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# STRAUSS AND LIONS TYPE RESULTS FOR A CLASS OF ORLICZ–SOBOLEV SPACES AND APPLICATIONS

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Abstract. The main goal of this work is to prove Strauss- and Lions-type results for Orlicz–Sobolev spaces. After, we use these results to study the existence of solutions for a class of quasilinear problems in $\mathbb{R}^N$.

## 1. Introduction

In recent years, a special attention has been given for quasilinear problems of the type

$$\text{(P)} \qquad \begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) + V(x)a(|u|)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^1L_A(\mathbb{R}^N) & \text{with } N \geq 2, \end{cases}$$

where $V$, $f$ are continuous functions satisfying some technical conditions and $a\colon [0,+\infty) \to [0,+\infty)$ is a $C^1$-function.

We cite the papers of Bonanno, Bisci and Radulescu [5], [6], Cerny [7], Clément, Garcia-Huidobro and Manásevich [8], Donaldson [11], Fuchs and Li [14], Fuchs and Osmolovski [15], Fukagai, Ito and Narukawa [16], [17], Gossez [18],

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Le and Schmitt [19], Mihailescu and Radulescu [21], [22], Mihailescu and Repovs [23], Mihailescu, Radulescu and Repovs [24], Orlicz [27], Santos [28] and references therein, where quasilinear problems like (P) have been considered in bounded and unbounded domains of $\mathbb{R}^N$. In some those papers, the authors have mentioned that this class of problem arises in a lot of applications, such as, nonlinear elasticity, plasticity and non-Newtonian fluids.

One of the most famous methods to get a solution for (P) is the variational method, where the weak solutions for (P) are precisely the critical points of the energy functional $J\colon X \to \mathbb{R}$ associated with (P), given by

$$J(u) = \int_{\mathbb{R}^N} A(|\nabla u|) + \int_{\mathbb{R}^N} V(x)A(|u|) - \int_{\mathbb{R}^N} F(u),$$

where $X$ is a convenient subspace of $W^1 L_A(\mathbb{R}^N)$, which depends of the hypotheses on the potential $V$.

In [16], Fukagai, Ito and Narukawa have used the variational method to show the existence of a solution for (P) by assuming that the function $a$ satisfies the following assumptions:

- The function $a(t)t$ is increasing in $(0,+\infty)$, that is,

$$(a_1) \qquad (a(t)t)' > 0 \quad \text{for all } t > 0.$$

- There exist $l, m \in (1, N)$ such that

$$(a_2) \qquad l \le \frac{a(|t|)t^2}{A(t)} \le m \quad \text{for all } t \neq 0,$$

  where

$$A(t) = \int_0^{|t|} a(s)s\,ds, \quad l \le m < l^*, \quad l^* = \frac{lN}{N-l} \quad \text{and} \quad m^* = \frac{mN}{N-m}.$$

Using these hypotheses, the authors showed that $A$ is a $N$-function satisfying the $\Delta_2$-condition. Moreover, in that paper, it is mentioned some examples of functions $A$, whose function $a(t)$ satisfies the conditions $(a_1)$–$(a_2)$. The examples are the following:

(i) $A(t) = |t|^p$ for $1 < p < N$.
(ii) $A(t) = |t|^p + |t|^q$ for $1 < p < q < N$ and $q \in (p, p^*)$ with $p^* = Np/(N-p)$.
(iii) $A(t) = (1+|t|^2)^\gamma - 1$ for $\gamma \in (1, N/(N-2))$.
(iv) $A(t) = |t|^p ln(1+|t|)$ for $1<p_0<p<N-1$ with $p_0 = (-1+\sqrt{1+4N})/2$.

Motivated by [16], more precisely, by hypotheses $(a_1)$–$(a_2)$ considered on function $a$, the main goal of the present paper is to prove that some results found in Strauss [29] and Lions [20] also hold in the Orlicz–Sobolev $W^1 L_A(\mathbb{R}^N)$ for $A(t) = \int_0^{|t|} a(s)s\,ds$, when the above conditions are assumed on $a$. Moreover, results of compactness have been proved for domains in $\mathbb{R}^N$, which are invariant by group $O(N)$.

It is well known in the literature, that if the energy functional is invariant by rotations, sometimes it is possible to find radial solutions for (P). In this case, Strauss-type results can be an interesting tool. Once that we did not find in the literature a Strauss-type result for Orlicz–Sobolev spaces, the first result of this article goes in this direction and it has the following statement:

THEOREM 1.1 (A Strauss-type result for Orlicz–Sobolev spaces). *Assume that* ($a_1$)–($a_2$) *hold and let* $v \in W^1L_A(\mathbb{R}^N)$ *be a radial function. Then*

$$|v(x)| \leq A^{-1}\bigg(\frac{C}{|x|^{N-1}}\int_{\mathbb{R}^N}[A(|v|)+A(|\nabla v|)]\bigg) \quad \text{a.e. in } \mathbb{R}^N,$$

*where* $A^{-1}$ *denotes the inverse function of* $A$ *restricted to* $[0,+\infty)$ *and* $C$ *is a positive constant independent of* $v$.

In the next result, we denote by $W^1L_{A,rad}(\mathbb{R}^N)$ the subspace of $W^1L_A(\mathbb{R}^N)$ consisting of radial functions and by $A_*$ the conjugate function of $A$.

THEOREM 1.2 (A compactness result for radial functions). *Assume that* ($a_1$)–($a_2$) *hold and let* $B$ *be a* $N$*-function verifying*

$$\lim_{t\to 0^+}\frac{B(t)}{A(t)} = 0 \tag{$B_1$}$$

*and*

$$\lim_{t\to +\infty}\frac{B(t)}{A_*(t)} = 0. \tag{$B_2$}$$

*Then, the embedding* $W^1L_{A,rad}(\mathbb{R}^N) \hookrightarrow L_B(\mathbb{R}^N)$ *is compact.*

The above theorem can be applied, when we intend to prove that some functional satisfies, for example, the well known Palais–Smale condition on the space of the radial functions.

In the proof of Theorems 1.1 and 1.2 the reader is invited to observe that they are true assuming that $A(t)=\int_0^{|t|}a(s)s\,ds$ is a $N$-function verifying the $\Delta_2$ condition. Here, we have used conditions ($a_1$)–($a_2$) in view of our applications, see Theorem 1.8 below.

Other important results are of Lions-type, however we did not find again results of this type for Orlicz–Sobolev spaces. Motivated by this fact, we prove also the following result

THEOREM 1.3 (A Lions-type result for Orlicz–Sobolev spaces). *Assume that* ($a_1$)–($a_2$) *hold and let* $(u_n) \subset W^1L_A(\mathbb{R}^N)$ *be a bounded sequence such that there exists* $R>0$ *satisfying*:

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}A(|u_n|)=0.$$

*Then, for any $N$-function $B$ verifying $\Delta_2$-condition with*

$$\text{(B}_1\text{)} \qquad \lim_{t\to 0}\frac{B(t)}{A(t)}=0$$

*and*

$$\text{(B}_2\text{)} \qquad \lim_{|t|\to +\infty}\frac{B(t)}{A_*(t)}=0,$$

*we have $u_n \to 0$ in $L_B(\mathbb{R}^N)$.*

Theorem 1.3 is interesting because it can be used to prove the existence of critical points for the energy functional $J$, when the potential $V$ is $\mathbb{Z}^N$-periodic.

Our next result can also be used to show compactness results for the space $W_0^1L_A(\Omega)$, when $\Omega \subset \mathbb{R}^N$ is invariant with respect to action of a subgroup of $O(N)$. Before to state it, we need to fix some definitions and notations. To this end, we follow the spirit of Willem's book [30].

Definition 1.4. Let $G$ be a subgroup of $O(N)$, $y \in \mathbb{R}^N$ and $r > 0$. We define,

$$m(y,r,G)=\sup\{n\in\mathbb{N}:\exists g_1,\ldots,g_n\in G: j\neq k\Rightarrow B_r(g_jy)\cap B_r(g_ky)=\emptyset\}.$$

An open set $\Omega \subset \mathbb{R}^N$ is said invariant when $g\Omega = \Omega$ for all $g \in G$. An invariant subset $\Omega \subset \mathbb{R}^N$ is compatible with $G$ if, for some $r > 0$,

$$\lim_{\substack{|y|\to+\infty\\ \operatorname{dist}(y,\Omega)\leq r}} m(y,r,G)=+\infty.$$

Definition 1.5. Let $G$ be a subgroup of $O(N)$ and let $\Omega \subset \mathbb{R}^N$ be an invariant set. The action of $G$ on $W_0^1L_A(\Omega)$ is defined by

$$gu(x)=u(g^{-1}x) \quad \text{for all } x\in\mathbb{R}^N.$$

The subspace of invariant functions is defined by

$$W_{0,G}^1L_A(\Omega)=\{u\in W_0^1L_A(\Omega): gu=u,\ \text{for all } g\in G\}.$$

Theorem 1.6 (A Compactness result involving the group $O(N)$). *If $\Omega$ is compatible with $G$ and* (a$_1$)–(a$_2$) *hold, the embedding $W_{0,G}^1L_A(\Omega)\hookrightarrow L_B(\Omega)$ is compact, for any $N$-function $B$ verifying $\Delta_2$-condition with*

$$\text{(B}_1\text{)} \qquad \lim_{t\to 0}\frac{B(t)}{A(t)}=0$$

*and*

$$\text{(B}_2\text{)} \qquad \lim_{|t|\to +\infty}\frac{B(t)}{A_*(t)}=0.$$

As an immediate consequence of the last result, we have the following corollary:

Corollary 1.7. *Let* $N_j \geq 2$, $j = 1, \dots, k$, $\sum_{j=1}^{k} N_j = N$ *and*

$$G = O(N_1) \times \dots \times O(N_k).$$

*Then, the compact embeddings of Theorem* 1.6 *occur with* $\Omega = \mathbb{R}^N$.

Related to the Theorems 1.2, 1.3, 1.6 and Corollary 1.7, we would like to cite the paper due to Fan, Zhao and Zhao [13], where results like above has been established for the space $W^{1,p(x)}(\mathbb{R}^N)$.

Motivated by the above results, we study the existence of solutions for some classes of quasilinear problems assuming that $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying

$$0 < V_0 = \inf_{x \in \mathbb{R}^N} V(x) \tag{V$_1$}$$

and $f\colon \mathbb{R} \to \mathbb{R}$ is a $C^1$-function satisfying the properties:

$$\lim_{|t| \to 0} \frac{f(t)}{a(|t|)|t|} = 0, \tag{f$_1$}$$

$$\lim_{|t| \to +\infty} \frac{f(t)}{a_*(|t|)|t|} = 0, \tag{f$_2$}$$

where $a_*(t)t$ is such that the Sobolev conjugate function $A_*$ of $A$ (see Section 2) is its primitive, that is, $A_*(t) = \int_0^{|t|} a_*(s)s\,ds$.

There exists $\theta > m$ such that

$$0 < \theta F(t) = \int_0^t f(s)ds \leq t f(t) \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{f$_3$}$$

Our main result concerning the existence of a solution for problem (P) is the following:

Theorem 1.8. *Suppose that* ($f_1$)–($f_3$), ($a_1$)–($a_2$) *and* ($V_1$) *hold. Moreover, assume that one of the following conditions hold*:

(a) $V$ *is a radial function, that is,* $V(x) = V(|x|)$, *for all* $x \in \mathbb{R}^N$,

*or*

(b) $V$ *is a* $\mathbb{Z}^N$ *periodic function, that is,* $V(x + y) = V(x)$, *for all* $x \in \mathbb{R}^N$ *and for all* $y \in \mathbb{Z}^N$.

*Then, problem* (P) *has a nontrivial solution.*

The plan of this paper is as follows. In Section 2, we review some proprieties of Orlicz and Orlicz–Sobolev spaces. In Section 3, we prove Theorems 1.1–1.3 and 1.6. In Section 4, we given a proof of Theorem 1.8.

## 2. A brief review about $N$-function and Orlicz–Sobolev spaces

In this section, we recall some properties of Orlicz and Orlicz–Sobolev spaces. The reader can find more properties of these spaces in the books of Adams and Fournier [1], Adams and Hedberg [2], Donaldson and Trundiger [12], Fuchs and Osmolovski [15], Musielak [25] and O'Neill [26].

First of all, we recall that a continuous function $\Phi\colon \mathbb{R} \to [0,+\infty)$ is a $N$-function if:

(a) $\Phi$ is convex.
(b) $\Phi(t) = 0$ if and only if $t = 0$.
(c) $\Phi(t)/t \xrightarrow{t\to 0} 0$ and $\Phi(t)/t \xrightarrow{t\to +\infty} +\infty$.
(d) $\Phi$ is even.

In what follows, we say that a $N$-function $\Phi$ verifies the $\Delta_2$-condition if

$$\Phi(2t) \le K\Phi(t) \quad \text{for all } t \ge 0,$$

for some constant $K > 0$. This condition can be rewritten in the following way: For each $s > 0$, there exists $M_s > 0$ such that

$$(\Delta_2) \qquad \Phi(st) \le M_s\Phi(t) \quad \text{for all } t \ge 0.$$

Fixed an open set $\Omega \subset \mathbb{R}^N$ and a N-function $\Phi$, the Orlicz space $L_\Phi(\Omega)$ is defined. When $\Phi$ satisfies $\Delta_2$-condition, the space $L_\Phi(\Omega)$ is the vectorial space of the measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi(|u|) < \infty.$$

The space $L_\Phi(\Omega)$ endowed with Luxemburg norm, that is, with the norm given by

$$\|u\|_\Phi = \inf\bigg\{\alpha > 0 : \int_\Omega \Phi\bigg(\frac{|u|}{\alpha}\bigg) \le 1\bigg\},$$

is a Banach space. The complement function of $\Phi$, denoted by $\widetilde{\Phi}(s)$, is given by the Legendre transformation, that is $\widetilde{\Phi}(s) = \max_{t\ge 0}\{st - \Phi(t)\}$ for $s \ge 0$. The functions $\Phi$ and $\widetilde{\Phi}$ are complementary each other. Moreover, we have the Young's inequality given by

$$st \le \Phi(t) + \widetilde{\Phi}(s) \quad \text{for all } t, s \ge 0. \tag{2.1}$$

Using the above inequality, it is possible to prove a Hölder type inequality, that is,

$$\bigg|\int_\Omega uv\bigg| \le 2\|u\|_\Phi\|v\|_{\widetilde{\Phi}}, \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\widetilde{\Phi}}(\Omega). \tag{2.2}$$

Another important function related to function $\Phi$, it is the Sobolev conjugate function $\Phi_*$ of $\Phi$ defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\,ds \quad \text{for } t > 0,$$

when

$$\int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds = +\infty.$$

If $\Phi(t) = |t|^p$ for $1 < p < N$, we have $\Phi_*(t) = p^{*p^*}|t|^{p^*}$, where $p^* = pN/(N-p)$.

The next lemma will be used in the proof of some results and its proof can be found in [18]

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\Phi\colon \mathbb{R} \to [0,\infty)$ be a $N$-function satisfying the $\Delta_2$-condition. If also the complementary function $\widetilde{\Phi}$ satisfies the $\Delta_2$-condition and $(\mathrm{f}_n)$ is a bounded sequence in $L_\Phi(\Omega)$ satisfying $f_n(x) \to f(x)$ almost everywhere in $\Omega$, then $f_n \rightharpoonup f$ in $L_\Phi(\Omega)$, that is,*

$$\int_\Omega f_n v\, dx \to \int_\Omega f v\, dx \quad \textit{for all } v \in L_{\widetilde{\Phi}}(\Omega).$$

Hereafter, we denote by $W_0^1 L_\Phi(\Omega)$ the Orlicz–Sobolev space obtained by the completion of $C_0^\infty(\Omega)$ with the norm $\|u\| = \|\nabla u\|_\Phi + \|u\|_\Phi$. When $\Omega = \mathbb{R}^N$, we use the symbol $W^1 L_\Phi(\mathbb{R}^N)$ to denote the space $W_0^1 L_\Phi(\mathbb{R}^N)$.

An important property that we must detach is: If $\Phi$ and $\widetilde{\Phi}$ satisfy $\Delta_2$-condition, the spaces $L_\Phi(\Omega)$ and $W^1 L_\Phi(\mathbb{R}^N)$ are reflexive and separable. Moreover, the $\Delta_2$-condition also implies that

$$u_n \to u \text{ in } L_\Phi(\Omega) \Leftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0, \tag{2.3}$$

$$u_n \to u \text{ in } W^1 L_\Phi(\Omega \Leftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0 \text{ and } \int_\Omega \Phi(|\nabla u_n - \nabla u|) \to 0. \tag{2.4}$$

In the literature, we find some important embeddings involving the Orlicz–Sobolev spaces, for example, it is possible to prove that embedding $W^1 L_\Phi(\mathbb{R}^N) \hookrightarrow L_B(\mathbb{R}^N)$ is continuous, if $B$ is a $N$-function satisfying

$$\limsup_{t\to 0} \frac{B(t)}{\Phi(t)} < +\infty \quad \text{and} \quad \limsup_{|t|\to+\infty} \frac{B(t)}{\Phi_*(t)} < +\infty.$$

When the space $\mathbb{R}^N$ is replaced by a bounded domain $D$ and the limits below hold

$$\limsup_{t\to 0} \frac{B(t)}{\Phi(t)} < +\infty \quad \text{and} \quad \limsup_{|t|\to+\infty} \frac{B(t)}{\Phi_*(t)} = 0, \tag{2.5}$$

the embedding

$$W^1 L_\Phi(D) \hookrightarrow L_B(D) \tag{2.6}$$

is compact.

The next four lemmas involve the functions $A$, $\widetilde{A}$ and $A_*$ and theirs proofs can be found in [16]. Hereafter, $A$ is the $N$-function given in the introduction and $\widetilde{A}$, $A_*$ are the complement and conjugate functions of $A$, respectively.

Lemma 2.2. *The functions $A$ and $\widetilde{A}$ satisfy the inequality*

$$\widetilde{A}(a(|t|)t) \leq A(2t) \quad \text{for all } t \geq 0. \tag{2.7}$$

Lemma 2.3. *Assume that* (a$_1$)–(a$_2$) *hold and let* $\xi_0(t) = \min\{t^l, t^m\}$, $\xi_1(t) = \max\{t^l, t^m\}$, *for all* $t \geq 0$. *Then*

$$\xi_0(\rho)A(t) \leq A(\rho t) \leq \xi_1(\rho)A(t) \qquad \text{for } \rho, t \geq 0, \tag{2.8}$$

$$\xi_0(\|u\|_A) \leq \int_{\mathbb{R}^N} A(|u|) \leq \xi_1(\|u\|_A) \quad \text{for } u \in L_A(\mathbb{R}^N). \tag{2.9}$$

Lemma 2.4. *The function $A_*$ satisfies the following inequality*

$$l^* \leq \frac{a_*(|t|)t^2}{A_*(t)} \leq m^* \quad \text{for } t \neq 0.$$

As an immediate consequence of the Lemma 2.4, we have the following result:

Lemma 2.5. *Assume that* (a$_1$)–(a$_2$) *hold and let* $\xi_2(t) = \min\{t^{l^*}, t^{m^*}\}$, $\xi_3(t) = \max\{t^{l^*}, t^{m^*}\}$, *for all* $t \geq 0$. *Then*

$$\xi_2(\rho)A_*(t) \leq A_*(\rho t) \leq \xi_3(\rho)A_*(t) \qquad \text{for } \rho, t \geq 0,$$

$$\xi_2(\|u\|_{A_*}) \leq \int_{\mathbb{R}^N} A_*(|u|) \leq \xi_3(\|u\|_{A_*}) \quad \text{for } u \in L_{A_*}(\mathbb{R}^N).$$

## 3. Strauss and Lions type results for Orlicz–Sobolev spaces

After the above brief review, we are able to prove our main results involving the Orlicz–Sobolev spaces.

Proof of Theorem 1.1 (Strauss' Theorem). First of all, we will establish the result for functions in $C_0^\infty(\mathbb{R}^N)$. After, by density, we establish the result for all radial functions in $W^1L_A(\mathbb{R}^N)$.

Consider $v \in C_0^\infty(\mathbb{R}^N)$, $|x| = r$ and $w(r) = v(x)$. Note that

$$A(w(b)) - A(w(r)) = \int_r^b \left(\frac{d}{ds}A(w)\right) ds \quad \text{for all } b > r > 0.$$

Since $w \in C_0^\infty([0,\infty))$, for $b$ large enough,

$$A(w(r)) = -\int_r^\infty a(|w|)ww'\, ds \leq \int_r^\infty a(|w|)|w||w'|\, ds.$$

Combining (2.1) with (2.7)

$$a(|w|)|w||w'| \leq \widetilde{A}(a(|w|)|w|) + A(|w'|) \leq A(2|w|) + A(|w'|),$$

then by $\Delta_2$-condition,

$$a(|w|)|w||w'| \leq KA(|w|) + A(|w'|).$$

Therefore,

$$A(w(r)) \leq (K+1)\int_r^\infty [A(|w(s)|) + A(|w'(s)|)]\, ds,$$

and we can conclude that

$$A(w(r)) \leq \frac{(K+1)}{r^{N-1}} \int_r^\infty [A(|w(s)|) + A(|w'(s)|)] s^{N-1}\, ds.$$

From this, there is $C > 0$ such that

$$A(v(x)) \leq \frac{C}{|x|^{N-1}} \int_{\mathbb{R}^N} [A(|v|) + A(|\nabla v|)].$$

Since $A$ is an even function, $A(v(x)) = A(|v(x)|)$ for all $x \in \mathbb{R}^N$, and so,

$$A(|v(x)|) \leq \frac{C}{|x|^{N-1}} \int_{\mathbb{R}^N} [A(|v|) + A(|\nabla v|)].$$

From this,

$$|v(x)| \leq A^{-1}\bigg( \frac{C}{|x|^{N-1}} \int_{\mathbb{R}^N} [A(|v|) + A(|\nabla v|)] \bigg) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

where $A^{-1}$ denotes the inverse function of $A$ restricted to $[0, +\infty)$. Now, the lemma follows from the density of $C_0^\infty(\mathbb{R}^N)$ in $W^1 L_A(\mathbb{R}^N)$. □

Now, we are able to prove the compactness result involving $W^1 L_{A,\mathrm{rad}}(\mathbb{R}^N)$.

Proof of Theorem 1.2 (Compactness Theorem ). Assume that $\{u_n\} \subset W^1 L_{A,\mathrm{rad}}(\mathbb{R}^N)$ is a sequence verifying $u_n \rightharpoonup 0$ in $W^1 L_{A,\mathrm{rad}}(\mathbb{R}^N)$. Without loss of generality, we can assume that $u_n \geq 0$ for all $n \in \mathbb{N}$. From $(\mathrm{B}_1)$–$(\mathrm{B}_2)$, for each $\varepsilon > 0$ and $q > 1$, there is $C > 0$ such that

$$B(t) \leq \varepsilon (A(t) + A_*(t)) + C|t|^q \quad \text{for all } t \geq 0. \tag{3.1}$$

Using Theorem 1.1, Lemma 2.3 and the boundedness of $\{u_n\}$ in $W^1 L_A(\mathbb{R}^N)$, for each $R > 0$, there is $C > 0$ such that

$$|u_n(x)|^q \leq C \bigg( \frac{1}{|x|^{(N-1)q/m}} + \frac{1}{|x|^{(N-1)g/l}} \bigg) \quad \text{in } [|x| \geq R] \text{ and for all } n \in \mathbb{N}.$$

Choosing $q$ large enough,

$$g(x) = C \bigg( \frac{1}{|x|^{(N-1)q/m}} + \frac{1}{|x|^{(N-1)q/l}} \bigg) \in L^1([|x| > \delta]) \quad \text{for all } \delta > 0.$$

The last inequality combined with Lebesgue's Theorem implies that

$$\int_{[|x| \geq R]} |u_n(x)|^q \to 0 \quad \text{as } n \to \infty.$$

This limit together with (3.1) leads to

$$\int_{[|x| \geq R]} B(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Observing that $(\mathrm{B}_1)$–$(\mathrm{B}_2)$ imply that (2.5)–(2.6) hold, one has that

$$W^1 L_A([|x| < R]) \hookrightarrow L_B([|x| < R])$$

is a compact embedding. Hence,

$$\int_{[|x|<R]} B(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

From (3.2) and (3.3),

$$\int_{\mathbb{R}^N} B(u_n) \to 0 \quad \text{as } n \to \infty,$$

and the proof of the theorem is complete. □

Proof of Theorem 1.3 (Lions' Theorem). First of all, we observe that

$$\int_{\mathbb{R}^N} B(|u_n|) = \int_{[|u_n|>k]} B(|u_n|) + \int_{[|u_n|\leq k]} B(|u_n|).$$

From ($B_2$), given $\varepsilon > 0$, there is $k > 0$ such that

$$B(t) = B(|t|) \leq \varepsilon A_*(|t|), \quad \text{if } |t| > k,$$

which yields

$$\int_{[|u_n|>k]} B(|u_n|) \leq \varepsilon \int_{[|u_n|>k]} A_*(|u_n|) \leq \varepsilon C,$$

and so,

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|u_n|) \leq \varepsilon C + \limsup_{n\to+\infty} \int_{[|u_n|\leq k]} B(|u_n|).$$

Claim 3.1.

$$\limsup_{n\to+\infty} \int_{[|u_n|\leq k]} B(|u_n|) = 0.$$

Using this claim,

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|u_n|) \leq \varepsilon C,$$

from where it follows that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|u_n|) = 0, \quad \text{then } u_n \xrightarrow{n\to+\infty} 0 \text{ in } L_B(\mathbb{R}^N).$$

Now, we will prove the Claim 3.1. Setting the function

$$v_n(x) = \chi_{[|u_n|\leq k]}(x) u_n(x),$$

it is sufficient to show that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|v_n|) = 0. \tag{3.4}$$

From ($\Delta_2$), there is $M_k > 0$ such that $A(|v_n/k|) \leq M_k A(|v_n|)$, for all $n \in \mathbb{N}$. This combined with Lemma 2.3 asserts

$$\int_{B_R(y)} A(|v_n|) \geq \frac{1}{M_k} \int_{B_R(y)} A\left(\left|\frac{v_n}{k}\right|\right) \geq C \int_{B_R(y)} \left|\frac{v_n}{k}\right|^m,$$

and so,

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \left|\frac{v_n}{k}\right|^m = 0.$$

Fixing $w_n = v_n/k$, ($|w_n|_\infty \leq 1$), we get

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |w_n|^m = 0. \tag{3.5}$$

Using again ($\Delta_2$), there is $\widehat{M}_k > 0$ such that

$$\int_{\mathbb{R}^N} B(|v_n|) = \int_{\mathbb{R}^N} B\bigg(k\frac{|v_n|}{k}\bigg) \leq \widehat{M}_k \int_{\mathbb{R}^N} B(|w_n|).$$

Consequently, the limit (3.4) follows if

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|w_n|) = 0.$$

Claim 3.2. *For all $\alpha > 1$ and $n \in \mathbb{N}$, $A(|w_n|^\alpha) \in W^{1,1}(\mathbb{R}^N)$.*

Indeed, since $|w_n|_\infty \leq 1$ and $w_n \in W^1L_A(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} A(|w_n|^\alpha) \leq \int_{\mathbb{R}^N} A(|w_n|) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^N} A(|\nabla w_n|) < +\infty. \tag{3.6}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla(A(|w_n|^\alpha))| &\leq \alpha \int_{\mathbb{R}^N} a(|w_n|^\alpha)|w_n|^\alpha |w_n|^{\alpha-1}|\nabla w_n| \\ &\leq \alpha \int_{\mathbb{R}^N} a(|w_n|^\alpha)|w_n|^\alpha|\nabla w_n|.\end{aligned}$$

Since by (2.1) and (2.7),

$$a(|w_n|^\alpha)|w_n|^\alpha|\nabla w_n| \leq \widetilde{A}(a(|w_n|^\alpha)|w_n|^\alpha) + A(|\nabla w_n|) \leq A(2|w_n|^\alpha) + A(|\nabla w_n|),$$

the $\Delta_2$-condition yields, $a(|w_n|^\alpha)|w_n|^\alpha|\nabla w_n| \leq KA(|w_n|^\alpha) + A(|\nabla w_n|)$, therefore, (3.6) gives

$$\int_{\mathbb{R}^N} |\nabla(A(|w_n|^\alpha))| < +\infty.$$

By Sobolev embedding, $W^{1,1}(B_R(y)) \hookrightarrow L^{N/(N-1)}(B_R(y))$. Therefore, there exists $C > 0$ such that

$$\bigg(\int_{B_R(y)} A(|w_n|^\alpha)^{N/(N-1)}\bigg)^{(N-1)/N} \leq C\int_{B_R(y)} (|\nabla A(|w_n|^\alpha)| + A(|w_n|^\alpha)).$$

Since by Lemma 2.3, $A(|t|) \geq c_0|t|^m$, for all $t \in [-1,1]$, it follows that

$$\bigg(\int_{B_R(y)} |w_n|^{\alpha m N/(N-1)}\bigg)^{N/(N-1)} \leq C\int_{B_R(y)} (a(|w_n|)|w_n||\nabla w_n| + A(|w_n|)).$$

Next, let us fix $\alpha > 0$ large enough and $p = m/N + m\alpha$. Thereby,

$$\begin{aligned}\int_{B_R(y)} |w_n|^p &= \int_{B_R(y)} |w_n|^{m/N}|w_n|^{m\alpha} \\ &\leq \bigg(\int_{B_R(y)} |w_n|^m\bigg)^{1/N}\bigg(\int_{B_R(y)} |w_n|^{m\alpha N/(N-1)}\bigg)^{(N-1)/N}.\end{aligned}$$

By (3.5),

$$\left(\int_{B_R(y)} |w_n|^m\right)^{1/N} < \varepsilon,$$

for $n$ large enough and for all $y \in \mathbb{R}^N$. Hence, there is $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(y)} |w_n|^p \leq \varepsilon c_1 \int_{B_R(y)} f_n, \quad n \geq n_0 \text{ and } y \in \mathbb{R}^N,$$

where $f_n = a(|w_n|)|w_n||\nabla w_n| + A(|w_n|)$.

Now, we set $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\mathbb{R}^N = \bigcup_{j\in\mathbb{N}} B_R(y_j)$ and each point of $\mathbb{R}^N$ is contained in at most $\kappa$ balls. Then,

$$\begin{aligned}\int_{\mathbb{R}^N} |w_n|^p &\leq \sum_{j\in\mathbb{N}} \int_{B_R(y_j)} |w_n|^p \leq \varepsilon c_1 \sum_{j\in\mathbb{N}} \int_{B_R(y_j)} f_n \\ &\leq \varepsilon c_1 \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^\mathbb{N}} f_n \chi_{B_R(y_j)} \leq \varepsilon c_1 \int_{\mathbb{R}^\mathbb{N}} f_n \sum_{j\in\mathbb{N}} \chi_{B_R(y_j)} \leq \varepsilon c_1 \kappa \int_{\mathbb{R}^N} f_n.\end{aligned}$$

As $\{u_n\}$ is bounded in $W^1L_A(\mathbb{R}^N)$, the sequence $\{f_n\}$ is bounded in $L^1(\mathbb{R}^N)$. In this way, the last inequality gives $w_n \xrightarrow{n\to+\infty} 0$ in $L^p(\mathbb{R}^N)$, for $p$ large enough. On the other hand,

$$|w_n|_m^m = \int_{\mathbb{R}^\mathbb{N}} |w_n|^m \leq c_0 \int_{\mathbb{R}^N} A(|w_n|) \leq C, \quad n \in \mathbb{N},$$

from where it follows that $\{w_n\}$ is bounded in $L^m(\mathbb{R}^N)$. Then, by interpolation, $w_n \xrightarrow{n\to+\infty} 0$ in $L^q(\mathbb{R}^N)$, for all $q > m$.

From ($a_2$), it follows that $l^*, m^* > m$, thus

$$w_n \xrightarrow{n\to+\infty} 0 \quad \text{in } L^{l^*}(\mathbb{R}^N) \text{ and } L^{m^*}(\mathbb{R}^N). \tag{3.7}$$

On the other hand, by Lemma 2.5, $A_*(t) \leq C(|t|^{m^*} + |t|^{l^*})$ for all $t \in \mathbb{R}^N$. This combined with (3.7) gives

$$\int_{\mathbb{R}^N} A_*(|w_n|) \to 0.$$

From ($B_1$)–($B_2$), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ verifying $B(|t|) \leq \varepsilon A(|t|) + C_\varepsilon A_*(|t|)$, $t \in \mathbb{R}$. Therefore,

$$\int_{\mathbb{R}^N} B(|w_n|) \leq \varepsilon \int_{\mathbb{R}^N} A(|w_n|) + C_\varepsilon \int_{\mathbb{R}^N} A_*(|w_n|) \leq \varepsilon C + C_\varepsilon \int_{\mathbb{R}^N} A_*(|w_n|),$$

from where it follows that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|w_n|) \leq \varepsilon C \quad \text{for all } \varepsilon > 0,$$

showing that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} B(|w_n|) = 0,$$

that is, $w_n \xrightarrow{n\to+\infty} 0$ in $L_B(\mathbb{R}^N)$, finishing the proof of lemma. □

Proof of Theorem 1.6 (compactness theorem involving the group $O(N)$). The proof follows the same arguments used in Willem [30, Theorem 1.24], when $|u_n|^2$ is replaced by $A(|u_n|)$. Here, we will make a sketch of the proof for convenience of the reader.

Let $\{u_n\}$ be a sequence in $W^1_{0,G}L_A(\Omega)$ with $u_n \rightharpoonup 0$ in $W^1_{0,G}L_A(\Omega)$. Without loss of generality, we can assume that $\{u_n\} \subset W^1_0 L_A(\mathbb{R}^N)$ by supposing that $u_n(x) = 0$ for all $x \in \Omega^c$.

From definition of $m(y, r, G)$,

$$\int_{B_r(y)} A(|u_n|) \leq \frac{\sup_n \int_{\mathbb{R}^N} A(|u_n|)}{m(y, r, G)} \quad \text{for all } n \in \mathbb{N} \text{ and } y \in \mathbb{R}^N.$$

Once that $\Omega$ is compatible with $G$, given $\varepsilon > 0$, there is $R > 0$ such that

$$\sup_{|y| \geq R} \int_{B_r(y)} A(|u_n|) \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

On the other hand, one has $B_r(y) \subset B_{R+r}(0)$ for all $y \in B_R(0)$ which implies that

$$\sup_{|y| < R} \int_{B_r(y)} A(|u_n|) \leq \int_{B_{R+r}(0)} A(|u_n|). \tag{3.9}$$

By (2.6), $u_n \to 0$ in $L_A(B_{R+r}(0))$ that is,

$$\int_{B_{R+r}(0)} A(|u_n|) \to 0. \tag{3.10}$$

Thereby, from (3.9) and (3.10), there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{|y| < R} \int_{B_r(y)} A(|u_n|) \leq \varepsilon \quad \text{for all } n \geq n_0.$$

Hence, from (3.8) and (3.10),

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} A(|u_n|) = 0.$$

Now, the result follows applying the Theorem 1.3. □

## 4. Existence of solutions for problem (P)

In this section, we will use the results obtained in the previous section to prove Theorem 1.8. Hereafter, let us denote by $J\colon X \to \mathbb{R}$ the energy functional related to (P) given by

$$J(u) = \int_{\mathbb{R}^N} A(|\nabla u|) + \int_{\mathbb{R}^N} V(x) A(|u|) - \int_{\mathbb{R}^N} F(u),$$

where $X = W^1 L_A(\mathbb{R}^N)$ when $V$ is periodic and

$$X = \left\{ u \in W^1 L_{A,\mathrm{rad}}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} V(x) A(|u|) < +\infty \right\}$$

when $V$ is a radial function. In both cases, $X$ will be endowed with the norm

$$\|u\| = \|\nabla u\|_A + \|u\|_{V,A}$$

where

$$\|u\|_{V,A} = \inf\bigg\{\alpha > 0;\ \int_{\mathbb{R}^N} V(x)A\bigg(\frac{|u|}{\alpha}\bigg) \le 1\bigg\}.$$

A simple computation gives that the above norm is equivalent to the usual norm of $W^1L_A(\mathbb{R}^N)$ when $V$ is a continuous periodic function satisfying $(\mathrm{V}_1)$. Moreover, it is possible to prove that $J \in C^1(X, \mathbb{R})$ with

$$J'(u)\phi = \int_{\mathbb{R}^N} a(|\nabla u|)\nabla u \nabla\phi + \int_{\mathbb{R}^N} V(x)a(|u|)u\phi - \int_{\mathbb{R}^N} f(u)\phi,$$

for all $\phi \in X$.

Our goal is looking for critical points of $J$, because its critical points are weak solutions for (P). Next, we will show three lemmas for the functional $J$, which are true when $V$ is radial or periodic. These lemmas will occur, because the below embeddings $X \hookrightarrow L_A(\mathbb{R}^N)$ and $X \hookrightarrow L_{A_*}(\mathbb{R}^N)$ are continuous. The first of them establishes that $J$ verifies the mountain pass geometry on $X$.

Lemma 4.1. *If* $(\mathrm{a}_1)$–$(\mathrm{a}_2)$, $(\mathrm{f}_1)$–$(\mathrm{f}_2)$ *and* $(\mathrm{V}_1)$ *hold, the functional $J$ satisfies the following conditions*:

(a) *There exist* $\rho, \eta > 0$, *such that* $J(u) \ge \eta$, *if* $\|u\| = \rho$.
(b) *For any* $\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, $J(t\phi) \to -\infty$ *as* $t \mapsto +\infty$.

Proof. (a) From assumptions $(\mathrm{f}_1)$–$(\mathrm{f}_2)$, given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 \le f(t)t \le \varepsilon a(|t|)|t|^2 + C_\varepsilon a_*(|t|)|t|^2 \quad \text{for all } t \in \mathbb{R}.$$

From $(\mathrm{a}_2)$ and Lemma 2.4,

$$0 \le f(t)t \le \varepsilon m A(|t|) + C_\varepsilon m^* A_*(|t|) \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

Using $(\mathrm{f}_3)$,

$$0 \le F(t) \le \frac{\varepsilon m}{\theta} A(|t|) + \widetilde{C} A_*(|t|) \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

From (4.2) and $(\mathrm{V}_1)$,

$$J(u) \ge \int_{\mathbb{R}^N} A(|\nabla u|) + \bigg(1 - \frac{\varepsilon m}{\theta V_0}\bigg)\int_{\mathbb{R}^N} V(x)A(|u|) - C\int_{\mathbb{R}^N} A_*(|u|).$$

Hence, for $\varepsilon$ small enough, the Lemmas 2.3 and 2.5 imply that

$$J(u) \ge C_1(\xi_0(\|\nabla u\|_A) + \xi_0(\|u\|_{V,A})) - C_2\xi_3(\|u\|_{A_*}).$$

Choosing $\rho > 0$ such that

$$\|u\| = \|\nabla u\|_A + \|u\|_{V,A} = \rho < 1 \quad \text{and} \quad \|u\|_{A_*} \le C(\|\nabla u\|_A + \|u\|_{V,A}) < \rho < 1,$$

we obtain

$$J(u) \ge C_1(\|\nabla u\|_A^m + \|u\|_{V,A}^m) - C_2\|u\|_{A_*}^{l^*},$$

which yields $J(u) \geq C_3\|u\|^m - C_4\|u\|^{l^*}$, for some positive constants $C_3$ and $C_4$. Since $0 < m < l^*$, there exists $\eta > 0$ such that $J(u) \geq \eta$ for all $\|u\| = \rho$.

(b) From ($f_3$), there exist $C_5$, $C_6 > 0$ such that $F(t) \geq C_5|t|^\theta - C_6$, for all $t \in \mathbb{R}$. Fixing $\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, the last inequality leads to

$$J(t\phi) \leq \xi_1(t)(\xi_1(\|\nabla\phi\|_A) + \xi_1(\|\phi\|_{V,A})) - C_5 t^\theta \int_{\mathbb{R}^N} |\phi|^\theta + C_6 \operatorname{supp}\phi.$$

Thus, for $t$ sufficient large,

$$J(t\phi) \leq t^m(\xi_1(\|\nabla\phi\|_A) + \xi_1(\|\phi\|_{V,A})) - C_5 t^\theta \int_{\mathbb{R}^N} |\phi|^\theta + C_6 \operatorname{supp}\phi.$$

Since $m < \theta$, the result follows. □

Now, in view of the last lemma, we can apply a version of Mountain Pass Theorem without the Palais–Smale condition found in [4] to get a sequence $\{u_n\} \subset X$ verifying

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{as } n \to \infty, \tag{4.3}$$

where the level $c$ is characterized by $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) > 0$ and $\Gamma = \{\gamma \in C([0,1], X) : J(0) = 0 \text{ and } J(\gamma(1)) < 0\}$.

Lemma 4.2. *Let $\{v_n\}$ be a $(\mathrm{PS})_d$ sequence for $J$. Then, $\{v_n\}$ is a bounded sequence in $X$.*

Proof. Since $\{v_n\}$ is a $(\mathrm{PS})_d$ sequence for the functional $J$, there is $C > 0$ such that

$$C(1 + \|v_n\|) \geq J(v_n) - \frac{1}{\theta}J'(v_n)v_n, \quad \text{for all } n \in \mathbb{N}.$$

From ($f_3$),

$$\begin{aligned} C(1 + \|v_n\|) &\geq \left(\frac{\theta - m}{\theta}\right) \int_{\mathbb{R}^N} A(|\nabla v_n|) + V(x)A(|v_n|) \\ &\geq \left(\frac{\theta - m}{\theta}\right)[\xi_0(\|\nabla v_n\|_A) + \xi_0(\|v_n\|_{V,A})]. \end{aligned}$$

Suppose for contradiction that, up to a subsequence, $\|v_n\| \to +\infty$. This way, we need to study the following situations:

(a) $\|\nabla v_n\|_A \to +\infty$ and $\|v_n\|_{V,A} \to +\infty$,

(b) $\|\nabla v_n\|_A \to +\infty$ and $\|v_n\|_{V,A}$ is bounded,

and

(c) $\|\nabla v_n\|_A$ is bounded and $\|v_n\|_{V,A} \to +\infty$.

In the first case, the Lemma 2.5 implies that

$$C(1 + \|v_n\|) \geq C_1[\|\nabla v_n\|_A^l + \|v_n\|_{V,A}^l] \geq C_2\|v_n\|^l,$$

for $n$ large enough, which is an absurd.

In case (b), we have for $n$ large enough

$$C_3(1 + \|\nabla v_n\|_A) \geq C(1 + \|v_n\|) \geq C_2 \|\nabla v_n\|_A^l,$$

which is an absurd again. The last case is similar to the case (b). $\square$

Using the fact that $X$ is reflexive, it follows from Lemma 4.2 that there exists a subsequence of $\{u_n\}$, still denoted by itself, and $u \in X$ such that $u_n \rightharpoonup u$ in $X$.

Lemma 4.3. *The sequence $\{u_n\}$ satisfies the following limit*

$$\nabla u_n(x) \xrightarrow{n \to +\infty} \nabla u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

*As a consequence, we deduce that $u$ is a critical point for $J$, that is, $J'(u) = 0$.*

Proof. We begin this proof observing that $(\text{a}_1)$ yields

$$(a(|x|)x - a(|y|)y)(x - y) > 0, \quad \text{for all } x, y \in \mathbb{R}^N \text{ with } x \neq y. \tag{4.4}$$

Given $R > 0$, let us consider $\xi = \xi_R \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$0 \leq \xi \leq 1, \quad \xi \equiv 1 \quad \text{in } B_R(0) \quad \text{and} \quad \operatorname{supp}(\xi) \subset B_{2R}(0).$$

Using the above information,

$$\begin{aligned} 0 &\leq \int_{B_R(0)} (a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u)(\nabla u_n - \nabla u) \\ &\leq \int_{B_{2R}(0)} (a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u)(\nabla u_n - \nabla u)\xi \\ &= \int_{B_{2R}(0)} a(|\nabla u_n|)\nabla u_n(\nabla u_n - \nabla u)\xi \\ &\quad - \int_{B_{2R}(0)} a(|\nabla u|)\nabla u(\nabla u_n - \nabla u)\xi. \end{aligned} \tag{4.5}$$

Now, combining the boundedness of $\{(u_n - u)\xi\}$ in $X$ with the limit $\|J'(u_n)\| = o_n(1)$, it follows that

$$\begin{aligned} o_n(1) = &\int_{B_{2R}(0)} a(|\nabla u_n|)\nabla u_n \nabla((u_n - u)\xi) \\ &+ \int_{B_{2R}(0)} V(x)a(|u_n|)u_n(u_n - u)\xi - \int_{B_{2R}(0)} f(u_n)(u_n - u)\xi. \end{aligned} \tag{4.6}$$

Note that $\{a(|u_n|)u_n\}$ is bounded in $L_{\widetilde{A}}(B_{2R}(0))$, because

$$\int_{B_{2R}(0)} \widetilde{A}(a(|u_n|)u_n) \leq \int_{B_{2R}(0)} A(2|u_n|) \leq K \int_{B_{2R}(0)} A(|u_n|) < +\infty.$$

From this

$$(4.7)\qquad \left|\int_{B_{2R}(0)} V(x)a(|u_n|)u_n(u_n-u)\xi\right| \le \int_{B_{2R}(0)} |V(x)||a(|u_n|)u_n||u_n-u| \le 2M\|a(|u_n|)|u_n|\|_{\widetilde{A},B_{2R}(0)}\|u_n-u\|_{A,B_{2R}(0)} \le C_1\|u_n-u\|_{A,B_{2R}(0)} \to 0.$$

where $M = \sup_{x\in B_{2R}(0)} |V(x)|$. On the other hand, using again the boundedness of $(u_n)$ in $X$ and (2.7),

$$\int_{B_{2R}(0)} \widetilde{A}_*(a_*(u_n)u_n) \le \int_{B_{2R}(0)} A_*(2u_n) \le C_2, \quad n\in\mathbb{N},$$

implying that $\{a_*(u_n)u_n\}$ is bounded in $L_{\widetilde{A}_*}(B_{2R}(0))$. Since

$$(4.8)\qquad \left|\int_{B_{2R}(0)} f(u_n)(u_n-u)\xi\right| \le \varepsilon\bigg(\int_{B_{2R}(0)} |a(|u_n|)u_n||u_n-u| + \int_{B_{2R}(0)} |a_*(u_n)u_n||u_n-u|\bigg) + c_1\int_{B_{2R}(0)} |u_n-u| \le \varepsilon\|a(|u_n|)u_n\|_{\widetilde{A},B_{2R}(0)}\|u_n-u\|_{A,B_{2R}(0)} + \varepsilon c_2\|a_*(u_n)u_n\|_{\widetilde{A}_*,B_{2R}(0)}\|u_n-u\|_{A_*,B_{2R}(0)} + c_3\|u_n-u\|_{A,B_{2R}(0)},$$

the boundedness of $\{u_n\}$, $\{a(u_n)u_n\}$ and $\{a_*(u_n)u_n\}$ in $L_A(B_{2R}(0))$, $L_{\widetilde{A}}(B_{2R}(0))$ and $L_{\widetilde{A}_*}(B_{2R}(0))$ respectively lead to

$$\left|\int_{B_{2R}(0)} f(u_n)(u_n-u)\xi\right| \le \varepsilon C_4 + c_3\|u_n-u\|_{A,B_{2R}(0)}.$$

Now, using the convergence of $\{u_n\}$ to $u$ in $L_A(B_{2R}(0))$, we get

$$(4.9)\qquad \left|\int_{B_{2R}(0)} f(u_n)(u_n-u)\xi\right| \to 0.$$

A similar idea can be used to establish the limit

$$(4.10)\qquad \int_{B_{2R}(0)} (u_n-u)a(|\nabla u_n|)\nabla u_n\nabla\xi \to 0.$$

Moreover, the weak convergence of $\{u_n\}$ to $u$ in $W^1L_A(\mathbb{R}^N)$ gives

$$(4.11)\qquad \int_{B_{2R}(0)} \xi a(|\nabla u|)\nabla u(\nabla u_n-\nabla u) \to 0.$$

From (4.5)–(4.11),

$$\int_{B_R(0)} (a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u)(\nabla u_n-\nabla u) \to 0.$$

Setting $\beta\colon \mathbb{R}^N \to \mathbb{R}^N$ by $\beta(x) = a(|x|)x$, $x \in \mathbb{R}^N$, and observing that $\beta$ is monotone by (4.4), the last limit imply that for some subsequence, still denoted by itself,

$$(\beta(\nabla u_n(x)) - \beta(\nabla u(x)))(\nabla u_n(x) - \nabla u(x)) \to 0 \quad \text{a.e. in } B_R(0).$$

Applying a result found in Dal Maso and Murat [9], it follows that $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in $B_R(0)$, for each $R > 0$. As $R$ is arbitrary, there is a subsequence of $\{u_n\}$, still denoted by itself, such that $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in $\mathbb{R}^N$.

Recalling that $\{a(|\nabla u_n|)\frac{\partial u_n}{\partial x_i}\}$ is bounded in $L_{\widetilde{A}}(\mathbb{R}^N)$, we get from Lemma 2.1

$$\int_{\mathbb{R}^N} a(|\nabla u_n|)\nabla u_n \nabla v \to \int_{\mathbb{R}^N} a(|\nabla u|)\nabla u \nabla v,$$

for all $v \in X_c = \{v \in X : v \text{ has compact support}\}$. On the other hand, once that $V$ is bounded on the support of $v$, $\{a(|u_n|)u_n\}$ is bounded in $L_{\widetilde{A}}(\mathbb{R}^N)$ and $\{a_*(|u_n|)u_n\}$ is bounded in $L_{\widetilde{A}_*}(\mathbb{R}^N)$, we have again by Lemma 2.1

$$\int_{\mathbb{R}^N} V(x)a(|u_n|)u_n v \to \int_{\mathbb{R}^N} V(x)a(|u|)uv$$

and

$$\int_{\mathbb{R}^N} f(u_n)v \to \int_{\mathbb{R}^N} f(u)v.$$

Therefore, $J'(u)v = 0$ for all $v \in X_c$. Now, the lemma follows using the fact that $X_c$ is dense in $X$. □

**4.1. Proof of Theorem 1.8.** The reader is invited to observe that the main difference between the radial and periodic case is the following: In the radial case, the Theorem 1.2 permits to prove that the energy functional $J$ verifies the (PS) condition, while in the periodic case, we do not have this condition and we overcome this difficulty by using the Theorem 1.3.

We will prove the Theorem 1.8 studying firstly the radial case, and after, the periodic case.

**The radial case.** For the radial case, we begin showing the following lemma:

Claim 4.4. *Let $\{u_n\}$ the sequence given in* (4.3). *If* $(\mathrm{f}_1)$–$(\mathrm{f}_2)$ *hold, one have*

$$\int_{\mathbb{R}^N} f(u_n)u_n \to \int_{\mathbb{R}^N} f(u)u.$$

Indeed, as $\{u_n\}$ is a bounded sequence in $W^1L_{A,\mathrm{rad}}(\mathbb{R}^N)$,

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} (A_*(u_n) + A(u_n)) < +\infty.$$

Moreover, by hypotheses $(\mathrm{f}_1)$–$(\mathrm{f}_2)$, the function $P(t) = f(t)t$ verifies the limit

$$\lim_{|t|\to 0} \frac{P(t)}{A(t) + A_*(t)} = 0 \quad \text{and} \quad \lim_{|t|\to +\infty} \frac{P(t)}{A(t) + A_*(t)} = 0.$$

Since by Theorem 1.1, $u_n(x) \to 0$ as $|x| \to +\infty$, uniformly with respect to $n$, it follows from [3, Theorem A.I],

$$\int_{\mathbb{R}^N} f(u_n)u_n \to \int_{\mathbb{R}^N} f(u)u,$$

proving the claim.

Recalling that $J'(u_n)u_n = o_n(1)$, or equivalently,

$$\int_{\mathbb{R}^N} (a(|\nabla u_n|)|\nabla u_n|^2 + V(x)a(|u_n|)|u_n|^2) = \int_{\mathbb{R}^N} f(u_n)u_n + o_n(1),$$

we derive from Claim 4.4

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (a(|\nabla u_n|)|\nabla u_n|^2 + V(x)a(|u_n|)|u_n|^2) = \int_{\mathbb{R}^N} f(u)u.$$

Using the fact that $J'(u)u = 0$, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (a(|\nabla u_n|)|\nabla u_n|^2 + V(x)a(|u_n|)|u_n|^2) = \int_{\mathbb{R}^N} (a(|\nabla u|)|\nabla u|^2 + V(x)a(|u|)|u|^2).$$

Once that $\nabla u_n(x) \to \nabla u(x)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$, we conclude that

$$\begin{aligned} a(|\nabla u_n|)|\nabla u_n|^2 &\to a(|\nabla u|)|\nabla u|^2 \quad \text{in } L^1(\mathbb{R}^N), \\ V(x)a(|u_n|)|u_n|^2 &\to V(x)a(|u|)|u|^2 \quad \text{in } L^1(\mathbb{R}^N). \end{aligned}$$

These limits combined with $(\mathrm{a}_2)$ yields

$$\int_{\mathbb{R}^N} A(|\nabla u_n - \nabla u|) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)A(|u_n - u|) \to 0.$$

Hence, by a similar arguments used in (2.4), we derive that $u_n \to u$ in $X$, and thus, $J(u) = c > 0$ and $J'(u) = 0$, showing that $u$ is a critical point of $J$ in $X$. Now, using a principle of symmetric criticality on reflexive Banach spaces due to de Morais Filho, Do Ó and Souto [10], we have that $u$ is a critical point of $J$ in $W^1L_A(\mathbb{R}^N)$, and so, $u$ is a nontrivial solution for problem (P).

**The periodic case.** By Lemma 4.3, we know that the weak limit $u$ of the sequence $\{u_n\}$ given in (4.3) is a critical point for $J$. If $u \neq 0$, the theorem is proved. However, if $u = 0$, we have the following claim:

Claim 4.5. *There is $R > 0$ such that*

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} A(u_n) > 0. \tag{4.12}$$

In fact, if the above claim does not hold, by using Theorem 1.3, we derive the limit

$$\int_{\mathbb{R}^N} B(|u_n|) \to 0, \tag{4.13}$$

for any $N$-function $B$ satisfying ($B_1$)–($B_2$). Fixing a $N$-function $B$ satisfying ($B_1$)–($B_2$), it follows from ($f_1$)–($f_2$) that given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(u_n)u_n| \le \varepsilon(A(|u_n|) + A_*(|u_n|)) + C_\varepsilon B(|u_n|) \quad \text{for all } n \in \mathbb{N}.$$

Thereby, the above inequality together with (4.13) gives

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0.$$

Recalling that $J'(u_n)u_n = o_n(1)$, that is,

$$\int_{\mathbb{R}^N} a(|\nabla u_n|)|\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)a(|u_n|)|u_n|^2 = \int_{\mathbb{R}^N} f(u_n)u_n + o_n(1),$$

we obtain

$$\int_{\mathbb{R}^N} a(|\nabla u_n|)|\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)a(|u_n|)|u_n|^2 \to 0.$$

The last limit together with ($a_2$) gives

$$\int_{\mathbb{R}^N} A(|\nabla u_n|) + \int_{\mathbb{R}^N} V(x)A(|u_n|) \to 0,$$

implying that $\{u_n\}$ converges strongly to zero in $W^1L_A(\mathbb{R}^N)$, leading to $c = 0$, which is an absurd. Thus, the limit (4.12) holds and the claim is proved.

Therefore, there are $R, \alpha > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\int_{B_R(y_n)} A(u_n) > \alpha. \tag{4.14}$$

Now, letting $\overline{u}_n(x) = u_n(x - y_n)$, since $V$ is $\mathbb{Z}^N$-periodic function, one has

$$\|\overline{u}_n\| = \|u_n\|, \quad J(\overline{u}_n) = J(u_n) \quad \text{and} \quad J'(\overline{u}_n) = o_n(1).$$

Then, there exists $\overline{u}$ such that $\overline{u}_n \rightharpoonup \overline{u}$ weakly in $W^1L_A(\mathbb{R}^N)$, and as before, it follows that $J'(\overline{u}) = 0$. Now, by (4.14),

$$\int_{B_R(0)} A(\overline{u}_n) \ge \alpha > 0,$$

which together with the compact embeddings yields

$$\int_{B_R(0)} A(\overline{u}) \ge \alpha > 0,$$

showing that $\overline{u} \ne 0$, and thereby, finishing the proof of the Theorem 1.8. □

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