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SADDLE POINT SOLUTIONS FOR NON-LOCAL ELLIPTIC OPERATORS

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Abstract. The paper deals with equations driven by a non-local integrodifferential operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. These equations have a variational structure and we find a solution for them using the Saddle Point Theorem. We prove this result for a general integrodifferential operator of fractional type and from this, as a particular case, one can derive an existence theorem for the fractional Laplacian, finding solutions of the equation

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where the nonlinear term f satisfies a linear growth condition.

1. Introduction

In this paper we deal with the following problem

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function whose properties will be introduced later and $\mathcal{L}_K$ is a non-local operator

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formally defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y)+u(x-y)-2u(x))K(y)\,dy, \tag{1.2}$$

for any $x \in \mathbb{R}^n$, where $K\colon \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a given function. In the case where $K(x) = |x|^{-(n+2s)}$, for a given $s \in (0,1)$, there are several studies about this problem (see [3] and references therein). In this case problem (1.1) becomes

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where $-(-\Delta)^s$ is the fractional Laplace operator which (up to normalization factors) may be formally defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \tag{1.4}$$

for any $x \in \mathbb{R}^n$.

As we said, here problems (1.1) and (1.3) are only expressed in a formal way. In classical terms, the definition in (1.4) makes sense if $u \in C_0^2(\Omega)$, for example. However, under suitable assumptions on f and K, we can express problems (1.1) and (1.3) in a variational form which allows us to give a simple and complete explanation and also to set the study of (1.1). In this way problem (1.1) becomes the Euler–Lagrange equation of a suitable functional defined in a suitable space.

For this, we assume that K satisfies the following conditions:

$$mK \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}; \tag{1.5}$$

$$\text{there exists } \theta > 0 \text{ and } s \in (0,1) \text{ such that } K(x) \geq \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.6}$$

The assumptions of the function f have a direct influence on the topological structure of the problem. When the function f satisfies superlinear and subcritical growth conditions, the functional associated to problem (1.1) satisfies the geometry of the Mountain Pass Theorem; see for example [8]. In [10] the right-hand side of equation (1.1) is equal to $f(x,u) + \lambda u$, where λ is a real parameter and the nonlinear term f satisfies superlinear and subcritical growth conditions. In this case critical points of the Euler–Lagrange functional can be obtained by using both the Mountain Pass Theorem and the Linking Theorem, depending on the value of λ.

In view of our problem we assume that, in addition to the usual Carathéodory conditions, f also satisfies the following condition:

$$\text{there exist } a \in L^2(\Omega) \text{ and } b \geq 0 \text{ such that } |f(x,t)| \leq a(x) + b|t| \text{ for any } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{1.7}$$

Now, we introduce the functional spaces. Here, the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (u(x)-u(y))^2 K(x-y) \text{ is in } L^1(Q,\, dx\, dy),$$

where $Q := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. The space X is endowed with the norm defined as

$$\|u\|_X = \bigg(\int_\Omega |u(x)|^2\, dx + \int_Q |u(x)-u(y)|^2 K(x-y)\, dx\, dy \bigg)^{1/2}. \tag{1.8}$$

It is immediate to observe that bounded and Lipschitz functions belong to X (see [7], [8] for further details on space X). Moreover, we denote with Z the closure of $C_0^\infty(\Omega)$ in X.

Now, we can state in a precise way problem (1.1) by writing it in the variational form:

$$\begin{cases} \displaystyle\int_Q (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\, dx\, dy \\ \displaystyle\qquad\qquad\qquad = \int_\Omega f(x,u(x))\varphi(x)\, dx \quad \text{for any } \varphi \in Z, \\ u \in Z. \end{cases} \tag{1.9}$$

Thanks to our assumptions on Ω, f and K, all the integrals in (1.9) are well defined if u, $\varphi \in Z$. We also point out that the odd part of function K gives no contribution to the integral of the left-hand side of (1.9). Therefore, it would be not restrictive to assume that K is even.

Now, we introduce the main result of the paper. Here, we denote with $\lambda_1, \lambda_2, \ldots$ the eigenvalues of $-\mathcal{L}_K$ which are briefly recalled in Proposition 2.2 (see also [10, Section 3]).

THEOREM 1.1. *Let Ω be a bounded subset of $\mathbb{R}^n$ and let K and f be two functions satisfying assumptions* (1.5)–(1.7). *Moreover, by setting*

$$\liminf_{|t|\to\infty} \frac{f(x,t)}{t} := \underline{\alpha}(x) \quad \textit{and} \quad \limsup_{|t|\to\infty} \frac{f(x,t)}{t} := \overline{\alpha}(x) \quad \textit{for a.e. } x \in \Omega, \tag{1.10}$$

we assume that one of the two following conditions is satisfied: either $\overline{\alpha}(x) < \lambda_1$ for almost every $x \in \Omega$, or there exists $k \in \mathbb{N}^*$ such that $\lambda_k < \underline{\alpha}(x) \le \overline{\alpha}(x) < \lambda_{k+1}$ for almost every $x \in \Omega$. Then, problem (1.9) *admits a solution $u \in Z$.*

REMARK 1.2. We notice that, in our framework, no solution of problem (1.9) is known from the beginning, unlike the cases treated in [8], [10] where the problems considered admit the trivial solution $u = 0$ (indeed, in our case, $f(x,0) + h(x)$ may not vanish and $u = 0$ may not be a solution).

The proof of Theorem 1.1 relies on the Saddle Point Theorem (see, for instance, [5]). In order to check the geometric assumptions needed for applying this result, we perform some energy estimates in fractional Sobolev spaces. Indeed, Theorem 1.1 is the fractional analog of a result valid for the classical Laplacian (see, e.g. [4, Theorem 4.1.1]). As a matter of fact, we plan to consider further applications of the Saddle Point Theorem for fractional operators for asymptotically linear terms in a forthcoming paper.

It is an interesting question if weak solutions of problem (1.9) solve also problem (1.1) in an appropriate strong sense. Some interesting results about this problem can be found in [11] (see also [1, Theorem 5]) and a more exhaustive answer will be provided in a forthcoming paper.

Moreover, it is worth pointing out that the solution found in Theorem 1.1 is unique, under a suitable condition on the nonlinearity.

COROLLARY 1.3. *Under the same assumptions of Theorem* 1.1 *and if in addition there exists a* $k \in \mathbb{N}^*$ *such that*

$$\lambda_k < \frac{f(x,s) - f(x,t)}{s-t} < \lambda_{k+1} \tag{1.11}$$

for any $s, t \in \mathbb{R}$ *with* $s \neq t$ *and almost every* $x \in \Omega$*, then the solution of problem* (1.9) *is unique.*

The paper is organized as follows. In Section 2 we introduce the functional setting we will work in and we recall some basic facts on the spectral theory of the operator $\mathcal{L}_K$. In Section 3 we prove Theorem 1.1 performing the classical Saddle Point Theorem.

2. The functional analytic setting and an eigenvalue problem

At first, we recall some preliminary results on the functional space Z, introduced on page 529.

LEMMA 2.1. *Let* $K\colon \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (1.5) *and* (1.6)*. Then, the following assertions hold true*:

(a) *Z is continuously embedded in* $W_0^{s,2}(\Omega)$ (*for a detailed description see* [3]) *which is the closure of* $C_0^\infty(\Omega)$ *in the space* $W^{s,2}(\Omega)$ *of functions u defined on Ω for which is well defined the so-called* Gagliardo norm

$$\|u\|_{W^{s,2}(\Omega)} = \left(\int_\Omega |u(x)|^2 \, dx + \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}.$$

(b) *Z is compactly embedded in $L^p(\Omega)$ for any $p \in [1, 2^*)$, where the fractional critical Sobolev exponent is defined as*

$$2^* := \begin{cases} \dfrac{2n}{n-2s} & \text{if } n > 2s, \\ +\infty & \text{if } n \leq 2s. \end{cases}$$

(c) *Z is a Hilbert space endowed with the following norm*

$$\|v\|_Z = \left(\int_Q |v(x) - v(y)|^2 K(x-y)dx\,dy\right)^{1/2}, \tag{2.1}$$

which is equivalent to the usual one defined in (1.8).

Proof. For part (a) we simply observe that by (1.6) we get

$$\theta \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le \int_Q |u(x) - u(y)|^2 K(x-y)\,dx\,dy \tag{2.2}$$

and so

$$\|u\|_{W^{s,2}(\Omega)} \le c(\theta)\|u\|_X,$$

with $c(\theta) = \max\{1, \theta^{-1/2}\}$.

Now, we prove part (b). Let Ω' be a regular, open subset of $\mathbb{R}^n$ such that $\Omega \subseteq \Omega'$. For any $u \in W_0^{s,2}(\Omega)$ we can define

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega' \setminus \Omega. \end{cases}$$

It is clear that $\widetilde{u} \in W_0^{s,2}(\Omega')$. Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in $C_0^\infty(\Omega)$ which converges to u in $W_0^{s,2}(\Omega)$ then $\{\widetilde{u}_j\}_{j\in\mathbb{N}}$ is a sequence in $C_0^\infty(\Omega')$ which converges to $\widetilde{u}$ in $W_0^{s,2}(\Omega')$. Moreover, we also have

$$\|\widetilde{u}\|_{W^{s,2}(\Omega')} = \|u\|_{W^{s,2}(\Omega)}.$$

Thus, $W_0^{s,2}(\Omega')$ is isometric embedded in $W_0^{s,2}(\Omega)$. The conclusion follows by remembering that $W_0^{s,2}(\Omega')$ is compactly embedded in $L^p(\Omega')$ with $1 \le p < 2^*$.

For the assertion (c) we claim that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C\left(\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} \tag{2.3}$$

for any $u \in W_0^{s,2}(\Omega)$. In fact, since Ω is bounded there is $R > 0$ such that $\Omega \subseteq B_R$ and $|B_R \setminus \Omega| > 0$. So, we get

$$\begin{aligned} &\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad \ge \int_{\mathcal{C}\Omega}\left(\int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dy\right)dx = \int_{\mathcal{C}\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s}}\,dy\right)dx \\ &\quad \ge \int_{B_R\setminus\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|2R|^{n+2s}}\,dy\right)dx = \frac{|B_R \setminus \Omega|}{(2R)^{n+2s}}\|u\|^2_{L^2(\Omega)} \end{aligned}$$

for any $u \in W_0^{s,2}(\Omega)$ (since $u = 0$ in $\mathbb{R}^n \setminus \Omega$), which proves our claim. Finally, by combining (2.2) and (2.3) we conclude the proof. $\square$

From now on, we take (2.1) as norm on Z. Now, we study some properties of eigenvalues and eigenfunctions of the non-local operator $-\mathcal{L}_K$ (for a more general and detailed study see [10]).

Proposition 2.2. *Let $K\colon \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (1.5) *and* (1.6). *Then, there exists an orthogonal complete basis of eigenvectors e_j $(j = 1, 2, \ldots)$ in Z normalized in $L^2(\Omega)$, by the quadratic form $\|\cdot\|^2_{L^2(\Omega)}$. The corrisponding eigenvalues λ_j^{-1} verify $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ and $\sup_{j\in\mathbb{N}^*} \lambda_j = +\infty$. Moreover, for any $k \in \mathbb{N}^*$, it follows that*

$$\int_Q (u(x) - u(y))^2 K(x-y)\, dx\, dy \le \lambda_k \|u\|^2_{L^2(\Omega)} \tag{2.4}$$

for any $u \in \operatorname{span}(e_1, \ldots e_k)$,

$$\int_Q (u(x) - u(y))^2 K(x-y)\, dx\, dy \ge \lambda_k \|u\|^2_{L^2(\Omega)} \quad \text{for any } u \in \mathbb{P}_k, \tag{2.5}$$

where $\mathbb{P}_k := \{u \in Z : \langle u, e_j\rangle_Z = 0$ for any $j = 1, \ldots, k-1\}$ $(\mathbb{P}_1 := Z)$.

Proof. The proof follows by the general theory of functional analysis and by the compact embedding of Z in $L^2(\Omega)$, proved in Lemma 2.1. Moreover, the fact that the eigenvalue λ_1 is simple is proved in [10, Proposition 9]. □

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we observe that problem (1.9) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J}\colon Z \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\int_Q |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \int_\Omega F(x, u(x))\, dx,$$

where F is the primitive of f with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)\, d\tau.$$

Moreover, note that the functional $\mathcal{J}$ is Fréchet differentiable in $u \in Z$ and for any $\varphi \in Z$

$$\mathcal{J}'(u)(\varphi) = \int_Q (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dx\, dy - \int_\Omega f(x, u(x))\varphi(x)\, dx.$$

Thus, critical points of $\mathcal{J}$ are solutions to problem (1.9). In order to find these critical points, we will divide the proof in two cases. At first, when $\overline{\alpha}(x) < \lambda_1$ the existence of the solution of problem (1.9) follows from the Weierstrass Theorem (i.e. by direct minimization). When $\lambda_k < \underline{\alpha}(x) \le \overline{\alpha}(x) < \lambda_{k+1}$ for some $k \in \mathbb{N}^*$, we will make use of the Saddle Point Theorem (see [5]). For this, we have to check that the functional $\mathcal{J}$ has a particular geometric structure (as stated, e.g.

in conditions (I_3) and (I_4) of [5, Theorem 4.6]) and that it satisfies the Palais–Smale compactness condition (see, for instance, [5, p. 3]).

3.1. The case $\overline{\alpha}(x) < \lambda_1$. In this subsection, in order to apply the Weierstrass Theorem we first verify that the functional $\mathcal{J}$ satisfy the following geometric feature.

PROPOSITION 3.1. *Let K and f be two functions satisfying assumptions* (1.5)–(1.7). *Moreover, let $\overline{\alpha}(x) < \lambda_1$ almost everywhere in Ω. Then, the functional $\mathcal{J}$ verifies*

$$\liminf_{\|u\|_Z \to +\infty} \frac{\mathcal{J}(u)}{\|u\|_Z^2} > 0. \tag{3.1}$$

PROOF. It is enough to show that if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in Z such that $\|u_j\|_Z \to +\infty$, then

$$\limsup_{j\to+\infty} \int_\Omega \frac{F(x,u_j(x))}{\|u_j\|_Z^2}\,dx < \frac{1}{2}. \tag{3.2}$$

By Lemma 2.1, up to a subsequence, there exists $u_0 \in Z$ such that $u_j/\|u_j\|_Z$ converges to u_0 strongly in $L^2(\Omega)$ and almost everywhere in Ω, as well as weakly in Z. So, $\|u_0\|_Z \le 1$. Now, by (1.7) we observe that

$$\frac{|F(x,u_j)|}{\|u_j\|_Z^2} \le \frac{a(x)|u_j| + b|u_j|^2/2}{\|u_j\|_Z^2},$$

where the sequence on the right-hand side converges in $L^1(\Omega)$. Moreover, we claim that

$$\limsup_{j\to+\infty} \frac{F(x,u_j(x))}{\|u_j t\|_Z^2} \le \frac{\overline{\alpha}(x)}{2}|u_0(x)|^2 \tag{3.3}$$

which follows by previous formula when $x \in \Omega$ such that $u_0(x) = 0$. While, for x such that $u_0(x) \ne 0$, (3.3) follows from the fact that in this case $|u_j(x)|^2 \to +\infty$ and so for j sufficiently large we get

$$\frac{F(x,u_j(x))}{\|u_j\|_Z^2} = \frac{F(x,u_j(x))}{|u_j(x)|^2}\frac{|u_j(x)|^2}{\|u_j\|_Z^2},$$

and also by (1.7) and (1.10) we have

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{t^2} \le \frac{\overline{\alpha}(x)}{2}.$$

Thus, by the generalized Fatou Lemma, (2.5) and (3.3) it follows that

$$\begin{aligned}\limsup_{j\to+\infty} \int_\Omega \frac{F(x,u_j(x))}{\|u_j\|_Z^2}\,dx &\le \int_\Omega \frac{\overline{\alpha}(x)}{2}|u_0(x)|^2\,dx \\ &\le \frac{\lambda_1}{2}\int_\Omega |u_0(x)|^2\,dx \le \frac{\|u_0\|_Z^2}{2} \le \frac{1}{2}.\end{aligned}$$

The second of these last inequalities is strict if $u_0 \neq 0$, while the last one is strict if $u_0 = 0$. $\square$

Proof of Theorem 1.1 (when $\overline{\alpha}(x) < \lambda_1$). As is well known, the map $u \mapsto \|u\|_Z^2$ is lower semicontinuous in the weak topology of Z, while the map $u \mapsto \int_\Omega F(x,u)$ is continuous in the weak topology of Z, since (1.7) implies that

$$|F(x,t)| \leq a(x)|t| + b\frac{|t|^2}{2}.$$

So, the functional $\mathcal{J}$ is lower semicontinuous and by using also (3.1) to obtain coerciveness we can apply the Weierstrass Theorem in order to find a minimum of $\mathcal{J}$ on Z, which is clearly a solution of problem (1.9). $\square$

3.2. The case $\lambda_k < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{k+1}$. At first, we recall that, in what follows, e_k will be the k-th eigenfunction corresponding to the eigenvalue λ_k of $-\mathcal{L}_K$ for any $k \in \mathbb{N}^*$, and we set

$$\mathbb{P}_{k+1} := \{u \in Z : \langle u, e_j \rangle_Z = 0 \text{ for any } j = 1, \dots, k\}$$

as defined in Proposition 2.2, while $H_k := \operatorname{span}\{e_1, \dots, e_k\}$ for any $k \in \mathbb{N}^*$. It is immediate to observe that $\mathbb{P}_{k+1} = H_k^\perp$ with respect to the scalar product in Z and $Z = H_k \oplus \mathbb{P}_{k+1}$.

Now, we prove that the functional $\mathcal{J}$ has the geometric features required by the Saddle Point Theorem.

Proposition 3.2. *Let K and f be two functions satisfying assumptions (1.5)–(1.7). Moreover, assume there exists $k \in \mathbb{N}^*$ such that $\lambda_k < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{k+1}$ almost everywhere in Ω. Then, the functional $\mathcal{J}$ verifies*

$$\limsup_{u \in H_k,\, \|u\|_Z \to +\infty} \frac{\mathcal{J}(u)}{\|u\|_Z^2} < 0. \tag{3.4}$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in H_k such that $\|u_j\|_Z \to +\infty$. Since H_k is finite dimensional, there exists $u_0 \in H_k$ such that $u_j/\|u_j\|_Z$ converges to u_0 strongly in Z and also $\|u_0\|_Z = 1$.

Now, by proceeding as in the proof of claim (3.3), it follows that

$$\liminf_{j\to+\infty} \frac{F(x,u_j(x))}{\|u_j\|_Z^2} \geq \frac{\underline{\alpha}(x)}{2}|u_0(x)|^2,$$

almost everywhere in Ω. So, by using also the Fatou Lemma and the fact that $\underline{\alpha}(x) > \lambda_k$, we have

$$\limsup_{j\to+\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_Z^2} \leq \frac{1}{2} - \int_\Omega \frac{\underline{\alpha}(x)}{2}|u_0(x)|^2\,dx < \frac{1}{2} - \frac{\lambda_k}{2}\int_\Omega |u_0(x)|^2\,dx.$$

By the last inequality, (2.4) and the fact that $\|u_0\|_Z = 1$, we get (3.4). $\square$

Also, Proposition 3.2 has the following counterpart.

Proposition 3.3. *Let K and f be two functions satisfying assumptions* (1.5)–(1.7). *Moreover, assume there exists $k \in \mathbb{N}^*$ such that $\lambda_k < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{k+1}$ almost everywherein Ω. Then, the functional $\mathcal{J}$ verifies*

$$\liminf_{u\in\mathbb{P}_{k+1},\, \|u\|_Z\to+\infty} \frac{\mathcal{J}(u)}{\|u\|_Z^2} > 0. \tag{3.5}$$

Proof. The proof is similar to the proof of Proposition 3.1. In this case we have $\overline{\alpha}(x) < \lambda_{k+1}$, for some $k \in \mathbb{N}^*$, instead of $\overline{\alpha}(x) < \lambda_1$. □

In order to prove the boundedness of a Palais–Smale sequence, we first introduce the following lemma.

Lemma 3.4. *Let K be a function satisfying* (1.5) *and* (1.6). *Moreover, assume there exist $k \in \mathbb{N}^*$ and a measurable function m on Ω such that $\lambda_k < m(x) < \lambda_{k+1}$ for almost every $x \in \Omega$. If $u_0 \in Z$ satisfies*

$$\langle u_0, \varphi\rangle_Z - \int_\Omega m(x)u_0(x)\varphi(x)\,dx = 0 \quad \textit{for any } \varphi \in Z, \tag{3.6}$$

then $u_0 = 0$.

Proof. We can write $u_0 = u_1 + u_2$, where $u_1 \in H_k$ and $u_2 \in \mathbb{P}_{k+1}$. By (3.6) we obtain

$$\begin{aligned}
\|u_1\|_Z^2 &= \int_\Omega m(x)(|u_1(x)|^2 + u_2(x)u_1(x))\,dx \\
&\geq \int_\Omega (\lambda_k|u_1(x)|^2 + m(x)u_2(x)u_1(x))\,dx, \\
\|u_2\|_Z^2 &= \int_\Omega m(x)(u_1(x)u_2(x) + |u_2(x)|^2)\,dx \\
&\leq \int_\Omega (m(x)u_2(x)u_1(x) + \lambda_{k+1}|u_2(x)|^2)\,dx.
\end{aligned}$$

If $u_0 \neq 0$, then at least one of the above inequalities is strict and so, by using also (2.4) and (2.5), it follows that

$$\|u_1\|_Z^2 - \|u_2\|_Z^2 > \int_\Omega (\lambda_k|u_1(x)|^2 - \lambda_{k+1}|u_2(x)|^2)\,dx \geq \|u_1\|_Z^2 - \|u_2\|_Z^2$$

which is a contradiction. □

Proposition 3.5. *Let K and f be two functions satisfying assumptions* (1.5)–(1.7). *Moreover, assume there exists $k \in \mathbb{N}^*$ such that $\lambda_k < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{k+1}$ almost everywhere in Ω. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z such that $\{\mathcal{J}'(u_j)\}_{j\in\mathbb{N}}$ is bounded. Then, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.*

Proof. *Step 1.* We argue by contradiction and suppose that $\{u_j\}_{j\in\mathbb{N}}$ is unbounded. By Lemma 2.1, up to a subsequence, there exists $u_0 \in Z$ such that $u_j/\|u_j\|_Z$ converges to u_0 strongly in $L^2(\Omega)$ and almost everywhere in Ω, as well as weakly in Z.

By our assumption on $\{\mathcal{J}'(u_j)\}_{j\in\mathbb{N}}$ there exists a costant $c>0$ such that

$$\frac{|\mathcal{J}'(u_j)(\varphi)|}{\|u_j\|_Z} = \left|\left\langle \frac{u_j}{\|u_j\|_Z}, \varphi\right\rangle_Z - \int_\Omega \frac{f(x,u_j(x))}{\|u_j\|_Z}\varphi(x)\,dx\right| \le \frac{c\|\varphi\|_Z}{\|u_j\|_Z} \tag{3.7}$$

for any $\varphi\in Z$ and $j\in\mathbb{N}$. By (1.7) we get

$$\frac{|f(x,u_j)|}{\|u_j\|_Z} \le \frac{a(x)}{\|u_j\|_Z} + b\frac{|u_j|}{\|u_j\|_Z},$$

where the sequence on the right-hand side is bounded in $L^2(\Omega)$. So, there exists $\beta\in L^2(\Omega)$ such that, up to a subsequence, $f(x,u_j)/\|u_j\|_Z$ converges weakly to β in $L^2(\Omega)$.

Now, we claim that

$$\begin{gathered}\beta(x) = m(x)u_0(x) \text{ with } m \text{ measurable}\\ \text{and such that } \underline{\alpha}(x)\le m(x)\le\overline{\alpha}(x) \text{ a.e. in } \Omega.\end{gathered} \tag{3.8}$$

As is well known

$$\liminf_{j\to+\infty}\frac{f(x,u_j(x))}{\|u_j\|_Z} \le \beta(x) \le \limsup_{j\to+\infty}\frac{f(x,u_j(x))}{\|u_j\|_Z} \quad \text{a.e. in } \Omega.$$

Moreover, if $x\in\Omega$ such that $u_0(x)\neq 0$, then for j sufficiently large

$$\frac{f(x,u_j(x))}{\|u_j\|_Z} = \frac{f(x,u_j(x))}{u_j(x)}\,\frac{u_j(x)}{\|u_j\|_Z}.$$

So, if $u_0(x)\ge 0$ we get $\underline{\alpha}(x)u_0(x)\le\beta(x)\le\overline{\alpha}(x)u_0(x)$, while if $u_0(x)<0$ the reversed inequalities hold true. This establishes (3.8), because when $x\in\Omega$ such that $u_0(x)=0$, we can set $m(x)=(\underline{\alpha}(x)+\overline{\alpha}(x))/2$. Thus, by sending $j\to+\infty$ in (3.7) and by using (3.8) we get

$$\langle u_0,\varphi\rangle_Z - \int_\Omega m(x)u_0(x)\varphi(x)\,dx = 0 \quad \text{for any } \varphi\in Z.$$

Thanks to this last formula and the fact that $\lambda_k<m(x)<\lambda_{k+1}$ we can use Lemma 3.4 by obtaining $u_0=0$.

Step 2. On the other hand, by using (3.7) with $\varphi=u_j/\|u_j\|_Z$ we get

$$\left|1-\int_\Omega \frac{f(x,u_j(x))}{\|u_j\|_Z}\,\frac{u_j(x)}{\|u_j\|_Z}\,dx\right| \le \frac{c}{\|u_j\|_Z},$$

for any $j\in\mathbb{N}$. But, since $f(x,u_j)/\|u_j\|_Z$ is bounded in $L^2(\Omega)$, $u_j/\|u_j\|_Z$ converges to 0 in $L^2(\Omega)$ and $c/\|u_j\|_Z$ goes to 0, we get a contradiction. □

PROOF OF THEOREM 1.1 (when $\lambda_k<\underline{\alpha}(x)\le\overline{\alpha}(x)<\lambda_{k+1}$). At first, we prove that $\mathcal{J}$ satisfies the geometric structure required by the Saddle Point Theorem. By Proposition 3.3 it follows that for any $M>0$ there exists $R>0$

such that if $u \in \mathbb{P}_{k+1}$ and $\|u\|_Z \geq R$, then $\mathcal{J}(u) \geq M$. If $u \in \mathbb{P}_{k+1}$ with $\|u\|_Z \leq R$, by applying (1.7), (2.5), and Hölder inequality we have

$$\begin{aligned}\mathcal{J}(u) &\geq -\int_\Omega F(x,u(x))\,dx \geq -\int_\Omega a(x)|u(x)|\,dx - \frac{b}{2}\int_\Omega |u(x)|^2\,dx \\ &\geq -\|a\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} - \frac{b}{2}\lambda_{k+1}^{-1}\|u\|_Z^2 \geq -C_R\end{aligned}$$

for some constant $C_R = C(R,\Omega) > 0$. So, we get

$$\mathcal{J}(u) \geq -C_R \quad \text{for any } u \in \mathbb{P}_{k+1}. \tag{3.9}$$

By Proposition 3.2 we can choose $T > 0$ in such way that, for any $u \in H_k$ with $\|u\|_Z = T$, we have

$$\sup_{u\in H_k,\,\|u\|_Z=T} \mathcal{J}(u) < -C_R \leq \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u). \tag{3.10}$$

We have thus proved that $\mathcal{J}$ has the geometric structure of the Saddle Point Theorem (see [5, Theorem 4.6]). Now, it remains to check the validity of the Palais–Smale condition. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z such that

$$\mathcal{J}(u_j) \to c, \tag{3.11}$$

$$\sup\{|\mathcal{J}'(u_j)(\varphi)| : \varphi \in Z,\ \|\varphi\|_Z = 1\} \to 0 \quad \text{for any } \varphi \in Z, \tag{3.12}$$

as $j \to +\infty$. By Proposition 3.5 $\{u_j\}_{j\in\mathbb{N}}$ is bounded, so by Lemma 2.1, up to a subsequence, there exists $u \in Z$ such that u_j converges to u strongly in $L^2(\Omega)$ and almost everywhere in Ω, as well as weakly in Z. Since, for any $\varphi \in Z$,

$$\mathcal{J}'(u_j)(\varphi) = \langle u_j, \varphi\rangle_Z - \int_\Omega f(x,u_j(x))\varphi(x)\,dx,$$

by using also (1.7) and (3.12) it follows that

$$0 = \|u\|_Z^2 - \int_\Omega f(x,u(x))u(x)\,dx \tag{3.13}$$

by taking $\varphi = u$, and also

$$\|u_j\|_Z^2 = \mathcal{J}'(u_j)(u_j) + \int_\Omega f(x,u_j(x))u_j(x)\,dx \to \int_\Omega f(x,u(x))u(x)\,dx \tag{3.14}$$

by taking $\varphi = u_j$ and sending $j \to +\infty$. Indeed, for the last formula we observe that $|f(x,u_j)u_j| \leq a(x)|u_j| + b|u_j|^2$, where the sequence on the right-hand side converges in $L^1(\Omega)$. Thus, by combining (3.13) and (3.14) we get $\|u_j\|_Z \to \|u\|_Z$ and so $\{u_j\}_{j\in\mathbb{N}}$ converges strongly to u in Z. □

Proof of Corollary 1.3. Let u_1, $u_2 \in Z$ be two solutions of problem (1.9). Then, we set $w := u_1 - u_2$ and

$$m(x) := \begin{cases} \dfrac{f(x,u_1(x)) - f(x,u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\ \dfrac{1}{2}(\lambda_k + \lambda_{k+1}) & \text{if } u_1(x) = u_2(x). \end{cases}$$

So, m is a measurable function which verifies $\lambda_k < m(x) < \lambda_{k+1}$ almost everywhere in Ω thanks to (1.11). Moreover, (1.9) implies that

$$\langle w, \varphi\rangle_Z - \int_\Omega m(x)w(x)\varphi(x)\,dx = 0 \quad \text{for any } \varphi \in Z.$$

Thus, by Lemma 3.4 it follows that $w = 0$. □

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