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ON THE SOLVABILITY OF NONLINEAR IMPULSIVE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we provide sufficient conditions for the existence of solutions to two-point boundary value problems for nonlinear ordinary differential equations subject to impulses. Our results depend on properties of the nonlinearities as well as on the solution space of the associated linear problem. Our approach is based on topological degree arguments in conjunction with the Lyapunov–Schmidt procedure.

1. Introduction

In this paper we provide criteria for the solvability of nonlinear, impulsive, two-point boundary value problems. We consider problems of the form

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0, 1] \setminus \{t_1, \ldots, t_k\}, \tag{1.1}$$

$$x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \quad i = 1, \ldots, k, \tag{1.2}$$

subject to boundary conditions

$$Bx(0) + Dx(1) = 0, \tag{1.3}$$

where the points t_i, $i = 1, \ldots, k$, are fixed with $0 < t_1 < \ldots < t_k < 1$.

Throughout the discussion we will assume that f, each J_i, and A are continuous. $x(t)$ is an element of $\mathbb{R}^n$, $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, $J_i\colon \mathbb{R}^n \to \mathbb{R}^n$, and for each $t \in [0, 1]$, $A(t)$ is an $n \times n$ matrix. The matrices B and D are $n \times n$ and, in order

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to avoid redundancies, we will assume that the augmented matrix $[B|D]$ has full row rank.

The main objective of this paper is the study of nonlinear impulsive boundary value problems at resonance; that is, systems where the associated linear homogeneous problem has nontrivial solutions. Our approach is based on the use of topological degree theory in conjunction with the Lyapunov–Schmidt procedure. The results we obtain depend on both properties of the nonlinearities and the solution space of the associated linear homogeneous problem.

There is an extensive literature regarding degree theory, the Lyapunov–Schmidt procedure, and projection schemes in nonlinear analysis. General theoretical results and applications to boundary value problems in differential equations can be found in [1], [3], [5], [10]–[12], [14]–[16], [18]. The solvability of discrete systems is considered in [2], [8], [13]. Those interested in the theory and application of impulsive systems may consult [4], [6], [7], [9], [17].

2. Preliminaries

We will formulate the nonlinear boundary value problem (1.1)–(1.3) as an operator equation. In order to do so, we introduce appropriate spaces and operators.

$\mathrm{PC}_{\{t_i\}}[0,1]$ will represent the set of $\mathbb{R}^n$-valued continuous functions on $[0,1]\setminus\{t_1,\ldots,t_k\}$ which have right and left-hand limits at each t_i, $i=1,\ldots,k$. On $\mathrm{PC}_{\{t_i\}}[0,1]$ we will use the supremum norm; that is,

$$\|\phi\| = \sup_{t\in[0,1]\setminus\{t_1,\ldots,t_k\}} |\phi(t)|,$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$. It is well known that when endowed with this norm, $\mathrm{PC}_{\{t_i\}}[0,1]$ is a Banach space. The subset of $\mathrm{PC}_{\{t_i\}}[0,1]$ consisting of continuously differentiable functions $\phi\colon [0,1]\setminus\{t_1,\ldots,t_k\}\to\mathbb{R}^n$ such that ϕ' has finite right and left-hand limits at each t_i, $i=1,\ldots,k$, will be denoted by $\mathrm{PC}^1_{\{t_i\}}[0,1]$. Finally, we define

$$X=\{\phi\in \mathrm{PC}_{\{t_i\}}[0,1] \mid B\phi(0)+D\phi(1)=0\}.$$

The norms on $\mathrm{PC}^1_{\{t_i\}}[0,1]$ and X will be the same as on $\mathrm{PC}_{\{t_i\}}[0,1]$. We would like to remark that with this norm $\mathrm{PC}^1_{\{t_i\}}[0,1]$ is not a Banach space.

We now introduce mappings $\mathcal{L}$ and $\mathcal{F}$. The domain of $\mathcal{L}$, written $\mathrm{dom}(\mathcal{L})$, is given by

$$\mathrm{dom}(\mathcal{L}) = \mathrm{PC}^1_{\{t_i\}}[0,1]\cap X.$$

The mapping $\mathcal{L}\colon \operatorname{dom}(\mathcal{L}) \subset X \to \mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ is defined by

$$\mathcal{L}x = \begin{bmatrix} x'(\,\cdot\,) - A(\,\cdot\,)x(\,\cdot\,) \\ x(t_1^+) - x(t_1^-) \\ \vdots \\ x(t_k^+) - x(t_k^-) \end{bmatrix}.$$

The nonlinear operator $\mathcal{F}\colon \mathrm{PC}_{\{t_i\}}[0,1] \to \mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ is given by

$$\mathcal{F}x = \begin{bmatrix} f(\,\cdot\,, x(\,\cdot\,)) \\ J_1(x(t_1^-)) \\ \vdots \\ J_k(x(t_k^-)) \end{bmatrix}.$$

We make $\mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ a Banach space by introducing the following norm:

$$\left\| \begin{bmatrix} h \\ v_1 \\ \vdots \\ v_k \end{bmatrix} \right\| = \max\{\|h\|, |v_1|, \dots, |v_k|\}.$$

Remark 2.1. With the definitions as above, it is clear that solving the nonlinear boundary value problem (1.1)–(1.3) is equivalent to solving $\mathcal{L}x = \mathcal{F}x$.

Before focusing on the nonlinear boundary problem (1.1)–(1.3), we analyze the linear homogeneous problem

$$x'(t) = A(t)x(t), \quad t \in [0,1] \setminus \{t_1, \dots, t_k\} \tag{2.1}$$

subject to boundary conditions (1.3), as well as the linear nonhomogeneous problem

$$\begin{aligned} &x'(t) = A(t)x(t) + h(t), && t \in [0,1] \setminus \{t_1, \dots, t_k\}, \\ &x(t_i^+) - x(t_i^-) = v_i, && i = 1, \dots, k, \end{aligned} \tag{2.2}$$

subject to the same boundary conditions. Here we assume $h \in \mathrm{PC}_{\{t_i\}}[0,1]$ and each v_i, $i = 1, \dots, k$, is an element of $\mathbb{R}^n$.

It is clear that a function x is a solution to the linear nonhomogeneous problem (2.2) subject to boundary conditions (1.3) if and only if $\mathcal{L}x = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$, where $v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. Taking $\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right] = 0$, we see that the solution space of the linear homogeneous problem (2.1) subject to the boundary conditions (1.3) is given by the $\operatorname{Ker}(\mathcal{L})$. We now characterize $\operatorname{Ker}(\mathcal{L})$ and $\operatorname{Im}(\mathcal{L})$.

Proposition 2.2. *A function x is a solution to the linear homogeneous problem* (2.1) *subject to the boundary conditions* (1.3) *if and only if $x(t) = \Phi(t)c$ for some $c \in \mathrm{Ker}(B + D\Phi(1))$. Here $\Phi(\,\cdot\,)$ is the principal fundamental matrix solution to $x' = A(\,\cdot\,)x$.*

Proof.

$$\begin{aligned}\mathcal{L}x = 0 &\Leftrightarrow x' = A(\cdot)x \text{ and } Bx(0) + Dx(1) = 0\\ &\Leftrightarrow x = \Phi(\,\cdot\,)x(0) \text{ and } Bx(0) + Dx(1) = 0\\ &\Leftrightarrow \text{there exists } c \in \mathbb{R}^n, \text{ such that } x = \Phi(\,\cdot\,)c \text{ and } Bc + D\Phi(1)c = 0. \quad \square\end{aligned}$$

Corollary 2.3. *The solution space of the linear homogeneous problem* (2.1) *subject to the boundary conditions* (1.3) *has the same dimension as the* $\mathrm{Ker}(B + D\Phi(1))$.

We now choose vectors $b_1, \ldots, b_p$, where $p \leq n$, from $\mathbb{R}^n$ which form a basis for $\mathrm{Ker}(B + D\Phi(1))$ and make the following definition:

Definition 2.4. We define $S(t)$ to be the $n \times p$ matrix whose ith column is $S_i(t) := \Phi(t)b_i$.

Corollary 2.5. *A function x is a solution to the linear homogeneous problem* (2.1) *with boundary conditions* (1.3) *if and only if $x(\,\cdot\,) = S(\,\cdot\,)\alpha$ for some $\alpha \in \mathbb{R}^p$.*

Proposition 2.6. *Let $\{c_1, \ldots, c_p\}$ be a basis for $\mathrm{Ker}((B + D\Phi(1))^T)$. Then the linear nonhomogeneous problem* (2.2) *subject to the boundary conditions* (1.3) *has a solution if and only if for each $i = 1, \ldots, p$, we have*

$$\bigg\langle c_i, D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\bigg)\bigg\rangle = 0.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Proof. It is well documented, see [4], [17], that $\mathcal{L}x = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$ if and only if x is given by the variation of parameters formula

$$x(t) = \Phi(t)\bigg(x(0) + \int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i < t}\Phi^{-1}(t_i)v_i\bigg) \tag{2.3}$$

and x satisfies the boundary conditions (1.3).

Imposing the boundary conditions, we have $\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right] \in \mathrm{Im}(\mathcal{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$Bw + D\bigg(\Phi(1)\bigg(w + \int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{l=1}^{k}\Phi^{-1}(t_l)v_l\bigg)\bigg)$$

$$\Leftrightarrow [B + D\Phi(1)]w = -D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{l=1}^{k} \Phi^{-1}(t_l)v_l\bigg)$$

$$\Leftrightarrow D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{l=1}^{k} \Phi^{-1}(t_l)v_l\bigg) \in \mathrm{Im}(B + D\Phi(1)). \tag{2.4}$$

Using the fact that $\mathrm{Im}(B+D\Phi(1))$ is the orthogonal complement of $\mathrm{Ker}((B+D\Phi(1))^{\mathrm{T}})$, we have that

$$\begin{bmatrix} h \\ v \end{bmatrix} \in \mathrm{Im}(\mathcal{L}) \Leftrightarrow \bigg\langle c, D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{l=1}^{k}\Phi^{-1}(t_l)v_l\bigg)\bigg\rangle = 0$$

for all $c \in \mathrm{Ker}((B + D\Phi(1))^{\mathrm{T}})$.

If we now define $W := [c_1, \dots, c_p]$ and $\Psi(t)^T := W^T D\Phi(1)\Phi^{-1}(t)$, we get the following corollary:

COROLLARY 2.7. *The linear nonhomogeneous problem* (2.2) *with boundary conditions* (1.3) *has a solution if and only if*

$$\int_0^1 \Psi^T(s)h(s)\,ds + \sum_{i=1}^{k}\Psi^T(t_i)v_i = 0.$$

REMARK 2.8. It is now clear that the linear nonhomogeneous boundary value problem (2.2) subject to the boundary conditions (1.3) has a unique solution if and only if $B + D\Phi(1)$ is invertible. If this is the case, $\mathcal{L}$ is a bijection. We then have, for each element $\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right] \in \mathrm{PC}_{\{t_i\}}[0,1]$, that the unique solution to $\mathcal{L}x = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$ is given by

$$\begin{aligned} x(t) =& \mathcal{L}^{-1}\begin{bmatrix} h \\ v \end{bmatrix}(t) \\ =& \Phi(t)\bigg(-[B + D\Phi(1)]^{-1}D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\bigg)\bigg) \\ &+ \Phi(t)\bigg(\int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t}\Phi^{-1}(t_i)v_i\bigg). \end{aligned}$$

3. Main Results

In this section we focus on the nonlinear boundary value problem

$$\begin{aligned} &x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, \dots, t_k\}, \\ &x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \qquad i = 1, \cdots, k, \end{aligned}$$

with boundary conditions

$$Bx(0) + Dx(1) = 0.$$

We are mainly interested in systems at resonance and our principle result in this regard is Theorem 3.7. In this theorem we establish conditions for the existence of solutions which are based on the interplay between the nonlinearities $f, J_1, \ldots, J_k$ and the solution space of the linear homogeneous problem (2.1) subject to the boundary conditions (1.3).

In Theorem 3.1 we present criteria for the solvability of (1.1)–(1.3) in the nonresonant case. The analysis in this case is simpler and the results obtained here are based on the growth rate of the nonlinearities.

THEOREM 3.1. *Suppose that the only solution to the linear homogeneous problem* (2.1) *subject to the boundary conditions* (1.3) *is the trivial solution. If there exist real numbers M_1, M_2 and α, with $0 \le \alpha < 1$, such that for all $t \in [0,1]$ and $y \in \mathbb{R}^n$, $|f(t,y)| \le M_1|y|^\alpha + M_2$ and $|J_i(y)| \le M_1|y|^\alpha + M_2$, then the nonlinear boundary value problem* (1.1)–(1.3) *has a solution.*

PROOF. Define $H\colon \mathrm{PC}_{\{t_i\}}[0,1] \to \mathrm{PC}_{\{t_i\}}[0,1]$ by

$$\begin{aligned}[H(x)](t) = \Phi(t)\bigg(&- [B + D\Phi(1)]^{-1} D\Phi(1) \\ &\cdot \bigg(\int_0^1 \Phi^{-1}(s) f(s, x(s))\, ds + \sum_{i=1}^{k} \Phi^{-1}(t_i) J_i(x(t_i^-)) \bigg)\bigg) \\ &+ \Phi(t)\bigg(\int_0^t \Phi^{-1}(s) f(s, x(s))\, ds + \sum_{t_i < t} \Phi^{-1}(t_i) J_i(x(t_i^-)) \bigg).\end{aligned}$$

From Remark 2.8, it is clear that the solutions of (1.1)–(1.3) are precisely the fixed points of H.

Using the fact that for all $t \in [0,1]$ and $y \in \mathbb{R}^n$

$$|f(t,y)| \le M_1|y|^\alpha + M_2 \quad \text{and} \quad |J_i(y)| \le M_1|y|^\alpha + M_2,$$

it follows that there exist B_1, B_2 such that

$$\|H(x)\| \le B_1\|x\|^\alpha + B_2.$$

Since $\alpha < 1$, we may choose r sufficiently large such that $B_1 r^\alpha + B_2 \le r$. With this in mind, we define $\mathcal{B} = \{x \in \mathrm{PC}_{\{t_i\}}[0,1] : \|x\| \le r\}$.

It is clear that $H(\mathcal{B}) \subset \mathcal{B}$. From basic properties of integral operators, it is evident that H is compact. The existence of a fixed point for H is now a consequence of Schauder's theorem. $\square$

We now turn our attention to the case in which the linear homogeneous problem (2.1) subject to the boundary conditions (1.3) has a nontrivial solution space. In this case we analyze (1.1)–(1.3) using a projection scheme known as the Lyapunov–Schmidt procedure. To do so we construct projections onto the $\mathrm{Ker}(\mathcal{L})$ and $\mathrm{Im}(\mathcal{L})$.

DEFINITION 3.2. Let $V\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $\operatorname{Ker}(B+D\Phi(1))$. Define $P\colon X \to X$ by

$$[Px](t) = \Phi(t)Vx(0).$$

PROPOSITION 3.3. *P is a projection onto* $\operatorname{Ker}(\mathcal{L})$.

PROOF. $[P^2x](t) = \Phi(t)V^2x(0) = \Phi(t)Vx(0) = [Px](t)$, thus P is a projection. From the characterization of $\operatorname{Ker}(\mathcal{L})$, it follows that $\operatorname{Im}(P) = \operatorname{Ker}(\mathcal{L})$. $\square$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $\operatorname{Ker}(W^T D\Phi(1))$. It follows from Corollary 2.7 that

$$\begin{bmatrix} h \\ v \end{bmatrix} \in \operatorname{Im}(\mathcal{L}) \Leftrightarrow [I-T]\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) = 0.$$

DEFINITION 3.4. Define $E\colon \mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk} \to \mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ by

$$E\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} h(\,\cdot\,) - \Phi(\,\cdot\,)[I-T]\bigg(\displaystyle\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) \\ v \end{bmatrix}.$$

PROPOSITION 3.5. *E is a projection onto* $\operatorname{Im}(\mathcal{L})$.

PROOF.

$$\begin{aligned}
&[I-T]\bigg(\int_0^1 \Phi^{-1}(s)\bigg[h(s) - \Phi(s)(I-T) \\
&\qquad \cdot \bigg(\int_0^1 \Phi^{-1}(u)h(u)\,du + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg)\bigg] + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) \\
&\quad = [I-T]\bigg(\int_0^1 \Phi^{-1}(s)h(s) + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) \\
&\qquad - [I-T]^2\bigg(\int_0^1 \Phi^{-1}(u)h(u)\,du + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) = 0.
\end{aligned}$$

It follows that $E^2 = E$ and that $\operatorname{Im}(E) \subset \operatorname{Im}(\mathcal{L})$.

To see that $\operatorname{Im}(\mathcal{L}) \subset \operatorname{Im}(E)$ note that if $\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right] \in \operatorname{Im}(\mathcal{L})$, then

$$[I-T]\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) = 0.$$

We then have

$$\Phi(\,\cdot\,)[I-T]\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg) = 0,$$

from which it follows that $E\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$. $\square$

For the sake of completeness, we now give a self-contained description of the Lyapunov–Schmidt projection procedure.

Proposition 3.6. *Solving $\mathcal{L}x = \mathcal{F}x$ is equivalent to solving the system*

$$\begin{cases} x = Px + M_p E\mathcal{F}x \\ \textit{and} \\ (I-E)\mathcal{F}x = 0 \end{cases}$$

where M_p is $\mathcal{L}^{-1}_{|\mathrm{Ker}(P)\cap \mathrm{dom}(\mathcal{L})}$.

Proof. We have

$$\mathcal{L}x = \mathcal{F}x \Leftrightarrow \begin{cases} E[\mathcal{L}x - \mathcal{F}x] = 0 \\ \text{and} \\ (I-E)[\mathcal{L}x - \mathcal{F}x] = 0 \end{cases} \Leftrightarrow \begin{cases} \mathcal{L}x - E\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} M_p\mathcal{L}x - M_pE\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0 \end{cases} \Leftrightarrow \begin{cases} (I-P)x - M_pE\mathcal{F}x = 0 \\ \text{and} \\ (I-E)\mathcal{F}x = 0. \end{cases} \qquad \square$$

We now come to our main result concerning the nonlinear boundary value problem (1.1)-(1.3). Before stating the result, we make some introductory assumptions and definitions.

In the following it will be assumed that for sufficiently large r, the map

$$(t,x) \to \begin{bmatrix} f(t,x) \\ J_1(x) \\ \vdots \\ J_k(x) \end{bmatrix}$$

is Lipschitz, in x, on the complement of $B(0,r)$. Here we use the standard convention of denoting, for any normed space Y, $\{y \in Y : \|y\| < r\}$ by $B(0,r)$. More specifically, we assume there exist real numbers R_0 and L, such that for all $t \in [0,1]$ and any x and $y \in \mathbb{R}^n$ with $|x| > R_0$ and $|y| > R_0$, we have

$$\left\| \begin{bmatrix} f(t,x) - f(t,y) \\ J_1(x) - J_1(y) \\ \vdots \\ J_k(x) - J_k(y) \end{bmatrix} \right\| \leq L|x-y|.$$

We let, for $r \geq R_0$, $L(r)$ denote the smallest Lipschitz constant on the complement of $B(0,r)$.

The following observation will be used in what follows. Let $M_{n\times p}$ denote the space of real-valued $n\times p$ matrices. The map $(C,y)\to Cy$ from $M_{n\times p}\times\mathbb{R}^p$ is obviously a continuous bilinear map. Combining this with the fact that $t\to S(t)$ is continuous, we see that $(t,\alpha)\to S(t)\alpha$ is the composition of continuous maps and therefore continuous. It follows that $(t,\alpha)\to|S(t)\alpha|$ attains its minimum on the compact set

$$\mathcal{O}:=[0,1]\times\{\alpha\in\mathbb{R}^p:|\alpha|=1\}.$$

For each $\alpha\neq 0$, $S(\,\cdot\,)\alpha$ is a nonzero solution to (2.1) and so $\eta:=\inf_{(t,\alpha)\in\mathcal{O}}|S(t)\alpha|>0$.

Theorem 3.7. *Suppose the following conditions hold:*

(C1) *The functions $f, J_1,\dots,J_k$, are bounded, say by b.*

(C2) *There exist real numbers R, $d>0$, and β such that for all $\alpha\in\mathbb{R}^p$ with $|\alpha|>R$,*

$$\left|\int_0^1\Psi^T(t)f(t,S(t)\alpha)\,dt+\sum_{i=1}^k\Psi^T(t_i)J_i(S(t_i)\alpha)\right|\geq d$$

and

$$\left\langle\alpha,\int_0^1\Psi^T(t)f(t,S(t)\alpha)\,dt+\sum_{i=1}^k\Psi^T(t_i)J_i(S(t_i)\alpha)\right\rangle\geq\beta>-d^2.$$

(C3) $\lim_{r\to\infty}L(r)(k+1)\,\|M_pE\|\,\left\|\Psi^T(\,\cdot\,)\right\|b<\min\{\sqrt{d^2+\beta}/\sqrt{2},d\}$, *where* $\left\|\Psi^T(\,\cdot\,)\right\|=\sup_{t\in[0,1]}\left\|\Psi^T(t)\right\|$, $\|M_pE\|$ *denotes the operator norm of* M_pE, *and* b *is as in* (C1).

Then there exists a solution to the nonlinear boundary value problem (1.1)–(1.3).

Proof. Since the functions $f,J_1,\dots,J_k$ are bounded by b, we have $\|\mathcal{F}x\|\leq b$ for each x in $\mathrm{PC}_{\{t_i\}}[0,1]$. For convenience, we assume $\{b_1,\dots,b_p\}$ (Definition 2.4) and $\{c_1,\dots,c_p\}$ (Proposition 2.6) have been chosen such that

$$\|S(\,\cdot\,)\|\leq 1\quad\text{and}\quad\left\|\Psi^T(\,\cdot\,)\right\|\leq 1.$$

From (C1)–(C3), there exists a positive real number, which we also denote by R, such that for all $\alpha\in\mathbb{R}^p$ with $|\alpha|\geq R$ and each real number $r\geq R$, we have the following:

(a) $\displaystyle\left|\int_0^1\Psi^T(t)f(t,S(t)\alpha)\,dt+\sum_{i=1}^k\Psi^T(t_i)J_i(S(t_i)\alpha)\right|\geq d.$

(b) $\displaystyle\left\langle\alpha,\int_0^1\Psi^T(t)f(t,S(t)\alpha)\,dt+\sum_{i=1}^k\Psi^T(t_i)J_i(S(t_i)\alpha)\right\rangle\geq\beta>-d^2.$

(c) $\displaystyle L(r)(k+1)\,\|M_pE\|\,\left\|\Psi^T(\,\cdot\,)\right\|b<\min\left\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}},d\right\}.$

We will establish the existence of a solution to (1.1)–(1.3) by showing the existence of a fixed point for an operator H.

We define the operator $H\colon \mathbb{R}^p \times \mathrm{Im}(I-P) \to \mathbb{R}^p \times \mathrm{Im}(I-P)$ by

$$H(\alpha,x)=\begin{bmatrix} \alpha - \displaystyle\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t))\,dt - \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-)) \\ \\ M_p E \mathcal{F}(S(\,\cdot\,)\alpha + x) \end{bmatrix}.$$

We use the max norm on the space $\mathbb{R}^p \times \mathrm{Im}(I-P)$; that is,

$$\|(\alpha,x)\| = \max\{|\alpha|, \|x\|\}.$$

For $h \in \mathrm{PC}_{\{t_i\}}[0,1]$ and $v \in \mathbb{R}^{nk}$ define

$$\begin{aligned} N_{h,v}(t) = \Phi(t)\bigg(- M_{BD} D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg)\bigg) \\ + \Phi(t)\bigg(\int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i\bigg), \end{aligned}$$

where M_{BD} denotes the right inverse of $B+D\Phi(1)$ when restricted to orthogonal complement of $\mathrm{Ker}(B+B\Phi(1))$. Since

$$N_{h,v}(0) = -M_{BD} D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg),$$

we have $VN_{h,v}(0) = 0$ and thus $P(N_{h,v}) = 0$. Further, from the characterization of the $\mathrm{Im}(\mathcal{L})$, (2.3)–(2.4), it follows that $\mathcal{L}(N_{h,v}) = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$. Since $M_p\left(\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]\right)$ is the unique element satisfying $P\left(M_p\left(\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]\right)\right) = 0$ and $\mathcal{L}\left(M_p\left(\left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]\right)\right) = \left[\begin{smallmatrix} h \\ v \end{smallmatrix}\right]$, it follows that

$$\begin{aligned} M_p\left(\begin{bmatrix} h \\ v \end{bmatrix}\right)(t) = \Phi(t)\bigg(- M_{BD} D\Phi(1)\bigg(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^k \Phi^{-1}(t_i)v_i\bigg)\bigg) \\ + \Phi(t)\bigg(\int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i\bigg). \end{aligned}$$

We now establish the compactness of M_p. Recall that for $\left[\begin{smallmatrix} h \\ v_1 \\ \vdots \\ v_k \end{smallmatrix}\right]$ in $\mathrm{PC}_{\{t_i\}}[0,1] \times \mathbb{R}^{nk}$ we use the norm

$$\left\| \begin{bmatrix} h \\ v_1 \\ \vdots \\ v_k \end{bmatrix} \right\| = \max\{\|h\|, |v_1|, \ldots, |v_k|\}.$$

Let

$$\mathcal{C} = \left\{ \begin{bmatrix} h \\ v_1 \\ \vdots \\ v_k \end{bmatrix} : \left\| \begin{bmatrix} h \\ v_1 \\ \vdots \\ v_k \end{bmatrix} \right\| \le c \right\}.$$

Writing $\left[\begin{smallmatrix} h \\ v \end{smallmatrix} \right]$ for an element $\left[\begin{smallmatrix} h \\ v_1 \\ \vdots \\ v_k \end{smallmatrix} \right]$ in $\mathcal{C}$, we have

$$\begin{aligned}
&\left| M_p\left(\begin{bmatrix} h \\ v \end{bmatrix} \right)(t) \right| \\
&= \left| \Phi(t)\Big(- M_{BD} D\Phi(1)\Big(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \Big)\Big) \right. \\
&\quad \left. + \Phi(t)\Big(\int_0^t \Phi^{-1}(s)h(s)\,ds + \sum_{t_i<t} \Phi^{-1}(t_i)v_i \Big) \right| \\
&\le \|\Phi(\,\cdot\,)\| \, \|M_{BD}D\Phi(1)\| \, \big\|\Phi^{-1}(\,\cdot\,)\big\| \, c(k+1) + \|\Phi(\,\cdot\,)\| \, \big\|\Phi^{-1}(\,\cdot\,)\big\| \, c(k+1).
\end{aligned}$$

Thus, $M_p(\mathcal{C})$ is a uniformily bounded family of functions in $\mathrm{PC}_{\{t_i\}}[0,1]$. Further, if $t_j < u < w < t_{j+1}$ for some $j = 0, \dots, k$, where $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$ then

$$\begin{aligned}
&\left| M_p\left(\begin{bmatrix} h \\ v \end{bmatrix} \right)(w) - M_p\left(\begin{bmatrix} h \\ v \end{bmatrix} \right)(u) \right| \\
&= \left| (\Phi(w) - \Phi(u))\Big(- M_{BD} D\Phi(1)\Big(\int_0^1 \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \Big)\Big) \right. \\
&\quad + (\Phi(w) - \Phi(u))\Big(\int_0^u \Phi^{-1}(s)h(s)\,ds + \sum_{i=1}^{j} \Phi^{-1}(t_i)v_i \Big) \\
&\quad \left. + \Phi(w) \int_u^w \Phi^{-1}(s)h(s)\,ds \right| \\
&\le \|\Phi(w) - \Phi(u)\| \, \big\|\Phi^{-1}(\,\cdot\,)\big\| \, c(\|M_{BD}D\Phi(1)\| \, k + j + 1) \\
&\quad + \|\Phi(\,\cdot\,)\| \, \big\|\Phi^{-1}(\,\cdot\,)\big\| \, c(w-u),
\end{aligned}$$

so that it is clear that $M_p(\mathcal{C})$ is an equicontinuous family on any subinterval of $[0,1] \setminus \{t_1, \dots, t_k\}$. From the Arzelá–Ascoli Theorem applied to each subinterval between jumps, we see that $M_p(\mathcal{C})$ is a relatively compact subset of $\mathrm{PC}_{\{t_i\}}[0,1]$. The compactness of M_p now follows.

Since the composition of a compact map with a bounded map is compact, the compactness of M_p and the boundedness of $E\mathcal{F}$ imply that H is a compact operator. Further, from Proposition 3.6, having a solution to (1.1)–(1.3) is equivalent to H having a fixed point.

We choose $R^* > \max\{(k+1)b, (R + \|M_pE\|\, b)/\eta\}$ and define

$$\Omega := B(0, R^*) \times B(0, \|M_pE\|\, b).$$

We will show that $\deg(I - H, \Omega, 0) \neq 0$. To this end, define

$$Q\colon [0,1] \times \overline{\Omega} \to \mathbb{R}^p \times \operatorname{Im}(I - P)$$

by

$$Q(\lambda, (\alpha, x)) =$$

$$\begin{bmatrix} (1-\lambda)\alpha + \lambda\bigg(\displaystyle\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t))\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-)\bigg) \\ x - \lambda M_p E\mathcal{F}(S(\,\cdot\,)\alpha + x) \end{bmatrix}$$

Using the fact that $\deg(Q(0, \,\cdot\,, \,\cdot\,), \Omega, 0) = \deg(I, \Omega, 0) = 1$, and that Q is clearly a homotopy between I and $I - H$, the result will follow once we show $0 \notin Q(\lambda, \partial(\Omega))$ for each $\lambda \in (0,1)$.

Now, it is clear that $(\alpha, x) \in \partial(\Omega)$ if and only if

$$|\alpha| = R^* \text{ and } \|x\| \leq \|M_pE\|\, b, \quad \text{or} \quad |\alpha| \leq R^* \text{ and } \|x\| = \|M_pE\|\, b.$$

With this in mind, let (α, x) be in $\partial(\Omega)$ and assume $|\alpha| \leq R^*$ with $\|x\| = \|M_pE\|\, b$. It follows that

$$\begin{aligned} \|x - \lambda M_pE\mathcal{F}(S(\,\cdot\,)\alpha + x)\| &\geq |\, \|x\| - \lambda \, \|M_pE\mathcal{F}(S(\,\cdot\,)\alpha + x)\| \,| \\ &\geq \|M_pE\|\, b - \lambda \, \|M_pE\|\, b > 0. \end{aligned}$$

Thus, $Q(\lambda, (\alpha, x)) \neq 0$.

Now suppose (α, x) is in $\partial(\Omega)$ and assume $|\alpha| = R^*$ with $\|x\| \leq \|M_pE\|\, b$. We then have

$$\begin{aligned} &\bigg|(1 - \lambda)\alpha + \lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t))\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-))\bigg)\bigg| \\ &\geq \bigg|(1 - \lambda)\alpha + \lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha)\bigg)\bigg| \\ &\quad - \bigg|\lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha)\bigg) \\ &\quad - \lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t))\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-))\bigg)\bigg|. \end{aligned}$$

Now

$$\bigg|(1 - \lambda)\alpha + \lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt + \sum_{i=1}^k \Psi^T(t_i) J_i(S(t_i)\alpha)\bigg)\bigg|^2$$

$$= (1-\lambda)^2|\alpha|^2 + \lambda^2\bigg|\bigg(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg)\bigg|^2$$

$$+ 2(1-\lambda)\lambda\bigg\langle \alpha, \int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg\rangle.$$

Since

$$|\alpha| = R^* > b(k+1) \geq \bigg|\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg|,$$

we have

$$(1-\lambda)^2|\alpha|^2 \geq (1-\lambda)^2\bigg|\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg|^2.$$

Combining this with the fact that (C2) holds, we have

$$\begin{aligned}(1-\lambda)^2|\alpha|^2 &+ \lambda^2\bigg|\bigg(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg)\bigg|^2 \\ &+ 2(1-\lambda)\lambda\bigg\langle \alpha, \int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg\rangle \\ &\geq ((1-\lambda)^2+\lambda^2)\bigg|\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg|^2 \\ &+ 2(1-\lambda)\lambda\bigg\langle \alpha, \int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg\rangle \\ &\geq ((1-\lambda)^2+\lambda^2)d^2 + 2(1-\lambda)\lambda\beta.\end{aligned}$$

For $\lambda \in [0,1]$, the function $\lambda \to ((1-\lambda)^2+\lambda^2)d^2 + 2(1-\lambda)\lambda\beta$ has a minimum of either $(d^2+\beta)/2$ or d^2. Thus,

$$\bigg|(1-\lambda)\alpha + \lambda\bigg(\int_0^1 \Psi^T(t)f(t,S(t)\alpha)\,dt + \sum_{i=1}^k \Psi^T(t_i)J_i(S(t_i)\alpha)\bigg)\bigg| \geq \min\left\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}}, d\right\}.$$

Using the fact that $|\alpha| \geq (R + \|M_pE\|\,b)/\eta$, we get

$$\inf_{t\in[0,1]} |S(t)\alpha| \geq \eta\bigg(\frac{R}{\eta}\bigg) = R,$$

and

$$\inf_{t\in[0,1]} |S(t)\alpha + x(t)| \geq \eta\bigg(\frac{R+\|M_pE\|\,b}{\eta}\bigg) - \|M_pE\|\,b = R.$$

It follows that

$$\bigg|(1-\lambda)\alpha + \lambda\bigg(\int_0^1 \Psi^T(t) f(t, S(t)\alpha + x(t))\,dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha + x(t_i^-))\bigg)\bigg|$$

$$\geq \min\bigg\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}}, d\bigg\} - \big\|\Psi^T(\,\cdot\,)\big\|\, L(R)(k+1)\,\|x\|$$

$$\geq \min\bigg\{\frac{\sqrt{d^2+\beta}}{\sqrt{2}}, d\bigg\} - \big\|\Psi^T(\,\cdot\,)\big\|\, L(R)(k+1)\,\|M_p E\|\, b > 0. \qquad \square$$

Remark 3.8. Theorem 3.7 is a considerable extension of the ideas appearing in [8], [15], [12] in many ways. First, it allows for continuous systems with impulses. Most importantly, it places no restriction on the dimension of the solution space of the linear homogeneous problem (2.1) with boundary conditions (1.3).

4. Examples

The following examples illustrate ways in which the hypothesis of the main result can be satisfied.

In our first example we analyze the solvability of

$$x'(t) = f(x(t)), \quad t \in [0,1] \setminus \bigg\{\frac{1}{4}\bigg\},$$

$$x\bigg(\frac{1}{4}^+\bigg) - x\bigg(\frac{1}{4}^-\bigg) = J\bigg(x\bigg(\frac{1}{4}^-\bigg)\bigg)$$

subject to

$$Bx(0) + Dx(1),$$

where

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $A = 0$, it follows that $\Phi(t) = I$ for all $t \in [0,1]$, and therefore

$$B + D\Phi(1) = B + D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We choose

$$W^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$\Psi^T(t) = W^T.$$

We now take

$$f(x_1, x_2, x_3) = \begin{bmatrix} (x_2 + \sin(x_2 + x_3))/(1 + \sqrt{x_1^2 + x_2^2 + x_3^2}) \\ x_3/(1 + \sqrt{x_1^2 + x_2^2 + x_3^2}) \\ f_3(x_1, x_2, x_3) \end{bmatrix}$$

and

$$J(x_1, x_2, x_3) = \begin{bmatrix} -x_3/(1 + \sqrt{x_1^2 + x_2^2 + x_3^2}) \\ (\cos(x_1 + x_2) + x_2)/(1 + \sqrt{x_1^2 + x_2^2 + x_3^2}) \\ J_3(x_1, x_2, x_3) \end{bmatrix},$$

where f_3 and J_3 are bounded continuous functions. We then have

$$\begin{aligned}\int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \\ = \begin{bmatrix} (\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \\ (\alpha_1 + \alpha_2 + \cos(\alpha_1))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \end{bmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}\left| \begin{bmatrix} (\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \\ (\alpha_1 + \alpha_2 + \cos(\alpha_1))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \end{bmatrix} \right|^2 \\ = \frac{2|\alpha|^2 + 2(\alpha_1 - \alpha_2)\sin(\alpha_1 + \alpha_2) + 2(\alpha_1 + \alpha_2)\cos(\alpha_1)}{(1 + |\alpha|)^2} \\ + \frac{\sin^2(\alpha_1 + \alpha_2) + \cos^2(\alpha_1)}{(1 + |\alpha|)^2}\end{aligned}$$

and

$$\begin{aligned}\left\langle \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} (\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \\ (\alpha_1 + \alpha_2 + \cos(\alpha_1))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \end{bmatrix} \right\rangle \\ = \frac{|\alpha|^2 + \alpha_1 \sin(\alpha_1 + \alpha_2) + \alpha_2 \cos(\alpha_1)}{1 + |\alpha|}.\end{aligned}$$

Thus, we may choose a real number R such that for each $\alpha \in \mathbb{R}^p$ with $|\alpha| \geq R$,

$$\left| \begin{bmatrix} (\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \\ (\alpha_1 + \alpha_2 + \cos(\alpha_1))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \end{bmatrix} \right| > 1$$

and

$$\left\langle \begin{bmatrix} \alpha_1 \\ \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} (\alpha_1 - \alpha_2 + \sin(\alpha_1 + \alpha_2))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \\ \\ (\alpha_1 + \alpha_2 + \cos(\alpha_1))/(1 + \sqrt{\alpha_1^2 + \alpha_2^2}) \end{bmatrix} \right\rangle > 0.$$

We now assume that, for $i = 1, 2, 3$, $\frac{\partial f_3}{\partial x_i}$ and $\frac{\partial J_3}{\partial x_i}$ exist and that

$$\lim_{r\to\infty} \sup_{|x|>r} \frac{\partial f_3}{\partial x_i}(x) < \infty \quad \text{and} \quad \lim_{r\to\infty} \sup_{|x|>r} \frac{\partial J_3}{\partial x_i}(x) < \infty.$$

An easy calculation shows

$$\begin{aligned} &Df(y_1, y_2, y_3) \\ &= c(y) \begin{bmatrix} -y_1y_2 - y_1 \sin(y_2+y_3) & -y_1y_3 & \frac{\partial f_3}{\partial x_1}(y_1, y_2, y_3) \\ d(y)(1+\cos(y_2+y_3)) - y_2^2 - y_2 \sin(y_2+y_3) & -y_2y_3 & \frac{\partial f_3}{\partial x_2}(y_1, y_2, y_3) \\ d(y)(\cos(y_2+y_3)) - y_2y_3 - y_3 \sin(y_2+y_3) & d(y) - y_3^2 & \frac{\partial f_3}{\partial x_3}(y_1, y_2, y_3) \end{bmatrix}^T, \end{aligned}$$

where $c(y) = 1/(|y|(1 + |y|)^2)$ and $d(y) = |y|(1 + |y|)$. It is then clear that

$$L_0^*(r) := \sup_{|x|>r} \|Df(x)\|$$

satisfies $\lim_{r\to\infty} L_0^*(r) = 0$. A simialr calculation shows the same is true for

$$L_i^*(r) := \sup_{|x|>r} \|DJ_i(x)\|.$$

An application of the integral mean value theorem then shows that (C3) is satisfied. Thus, by Theorem 3.7, the nonlinear boundary value problem has a solution.

Remark 4.1. We have chosen the matrix A to be 0 in order to convey the essential ideas of Theorem 3.7; that is, the relationship between the behavior of the nonlinearities and the solution space of the associated linear homogeneous boundary value problem. It should be clear that a similar analysis can be carried out when the matrix A is nonzero.

For our second example we focus on the solvability of

$$\begin{aligned} &x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in [0,1] \setminus \{t_1, \ldots, t_k\}, \\ &x(t_i^+) - x(t_i^-) = J_i(x(t_i^-)), \qquad i = 1, \ldots, k \end{aligned}$$

subject to

$$Bx(0) + Dx(1) = 0,$$

when, for large α, $\sum_{i=1}^{k} \Psi^T(t_i)J_i(S(t_i)\alpha)$ is bounded away from 0. That is, we assume that there exists positive real numbers R_1 and d, such that for all $\alpha \in \mathbb{R}^p$

with $|\alpha| > R_1$,

$$\left|\sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha)\right| > d.$$

If we assume the following:

(1) There exists a real number R_2 such that for all $\alpha \in \mathbb{R}^p$ with $|\alpha| > R_2$,

$$\left\langle \alpha, \int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt + \sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha) \right\rangle \geq 0.$$

(2) $\lim_{r\to\infty} L(r) = 0$,

then Theorem 3.7 guarantees that the nonlinear boundary value problem has a solution provided, for large α,

$$\left| \int_0^1 \Psi^T(t) f(t, S(t)\alpha)\, dt \right| < d.$$

We would like to point out the relative simplicity of computing

$$\left|\sum_{i=1}^{k} \Psi^T(t_i) J_i(S(t_i)\alpha)\right|.$$

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