider a family of perturbed functionals

$$I_\mu(u) = \frac{\mu}{4} \int_\Omega |\nabla u|^4 \, dx + I_0(u) \tag{1.5}$$

where $\mu \in (0, 1]$ is a parameter. Obviously, I_μ is a C^1-functional on $W_0^{1,4}(\Omega)$. For all $\phi \in W_0^{1,4}(\Omega)$,

$$\langle I_\mu'(u), \phi\rangle = \mu \int_\Omega |\nabla u|^2 \nabla u \nabla\phi \, dx + \langle I_0'(u), \phi\rangle. \tag{1.6}$$

We have the following existence results for (P).

Theorem 1.1. *Assume $4 < q < 22^*$.*

(a) *For every $\beta > 0$, $\alpha \in \mathbb{R}$, the problem* (P) *has a sequence of weak solutions $\{u_n\}$ such that $I_0(u_n) \to \infty$ as $n \to \infty$.*
(b) *For every $\alpha > 0$, $\beta \in \mathbb{R}$, the problem* (P) *has a sequence of weak solutions $\{v_n\}$ with $I_0(v_n) < 0$ such that $I_0(v_n) \to 0$ as $n \to \infty$.*

Theorem 1.2. *Assume $q = 22^*$ and $\beta > 0$. Then there exists $\alpha^* > 0$ such that, for every $0 < \alpha < \alpha^*$, the problem* (P) *has a sequence of weak solutions $\{v_n\}$ with $I_0(v_n) < 0$ such that $I_0(v_n) \to 0$ as $n \to \infty$.*

Remark 1.3. In fact, our results can be generalized to the more general case

$$- \sum_{i,j=1}^N D_j(a_{ij}(x,u) D_i u) + \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(x,u) D_i u D_j u = \alpha |u|^{p-2} u + \beta |u|^{q-2} u,$$

for $x \in \Omega$, where

$$D_i = \frac{\partial}{\partial x_i} \quad \text{and} \quad D_s a_{ij}(x,s) = \frac{\partial}{\partial s} a_{ij}(x,s).$$

For $a_{ij}(x,u) = (1 + u^2)\delta_{ij}$, the equation is reduced to (P).

Notations. We denote by $\|\cdot\|$ the norm of $W_0^{1,4}(\Omega)$, by $\|\cdot\|_2$ the norm of $W_0^{1,2}(\Omega)$ and by $|\cdot|_s$ the norm of $L^s(\Omega)(1 < s < +\infty)$, C and C_i stand for different positive constants.

2. Proof of Theorem 1.1

First, similar to [22], we have the following convergence results for (P).

Lemma 2.1. *Let $\mu_n \to 0$ and $q \le 22^*$. Suppose $\{u_n\} \subset W_0^{1,4}(\Omega)$ satisfies $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \le C$ for some $C \in \mathbb{R}$ independent of n. Then there is $u \in W_0^{1,4}(\Omega)$ such that, up to a subsequence, $u_n \to u$ in $W_0^{1,2}(\Omega)$, $u_n\nabla u_n \to u\nabla u$ in $L^2(\Omega)$, $\mu_n \int_\Omega |\nabla u_n|^4\,dx \to 0$ and $I_{\mu_n}(u_n) \to I_0(u)$ as $n \to \infty$, and u is a critical point of I_0.*

Proof. The proof is similar to [22]. We sketch it for completeness. By $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \le C$, we obtain

$$
\begin{aligned}
C &\ge I_{\mu_n}(u_n) - \frac{1}{q}\langle I'_{\mu_n}(u_n), u_n\rangle \\
&= \left(\frac{1}{4} - \frac{1}{q}\right)\mu_n \int_\Omega |\nabla u_n|^4\,dx + \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |\nabla u_n|^2\,dx \\
&\quad + \left(\frac{1}{2} - \frac{2}{q}\right)\int_\Omega u_n^2|\nabla u_n|^2\,dx + \alpha\left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega |u_n|^p\,dx.
\end{aligned} \tag{2.1}
$$

We claim that there exists $C_0 > 0$ such that

$$\int_\Omega |u_n|^p dx \le C_0.$$

If not, without loss of generality, we may assume $|u_n|_p \to \infty$ as $n \to \infty$. By (2.1), we have

$$
\begin{aligned}
C &\ge \left(\frac{1}{2} - \frac{2}{q}\right)\int_\Omega u_n^2|\nabla u_n|^2\,dx + \alpha\left(\frac{1}{q} - \frac{1}{p}\right)|u_n|_p^p \\
&= \left(\frac{1}{8} - \frac{1}{2q}\right)\int_\Omega |\nabla u_n^2|^2\,dx + \alpha(\frac{1}{q} - \frac{1}{p})|u_n|_p^p \\
&\ge \left(\frac{1}{8} - \frac{1}{2q}\right)C_1|u_n|_4^4 + \alpha\left(\frac{1}{q} - \frac{1}{p}\right)|u_n|_p^p \\
&\ge \left(\frac{1}{8} - \frac{1}{2q}\right)C_2|u_n|_p^4 - \alpha\left(\frac{1}{p} - \frac{1}{q}\right)|u_n|_p^p,
\end{aligned} \tag{2.2}
$$

which is impossible since $p < 2$ and $q > 4$.

Using (2.1) again, we have

$$\mu_n \int_\Omega |\nabla u_n|^4\,dx + \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega u_n^2|\nabla u_n|^2\,dx \le C_3, \tag{2.3}$$

where C_3 is independent of n. Then we have $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, $u_n\nabla u_n \rightharpoonup u\nabla u$ in $L^2(\Omega)$ and $u_n(x) \to u(x)$ for almost every $x \in \Omega$. Note that u_n satisfies

$$(2.4)\quad \mu_n\int_\Omega|\nabla u_n|^2\nabla u_n\nabla\phi\,dx+\int_\Omega[(1+u_n^2)\nabla u_n\nabla\phi+u_n|\nabla u_n|^2\phi]\,dx \\ -\alpha\int_\Omega|u_n|^{p-2}u_n\phi\,dx-\beta\int_\Omega|u_n|^{q-2}u_n\phi\,dx=0,$$

for all $\phi\in W_0^{1,4}(\Omega)$. Since

$$\bigg(\int_\Omega|u_n|^{4N/(N-2)}\,dx\bigg)^{(N-2)/N}\le C_4\int_\Omega u_n^2|\nabla u_n|^2\,dx\le C_5,$$

by Moser's iteration we obtain

$$(2.5)\qquad |u_n|_{L^\infty(\Omega)}\le C_6,$$

and hence $|u|_{L^\infty(\Omega)}\le C_6$, where C_6 is independent of n. Now, similar to the arguments in [10] (see also [22]), one can show that u is a critical point of I_0. In fact, we choose $\phi=\psi e^{-u_n}$ in (2.4), where $\psi\in C_0^\infty(\Omega)$ satisfies $\psi\ge 0$. It follows from (2.4) that

$$\begin{aligned}(2.6)\qquad 0=&\,\mu_n\int_\Omega|\nabla u_n|^2\nabla u_n(\nabla\psi e^{-u_n}-\psi\nabla u_n e^{-u_n})\,dx\\ &+\int_\Omega(1+u_n^2)\nabla u_n(\nabla\psi e^{-u_n}-\psi\nabla u_n e^{-u_n})\,dx\\ &+\int_\Omega u_n|\nabla u_n|^2\psi e^{-u_n}\,dx-\alpha\int_\Omega|u_n|^{p-2}u_n\psi e^{-u_n}\,dx\\ &-\beta\int_\Omega|u_n|^{q-2}u_n\psi e^{-u_n}\,dx\\ \le&\,\mu_n\int_\Omega|\nabla u_n|^2\nabla u_n\nabla\psi e^{-u_n}\,dx+\int_\Omega(1+u_n^2)\nabla u_n\nabla\psi e^{-u_n}\,dx\\ &-\int_\Omega(1+u_n^2-u_n)|\nabla u_n|^2\psi e^{-u_n}\,dx\\ &-\alpha\int_\Omega|u_n|^{p-2}u_n\psi e^{-u_n}\,dx-\beta\int_\Omega|u_n|^{q-2}u_n\psi e^{-u_n}\,dx.\end{aligned}$$

By Fatou's lemma, the weak convergence of u_n and (2.3) we have

$$\begin{aligned}(2.7)\qquad 0\le&\int_\Omega(1+u^2)\nabla u\nabla\psi e^{-u}\,dx-\int_\Omega(1+u^2-u)|\nabla u|^2\psi e^{-u}\,dx\\ &-\alpha\int_\Omega|u|^{p-2}u\psi e^{-u}\,dx-\beta\int_\Omega|u|^{q-2}u\psi e^{-u}\,dx\\ =&\int_\Omega(1+u^2)\nabla u\nabla(\psi e^{-u})\,dx+\int_\Omega u|\nabla u|^2\psi e^{-u}\,dx\\ &-\alpha\int_\Omega|u|^{p-2}u\psi e^{-u}\,dx-\beta\int_\Omega|u|^{q-2}u\psi e^{-u}\,dx.\end{aligned}$$

Let $\chi\ge 0$, $\chi\in C_0^\infty(\Omega)$. We may choose a sequence of nonnegative functions $\psi_n\to\chi e^u$ in $W_0^{1,2}(\Omega)$, $\psi_n\to\chi e^u$ for almost every $x\in\Omega$ and $\{\psi_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Then, by approximations in (2.7), we have

$$\int_\Omega (1+u^2)\nabla u \nabla \chi\,dx + \int_\Omega u|\nabla u|^2\chi\,dx - \alpha\int_\Omega |u|^{p-2}u\chi\,dx - \beta\int_\Omega |u|^{q-2}u\chi\,dx \geq 0. \tag{2.8}$$

Similarly, we can obtain an opposite inequality. Thus, we have for all $\chi \in C_0^\infty(\Omega)$,

$$\int_\Omega (1+u^2)\nabla u \nabla \chi\,dx + \int_\Omega u|\nabla u|^2\chi\,dx - \alpha\int_\Omega |u|^{p-2}u\chi\,dx - \beta\int_\Omega |u|^{q-2}u\chi\,dx = 0. \tag{2.9}$$

That is, u is a critical point of I_0 and a solution of (P). Replacing χ with u in (2.9) and doing approximations again we have

$$\int_\Omega (1+u^2)|\nabla u|^2\,dx - \alpha\int_\Omega |u|^p\,dx - \beta\int_\Omega |u|^q\,dx = 0. \tag{2.10}$$

Setting $\phi = u_n$ in (2.4), we have

$$\mu_n\int_\Omega |\nabla u_n|^4\,dx + \int_\Omega (1+u_n^2)|\nabla u_n|^2\,dx - \alpha\int_\Omega |u_n|^p\,dx - \beta\int_\Omega |u_n|^q\,dx = 0. \tag{2.11}$$

Using

$$\int_\Omega |u_n|^p\,dx \to \int_\Omega |u|^p\,dx, \qquad \int_\Omega |u_n|^q\,dx \to \int_\Omega |u|^q\,dx,$$

(2.10), (2.11) and the lower semi-continuity we obtain

$$\int_\Omega |\nabla u_n|^2\,dx \to \int_\Omega |\nabla u|^2\,dx, \qquad \int_\Omega u_n^2|\nabla u_n|^2\,dx \to \int_\Omega u^2|\nabla u|^2\,dx,$$
$$\mu_n\int_\Omega |\nabla u_n|^4\,dx \to 0.$$

In particular, we have $u_n \to u$ in $W_0^{1,2}(\Omega)$, $u_n\nabla u_n \to u\nabla u$ in $L^2(\Omega)$ and $I_{\mu_n}(u_n) \to I_0(u)$. □

Let $\{e_j\}$ be a Schauder basis of $W_0^{1,4}(\Omega)$ (see [16] and [7]). Define $X_j := \mathbb{R}e_j$. Note that for each $\mu \in (0,1]$, I_μ is even. Now, some notations are in order. Set

$$Y_k := \bigoplus_{j=0}^{k} X_j, \qquad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j},$$
$$B_k := \Big\{u \in Y_k : \int_\Omega (1+u^2)|\nabla u|^2\,dx \leq \rho_k^2\Big\},$$
$$N_k := \Big\{u \in Z_k : \int_\Omega (1+u^2)|\nabla u|^2\,dx = r_k^2\Big\},$$

where $\rho_k > r_k > 0$.

The following intersection property is similar to Lemma 3.4 in [26].

Lemma 2.2. *If $\gamma \in C(B_k, W_0^{1,4}(\Omega))$ is odd and $\gamma|_{\partial B_k} = \mathrm{id}$, then*

$$\gamma(B_k) \cap N_k \neq \emptyset.$$

Proof. Define

$$U := \left\{ u \in B_k : \int_\Omega (1+\gamma^2(u))|\nabla \gamma(u)|^2\,dx < r_k^2 \right\}.$$

Denote by P_k the projector onto Y_{k-1} such that $P_k Z_k = \{0\}$. By the Borsuk–Ulam Theorem, there is $u_0 \in B_k$ with

$$\int_\Omega (1+\gamma^2(u_0))|\nabla \gamma(u_0)|^2\,dx = r_k^2$$

such that $P_k \gamma(u_0) = 0$. Hence $u_0 \in \gamma(B_k) \cap N_k$. □

Proof of Theorem 1.1. (a) Assume $\alpha \in \mathbb{R}$ and $\beta > 0$. The proof is divided into several steps.

Step 1. For each k, there is $r_k > 0$ independent of $\mu \in (0,1]$ such that

$$\inf_{u \in N_k} I_\mu(u) \to \infty, \quad \text{as } k \to \infty.$$

In fact, define

$$\theta_k := \sup_{\substack{u \in Z_k \\ u \neq 0}} \frac{|u|_q^2}{\displaystyle\int_\Omega (1+u^2)|\nabla u|^2\,dx}.$$

It is clear that $0 < \theta_{k+1} \le \theta_k$, thus $\theta_k \to \theta \ge 0$ as $k \to \infty$. For each k, there exists $u_k \in Z_k$ such that

$$\left(\int_\Omega (1+u_k^2)|\nabla u_k|^2\,dx \right)^{1/2} = 1 \quad \text{and} \quad |u_k|_q > \frac{\theta_k}{2}.$$

By the definition of Z_k, $u_k \rightharpoonup 0$ in $W_0^{1,4}(\Omega)$ (see p. 182–183 in [15]). The Sobolev imbedding theorem implies that $u_k \to 0$ in $L^q(\Omega)$. Therefore, $\theta = 0$, i.e.

$$\theta_k \to 0 \quad \text{as} \quad k \to \infty. \tag{2.12}$$

Observe that there exists $C_1 > 0$ such that $|u|^p \le C_1(1+|u|^q)$, which yields that

$$\begin{aligned}
I_0(u) &= \frac{1}{2}\int_\Omega (1+u^2)|\nabla u|^2\,dx - \frac{\alpha}{p}|u|_p^p - \frac{\beta}{q}|u|_q^q \\
&\ge \frac{1}{2}r_k^2 - \frac{|\alpha|C_1|\Omega|}{p} - \left(\frac{\beta}{q} + \frac{|\alpha|C_1}{p}\right)\theta_k^q \left(\int_\Omega (1+u^2)|\nabla u|^2\,dx \right)^{q/2} \\
&= \frac{1}{2}r_k^2 - \frac{|\alpha|C_1|\Omega|}{p} - \left(\frac{\beta}{q} + \frac{|\alpha|C_1}{p}\right)\theta_k^q r_k^q \\
&= r_k^2\left[\frac{1}{2} - \left(\frac{\beta}{q} + \frac{|\alpha|C_1}{p}\right)\theta_k^q r_k^{q-2}\right] - \frac{|\alpha|C_1|\Omega|}{p}
\end{aligned}$$

for $u \in N_k$. Choosing $r_k = [4(\beta/q + |\alpha|C_1/p)]^{-1/(q-2)}\theta_k^{-q/(q-2)}$, we have $r_k \to \infty$ as $k \to \infty$. Thus

$$\inf_{u \in N_k} I_\mu(u) \ge \inf_{u \in N_k} I_0(u) \ge \frac{1}{4}r_k^2 - \frac{|\alpha|C_1|\Omega|}{p} \to \infty \quad \text{as } k \to \infty. \tag{2.13}$$

Step 2. For each k, there is $\rho_k > r_k$ independent of $\mu \in (0,1]$ such that

$$a_k := \max_{\substack{u \in Y_k \\ \int_\Omega (1+u^2)|\nabla u|^2\,dx = \rho_k^2}} I_\mu(u) \leq 0. \tag{2.14}$$

In fact, for $u \in Y_k$, we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{4}\mu \int_\Omega |\nabla u|^4\,dx + \frac{1}{2}\int_\Omega (1+u^2)|\nabla u|^2\,dx \\ &\quad - \frac{\alpha}{p}\int_\Omega |u|^p\,dx - \frac{\beta}{q}\int_\Omega |u|^q\,dx \\ &\leq \frac{1}{4}\mu \int_\Omega |\nabla u|^4\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\bigg(\int_\Omega |u|^4\,dx\bigg)^{1/2}\bigg(\int_\Omega |\nabla u|^4\,dx\bigg)^{1/2} \\ &\quad + \frac{|\alpha|}{p}\int_\Omega |u|^p\,dx - \frac{\beta}{q}\int_\Omega |u|^q\,dx \\ &\leq \frac{1}{4}\mu\|u\|^4 + C_2\|u\|^2 + C_3\|u\|^4 + C_4\|u\|^p - C_5\|u\|^q \\ &\leq \frac{1}{4}\|u\|^4 + C_2\|u\|^2 + C_3\|u\|^4 + C_4\|u\|^p - C_5\|u\|^q, \end{aligned}$$

since all norms are equivalent on the finite dimensional space Y_k, which implies that we can choose $\rho_k > r_k$ independent of $\mu \in (0,1]$ such that $b_k \leq 0$.

Step 3. We claim that I_μ satisfies the $(\mathrm{PS})_c$ condition for every $c > 0$. In fact, let $\{u_n\} \subset W_0^{1,4}(\Omega)$ be such that $I_\mu(u_n) \to c$, $I'_\mu(u_n) \to 0$ as $n \to \infty$. For n large enough, we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq I_\mu(u_n) - \frac{1}{q}\langle I'_\mu(u_n), u_n\rangle \\ &= \bigg(\frac{1}{4} - \frac{1}{q}\bigg)\mu \int_\Omega |\nabla u_n|^4\,dx + \bigg(\frac{1}{2} - \frac{1}{q}\bigg)\int_\Omega |\nabla u_n|^2\,dx \\ &\quad + \bigg(\frac{1}{2} - \frac{2}{q}\bigg)\int_\Omega u_n^2|\nabla u_n|^2\,dx + \bigg(\frac{\alpha}{q} - \frac{\alpha}{p}\bigg)\int_\Omega |u_n|^p\,dx \\ &\geq \bigg(\frac{1}{4} - \frac{1}{q}\bigg)\mu\|u_n\|^4 - C_6\|u_n\|^p. \end{aligned} \tag{2.15}$$

Thus $\{u_n\}$ is bounded in $W_0^{1,4}(\Omega)$. Up to a subsequence, we may assume $u_n \rightharpoonup u$ in $W_0^{1,4}(\Omega)$ and $u_n \to u$ in $L^s(\Omega)$ for $1 < s < 22^*$. Choosing $\phi = u_n - u_m$ in (1.6) we have

$$\begin{aligned} o(1)\|u_n - u_m\| &= \langle I'_\mu(u_n) - I'_\mu(u_m), u_n - u_m\rangle \\ &= \mu \int_\Omega (|\nabla u_n|^2\nabla u_n - |\nabla u_m|^2\nabla u_m)(\nabla u_n - \nabla u_m)\,dx \\ &\quad + \int_\Omega |\nabla u_n - \nabla u_m|^2\,dx \\ &\quad + \int_\Omega (u_n^2\nabla u_n - u_m^2\nabla u_m)(\nabla u_n - u_m)\,dx \end{aligned} \tag{2.16}$$

$$+\int_\Omega (u_n|\nabla u_n|^2 - u_m|\nabla u_m|^2)(u_n-u_m)\,dx$$
$$-\alpha\int_\Omega (|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n-u_m)\,dx$$
$$-\beta\int_\Omega (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m)(u_n-u_m)\,dx.$$

Now we estimate the above terms appeared on the right hand one by one.

$$\begin{aligned}(2.17)\qquad &\int_\Omega (u_n^2\nabla u_n - u_m^2\nabla u_m)(\nabla u_n - u_m)\,dx\\ &= \int_\Omega u_n^2|\nabla u_n - \nabla u_m|^2\,dx + \int_\Omega (u_n^2-u_m^2)\nabla u_m(\nabla u_n - \nabla u_m)\,dx\\ &\geq -|u_n-u_m|_4(|u_n|_4+|u_m|_4)\|u_m\|(\|u_n\|+\|u_m\|)\to 0.\end{aligned}$$

$$\begin{aligned}(2.18)\qquad &\left|\int_\Omega (u_n|\nabla u_n|^2 - u_m|\nabla u_m|^2)(u_n-u_m)\,dx\right|\\ &\qquad\leq (|u_n|_4\|u_n\|^2 + |u_m|_4\|u_m\|^2)(|u_n-u_m|_4)\to 0.\end{aligned}$$

$$\begin{aligned}(2.19)\qquad &\left|\alpha\int_\Omega (|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n-u_m)\,dx\right|\\ &\qquad\leq |\alpha|(|u_n|_p^{p-1}+|u_m|_p^{p-1})|u_n-u_m|_p\to 0.\end{aligned}$$

$$\begin{aligned}(2.20)\qquad &\left|\beta\int_\Omega (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m)(u_n-u_m)\,dx\right|\\ &\qquad\leq |\beta|\int_\Omega (|u_n|^{q-1}+|u_m|^{q-1})|u_n-u_m|\,dx\\ &\qquad\leq |\beta|(|u_n|_q^{q-1}+|u_m|_q^{q-1})|u_n-u_m|_q\to 0.\end{aligned}$$

$$\begin{aligned}(2.21)\qquad &\mu\int_\Omega (|\nabla u_n|^2\nabla u_n - |\nabla u_m|^2\nabla u_m)(\nabla u_n-\nabla u_m)\,dx\\ &\qquad\geq C_7\int_\Omega |\nabla u_n-\nabla u_m|^4\,dx\end{aligned}$$

for some $C_7>0$. Combining (2.16)–(2.20) together we obtain

$$C_7\int_\Omega |\nabla u_n - \nabla u_m|^4\,dx \leq o(1)\|u_n-u_m\| + o(1),$$

which impliese that $\{u_n\}$ is a Cauchy sequence in $W_0^{1,4}(\Omega)$, and hence there is $u\in W_0^{1,4}(\Omega)$ such that $u_n\to u$ in $W_0^{1,4}(\Omega)$.

Step 4. Define for $k\geq 2$, $c_k(\mu) := \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} I_\mu(\gamma(u))$, where

$$\Gamma_k := \{\gamma\in C(B_k,X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = \mathrm{id}\}.$$

Obviously, for each k, there holds $c_k(\mu) \le c_k(1) \le \max_{u\in B_k} I_1(u) < \infty$, since I_μ is increasing in μ. On the other hand, by Lemma 2.2, $\gamma(B_k)\cap N_k \ne \emptyset$ for $\gamma \in \Gamma_k$. Therefore, $c_k(\mu) \ge \inf\limits_{u\in N_k} I_\mu(u) \ge \inf\limits_{u\in N_k} I_0(u)$. It follows from (2.13) that $c_k(\mu) \to \infty$ as $k\to\infty$.

Step 5. For each k, there is a sequence $u_{n,k}$ such that $I_\mu(u_{n,k}) \to c_k(\mu)$ and $I'_\mu(u_{n,k}) \to 0$ as $n\to\infty$. Otherwise, for every $\varepsilon \in (0,(c_k-a_k)/2)$ and $\gamma\in\Gamma_k$ with $\max\limits_{u\in B_k} I_\mu(\gamma(u)) \le c_k(\mu)+\varepsilon$, where a_k is given in (2.14), it follows from the equivariant deformation lemma (see Lemma 3.1 in [26]) that there exists $\eta \in C([0,1]\times W_0^{1,4}(\Omega), W_0^{1,4}(\Omega))$ such that $\eta(1, I_\mu^{c_k(\mu)+\varepsilon}) \subset I_\mu^{c_k(\mu)-\varepsilon}$ and $\varphi(u) := \eta(1,\gamma(u)) \in \Gamma_k$. Thus $c_k(\mu) \le \max\limits_{u\in B_k} I_\mu(\gamma(u)) \le c_k(\mu)-\varepsilon$, which is absurd.

By Step 3, $c_k(\mu)$ is a critical value of I_μ. Thus I_μ has an unbounded sequence of critical values and hence I_μ has a sequence of critical points $\{u_{\mu,k}\}$. By Lemma 2.1, passing to the limit for $\mu\to 0$, I_0 has a sequence of critical points $\{u_k\}$ such that $I_0(u_k)\to\infty$ as $k\to\infty$. This completes the proof of (a).

(b) Assume $\alpha>0$ and $\beta\in\mathbb{R}$. The proof is also divided into several steps.

Step 1. We claim there exists k_0 such that, for each $k\ge k_0$, there is $\rho_k>0$ independent of $\mu\in(0,1]$ such that

$$\inf_{\substack{u\in Z_k\\ \int_\Omega(1+u^2)|\nabla u|^2\,dx=\rho_k^2}} I_\mu(u) \ge 0.$$

In fact, define

$$\vartheta_k := \sup_{\substack{u\in Z_k\\ u\ne 0}} \frac{|u|_p^2}{\displaystyle\int_\Omega (1+u^2)|\nabla u|^2\,dx}.$$

Similar to (2.12), $\vartheta_k\to 0$ as $k\to\infty$. Since

$$|u|_q \le C_0\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)^{1/2} \le C_0\bigg(\int_\Omega (1+u^2)|\nabla u|^2\,dx\bigg)^{1/2}$$

and $q>4$, there exists $R>0$ such that

$$\int_\Omega (1+u^2)|\nabla u|^2\,dx \le R \Rightarrow \frac{\beta}{q}|u|_q^q \le \frac{1}{4}\int_\Omega (1+u^2)|\nabla u|^2\,dx,$$

which yields

$$\begin{aligned} I_0(u) &= \frac12\int_\Omega (1+u^2)|\nabla u|^2\,dx - \frac{\alpha}{p}|u|_p^p - \beta/q|u|_q^q \\ &\ge \frac14\int_\Omega (1+u^2)|\nabla u|^2\,dx - \alpha/p|u|_p^p \\ &\ge \frac14\int_\Omega (1+u^2)|\nabla u|^2\,dx - \frac{\alpha}{p}\vartheta_k^p\bigg(\int_\Omega (1+u^2)|\nabla u|^2\,dx\bigg)^{p/2}, \end{aligned} \tag{2.22}$$

for $u\in W_0^{1,4}(\Omega)$ with $\int_\Omega (1+u^2)|\nabla u|^2\,dx \le R$.

Choosing $\rho_k = (8\alpha/p)^{1/(2-p)}\vartheta_k^{p/(2-p)}$ we have $\rho_k \to 0$ as $k \to \infty$. Then there exists k_0 such that $\rho_k \le R$ when $k \ge k_0$. Thus, for $k \ge k_0$ and $u \in Z_k$ with $\int_\Omega (1+u^2)|\nabla u|^2\,dx = \rho_k^2$, it follows from (2.22) that

$$I_\mu(u) \ge I_0(u) \ge \rho_k^2\left[\frac{1}{4} - \frac{\alpha}{p}\vartheta_k^p\rho_k^{p-2}\right] \ge 0,$$

which implies that the claim is true.

Step 2. For each $k \ge k_0$, there is $0 < r_k < \rho_k$ independent of $\mu \in (0,1]$ such that

$$\bar{b}_k := \max_{\substack{u\in Y_k\\ \int_\Omega(1+u^2)|\nabla u|^2\,dx = r_k^2}} I_\mu(u) < 0.$$

In fact, for $u \in Y_k$, we have

$$\begin{aligned}
I_\mu(u) &= \frac{\mu}{4}\int_\Omega |\nabla u|^4\,dx + \frac{1}{2}\int_\Omega (1+u^2)|\nabla u|^2\,dx - \frac{\alpha}{p}\int_\Omega |u|^p\,dx - \frac{\beta}{q}\int_\Omega |u|^q\,dx \\
&\le \frac{\mu}{4}|\nabla u|_4^4 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2}|u|_4^2|\nabla u|_4^2 - \frac{\alpha}{p}|u|_p^p + \frac{|\beta|}{q}|u|_q^q \\
&\le \frac{1}{4}\|u\|^4 + C_1\|u\|^2 + C_2\|u\|^4 - C_3\|u\|^p + C_4\|u\|^q,
\end{aligned}$$

since all norms are equivalent on the finite dimensional space Y_k, which implies that we can choose $0 < r_k < \rho_k$ independent of $\mu \in (0,1]$ such that $\bar{b}_k < 0$.

Step 3. We obtain from (2.22), for $k \ge k_0$ and $u \in B_k$

$$I_\mu(u) \ge I_0(u) \ge -\frac{\alpha}{p}\vartheta_k^p\left(\int_\Omega (1+u^2)|\nabla u|^2\,dx\right)^{p/2} \ge -\frac{\alpha}{p}\,\vartheta_k^p\rho_k^p.$$

Then $\bar{a}_k := \inf_{u\in B_k} I_\mu(u) \to 0$ as $k \to \infty$ since $\vartheta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$.

Step 4. Now we prove that I_μ satisfies the $(\mathrm{PS})_c^*$ condition for each $c < 0$ with respect to $\{Y_k\}$. Consider a sequence $\{u_{n_k}\} \subset W_0^{1,4}(\Omega)$ such that $n_k \to \infty$, $u_{n_k} \in Y_{n_k}$, $I_\mu(u_{n_k}) \to c$ and $I_\mu|'_{Y_{n_k}}(u_{n_k}) \to 0$. For k large enough, we have

$$\begin{aligned}
c + 1 + \|u_{n_k}\| &\ge I_\mu(u_{n_k}) - \frac{1}{q}\langle I'_\mu(u_{n_k}), u_{n_k}\rangle \\
&= \left(\frac{1}{4} - \frac{1}{q}\right)\mu\int_\Omega |\nabla u_{n_k}|^4\,dx + \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |\nabla u_{n_k}|^2\,dx \\
&\quad + \left(\frac{1}{2} - \frac{2}{q}\right)\int_\Omega u_{n_k}^2|\nabla u_{n_k}|^2\,dx + \alpha\left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega |u_{n_k}|^p\,dx \\
&\ge \left(\frac{1}{4} - \frac{1}{q}\right)\mu\|u_{n_k}\|^4 + \alpha\left(\frac{1}{q} - \frac{1}{p}\right)C_7\|u_{n_k}\|^p.
\end{aligned}$$

Thus $\{u_{n_k}\}$ is bounded in $W_0^{1,4}(\Omega)$. Similar to the proof of part (a), one can show that $\{u_{n_k}\}$ has a convergent subsequence in $W_0^{1,4}(\Omega)$.

Step 5. We fix $n \geq k \geq k_0$ and define

$$\begin{aligned} Z_k^n &:=, \bigoplus_{j=k}^{n} X_j, \\ B_k^n &:= \left\{ u \in Z_k^n : \int_\Omega (1+u^2)|\nabla u|^2\,dx \leq \rho_k^2 \right\}, \\ \Gamma_k^n &:= \{\gamma \in C(B_k^n, Y_n) : \gamma \text{ is odd and } \gamma|_{\partial B_k^n} = \mathrm{id}\}, \\ \overline{c}_k^n(\mu) &:= \sup_{\gamma \in \Gamma_k^n} \min_{u \in B_k^n} I_\mu(\gamma(u)). \end{aligned}$$

Then $\overline{c}_k^n(\mu) \in [\overline{a}_k, \overline{b}_k]$. Now, repeating the arguments in (a) to the functional $-I_\mu$ defined on the space Y_n, there exists $u_n \in Y_n$ such that

$$\overline{c}_k^n(\mu) - \frac{2}{n} \leq I_\mu(u_n) \leq \overline{c}_k^n(\mu) + \frac{2}{n}, \qquad \|I_\mu|'_{X_n}(u_n)\| \leq \frac{8}{n}.$$

Since I_μ satisfies the $(\mathrm{PS})_c^*$ condition, we see that $\{\overline{c}_k^n(\mu)\}$ converges along a subsequence to a critical value $\overline{c}_k(\mu) \in [\overline{a}_k, \overline{b}_k]$ of I_μ as $n \to \infty$. Moreover, by Step 3, $\overline{c}_k(\mu) \to 0_-$ as $k \to \infty$. Using Lemma 2.1 and passing to the limit for $\mu \to 0$ we have that I_0 has a sequence of negative critical values going to 0. This completes the proof of (b). □

3. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. The following lemma plays a key role in the proof.

Lemma 3.1. *There is $\lambda > 0$ such that, for any $\alpha > 0$ and*

$$c < \frac{S^{N/2}}{2N2^{N/2}\beta^{(N-2)/2}} - \lambda\alpha^{22^*/(22^*-p)}, \tag{3.1}$$

the functional I_μ satisfies the $(\mathrm{PS})_c^*$ *condition.*

Proof. Consider a sequence $\{u_{n_k}\} \subset W_0^{1,4}(\Omega)$ such that $n_k \to \infty$, $u_{n_k} \in Y_{n_k}$, $I_\mu(u_{n_k}) \to c$, $I_\mu|'_{Y_{n_k}}(u_{n_k}) \to 0$. As in the proof of Theorem 1.1, $\{u_{n_k}\}$ is bounded in $W_0^{1,4}(\Omega)$. Going if necessary to a subsequence, we can assume that $u_{n_k} \rightharpoonup u$ in $W_0^{1,4}(\Omega)$, $u_{n_k} \to u$ in $L^s(\Omega)$ for $1 < s < 22^*$ and $u_{n_k} \to u$ almost everywhere on Ω. Since $\{u_{n_k}\}$ is bounded in $L^{22^*}(\Omega)$, $\{|u_{n_k}|^{22^*-2}u_{n_k}\}$ is bound in $L^{4N/(3N+2)}(\Omega)$ and so

$$|u_{n_k}|^{22^*-2}u_{n_k} \rightharpoonup |u|^{22^*-2}u \quad \text{in } L^{4N/(3N+2)}(\Omega).$$

Then a standard argument shows that u is a critical point of I_μ (see [26]).

We write $v_{n_k} := u_{n_k} - u$. The Brezis–Lieb lemma (see [8]) leads to

$$\begin{aligned} |\nabla u_{n_k}^2|_2^2 &= |\nabla u^2|_2^2 + |\nabla v_{n_k}^2|_2^2 + o(1), \\ |u_{n_k}|_{22^*}^{22^*} &= |u|_{22^*}^{22^*} + |v_{n_k}|_{22^*}^{22^*} + o(1). \end{aligned}$$

Since $\langle I'_\mu(u_{n_k}), u_{n_k}\rangle \to 0$, so we have

$$\begin{aligned}&\mu|\nabla v_{n_k}|_4^4 + |\nabla v_{n_k}|_2^2 + \frac{1}{2}|\nabla v_{n_k}^2|_2^2 - \beta|v_{n_k}|_{22^*}^{22^*}\\ &\to -\mu\int_\Omega |\nabla u|^4\,dx - 2\int_\Omega u^2|\nabla u|^2\,dx - \int_\Omega |\nabla u|^2\,dx + \alpha\int_\Omega |u|^p\,dx + \beta\int_\Omega |u|^{22^*}\,dx\\ &= -\langle I'_\mu(u), u\rangle = 0.\end{aligned}$$

Therefore, we may assume that

$$\mu|\nabla v_{n_k}|_4^4 + |\nabla v_{n_k}|_2^2 + \frac{1}{2}|\nabla v_{n_k}^2|_2^2 \to b, \qquad \beta|v_{n_k}|_{22^*}^{22^*} \to b.$$

By the Soblev inequality, we have

$$b \geq \frac{1}{2}|\nabla v_{n_k}^2|_2^2 \geq \frac{1}{2}S|v_{n_k}^2|_{2^*}^2 = \frac{1}{2}S|v_{n_k}|_{22^*}^4,$$

and so $b \geq S(b/\beta)^{2/2^*}/2$. Then $b = 0$ or $b \geq S^{N/2}/(2^{N/2}\beta^{(N-2)/2})$.

Assume $b \geq S^{N/2}/(2^{N/2}\beta^{(N-2)/2})$. We have

$$\begin{aligned}c + o(1) &= I_\mu(u_{n_k}) - \frac{1}{4}\langle I'_\mu(u_{n_k}), u_{n_k}\rangle\\ &= \frac{1}{4}|\nabla u_{n_k}|_2^2 + \alpha\left(\frac{1}{4} - \frac{1}{p}\right)|u_{n_k}|_p^p + \beta\left(\frac{1}{4} - \frac{1}{22^*}\right)|u_{n_k}|_{22^*}^{22^*}\\ &\geq \alpha\left(\frac{1}{4} - \frac{1}{p}\right)|u_{n_k}|_p^p + \frac{\beta}{2N}|u_{n_k}|_{22^*}^{22^*}\\ &= \alpha\left(\frac{1}{4} - \frac{1}{p}\right)|u|_p^p + \frac{\beta}{2N}\left(\frac{b}{\beta} + |u|_{22^*}^{22^*}\right) + o(1)\\ &\geq \frac{S^{N/2}}{2N2^{N/2}\beta^{(N-2)/2}} + \frac{\beta}{2N}|u|_{22^*}^{22^*} - C_1\alpha|u|_{22^*}^p\end{aligned}$$

for some $C_1 > 0$. A direct computation shows that

$$\min_{t>0}\left(\frac{\beta}{2N}t^{22^*} - C_1\alpha t^p\right) = -\left(1 - \frac{p}{22^*}\right)C_1^{22^*/(22^*-p)}(pN)^{p/(22^*-p)}\alpha^{22^*/(22^*-p)}$$

Setting $\lambda := (1 - p/22^*)C_1^{22^*/(22^*-p)}(plN)^{p/(22^*-p)} > 0$ we have

$$c \geq \frac{S^{N/2}}{2N2^{N/2}\beta^{(N-2)/2}} - \lambda\alpha^{22^*/(22^*-p)}, \tag{3.2}$$

which contradicts (3.1). So $b = 0$, and therefore $u_{n_k} \to u$ in $W_0^{1,4}(\Omega)$. □

PROOF OF THEOREM 1.2. By Lemma 3.1, there exists $\alpha^* > 0$ such that for every $0 < \alpha < \alpha^*$ and $c < 0$, the functional $I_\mu(u)$ satisfies the $(\mathrm{PS})_c^*$ condition. Now, repeating the proof of the part (b) of Theorem 1.1 we can obtain the conclusion. □

Acknowledgments. F. Zhao thanks the College of William and Mary for the invitation and the hospitality during his visit.

References

[1] C.O. Alves, G.M. Figueiredo and U.B. Severo, *Multiplicity of positive solutions for a class of quasilinear problems*, Adv. Differential Equations **14** (2009), 911–942.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[3] A. Ambrosetti and Z.-Q. Wang, *Positive solutions to a class of quasilinear elliptic equations on* $\mathbb{R}$, Discrete Contin. Dyn. Syst. **9** (2003), 55–66.

[4] T. Bartsch, *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal. **20** (1993), 1205–1216.

[5] T. Bartsch and M. Willem, *Periodic solutions of nonautonomous Hamiltonian systems with symmetries*, J. Reine Angew. Math. **451** (1994), 149–159.

[6] ———, *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc. **123** (1995), 3555–3561.

[7] H. Bellout, *On a special Schauder basis for the Sobolev spaces* $W_0^{1,p}(\Omega)$, Illinois J. Math. **39** (1995), 187–195.

[8] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[9] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[10] A. Canino and M. Degiovanni, *Nonsmooth critical point theory and quaslinear Schrödinger elliptic equations*, Topol. Methods in Differential Equations and Equations and Inclusions (Montreal, PQ, 1994), **472** (1995), 1–50.

[11] M. Colin and L. Jeanjean, *Solutions for quasilinear Schrödinger equations: a dual approach*, Nonlinear Anal. **56** (2002), 392–344.

[12] J.M. do Ó, O. Miyagaki and S. Soares, *Soliton solutions for quasilinear Schrödinger equations with critical growth*, J. Differential Equations **248** (2010), 722–744.

[13] J.M. do Ó and U. Severo, *Quasilinear Schrödinger equations involving concave and convex nonlinearities*, Commun. Pure Appl. Anal. **8** (2009), 621–644.

[14] ———, *Solitary wave for a class of quasilinear quasilinear Schrödinger equations in dimension two*, Calc. Var. Partial Differential Equations **69** (2010), 397–408.

[15] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.

[16] S. Fučík, O. John and J. Nečas, *On the existence of Schauder bases in Sobolev spaces*, Comment. Math. Univ. Carolinae **13** (1972), 163–175.

[17] A. J. García and A.I. Peral, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877–895.

[18] H.F. Lins and E.A.B. Silva, *Quasilinear asymptotically periodic elliptic equations with critical growth*, Nonlinear Anal. **71** (2009), 2890–2905.

[19] J. Liu and Z.-Q. Wang, *Solition solutions for quasilinear Schrödinger equations*, Proc. Amer. Math. Soc. **131** (2003), 329–344.

[20] J. Liu, Y. Wang and Z.-Q. Wang, *Solutions for quasilinear Schrödinger equations via the Nehari method*, Comm. Partial Differential Equations **29** (2004), 879–901.

[21] ———, *Solition solutions for quasilinear Schrödinger equations* II, J. Differential Equations **187** (2003), 473–493.

[22] X. Liu, J. Liu and Z.-Q. Wang, *Quasilinear elliptic equations via perturbation method*, Proc. Amer. Math. Soc. **141** (2013), 253–263.

[23] A. Moameni, *Existence of solition solutions for a quasilinear Schrödinger equations involing critical growth in* $\mathbb{R}^N$, J. Differential Equations **229** (2006), 570–587.
[24] M. Poppenburg, K.Schmitt and Z.-Q. Wang, *On the existence of solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Differential Equations **14** (2002), 392–344.
[25] E.A.B. Silva and G.F. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with critical growth*, Nonlinear Anal. **72** (2010), 2935–2949.
[26] M. Willem, *Minimax Theorem, Progress in Nonlinear Differential Equations and their Applications*, Vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

Manuscript received January 1, 2014

Leran Xia and Fukun Zhao
Department of Mathematics
Yunnan Normal University
Kunming 650500 Yunnan, P.R.CHINA
E-mail address: fukunzhao@163.com

Minbo Yang
Department of Mathematics
Zhejiang Normal University
Jinhua 321004 Zhejiang, P.R.CHINA