Topological **M**ethods in **N**onlinear **A**nalysis
Volume 44, No. 2, 2014, 369–379

© 2014 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

SOME REMARKS ON PARK'S ABSTRACT CONVEX SPACES

Władysław Kulpa — Andrzej Szymanski

Abstract. We discuss S. Park's abstract convex spaces and their relevance to classical convexieties and L^*-operators. We construct an example of a space satisfying the partial KKM principle that is not a KKM space. The existence of such a space solves a problem by S. Park.

1. Introduction

The *KKM theorem* refers to the following celebrated theorem due to B. Knaster, K. Kuratowski, and S. Mazurkiewicz [4].

Theorem 1.1. *If Δ_n is the unit simplex in $\mathbf{R}^{n+1}$ and $F_1, \ldots, F_{n+1}$ are closed subsets of Δ_n such that $\Delta_J \subseteq \bigcup\{F_i : i \in J\}$ for each $J \subseteq \{1, \ldots, n+1\}$, then $F_1 \cap \ldots \cap F_{n+1} \neq \emptyset$.*

Since its publication in 1929, the KKM theorem has found numerous applications in various branches of mathematics. It is also considered to be fundamental in the development of many areas of mathematics (see e.g. [25]). In fact, the KKM theorem is one of equivalent versions of the Brouwer fixed point theorem.

Many aspects of modern mathematics, e.g. non-convex global optimization theory, mathematical economics, or approximation theory, to list only a few, are dealt with on spaces not admitting linear structures let alone simplexes (see [5]).

2000 *Mathematics Subject Classification.* Primary 46A19, 54H25; Secondary 46A03, 47H10.

Key words and phrases. Abstract convex space, KKM principle.

Instead, abstract convex structures have become indispensable. Versions of the KKM theorem for such setting followed (see e.g., [1], [9], [3]).

In 2006 Sehie Park initiated and has developed since, the KKM theory. In his own words, "*the KKM theory is the study on the equivalent formulations of the KKM theorem and their applications*". For this purpose, Park introduced the concepts of an abstract convex space and a (partial) KKM space. Several results of basic importance for the KKM theory have been established within those classes of spaces (see [9]–[19]).

In the paper, we construct an example of a partial KKM space that is not a KKM space (cf. Example 2.8), thus showing that the class of partial KKM spaces is a proper subclass of the class of KKM spaces. This solves a problem posed by Park (see e.g. [13], [16]). It also shows that some results from e.g. [21] or [22] are false as stated and need to be corrected. We also construct a class of partial KKM spaces that do not admit L-structures in the sense of H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, [8] (see Corollary 3.7). The long line was the only one example of this sort known before.

We refer to [2] for all undefined topological notions.

2. Deconstructing Park's abstract convex spaces

One the purposes of this papers is to construct an example to distinguish the class of partial KKM spaces from the class of KKM spaces. Towards this goal, we investigate abstract convex structures in the sense of Park. We show that those structures can be identified with classic convex structures in the sense of van de Vel (see Theorem 2.5). This fact, in turn, enables us to identify abstract convex spaces satisfying the partial KKM principle with topological spaces admitting L^*-operators in the sense of Kulpa and Szymanski (see Theorem 3.4). To get a required example, we construct a special abstract convex structure on the unit interval $[0,1]$ (see Example 2.8).

Following M. van de Vel [24], a *convexity* on a *set* E is a collection $\mathcal{G}$ of subsets of E satisfying the following conditions:

(c1) $\emptyset, E \in \mathcal{G}$;
(c2) If $\emptyset \neq \mathcal{R} \subseteq \mathcal{G}$, then $(\bigcap \mathcal{R}) \in \mathcal{G}$;
(c3) If $\emptyset \neq \mathcal{R} \subseteq \mathcal{G}$ is a chain with respect to inclusion, then $(\bigcup \mathcal{R}) \in \mathcal{G}$.

The members of $\mathcal{G}$ are called *convex sets* and the pair $(E,\mathcal{G})$ is called a *convexity space*. The *convex hull* of a set $X \subseteq E$ is $\operatorname{co} X = \bigcap\{R \in \mathcal{G} : X \subseteq R\}$. Then $X \subseteq \operatorname{co} X \in \mathcal{G}$, and $A \subseteq B$ implies that $\operatorname{co} A \subseteq \operatorname{co} B$. Moreover,

Proposition 2.1. $\operatorname{co} X = \bigcup\{\operatorname{co} N : N$ *is a finite subset of* $X\}$.

See [24] for a proof.

To facilitate further discussion, we introduce some additional symbols. For any set E, let $\langle E\rangle$ and $\exp(E)$ denote, respectively, the set of all finite non-empty subsets of X, and the set of all non-empty subsets of X. Now, for any map $\Upsilon\colon \langle E\rangle \to \exp(E)$, let

$$\mathcal{G}_\Upsilon = \{X \subseteq E : \text{for all } N \in \langle X\rangle,\ \Upsilon(N) \subseteq X\}.$$

Lemma 2.2. *If* $\Upsilon\colon \langle E\rangle \to \exp(E)$, *then* $\mathcal{G}_\Upsilon$ *is a convexity on the set* E.

Proof. The verification of conditions (c1)–(c3) for the family $\mathcal{G}_\Upsilon$ is straightforward. □

The converse statement, that is, that any convexity can be represents as $\mathcal{G}_\Upsilon$ for some map Υ, is also true. More specifically,

Lemma 2.3. *If* $\mathcal{G}$ *is a convexity on the set* E *and* Υ *is the convex hull operator restricted to* $\langle E\rangle$, *then* $\mathcal{G}_\Upsilon = \mathcal{G}$.

Proof. If $X \in \mathcal{G}_\Upsilon$, then for each $N \in \langle X\rangle$, $\Upsilon(N) = \operatorname{co} N \subseteq X$. Hence, by Proposition 2.1, $X = \bigcup\{\operatorname{co} N : N \in \langle X\rangle\} = \operatorname{co} X \in \mathcal{G}$. This shows that $\mathcal{G}_\Gamma \subseteq \mathcal{G}$. The converse inclusion is obvious. □

We follow S. Park (see, e.g. [10]) in the terminology and (most of) symbols concerning abstract convex spaces and related concepts.

An *abstract convex space* $(E, D; \Gamma)$ consists of non-empty sets E, D, and a map $\Gamma\colon \langle D\rangle \to \exp(E)$. If $D \subset E$, then the abstract convex space $(E, D; \Gamma)$ is denoted by $(E \supset D; \Gamma)$. If $D = E$, it is denoted by $(E; \Gamma)$.

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ *relative to* $D' \subseteq D$ if for any $N \in \langle D'\rangle$, $\Gamma(N) \subseteq X$.

Lemma 2.4. *If* $(E, D; \Gamma)$ *is an abstract convex space and* $D' \subseteq D$, *then the family* $\mathcal{G}_{DD'\Gamma} = \{X \subseteq E : X$ *is a* Γ-*convex subset of* $(E, D; \Gamma)$ *relative to* $D'\}$ *is a convexity on the set* E.

Proof. The verification of conditions (c1)–(c3) for the family $\mathcal{G}_{DD'\Gamma}$ is straightforward again. □

A subset X of E is said to be a Γ-*convex subset* of $(E \supset D; \Gamma)$ if X is Γ-convex relative to $D' = X \cap D$. Let

$$\mathcal{G}_{D\Gamma} = \{X \subseteq E : X \text{ is a } \Gamma\text{-convex subset of } (E \supset D; \Gamma)\}.$$

Theorem 2.5. *If* $(E \supset D; \Gamma)$ *is an abstract convex space, then there exists a map* $\Upsilon\colon \langle E\rangle \to \exp(E)$ *such that* $\mathcal{G}_{D\Gamma} = \mathcal{G}_\Upsilon$. *In particular, if* $(E \supset D; \Gamma)$ *is an abstract convex space, then the family* $\mathcal{G}_{D\Gamma}$ *is a convexity on* E.

Proof. Let $(E \supset D; \Gamma)$ be an abstract convex space. Let us define a map $\Upsilon\colon \langle E\rangle \to \exp(E)$ by the following rule:

$$\Upsilon(N) = \begin{cases} \Gamma(N) & \text{if } N \subseteq D, \\ N & \text{if } N \nsubseteq D. \end{cases}$$

We shall show that for arbitrary subset X of E, X is a Γ-convex subset of $(E \supset D; \Gamma)$ if and only if $X \in \mathcal{G}_\Upsilon$.

Assume that X is a Γ-convex subset of $(E \supset D; \Gamma$. Let $N \in \langle X\rangle$. If $N \subseteq D$, then $N \in \langle D'\rangle$, where $D' = X \cap D$. Since X is a Γ-convex subset of $(E \supset D; \Gamma)$, $\Upsilon(N) = \Gamma(N) \subseteq X$. If $N \nsubseteq D$, then $\Upsilon(N) = N \subseteq X$. This shows that $\mathcal{G}_{D\Gamma} \subseteq \mathcal{G}_\Upsilon$.

Assume that $X \in \mathcal{G}_\Upsilon$. Let $N \in \langle D'\rangle$, where $D' = X \cap D$. Then $N \in \langle D'\rangle \cap \langle X\rangle$ and therefore $\Gamma(N) = \Upsilon(N) \subseteq X$. Thus $\mathcal{G}_{D\Gamma} \supseteq \mathcal{G}_\Upsilon$.

Now, the second part of the theorem follows from Lemma 2.2. □

Remark 2.6. (a) Park's general notion of *abstract convex space* $(E, D; \Gamma)$ seems to be a bit superfluous. For if $|D| \leq |E|$, then we may rename the elements of D by elements of a subset of E. If so, then the original abstract convex space $(E, D; \Gamma)$ can be regarded as given in the form $(E \supset D; \Gamma)$. No example of abstract convex space $(E, D; \Gamma)$ with $|D| > |E|$ has ever been considered.

(b) By Lemma 2.2 and Theorem 2.5, any convexity space can be considered as an abstract convex space and vice versa. Consequently, classifying (known and previously considered) convexity spaces as abstract convex spaces (cf. [9] through [23]) is to some extend obsolete, unless one wants to distinguish a special multifunction Γ.

In our forthcoming considerations involving Park's abstract convex spaces we are going to assume that they are given in the form (E, Γ) or $(E, \mathcal{G}_\Gamma)$.

Let $(E; \Gamma)$ be an abstract convex space. Following Park (see e.g. [9]) if a map $G\colon E \to \exp(E)$ satisfies

$$\Gamma(A) \subseteq G(A) = \bigcup_{x \in A} G(x)$$

for each $A \in \langle E\rangle$, then G is called a *KKM map*.

Let $(E; \Gamma)$ be an abstract convex space, where E is a topological space. $(E; \Gamma)$ satisfies the *partial KKM principle* if

(∗) For any closed-valued KKM map $G\colon E \to \exp(E)$, the family $\{G(x) : x \in E\}$ has the finite intersection property.

The *KKM principle* for the abstract convex space $(E; \Gamma)$ is the statement that the property (∗) also holds for any open-valued KKM map. An abstract convex space is called a *KKM space* if it satisfies the KKM principle. It's been an open problem, due to S. Park, whether there is a space satisfying the partial KKM principle that is not a KKM space (see e.g. [13], [16], [20] or [22]). Example 2.8, below, provides an affirmative answer to this problem. To show the "sharpness" of our example we first state conditions under which any partial KKM space is a KKM space.

Let $\mathcal{F}$ be a family of subsets of a set X. A family $\{W_F : F \in \mathcal{F}\}$ is said to be a *swelling* of the family $\mathcal{F}$ if for every $F \in \mathcal{F}$, $W_F \subseteq F \subseteq X$, and for each non-empty finite subfamily $\mathcal{R}$ of $\mathcal{F}$, $\bigcap\{W_F : F \in \mathcal{R}\} \neq \emptyset$ if and only if $\bigcap\{F : F \in \mathcal{R}\} \neq \emptyset$.

Let us notice that if the family $\{W_F : F \in \mathcal{F}\}$ is a swelling of the family $\mathcal{F}$ and $W_F \subseteq E_F \subseteq F$ for each $F \in \mathcal{F}$, then $\{E_F : F \in \mathcal{F}\}$ is also a swelling of the family $\mathcal{F}$. It is known that if E is a normal Hausdorff space, then any finite family of closed subsets of X has an open swelling (cf. [2]).

Proposition 2.7. *Let $(E;\Gamma)$ be an abstract convex space that satisfies the partial KKM principle. If E is a normal Hausdorff space and Γ is a closed-valued map, then $(E;\Gamma)$ is a KKM space.*

Proof. We shall show that

$(***)$ If $S\colon E \to \exp(E)$ is a closed-valued map and $E = \bigcup_{x\in A} S(x)$ for some $A \in \langle E\rangle$, then there exists a $B \in \langle A\rangle$ such that

$$\Gamma(B) \cap \bigcap\{S(x) : x \in B\} \neq \emptyset.$$

Let $T(x) = S(x) - \operatorname{Int}(S(x))$ for each $x \in A$, let $\mathcal{F}_1 = \{S(x) : x \in A\}$, let $\mathcal{F}_2 = \{T(x) : x \in A\}$, and let $\mathcal{F}_3 = \{\Gamma(B) : B \in \langle A\rangle\}$. Take an open swelling $\{U_F : F \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3\}$ of the finite closed family $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ satisfying additionally that $U_{T(x)} \subseteq U_{S(x)}$ for each $x \in A$.

Notice that $\{\operatorname{Int}(S(x)) \cup U_{T(x)} : x \in A\}$ is an open cover of E. Since $(E;\Gamma)$ satisfies the partial KKM principle, there exists $B \in \langle A\rangle$ such that $\Gamma(B) \cap \bigcap\{\operatorname{Int}(S(x)) \cup U_{T(x)} : x \in B\} \neq \emptyset$. Consider the following three possible cases.

Case 1. $\Gamma(B) \cap \bigcap\{U_{T(x)} : x \in B\} \neq \emptyset$. Since $U_{\Gamma(B)} \cap \bigcap\{U_{T(x)} : x \in B\} \neq \emptyset$ if and only if $\Gamma(B) \cap \bigcap\{T(x) : x \in B\} \neq \emptyset$, $\emptyset \neq \Gamma(B) \cap \bigcap\{T(x) : x \in B\} \subseteq \Gamma(B) \cap \bigcap\{S(x) : x \in B\}$, we are done.

Case 2. $\Gamma(B) \cap \bigcap\{\operatorname{Int}(S(x)) : x \in B\} \neq \emptyset$. Then $\Gamma(B) \cap \bigcap\{S(x) : x \in B\} \neq \emptyset$, so we are done.

Case 3. Cases 1 and 2 do not occur. Thus there exists $C \subseteq B$ such that $C \neq B$ and $\Gamma(B) \cap \bigcap\{\operatorname{Int}(S(x)) : x \in C\} \cap \bigcap\{U_{T(x)} : x \in B - C\} \neq \emptyset$. Since $U_{T(x)} \subseteq U_{S(x)}$ for each x, $\Gamma(B) \cap \bigcap\{\operatorname{Int}(S(x)) : x \in C\} \cap \bigcap\{U_{S(x)} : x \in B - C\} \neq \emptyset$. It follows that $U_{\Gamma(B)} \cap \bigcap\{U_{S(x)} : x \in B\} \neq \emptyset$. Consequently, $\Gamma(B) \cap \bigcap\{S(x) : x \in B\}$ and we are done.

Since the statement $(***)$ is the contrapositive version of the statement asserting that $(E;\Gamma)$ satisfies the KKM principle, where the open sets $G(x)$ have been replaced by their complements $S(x)$, our proposition holds. □

Example 2.8. Let $0 < p < 0.5 < q < 1$ and let $\Gamma\colon [0,1] \to \exp([0,1])$ be given by

$$\Gamma(A) = \begin{cases} \{p\} & \text{if } A = \{p\}, \\ \{q\} & \text{if } A = \{q\}, \\ [0,1] - \{0.5\} & \text{if } A = \{p,q\}, \\ [0,1] & \text{otherwise.} \end{cases}$$

Let $G\colon$: $[0,1] \to \exp([0,1])$ be a closed-valued multimap such that $\Gamma(A) \subseteq \bigcup_{x\in A} G(x)$ for each $A \in \langle E\rangle$. Take any $B \in \langle E\rangle$ and consider $\bigcap\{G(x) : x \in B\}$. Since $G(x) = [0,1]$ if $x \notin \{p,q\}$, $\bigcap\{G(x) : x \in B\} = G(p) \cap G(q)$.

Since $[0,1] - \{0.5\} = \Gamma(\{p,q\}) \subseteq G(p) \cup G(q)$, $G(p) \cup G(q) = [0,1]$. Hence $G(p) \cap G(q) \neq \emptyset$. Thus $\bigcap\{G(x) : x \in B\} \neq \emptyset$ showing that the abstract convex space $([0,1];\Gamma)$ satisfies the partial KKM principle.

Let $F\colon [0,1] \to \exp([0,1])$ be given by

$$F(x) = \begin{cases} [0,0.5) & \text{if } x = p, \\ (0.5,1] & \text{if } x = q, \\ [0,1] & \text{if } x \in [0,1] - \{p,q\}. \end{cases}$$

Then F is an open-valued multifunction on $[0,1]$. Let us check that F is a KKM map.

Clearly, $\Gamma(A) \subseteq F(A) = \bigcup_{x\in A} F(x)$ whenever $A \nsubseteq \{p,q\}$. If $A = \{p,q\}$, then $\Gamma(A) = [0,1] - \{0.5\} = [0,0.5) \cup (0.5,1] = F(p) \cup F(q)$. Finally, if $A = \{p\}$ or $A = \{q\}$, then $\Gamma(A) \subseteq F(A)$ too. Since the sets $F(p)$ and $F(q)$ are disjoint, the family $\{F(x) : x \in \{p,q\}\}$ does not have the finite intersection property.

Remark 2.9. In the two examples constructed in Example 2.8, the underlying space E $(= [0,1])$ is metric compact (and thus normal) and the map Γ assumes closed values everywhere but one point.

Remark 2.10. Our Example 2.8 along with Proposition 2.7 show that, e.g. Corollary 3.4 from [21] or Theorem 4.2 from [22] are false as stated and need to be corrected.

3. L-spaces and L^*-operators

The contrapositive version of the KKM theorem, where the sets F_i have been replaced by their complements U_i, takes the following form:

Theorem 3.1. *If the open sets* $U_1, \dots, U_{n+1}$ *cover* Δ_n*, then there exists* $J \subseteq \{1, \dots, n+1\}$ *such that* $\Delta_J \cap \bigcap_{i \in J} U_i \neq \emptyset$.

The latter version of the KKM theorem (in a slightly more general form) is the Theorem on Indexed Families, due to Kulpa [6].

Theorem 3.2. *Let* $\sigma\colon \Delta_n \to X$ *be a continuous function. If* $V_1, \dots, V_{n+1}$ *are open subsets of* X *that cover* $\sigma(\Delta_n)$*, then there exists* $J \subseteq \{1, \dots, n+1\}$ *such that* $\sigma(\Delta_J) \cap \bigcap_{i \in J} V_i \neq \emptyset$.

In particular,

Corollary 3.3. *Let* A *be a non-empty finite subset of a linear topological space* X*. If* V_x*,* $x \in A$*, are open subsets of* X *that cover* $\operatorname{co} A$*, then there exists* $B \subseteq A$ *such that* $\operatorname{co} B \cap \bigcap\{V_x : x \in B\} \neq \emptyset$.

The property of the convex hull operator in linear topological spaces exhibited in the corollary was the primary motivation for us to introduce L^*-operators (cf. [7]).

An L^*-*operator* on X is any map $\Lambda\colon \langle X \rangle \to \exp(X)$ that satisfies the following condition:

($*$) If $A \in \langle X \rangle$ and $\{U_x : x \in A\}$ is a cover of X by non-empty open sets, then there exists $B \subseteq A$ such that $\Lambda(B) \cap \bigcap\{U_x : x \in B\} \neq \emptyset$.

A topological space X together with an L^*-operator Λ is referred to as an L^*-*space* and it is denoted by (X, Λ).

Thus Corollary 3.3 states that the convex hull operator on a linear topological space is an L^*-operator on that space. Examples of L^*-operators, and thus of L^*-spaces, abound. In fact, one can define an L^*-operator on arbitrary topological space X. Simply set $\Lambda(A)$ to be an any dense subset of X. Notwithstanding their triviality those examples at least witness the fact that L^*-operators may not be monotone, i.e. that $B \subseteq A$ implies $\Lambda(B) \subseteq \Lambda(A)$, nor that A has to be a subset of $\Lambda(A)$. Notice also that if $\Lambda\colon \langle X \rangle \to \exp(X)$ is an L^*-operator and $\Upsilon\colon \langle X \rangle \to \exp(X)$ verifies that $\Lambda(A) \cap \Upsilon(A)$ is a dense subset of $\Lambda(A)$ for each $A \in \langle X \rangle$, then Υ is an L^*-operator too.

Let us observe that the contrapositive version of the statement asserting that $(E;\Gamma)$ satisfies the partial KKM principle, where the closed sets $G(x)$ have been replaced by their complements $S(x)$, has the following form:

($**$) If $S\colon E \to \exp(E)$ is an open-valued map and $E = \bigcup_{x \in A} S(x)$ for some $A \in \langle E \rangle$, then there exists a $B \in \langle A \rangle$ such that

$$\Gamma(B) \cap \bigcap\{S(x) : x \in B\} \neq \emptyset.$$

In Park's terminology, an abstract convex space $(E;\Gamma)$ satisfying ($**$) is referred to as possessing the *Fan type matching property* (see [11]).

Theorem 3.4. *Let $(E;\Gamma)$ be an abstract convex space, where E is a topological space. $(E;\Gamma)$ satisfies the partial KKM principle if and only if Γ is an L^*-operator on E.*

Proof. Assume that $(E;\Gamma)$ satisfies the partial KKM principle. Let $\{U_x : x \in A\}$, $A \in \langle E\rangle$, be an open cover of E. Suppose to the contrary that $\Gamma(B) \cap \bigcap\{U_x : x \in B\} = \emptyset$ for each $B \subseteq A$. Define $G\colon E \to \exp(E)$ by setting

$$G(x) = \begin{cases} E - U_x & \text{if } x \in A,\\ E & \text{if } x \notin A.\end{cases}$$

Then G a closed-valued map KKM map on E. Since $\bigcap\{G(x) : x \in A\} = \emptyset$, we get a contradiction.

Assume that Γ is an L^*-operator on E. Let $G\colon E \to \exp(E)$ be a closed-valued KKM map on the abstract convex space $(E;\Gamma)$. Suppose to the contrary that $\bigcap\{G(x) : x \in A\} = \emptyset$ for some $A \in \langle E\rangle$. Since Γ is an L^*-operator on E, there exists $B \subseteq A$ such that $\Gamma(B) \cap \bigcap\{E - G(x) : x \in B\} \neq \emptyset$, i.e. $\Gamma(A) \nsubseteq \bigcup_{x\in A} G(x)$. So that G is not a KKM map, a contradiction. □

In considerations involving general convexities it is often postulated that all singletons be convex sets. A convexity with this property is referred to as a T_1 *convexity* [24]. In the setting of an abstract convex space $(E;\Gamma)$, this amounts to requiring that $\Gamma(\{x\}) = \{x\}$ for each $x \in E$. In terms of an L^*-operator Λ, it means also that $\Lambda(\{x\}) = \{x\}$ for each $x \in E$.

Lemma 3.5. *A topological space E is connected if and only if E admits an L^*-operator such that $\Lambda(\{x\}) = \{x\}$ for each $x \in E$.*

Proof. Let E be a connected space and let $A \in \langle E\rangle$. Set

$$\Lambda(A) = \begin{cases} A & \text{if } |A| = 1,\\ \text{any dense set} & \text{if } |A| > 1.\end{cases}$$

By the definition, $\Lambda(\{x\}) = \{x\}$ for each $x \in E$. To check the condition $(*)$ for Λ, take an $A \in \langle E\rangle$ and an open cover $\{U_x : x \in A\}$ of E. If $x \in U_x$ for some $x \in A$, then $\Lambda(\{x\}) \cap U_x = \{x\} \cap U_x \neq \emptyset$. If $x \notin U_x$ for all $x \in A$, then, $|A| \geq 2$. Since E is connected, there exist $a, b \in A$, $a \neq b$, such that $U_a \cap U_b \neq \emptyset$. Hence $\Lambda(\{a,b\}) \cap U_a \cap U_b \neq \emptyset$ because $\Lambda(\{a,b\})$ is dense.

Let Λ be an L^*-operator on the space E such that $\Lambda(\{x\}) = \{x\}$ for each $x \in E$. Assume to the contrary that E is not connected, say $X = U \cup V$ for some non-empty disjoint open sets. Pick $a \in U$, $b \in V$, and set $U_a = V$ and $U_b = U$. Since $\Lambda(\{x\}) \cap U_x = \emptyset$ for each $x \in \{a,b\}$, we have to have $\Lambda(\{a,b\}) \cap U_a \cap U_b \neq \emptyset$, which is a contradiction. □

Recall that $\Delta_m \subseteq \mathbb{R}^{m+1}$ denotes the unit simplex in $\mathbb{R}^{m+1}$. The following definitions are due to Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [8].

An *L-structure* on a topological space E is given by a function $\Gamma\colon \langle E\rangle \to \exp(E)$ verifying:

(✠) For every $A \in \langle E\rangle$, say $A = \{x_0, \dots, x_n\}$, there exists a continuous function $f^A\colon \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \dots, n\}$,

$$f^A(\Delta_J) \subseteq \Gamma(\{x_i : i \in J\}).$$

The pair (E, Γ) is then called an *L-space*, and $Y \subseteq E$ is said to be *L-convex* if $\Gamma(A) \subseteq Y$ for each $A \in \langle Y\rangle$.

L-spaces defined by monotone functions Γ were introduced and studied earlier by Park and Kim [23]. Presently, a generalization of L-spaces to abstract convex spaces is referred to as a *G-convex space* (see [13]). Let us point out that a (trivial) L-structure can be defined on arbitrary topological space E: set $\Gamma(A)$ to be the whole space E for each $A \in \langle E\rangle$, and f^A to be a constant map.

The class of L-spaces constitutes a generalization of some known classes of special convexity structures such as Bielawski's *B-simplicial convexity* or Horvath's *c-structures* (see [8]). Park [17] showed that the class of (partial) KKM spaces constitutes a generalization of the class of L-spaces, and later on, that the inclusion is proper by showing that the long line can be made into a KKM space and that the KKM space cannot be an L-space (see [18]). It appears this is the only known example that distinguishes the two classes. The corollary, below, enables to get a class of examples of (partial) KKM spaces that are not L-spaces (cf. Corollary 3.7).

Let us recall that a space X is *pathway connected* if for every pair a, b of points of X there exists a continuous function $f\colon [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$.

Proposition 3.6. *If (E, Γ) is an L-space such that $\Gamma(\{x\}) = \{x\}$ for each $x \in E$, then E is a pathway connected space.*

Proof. Let $A = \{x_0, x_1\}$ be a pair of distinct points of E and let $f^A\colon \Delta_1 \to \Gamma(A)$ satisfies (✠). Without loss of generality, we may think of Δ_1 as the unit segment $[0,1]$. Since

$$f^A(\{0\}) \subseteq \Gamma(\{x_0\}) = \{x_0\} \quad \text{and} \quad f^A(\{1\}) \subseteq \Gamma(\{x_1\}) = \{x_1\},$$

the function f^A is as required. □

Corollary 3.7. *Let Λ be any L^*-operator on a connected space E verifying $\Lambda(\{x\}) = \{x\}$ for each $x \in E$. If E is not pathway connected, then the abstract convex space (E, Λ) is a partial KKM space that is not an L-space.*

Acknowledgements. The authors would like to thank an anonymous referee whose comments improved readability of the paper.

References

[1] X.P. Ding, *Generalized G-KKM theorems in generalized convex spaces and their applications*, J. Math. Anal. Appl. **266** (2002), 21–37.

[2] R. Engelking, *General Topology*, Heldermann, Berlin 1989.

[3] L. Gonzáles, S. Kilmer and J. Rebaza, *From a KKM theorem to Nash equilibria in L-spaces*, Topology Appl. **155** (2007), 165–170.

[4] B. Knaster, K. Kuratowski and S.Mazurkiewicz, *Ein beweis des fixpunktsatzes für n-dimensionale simplexe*, Fund. Math. **14** (1929), 132–137.

[5] I. Komov, Dinc The Luc and A. Rubinov, *Generalized convexity and related topics*, Lecture Notes in Economics and Mathematical Systems # 583, Springer.

[6] W. Kulpa, *Convexity and the Brouwer Fixed Point Theorem*, Topology Proc. **22** (1997), 211–235.

[7] W. Kulpa and A. Szymanski, *Applications of general infimum principles to fixed-point theory and game theory*, Set Valued Anal. **16** (2008), 375–398.

[8] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.V. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–150.

[9] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45** (1) (2008), 1–27.

[10] ———, *A genesis of general KKM theorems for abstract convex spaces*, J. Nonlinear Anal. Optim. **2** (2011), 133–146.

[11] ———, *Foundations of the KKM theory*, J. Nonlinear Convex Anal. **9** (2008), 1–20.

[12] ———, *New generalizations of basic theorems in the KKM theory*, Nonlinear Anal. **74** (2011), 3000–3010.

[13] ———, *The KKM principle in abstract convex spaces: equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[14] ———, *A genesis of general KKM theorems for abstract convex spaces*, J. Nonlinear Anal. Optim. **2** (1) (2011), 121–132.

[15] ———, *Remarks on simplicial spaces and L^*-spaces of Kulpa and Szymanski*, Comm. Appl. Nonlinear Anal. **19** (1) (2012), 59–69.

[16] ———, *Generalizations of the Nash Equilibrium Theorem in the KKM Theory*, Hindawi Publishing Corporation, Fixed Point Theory and Applications 2010, Article ID 234706.

[17] ———, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. Appl. Math. **7** (2000), 1–28.

[18] ———, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. **69** (2008), 4352–4364.

[19] ———, *Examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps on abstract convex spaces*, Soochow J. Math. **33** (2007), 477–486.

[20] ———, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11** (2006), 67–77.

[21] ———, *Remarks on the partial KKM principle*, Nonlinear Anal. Forum **14** (2009), 51–62.

[22] ———, *Remarks on KC-maps and KO-maps in abstract convex spaces*, Nonlinear Anal. Forum **12** (1) (2007), 29–40.

[23] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU **18** (1993), 1–21.

[24] M.L.J. van de Vel, *Theory of Convex Structures*, North-Holland Mathematical Library, **50**, 1993.

[25] E. Zeidler, *Applied Functional Analysis – Applications to Mathematical Physics*, Springer–Verlag, New York, 1995.

Manuscript received December 3, 2012

Władysław Kulpa
Faculty of Mathematics and Natural Sciences
College of Sciences
Cardinal Stefan Wyszyński University
ul. Dewaitis
01-815, Warszawa, POLAND
E-mail address: w.kulpa@uksw.edu.pl

Andrzej Szymanski
Department of Mathematics
Slippery Rock University
Slippery Rock, PA 16057, USA
E-mail address: andrzej.szymanski@sru.edu