

ON A CLASS OF NONHOMOGENEOUS ELLIPTIC PROBLEMS INVOLVING EXPONENTIAL CRITICAL GROWTH

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ABSTRACT. In this paper we establish the existence of solutions for elliptic equations of the form $-\operatorname{div}(|\nabla u|^{n-2}\nabla u) + V(x)|u|^{n-2}u = g(x, u) + \lambda h$ in \mathbb{R}^n with $n \geq 2$. Here the potential $V(x)$ can change sign and the nonlinearity $g(x, u)$ is possibly discontinuous and may exhibit exponential growth. The proof relies on the application of a fixed point result and a version of the Trudinger–Moser inequality.

1. Introduction and the main result

In this paper we consider the existence of nontrivial weak solutions for the following class of elliptic problems:

$$(1.1) \quad -\operatorname{div}(|\nabla u|^{n-2}\nabla u) + V(x)|u|^{n-2}u = g(x, u) + \lambda h \quad \text{in } \mathbb{R}^n,$$

where $n \geq 2$, λ is a positive parameter, g and V are functions satisfying mild conditions and h belongs to the dual of an appropriated function space.

When $n = 2$, problem (1.1) is related to the existence of standing wave solutions for the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m} \Delta_x \psi + V(x)\psi - g(x, |\psi|)\psi - h(x)e^{i\omega t},$$

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where m and h are positive constants, $\omega \in \mathbb{R}$ and $\psi: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{C}$. Such equations arise in various branches of mathematical physics (see for example [3], [4], [27]) and they have been the subject of extensive study during the last years.

We point out that the existence of solutions for problem (1.1) has been discussed under various conditions on the potential V . We refer the reader to [13], [15], [18], [30] and references therein. It is worthwhile to remark that in these works different hypotheses are assumed on V in order to overcome the problem of “lack of compactness”, typical for elliptic problems in unbounded domains. More precisely, in many papers it is usually assumed that the potential is uniformly positive, that is, $V(x) \geq V_0 > 0$ for any $x \in \mathbb{R}^n$ and satisfies one of the following assumptions:

- (1) $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$;
- (2) $1/V \in L^1(\mathbb{R}^n)$;
- (3) for every $M > 0$ the Lebesgue measure $\mu(\{x \in \mathbb{R}^n : V(x) \leq M\}) < \infty$.

Each of these conditions guarantee that the space $E := \{u \in W^{1,n}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V|u|^n dx < \infty\}$ is compactly embedded in the Lebesgue space $L^p(\mathbb{R}^n)$ for all $p \geq n$.

We point out that just a few results are available when $V(x)$ changes sign, see [11], [26], [31]. However, we stress that in all the above quoted papers, the authors require the potential V to be continuous and bounded below. Here, we concentrate our efforts to treat the case in which V can change sign without requiring any additional condition in order to get compactness. In particular we do not require a uniform bound on the potential which may develop singularities near zero. Physically, this correspond to collision of particle with the center of force, see for instance [22] for more details. Throughout the paper, we assume the following hypotheses on V :

(V₁) $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable and there exist $b_0, R_0 > 0$ such that

$$V(x) \geq b_0, \quad \text{for } |x| \geq R_0;$$

(V₂) $V^- \in L^p(B_{R_0})$ for some $1 < p \leq \infty$;

where $V^\pm = \max\{\pm V, 0\}$ and B_R denotes the open ball centered at the origin in \mathbb{R}^n .

A typical example of function which satisfies the hypotheses (V₁)–(V₂) is given by

$$(1.2) \quad V_\varepsilon(x) = \begin{cases} 1 + \eta(x) & \text{for } |x| \geq 1, \\ -\frac{\varepsilon}{|x|^{n/(2p)}} & \text{for } |x| < 1, \end{cases}$$

where $\varepsilon > 0$, $p > 1$ and η is a nonnegative measurable function for $|x| \geq 1$.

Our second aim in this paper is treat nonlinearities $g(x, s)$ possibly discontinuous and that may exhibit exponential growth. In order to better describe

the hypotheses on $g(x, s)$ we recall some well known facts about the limiting Sobolev embedding theorem.

Let Ω be a smooth domain in \mathbb{R}^n and $W^{1,p}(\Omega)$ be the Sobolev space endowed with its usual norm

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_{\Omega} (|\nabla u|^p + |u|^p) dx \right]^{1/p}.$$

We define $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. After the works [21], [24] and [29], the limiting case $p = n$ of the Sobolev embedding has received considerable attention in recent years. Roughly speaking, their results state that if Ω is bounded and $\alpha > 0$ then there holds the embedding

$$W_0^{1,n}(\Omega) \hookrightarrow L_A(\Omega),$$

where $L_A(\Omega)$ is the Orlicz space based on the N -function $A(s) = e^{|s|^{n/(n-1)}} - 1$. In fact, this embedding is the “best possible” within the Orlicz framework, that is, if there exists any embedding of the form

$$W_0^{1,n}(\Omega) \hookrightarrow L_B(\Omega),$$

then A dominates the N -function B near infinity, see [16] (for others improvements, see e.g. [6]). This fact was the main motivation for the notion of exponential critical growth as introduced in the papers [2], [9].

When $\Omega = \mathbb{R}^n$, Adams (see [1]) proved that if $n \geq 2$, $\alpha > 0$ and $u \in W^{1,n}(\mathbb{R}^n)$ then

$$(1.3) \quad \int_{\mathbb{R}^n} \Phi_\alpha(u) dx < \infty,$$

where

$$\Phi_\alpha(s) := e^{\alpha|s|^{n/(n-1)}} - \sum_{j=0}^{n-2} \frac{\alpha^j |s|^{jn/(n-1)}}{j!}.$$

This result has motivated the study of semilinear and quasilinear elliptic equations in the whole space \mathbb{R}^n , when the nonlinearity $g(x, s)$ behaves as $\Phi_\alpha(s)$ at infinity, among others see [7], [12], [15], [23] and the references therein.

We emphasize that if $g(x, s)$ is a Carathéodory function, for instance, as in the works [10], [13], [14], [18], [28], [30], one can perform minimization arguments (see for example [20]), based on Ekeland variational principle and Lemma 2.1 of [9] to find a solution for nonhomogeneous problems of the type (1.1).

In this paper, we consider a more general class of nonlinearities $g(x, s)$, namely:

- (G₁) for each $u: \mathbb{R}^n \rightarrow \mathbb{R}$ measurable, the Nemytskiĭ function $N_g: \mathbb{R}^n \rightarrow \mathbb{R}$, $N_g(x) = g(x, u(x))$ is measurable;

(G₂) for each $x \in \mathbb{R}^n$, $g(x, s)$ is nondecreasing in s and

$$|g(x, s)| \leq c_1 k(x) |s|^\rho + c_2 \Phi_{\alpha_0}(s) |s|^\mu, \quad \text{for all } (x, s) \in \mathbb{R}^n \times \mathbb{R},$$

where $k \in L^\sigma(\mathbb{R}^n)$ for some $1 < \sigma \leq \infty$, $c_1, c_2 > 0$, $\alpha_0 > 0$, $\rho > n - 1$ and $\mu > n - 1$.

EXAMPLE 1.1. As an example of a discontinuous function $g(x, s)$ satisfying the conditions above, consider

$$g(x, s) = [k(x) |s|^{\rho-1} s + |s|^{\mu-1} s \Phi_{\alpha_0}(s)] H(s),$$

where k is a nonnegative function in $L^\sigma(\mathbb{R}^n)$ for some $1 < \sigma \leq \infty$, $\rho > n - 1$, $\mu > n - 1$ and H is such that $H(s) = 1$ if $s < 1$ and $H(s) = 2$ if $s \geq 1$.

EXAMPLE 1.2. Let $g: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function, that is, for any bounded interval $I \subset \mathbb{R}$, there exists $C > 0$ such that $|g(x, s)| \leq C$ for every $(x, s) \in \mathbb{R}^n \times I$. If $g(x, s)$ is nondecreasing in s , behaves at infinity as $\Phi_\alpha(s)$ and it holds

$$\lim_{s \rightarrow 0} \frac{g(x, s)}{|s|^{\rho-1} s} = b \in \mathbb{R}, \quad \text{uniformly in } x \in \mathbb{R}^n,$$

with $\rho > n - 1$, then conditions (G₁)–(G₂) are satisfied.

In order to obtain the existence of solutions for (1.1), we work in the subspace of $W^{1,n}(\mathbb{R}^n)$ defined by

$$X = \left\{ u \in W^{1,n}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V^+(x) |u|^n dx < \infty \right\}$$

and we consider in X the norm

$$\|u\| := \left[\int_{\mathbb{R}^n} (|\nabla u|^n + V^+(x) |u|^n) dx \right]^{1/n}.$$

Using condition (V₁), we show that the embedding $X \hookrightarrow W^{1,n}(\mathbb{R}^n)$ is continuous (see Lemma 2.1 below). Observing that

$$\|u\| = (\|\nabla u\|_n^n + \|(V^+(x))^{1/n} u\|_n^n)^{1/n}$$

and since $W^{1,n}(\mathbb{R}^n)$ and $L^n(\mathbb{R}^n)$ are complete spaces, one can see that $(X, \|\cdot\|)$ is a Banach space. Moreover, using the Clarkson’s first inequality (see [5, p. 95]) it follows that X is uniformly convex, and thus reflexive.

Next, for all $n \leq t < \infty$ we set

$$(1.4) \quad S_t := \inf_{\substack{u \in X \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} (|\nabla u|^n + V^+(x) |u|^n) dx}{\left(\int_{\mathbb{R}^n} |u|^t dx \right)^{n/t}}.$$

It follows from the continuous embedding $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^t(\mathbb{R}^n)$ that $S_t > 0$. In this context, we shall assume that $h \in X'$, the dual space of X , and we say that $u \in X$ is a weak solution of problem (1.1) if the following holds

$$\int_{\mathbb{R}^n} |\nabla u|^{n-2} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n} V(x) |u|^{n-2} uv \, dx = \int_{\mathbb{R}^n} g(x, u) v \, dx + \lambda \langle h, v \rangle,$$

for all $v \in X$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its dual X' .

At this point, in addition to the hypotheses on V , we assume the condition

$$(V_3) \quad \|V^-\|_{L^p(B_{R_0})} < S_{t_0}, \text{ where } t_0 := pn/(p-1) > n.$$

Note that the potentials V_ε defined in (1.2) satisfy hypothesis (V_3) provided ε is sufficiently small.

The main result in this work can be stated as follows:

THEOREM 1.3. *Suppose that (V_1) – (V_3) and (G_1) – (G_2) hold. Then, there exists $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, problem (1.1) has a weak solution.*

The main features of this class of problems are that they are set in unbounded domains and involve a nonlinear operator (for $n \geq 3$). Furthermore, the nonlinearity $g(x, s)$ may develop jump discontinuities in s and behave like $\Phi_\alpha(s)$ at infinity. For the authors knowledge the the result is new even for the semilinear case $n = 2$.

REMARK 1.4. In certain cases, it is possible to obtain nonexistence results for problem (1.1) when λ is large. For example, in addition to the hypothesis (V_1) , if we assume that

$$\int_{|x| \geq R_0} \frac{1}{V(x)^{1/(n-1)}} \, dx < \infty,$$

one can show that the embedding $X \hookrightarrow L^t(\mathbb{R}^n)$ is compact for each $1 \leq t < \infty$ (see for instance [30, Lemma 2.4]). Consequently, the infimum S_t in (1.4) is attained in a nonnegative function $\varphi_t \in X$. Hence, in particular for $t = n = 2$ we have

$$(1.5) \quad \int_{\mathbb{R}^2} \nabla \varphi_2 \nabla v \, dx + \int_{\mathbb{R}^2} V^+(x) \varphi_2 v \, dx = S_2 \int_{\mathbb{R}^2} \varphi_2 v \, dx, \quad \text{for all } v \in X.$$

By the strong maximum principle, it follows that $\varphi_2 > 0$ in \mathbb{R}^2 . Next, setting

$$M = \sup\{S_2 r - g(x, r) : (x, r) \in \mathbb{R}^2 \times \mathbb{R}\},$$

if we also assume $0 < M < \infty$ and $h \in L^\nu(\mathbb{R}^2)$ for some $1 < \nu \leq \infty$, with $h \geq 0$, $h \neq 0$, then there exists $0 < \Lambda < \infty$ such that problem (1.1) does not have nonnegative solution for $\lambda > \Lambda$. In fact, if u is a nonnegative solution of (1.1), we have

$$\int_{\mathbb{R}^2} \nabla u \nabla \varphi_2 \, dx + \int_{\mathbb{R}^2} [V^+(x) - V^-(x)] u \varphi_2 \, dx = \int_{\mathbb{R}^2} g(x, u) \varphi_2 \, dx + \lambda \int_{\mathbb{R}^2} h(x) \varphi_2 \, dx.$$

Using this, and taking $v = u$ in (1.5) we get

$$\begin{aligned} \lambda \int_{\mathbb{R}^2} h(x)\varphi_2 dx &= - \int_{\mathbb{R}^2} V^-(x)u\varphi_2 dx + \int_{\mathbb{R}^2} [S_2u - g(x, u)]\varphi_2 dx \\ &\leq \int_{\mathbb{R}^2} [S_2u - g(x, u)]\varphi_2 dx \end{aligned}$$

and therefore we must have

$$\lambda \leq \Lambda := \frac{M \int_{\mathbb{R}^2} \varphi_2 dx}{\int_{\mathbb{R}^2} h(x)\varphi_2 dx}.$$

REMARK 1.5. Motivated by the classical Trudinger–Moser inequality, the authors in [2], [12], [9] have introduced the notion of subcriticality and criticality, namely, we say that a function $g(x, s)$ has *subcritical growth* if for all $\alpha > 0$,

$$\lim_{|s| \rightarrow \infty} g(x, s)e^{-\alpha|s|^{n/(n-1)}} = 0 \quad \text{uniformly in } x \in \mathbb{R}^n,$$

and has *critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \rightarrow \infty} g(x, s)e^{-\alpha|s|^{n/(n-1)}} = \begin{cases} 0 & \text{for all } \alpha > \alpha_0, \\ +\infty & \text{for all } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^n$. We point out that hypothesis (G₂) includes nonlinearities with this kind of growth.

REMARK 1.6. Using the regularity result due to J. Serrin [25] and (1.3) one can prove that weak solutions to problem (1.1) actually belong to $C_{\text{loc}}^{1,\theta}(\mathbb{R}^n)$.

REMARK 1.7. We mention that recently nonhomogeneous problems involving the critical exponential growth for singular elliptic equations in unbounded domains have been investigated, among others, by J.M. do Ó and M. de Souza [10], N. Lam and G. Lu [18] and Y. Yang [30]. It is important to mention that our results also apply, with slight modification, to the singular case and for a class of more general operators, such as, Leray–Lions operators.

The outline of the paper is as follows: Section 2 contains some preliminary results which are need to the proof of our main result and in Section 3 we prove Theorem 1.3.

Hereafter, C, C_0, C_1, C_2, \dots will denote positive (possibly different) constants and we shall use the notation $\|\cdot\|_p$ for the norm in $L^p(\mathbb{R}^n)$.

2. Preliminary results

In this section, we obtain some technical results and we establish the appropriate setting to prove Theorem 1.3.

LEMMA 2.1. *By condition (V₁) the space X is continuously embedded in W^{1,n}(ℝⁿ). In particular, the embedding X ↪ L^s(ℝⁿ) is continuous for each s ∈ [n, ∞).*

PROOF. It is enough to check that there exists a constant C > 0 such that

$$(2.1) \quad \int_{\mathbb{R}^n} (|\nabla u|^n + V^+(x)|u|^n) dx \geq C \int_{\mathbb{R}^n} |u|^n dx, \quad \text{for all } u \in X.$$

Suppose, by contradiction, that (2.1) does not hold. Then for each k ∈ ℕ there exists u_k ∈ X such that

$$\int_{\mathbb{R}^n} |u_k|^n dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} (|\nabla u_k|^n + V^+(x)|u_k|^n) dx < 1/k.$$

So ‖∇u_k‖_n → 0, (u_k) is bounded in W^{1,n}(ℝⁿ) and, up to a subsequence,

$$(2.2) \quad u_k \rightharpoonup u \quad \text{in } W^{1,n}(\mathbb{R}^n).$$

Now, by using the Gagliardo–Nirenberg inequality (see [17, Proposition 8.12]) we reach

$$\|u_k\|_{2n} \leq C \|\nabla u_k\|_n^{1/2},$$

from which it follows that ‖u_k‖_{2n} → 0. On the other hand, by (2.2) u_k → u in L²ⁿ(B_r) for all r > 0. Thus, u = 0. By (V₁) we have

$$\int_{|x| \geq R_0} |u_k|^n dx \leq \frac{1}{b_0} \int_{|x| \geq R_0} V^+(x)|u_k|^n dx < \frac{1}{kb_0} \rightarrow 0.$$

From this convergence and since u_k → 0 in Lⁿ(B_{R₀}), taking the limit as k → ∞ in the equality

$$1 = \int_{\mathbb{R}^n} |u_k|^n dx = \int_{B_{R_0}} |u_k|^n dx + \int_{|x| \geq R_0} |u_k|^n dx,$$

we obtain a contradiction. Therefore, (2.1) holds and the lemma is proved. □

We next recall a version of the Trudinger–Moser inequality which holds in the whole space for which we refer to [1], [7], [12], [19], [23]. We denote the measure of the (n − 1)-sphere in ℝⁿ by ω_{n−1} and α_n := nω_{n−1}^{1/(n−1)}.

LEMMA 2.2. *Let n ≥ 2 and u ∈ W^{1,n}(ℝⁿ). If ‖∇u‖_n ≤ 1, ‖u‖_n ≤ M and 0 < α < α_n, then there exists a constant C = C(n, M, α) such that*

$$(2.3) \quad \int_{\mathbb{R}^n} \Phi_\alpha(u) dx \leq C(n, M, \alpha).$$

As a consequence of (2.3), we establish in our function space framework X an estimate that will be essential in our arguments.

LEMMA 2.3. *Suppose that (V₁) holds. Let $u, v \in X$ and $\alpha, \mu > 0$. If $\|u\| \leq M$ and $\alpha M^{n/(n-1)} < \alpha_n$, then there exists a constant $C = C(\alpha, M, \mu) > 0$ such that*

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu|v| \, dx \leq C\|u\|^\mu\|v\|.$$

PROOF. First, we choose $q > 1$ sufficiently close to 1 satisfying

$$q\alpha M^{n/(n-1)} < \alpha_n \quad \text{and} \quad \sigma := \frac{2q}{q-1} > \max\{n, n/\mu\}.$$

Thus, $1/q + 1/\sigma + 1/\sigma = 1$ and applying the generalized Hölder inequality and Lemma 2.1 one has

$$\begin{aligned} (2.4) \quad \int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu|v| \, dx &\leq \left(\int_{\mathbb{R}^n} [\Phi_\alpha(u)]^q \, dx \right)^{1/q} \|u\|_{\mu\sigma}^\mu \|v\|_\sigma \\ &\leq C_1 \left(\int_{\mathbb{R}^n} \Phi_{q\alpha}(u) \, dx \right)^{1/q} \|u\|^\mu \|v\| \\ &\leq C_1 \left[\int_{\mathbb{R}^n} \Phi_{q\alpha M^{n/(n-1)}} \left(\frac{u}{\|\nabla u\|_n} \right) \, dx \right]^{1/q} \|u\|^\mu \|v\|. \end{aligned}$$

Since $q\alpha M^{n/(n-1)} < \alpha_n$, in view of the Trudinger–Moser inequality (2.3) we conclude that

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)|u|^\mu|v| \, dx \leq C\|u\|^\mu\|v\|$$

and the proof is finished. □

For the convenience of the reader, in the sequel, we recall some basic concepts and notations. Let X be a real Banach space. A nonempty subset $X_+ \neq \{0\}$ of X is called an order cone if the following hold:

- (i) X_+ is closed and convex;
- (ii) if $u \in X_+$ and $\alpha \geq 0$, then $\alpha u \in X_+$;
- (iii) if $u \in X_+$ and $-u \in X_+$, then $u = 0$.

We observe that an order cone X_+ induces in a natural way a partial order in X as follows: $x \preceq y$ if and only if $y - x \in X_+$, and (X, \preceq) is called an ordered Banach space. If in addition, $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in X$ with respect to \preceq then we say that $(X, \|\cdot\|)$ is a *lattice*. Furthermore, if $\|x^\pm\| \leq \|x\|$ for all $x \in X$, with $x^+ := \sup\{0, x\}$ and $x^- := -\inf\{0, x\}$ then $(X, \|\cdot\|)$ is called a *Banach semilattice*.

Special examples of Banach semilattices are the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$ (Ω is a domain of \mathbb{R}^n) when we consider the natural partial order ($u \preceq v$ if and only if $u \leq v$ almost everywhere in Ω).

Let (X, \preceq) and $(\tilde{X}, \triangleleft)$ be ordered Banach spaces. We say that an operator $G: X \rightarrow \tilde{X}$ is increasing if and only if for all $x, y \in X$, $x \preceq y$ implies $Gx \triangleleft Gy$.

A subset B of X is said to have the *fixed point property* if every increasing operator $S: B \rightarrow B$ has a fixed point.

Now, we present a version of the fixed point result due to S. Carl and S. Heikkilä (see Corollary 2.2 in [8]) which we use to prove Theorem 1.3.

LEMMA 2.4. *Let X be a Banach semilattice which is reflexive. Then any closed ball of X has the fixed point property.*

For more details with respect to definitions and results about ordered Banach spaces, we refer to [8] and references therein.

3. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. To do this, we need to introduce some appropriate operators in order to apply Lemma 2.4. First, we consider the operator $L: X \rightarrow X'$ defined by

$$\langle Lu, v \rangle = \int_{\mathbb{R}^n} |\nabla u|^{n-2} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n} V^+(x) |u|^{n-2} uv \, dx, \quad u, v \in X.$$

Note that for each $u \in X$, Lu is a linear map. Moreover, we deduce from Hölder inequality that

$$|\langle Lu, v \rangle| \leq 2 \|u\|^{n-1} \|v\|,$$

which shows that $Lu \in X'$ and therefore L is well defined. Furthermore, we have

LEMMA 3.1. *Under the hypothesis (V_1) , the operator $L: X \rightarrow X'$ is continuous and invertible.*

POOF. Let (u_k) in X such that $u_k \rightarrow u$ in X . Using Hölder inequality, for $v \in X$ with $\|v\| \leq 1$ we obtain

$$\begin{aligned} |\langle Lu_k - Lu, v \rangle| &\leq \int_{\mathbb{R}^n} \left| |\nabla u_k|^{n-2} \nabla u_k - |\nabla u|^{n-2} \nabla u \right| |\nabla v| \, dx \\ &\quad + \int_{\mathbb{R}^n} V^+(x) \left| |u_k|^{n-2} u_k - |u|^{n-2} u \right| |v| \, dx \\ &\leq \left(\int_{\mathbb{R}^n} \left| |\nabla u_k|^{n-2} \nabla u_k - |\nabla u|^{n-2} \nabla u \right|^{n/(n-1)} \, dx \right)^{(n-1)/n} \\ &\quad + \left(\int_{\mathbb{R}^n} V^+(x) \left| |u_k|^{n-2} u_k - |u|^{n-2} u \right|^{n/(n-1)} \, dx \right)^{(n-1)/n}. \end{aligned}$$

Since $u_k \rightarrow u$ in X , up to a subsequence, $\nabla u_k \rightarrow \nabla u$ and $u_k \rightarrow u$ almost everywhere in \mathbb{R}^n . Moreover, there exist $w_1, w_2 \in L^1(\mathbb{R}^n)$ such that $|\nabla u_k|^n \leq w_1 \in L^1(\mathbb{R}^n)$ and $V^+(x) |u_k|^n \leq w_2 \in L^1(\mathbb{R}^n)$ almost everywhere in \mathbb{R}^n . Hence, in view of the Lebesgue dominated convergence theorem and from the last inequality

$$\|Lu_k - Lu\|_{X'} = \sup_{v \in X, \|v\| \leq 1} |\langle Lu_k - Lu, v \rangle| \rightarrow 0$$

and the continuity of L is proved. Furthermore, from $\langle Lu, u \rangle = \|u\|^n$ for all $u \in X$ and $n \geq 2$ we get

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle Lu, u \rangle}{\|u\|} = \infty.$$

On the other hand, using the well known inequality

$$(3.1) \quad (|a|^{n-2}a - |b|^{n-2}b)(a - b) \geq C_n|a - b|^{n-2}, \quad a, b \in \mathbb{R}^n,$$

one has

$$\langle Lv_1 - Lv_2, v_1 - v_2 \rangle \geq C_n\|v_1 - v_2\|^n > 0$$

for all $v_1, v_2 \in X$, $v_1 \neq v_2$. Thus, by the Minty-Browder theorem (see for instance Theorem 5.16 in [5]), the operator L is invertible and this completes the proof. \square

At this point, we consider another operator $T : X \rightarrow X'$ given by

$$\langle Tu, v \rangle = \int_{\mathbb{R}^n} [V^-(x)|u|^{n-2}u + g(x, u)]v \, dx + \lambda \langle h, v \rangle, \quad u, v \in X.$$

It is clear that for each fixed $u \in X$ fixed Tu is a linear map. As a consequence of Lemma 2.3 we have the following estimate:

LEMMA 3.2. *Assume (V₁)–(V₂) and (G₁)–(G₂). Let $M > 0$ be such that $\alpha_0 M^{n/(n-1)} < \alpha_n$. There exist positive constants C_1, C_2 such that if $\|u\| \leq M$, then*

$$|\langle Tu, v \rangle| \leq (S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})})\|u\|^{n-1} + C_1\|u\|^\rho + C_2\|u\|^\mu + \lambda\|h\|_{X'}\|v\|,$$

for all $v \in X$. In particular,

$$\|Tu\|_{X'} \leq S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})}\|u\|^{n-1} + C_1\|u\|^\rho + C_2\|u\|^\mu + \lambda\|h\|_{X'}.$$

PROOF. Here, let us consider $1 < p < \infty$ and $1 < \sigma < \infty$. The case $p = \infty$ or $\sigma = \infty$ are more simple and they are treated similarly. Note that

$$\frac{1}{p} + \frac{n-1}{t_0} + \frac{1}{t_0} = 1 \Leftrightarrow t_0 = \frac{np}{p-1}.$$

Since $V^-(x) = 0$ for $|x| \geq R_0$, using the generalized Hölder inequality together with the inequality $\|u\|_{t_0}^n \leq S_{t_0}^{-1}\|u\|^n$ one has

$$(3.2) \quad \left| \int_{\mathbb{R}^n} V^-(x)|u|^{n-2}uv \, dx \right| \leq S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})}\|u\|^{n-1}\|v\|.$$

Analogously, setting $\beta := \sigma(\rho+1)/(\sigma-1) > n$ we see that

$$\frac{1}{\sigma} + \frac{\rho}{\beta} + \frac{1}{\beta} = 1,$$

from which follows that

$$(3.3) \quad \left| \int_{\mathbb{R}^n} k(x)|u|^\rho v \, dx \right| \leq \|k\|_\sigma \|u\|_\beta^\rho \|v\|_\beta \leq C \|k\|_\sigma \|u\|^\rho \|v\|.$$

On the other hand, Lemma 2.3 with $\alpha = \alpha_0$ yields

$$(3.4) \quad \int_{\mathbb{R}^n} \Phi_{\alpha_0}(u)|u|^\mu|v| \, dx \leq C\|u\|^\mu\|v\|.$$

Since $|\langle h, v \rangle| \leq \|h\|_{X'}\|v\|$, using estimates (3.2)–(3.4) and condition (G_2) , we obtain the desired result. \square

Analyzing the previous proof, we observe that (3.2) and (3.3) holds for each $u \in X$ as well as estimate (2.4). This fact together with hypothesis (G_2) shows that the operator T is well defined.

Now, we define the operator $S: X \rightarrow X$ by $S = L^{-1} \circ T$.

In order to address the existence of a fixed point for S , we need the following lemma:

LEMMA 3.3. *Under the hypotheses of Theorem 1.3, there exists $R > 0$ such that $S(B_X[0, R]) \subset B_X[0, R]$, where $B_X[0, R] = \{u \in X : \|u\| \leq R\}$.*

PROOF. Let $u \in X$ and set $v = (L^{-1} \circ T)u = Su$. Since $\langle Lv, v \rangle = \|v\|^n$ we get

$$\|Su\|^n = \langle Tu, Su \rangle \leq \|Tu\|_{X'}\|Su\|.$$

Choosing $R \leq M$, if $\|u\| \leq R$ then by Lemma 3.2 one has

$$\begin{aligned} \|Su\|^{n-1} &\leq \|Tu\|_{X'} \leq S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})}\|u\|^{n-1} + C_1\|u\|^\rho + C_2\|u\|^\mu + \lambda\|h\|_{X'} \\ &\leq S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})}R^{n-1} + C_1R^\rho + C_2R^\mu + \lambda\|h\|_{X'}, \end{aligned}$$

which implies

$$(3.5) \quad \frac{\|Su\|^{n-1}}{R^{n-1}} \leq S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})} + C_1R^{\rho-n+1} + C_2R^{\mu-n+1} + \lambda\frac{\|h\|_{X'}}{R^{n-1}}.$$

Now, choose $R > 0$ sufficiently small so that

$$C_1R^{\rho-n+1} + C_2R^{\mu-n+1} \leq \frac{1 - S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})}}{2}.$$

Setting

$$\lambda_0 := \frac{R^{n-1}(1 - S_{t_0}^{-1}\|V^-\|_{L^p(B_{R_0})})}{2\|h\|_{X'}},$$

for all $0 < \lambda \leq \lambda_0$, we deduce from (3.5) that $\|Su\|^{n-1}/R^{n-1} \leq 1$. Therefore, $S(B_X[0, R]) \subset B_X[0, R]$ and the proof is complete. \square

At this point, we are ready to prove our main result.

PROOF OF THEOREM 1.3. In order to apply Lemma 2.4, we consider the following partial order in X :

$$(3.6) \quad u_1, u_2 \in X, \quad u_1 \preceq u_2 \Leftrightarrow u_1 \leq u_2 \text{ a.e. in } \mathbb{R}^n.$$

It is clear that (X, \preceq) is an ordered Banach space and for all $u, v \in X$, there exist $\sup\{u, v\}$ and $\inf\{u, v\}$ with respect to the order \preceq . Moreover, recalling

that $u^+ = \sup\{u, 0\}$ and $u^- = -\inf\{u, 0\}$, by the order (3.6), u^+ and u^- are the positive and negative parts of u . Since $|\nabla u^\pm| \leq |\nabla u|$ and $|u^\pm| \leq |u|$ almost everywhere in \mathbb{R}^n we see that $\|u^\pm\| \leq \|u\|$. Hence (X, \preceq) is a Banach semilattice which is reflexive. We also observe that the dual space X' , endowed with the order:

$$\varphi_1, \varphi_2 \in X', \varphi_1 \triangleleft \varphi_2 \Leftrightarrow \langle \varphi_1, v \rangle \leq \langle \varphi_2, v \rangle, \text{ for all } v \in X_+$$

where $X_+ := \{v \in X : v \geq 0 \text{ almost everywhere in } \mathbb{R}^n\}$, is an ordered Banach space.

We claim that $L^{-1}: (X', \triangleleft) \rightarrow (X, \preceq)$ is an increasing operator. Indeed, let $\varphi_1, \varphi_2 \in X'$ such that $\varphi_1 \triangleleft \varphi_2$. Setting $u_1 = L^{-1}\varphi_1$ and $u_2 = L^{-1}\varphi_2$, for $v \in X_+$ one has

$$\begin{aligned} 0 \leq \langle \varphi_2 - \varphi_1, v \rangle &= \langle Lu_2 - Lu_1, v \rangle \\ &= \int_{\mathbb{R}^n} (|\nabla u_2|^{n-2} \nabla u_2 - |\nabla u_1|^{n-2} \nabla u_1) \nabla v \, dx \\ &\quad + \int_{\mathbb{R}^n} V^+(x) (|u_2|^{n-2} u_2 - |u_1|^{n-2} u_1) v \, dx. \end{aligned}$$

From this, taking $v = (u_2 - u_1)^- \in X_+$ and using (3.1) we reach

$$0 \leq \langle \varphi_2 - \varphi_1, (u_2 - u_1)^- \rangle \leq -C_n \|(u_2 - u_1)^-\|^n \leq 0$$

whence we conclude that $(u_2 - u_1)^- = 0$ and so $u_1 \leq u_2$ almost everywhere in \mathbb{R}^n , that is, $L^{-1}\varphi_1 \preceq L^{-1}\varphi_2$. Now, we prove that $T: (X, \preceq) \rightarrow (X', \triangleleft)$ is increasing. To this, let $u_1, u_2 \in X$ such that $u_1 \leq u_2$ almost everywhere in \mathbb{R}^n . Since the function $f(s) := |s|^{n-2}s$ is increasing it follows from assumption (G₂) that

$$\begin{aligned} \langle Tu_1, v \rangle &= \int_{\mathbb{R}^n} [V^-(x)|u_1|^{n-2}u_1 + g(x, u_1)]v \, dx + \lambda \langle h, v \rangle \\ &\leq \int_{\mathbb{R}^n} [V^-(x)|u_2|^{n-2}u_2 + g(x, u_2)]v \, dx + \lambda \langle h, v \rangle = \langle Tu_2, v \rangle, \end{aligned}$$

for all $v \in X_+$, that is, $Tu_1 \triangleleft Tu_2$. Consequently, by definition, the operator $S: (X, \preceq) \rightarrow (X, \preceq)$ is also increasing. By Lemma 2.4, $B_X[0, R]$ has the *fixed point property* and in view of Lemma 3.3 there exists $u_0 \in B_X[0, R]$ such that $Su_0 = u_0$. Since $S = L^{-1} \circ T$ we have

$$\langle Lu_0, v \rangle = \langle Tu_0, v \rangle, \text{ for all } v \in X.$$

Thus, by the definitions of L and T , u_0 is a weak solution of (1.1) and the theorem is proved. \square

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