

PERIODIC SOLUTIONS OF A KIND OF LIÉNARD EQUATIONS WITH TWO DEVIATING ARGUMENTS

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ABSTRACT. In this paper, we deal with the existence of periodic solutions of a kind of Liénard equations with two deviating arguments

$$x'' + f(t, x(t - \sigma(t)))x'(t) + g(t, x(t - \tau(t))) = p(t).$$

Some new results on the existence of periodic solutions of the given equations are proved by using the continuation theorem.

1. Introduction

In recent years, the periodic problem of Liénard equations with deviating arguments has been widely studied because of its background in applied sciences (see [2], [5], [7], [9]–[13] and the references cited therein). In [7], Huang and Xiang studied the periodic solutions of Duffing equations with a single constant deviating argument

$$(1.1) \quad x'' + g(x(t - \tau)) = p(t).$$

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When g satisfies one sign condition $xg(x) > 0$, $|x| > M$ and a one-sided boundedness condition, the authors proved the existence of periodic solutions of equation (1.1). In [9], Liu and Huang studied the periodic solutions of Liénard equations with one deviating argument

$$(1.2) \quad x'' + f(x(t))x'(t) + g(t, x(t - \tau(t))) = p(t).$$

When f, g are Lipschitz continuous, f is bounded and g satisfies one sign condition $x(g(t, x) - p(t)) > 0$ (or < 0) for $t \in \mathbb{R}$, $|x| \geq d$, they obtained the existence and uniqueness of periodic solution of equation (1.2).

In [11], Lu and Ge discussed the existence of periodic solutions of Liénard equations with two deviating arguments

$$(1.3) \quad x'' + f(t, x(t - \sigma(t)))x'(t) + \beta(t)g(x(t - \tau(t))) = p(t),$$

where $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and 2π periodic with respect to the first variable, $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\sigma, \beta, \tau, p: \mathbb{R} \rightarrow \mathbb{R}$ are continuous and 2π periodic.

Assume that the following conditions are satisfied:

$$(1.4) \quad \lim_{|x| \rightarrow +\infty} \operatorname{sgn}(x)g(x) = +\infty,$$

$$(1.5) \quad \limsup_{|x| \rightarrow +\infty} \left| \frac{g(x)}{x} \right| \leq r.$$

Moreover,

$$(1.6) \quad 0 < \beta_0 = \min_{t \in [0, 2\pi]} \beta(t) \leq \beta_1 = \max_{t \in [0, 2\pi]} \beta(t).$$

When the conditions (1.4), (1.5) and (1.6) hold, it was proved in [11] that, if $|f(t, x)| \leq \varrho$ (ϱ is a positive constant) for $(t, x) \in \mathbb{R}^2$, then equation (1.3) has at least one 2π periodic solution provided that the inequality $2\pi(\varrho + 2\pi r\beta_1) < 1$ holds and if $|f(t, x)| \geq \varrho$ for $(t, x) \in \mathbb{R}^2$, then equation (1.3) has at least one 2π periodic solution provided that the inequality $2\pi r\beta_1 < \varrho$ holds.

In this paper, we deal with the existence of periodic solutions of a kind of Liénard equations with two deviating arguments

$$(1.7) \quad x'' + f(t, x(t - \sigma(t)))x'(t) + g(t, x(t - \tau(t))) = p(t),$$

where $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous and 2π periodic with respect to the first variable, $\sigma, \tau, p: \mathbb{R} \rightarrow \mathbb{R}$ are continuous and 2π periodic. By taking a new estimating method, we give some new conditions to guarantee the existence of periodic solutions of equation (1.7). Main results of this paper are the following theorems.

THEOREM 1.1. *Assume that there exist constants $\varrho \geq 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:*

$$(f_1) \quad |f(t, x)| \leq \varrho \text{ for } (t, x) \in \mathbb{R}^2;$$

(g₁) $\operatorname{sgn}(x)g(t, x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;

(g₂) $g(t, x) \leq rx + c$ for $t \in \mathbb{R}$, $|x| \geq d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $2\pi(\varrho + \pi r) < 1$ holds.

THEOREM 1.2. Assume that there exist constants $\varrho \geq 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:

(f₁) $|f(t, x)| \leq \varrho$ for $(t, x) \in \mathbb{R}^2$;

(g₁) $\operatorname{sgn}(x)g(t, x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;

(g₃) $|g(t, x)| \leq r|x| + c$ for $t \in \mathbb{R}$, $|x| \geq d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $\varrho + \pi r < 1$ holds.

THEOREM 1.3. Assume that there exist constants $\varrho > 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:

(f₂) $|f(t, x)| \geq \varrho$ for $(t, x) \in \mathbb{R}^2$;

(g₁) $\operatorname{sgn}(x)g(t, x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;

(g₃) $|g(t, x)| \leq r|x| + c$ for $t \in \mathbb{R}$, $|x| \geq d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $\pi r < \varrho$ holds.

REMARK 1.4. The main Theorems 1.1–1.3 improve and complement the results in [11]. In fact, if we take $g(t, x) = \beta(t)g(x)$, then we have

$$\limsup_{|x| \rightarrow +\infty} \left| \frac{g(t, x)}{x} \right| \leq \beta_1 r$$

provided that the conditions (1.5) and (1.6) hold. When the conditions (1.4)–(1.6) hold, we know from Theorem 1.2 that, if $|f(t, x)| \leq \varrho$ (ϱ is a positive constant) for $(t, x) \in \mathbb{R}^2$, then equation (1.3) has at least one 2π periodic solution provided that the inequality $\varrho + \pi r\beta_1 < 1$ holds and we know from Theorem 2.3 that, if $|f(t, x)| \geq \varrho$ for $(t, x) \in \mathbb{R}^2$, then equation (1.3) has at least one 2π periodic solution provided that the inequality $\pi r\beta_1 < \varrho$ holds. Therefore, Theorems 1.2 and 1.3 improve the results in [11].

Throughout this paper, for any continuous 2π periodic function $x(t)$, we always use notations as follows,

$$\|x\|_\infty = \max_{t \in [0, 2\pi]} \{|x(t)|\}, \quad \|x\|_2 = \left(\int_0^{2\pi} x^2(t) dt \right)^{1/2}.$$

2. Preliminary lemmas

As is well known that continuation theorems play an important role in studying the existence of periodic solutions of the second order differential equations. We now introduce a continuation theorem given by Gaines and Mawhin ([4]).

LEMMA 2.1. *Let X and Y be two Banach spaces. Suppose that $L: D \subset X \rightarrow Y$ is a Fredholm operator with index zero and $N: X \rightarrow Y$ is L -compact on $\bar{\Omega}$, where Ω is an open bounded subset of X . Moreover, assume that all the following conditions are satisfied:*

- (a) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
- (b) $Nx \notin \text{Im } L$, for all $x \in \partial\Omega \cap \text{Ker } L$;
- (c) The Brower degree $\text{deg}\{QN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then equation $Lx = Nx$ has at least one solution in $\bar{\Omega}$.

Next, we shall give two lemmas which will be used in getting the prior bounds.

LEMMA 2.2. *Let $x(t)$ be a continuously differentiable 2π periodic function. Then, for any $t \in [0, 2\pi]$,*

$$\|x\|_{\infty} \leq |x(t)| + \pi \|x'\|_{\infty}.$$

PROOF. Let $\bar{t} \in [t, t + 2\pi]$ be such that $|x(\bar{t})| = \|x\|_{\infty}$. Then we have

$$|x(\bar{t})| = \left| x(t) + \int_t^{\bar{t}} x'(s) ds \right| \leq |x(t)| + \int_t^{\bar{t}} |x'(s)| ds.$$

On the other hand, we have

$$|x(\bar{t})| = |x(\bar{t} - 2\pi)| = \left| x(t) + \int_t^{\bar{t}-2\pi} x'(s) ds \right| \leq |x(t)| + \int_{\bar{t}-2\pi}^t |x'(s)| ds.$$

Therefore,

$$(2.1) \quad \|x\|_{\infty} \leq |x(t)| + \frac{1}{2} \int_0^{2\pi} |x'(t)| dt,$$

which implies

$$\|x\|_{\infty} \leq |x(t)| + \pi \|x'\|_{\infty}. \quad \square$$

LEMMA 2.3. *Let $x(t)$ be a twice continuously differentiable 2π periodic function. Then*

$$\int_0^{2\pi} |x'(t)|^2 dt \leq \int_0^{2\pi} |x''(t)|^2 dt.$$

PROOF. The proof follows directly from the Wirtinger inequality ([6]). \square

3. Main theorems

In this section, we shall use the continuation theorem introduced in Section 2 to prove the existence of periodic solutions of equation (1.7). To this end, we first quote some notations and definitions.

Let X and Y be two Banach spaces defined by

$$\begin{aligned} X &= \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t + 2\pi) = x(t), \text{ for all } t \in \mathbb{R}\}, \\ Y &= \{y \in C(\mathbb{R}, \mathbb{R}) : y(t + 2\pi) = y(t), \text{ for all } t \in \mathbb{R}\} \end{aligned}$$

with the norms

$$\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}, \quad \|y\|_Y = \|y\|_\infty.$$

Define a linear operator

$$L: D(L) \subset X \rightarrow Y, \quad Lx = x'',$$

where $D(L) = \{x \in X : x'' \in C(\mathbb{R}, \mathbb{R})\}$, and a nonlinear operator

$$N: X \rightarrow Y, \quad (Nx)(t) = -f(t, x(t - \sigma(t)))x'(t) - g(t, x(t - \tau(t))) + p(t).$$

It is easy to see that

$$\text{Ker } L = \mathbb{R} \quad \text{and} \quad \text{Im } L = \left\{ y \in Y : \int_0^{2\pi} y(t) dt = 0 \right\}.$$

It follows that L is a Fredholm mapping of index zero.

Let us define two continuous projectors $P: X \rightarrow \text{Ker } L$ and $Q: Y \rightarrow Y$ by setting

$$Px = x(0), \quad Qy = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt.$$

Set $L_P = L|_{D(L) \cap \text{Ker } P} \rightarrow \text{Im } L$. Then L_P is an algebraic isomorphism and we define $K_P: \text{Im } L \rightarrow D(L)$ by

$$K_P = L_P^{-1}.$$

Clearly, we have that, for any $y \in \text{Im } L$,

$$(K_P y)(t) = -\frac{t}{2\pi} \int_0^{2\pi} (t-s)y(s) ds + \int_0^t (t-s)y(s) ds.$$

For any open bounded set $\Omega \subset X$, we can prove by standard arguments that $K_P(I - Q)N$ and QN are relatively compact on the closure $\bar{\Omega}$. Therefore, N is L -compact on $\bar{\Omega}$.

It is noted that equation (1.7) is equivalent to the operator equation

$$Lx = Nx.$$

To use Lemma 2.1, we embed this operator equation into an equation family with a parameter $\lambda \in (0, 1)$,

$$Lx = \lambda Nx,$$

which is equivalent to the equation as follows,

$$(3.1) \quad x'' + \lambda f(t, x(t - \sigma(t)))x'(t) + \lambda g(t, x(t - \tau(t))) = \lambda p(t), \quad \lambda \in (0, 1).$$

In the following, we shall prove some new theorems on the existence of periodic solutions of equation (1.7) by using the continuation theorem. By standard arguments [4], it suffices to prove that there exist positive constants M_1 and M_2 such that, for any 2π periodic solution $x(t)$ of equation (3.1),

$$\|x\|_\infty < M_1, \quad \|x'\|_\infty < M_2.$$

THEOREM 3.1. *Assume that there exist constants $\varrho \geq 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:*

$$(f_1) \quad |f(t, x)| \leq \varrho \text{ for } (t, x) \in \mathbb{R}^2;$$

$$(g_1) \quad \operatorname{sgn}(x)g(t, x) > l \text{ for } t \in \mathbb{R}, |x| > d, \text{ where } l = \max\{|p(t)| : t \in [0, 2\pi]\};$$

$$(g_2) \quad g(t, x) \leq rx + c \text{ for } t \in \mathbb{R}, x \geq d.$$

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $2\pi(\varrho + \pi r) < 1$ holds.

PROOF. We shall prove that all conditions of Lemma 2.1 are satisfied. Consider the auxiliary equation (3.1). Let $x(t)$ be any 2π periodic solution of equation (3.1). Then there exist $t_*, t^* \in [0, 2\pi]$ such that

$$x(t_*) = \min_{t \in [0, 2\pi]} x(t), \quad x(t^*) = \max_{t \in [0, 2\pi]} x(t).$$

It follows that $x'(t_*) = 0$, $x''(t_*) \geq 0$; $x'(t^*) = 0$, $x''(t^*) \leq 0$. Therefore,

$$g(t_*, x(t_* - \tau(t_*))) \leq l, \quad g(t^*, x(t^* - \tau(t^*))) \geq -l.$$

According to (g_1) , we get

$$x(t_* - \tau(t_*)) \leq d, \quad x(t^* - \tau(t^*)) \geq -d.$$

Consequently, there exists $\bar{t} \in [0, 2\pi]$ such that $|x(\bar{t} - \tau(\bar{t}))| \leq d$. Since $x(t)$ and $\tau(t)$ are 2π periodic, there exists $\tilde{t} \in [0, 2\pi]$ such that

$$(3.2) \quad |x(\tilde{t})| \leq d.$$

From Lemma 2.2 and (3.2) we have

$$(3.3) \quad \|x\|_\infty \leq d + \pi \|x'\|_\infty.$$

Write $S_1 = \{t \in [0, 2\pi] : x(t - \tau(t)) > d\}$, $S_2 = \{t \in [0, 2\pi] : x(t - \tau(t)) < -d\}$, $S_3 = \{t \in [0, 2\pi] : |x(t - \tau(t))| \leq d\}$. Since

$$\int_0^{2\pi} f(t, x(t - \sigma(t)))x'(t) dt + \int_0^{2\pi} g(t, x(t - \tau(t))) dt = \int_0^{2\pi} p(t) dt,$$

we get

$$\begin{aligned} - \int_{S_2} g(t, x(t - \tau(t))) dt &= \int_0^{2\pi} f(t, x(t - \sigma(t)))x'(t) dt \\ &\quad + \left(\int_{S_1} + \int_{S_3} \right) g(t, x(t - \tau(t))) dt - \int_0^{2\pi} p(t) dt. \end{aligned}$$

Furthermore,

$$(3.4) \quad \begin{aligned} \int_{S_2} |g(t, x(t - \tau(t)))| dt &= - \int_{S_2} g(t, x(t - \tau(t))) dt \\ &\leq \int_0^{2\pi} |f(t, x(t - \sigma(t)))x'(t)| dt \end{aligned}$$

$$+ \left(\int_{S_1} + \int_{S_3} \right) |g(t, x(t - \tau(t)))| dt + \int_0^{2\pi} |p(t)| dt.$$

From (2.1) and (3.4) we have

$$\begin{aligned} \|x'\|_\infty &\leq \frac{1}{2} \int_0^{2\pi} |x''(t)| dt \leq \frac{1}{2} \int_0^{2\pi} |f(t, x(t - \sigma(t)))x'(t)| dt \\ &\quad + \frac{1}{2} \int_0^{2\pi} |g(t, x(t - \tau(t)))| dt + \frac{1}{2} \int_0^{2\pi} |p(t)| dt \\ &\leq \int_0^{2\pi} |f(t, x(t - \sigma(t)))x'(t)| dt \\ &\quad + \left(\int_{S_1} + \int_{S_3} \right) |g(t, x(t - \tau(t)))| dt + \int_0^{2\pi} |p(t)| dt. \end{aligned}$$

Furthermore, from (f₁), (g₂) and (3.3) that

$$\begin{aligned} \|x'\|_\infty &\leq 2\varrho\pi \|x'\|_\infty + 2r\pi \|x\|_\infty + 2\pi(c + M_d + l) \\ &\leq 2\varrho\pi \|x'\|_\infty + 2r\pi(d + \pi \|x'\|_\infty) + 2\pi(c + M_d + l) \\ &= 2\pi(\varrho + r\pi) \|x'\|_\infty + 2\pi(rd + M_d + c + l), \end{aligned}$$

where $M_d = \max\{|g(x)| : |x| \leq d\}$.

Since $2\pi(\varrho + r\pi) < 1$, there exists a constant $M_1 > 0$ such that $\|x'\|_\infty \leq M_1$. According to (3.3), we obtain $\|x\|_\infty \leq d + \pi M_1$. □

Similarly, we have the following result.

THEOREM 3.1'. *Assume that there exist constants $\varrho \geq 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:*

- (f₁) $|f(t, x)| \leq \varrho$ for $(t, x) \in \mathbb{R}^2$;
- (g₁) $\text{sgn}(x)g(t, x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;
- (g'₂) $g(t, x) \geq rx - c$ for $t \in \mathbb{R}$, $x \leq -d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $2\pi(\varrho + \pi r) < 1$ holds.

REMARK 3.2. The conclusion of Theorem 3.1 (or Theorem 3.1') still holds if the condition (g₁) is replaced by the condition:

$$(g'_1) \text{sgn}(x)g(t, x) < -l \text{ for } t \in [0, 2\pi], |x| > d, \text{ where } l = \max\{|p(t)| : t \in [0, 2\pi]\},$$

whereas condition (g₂) (or (g'₂)) is replaced by the condition:

$$(g''_2) g(t, x) \geq -rx - c \text{ for } t \in \mathbb{R}, x \geq d, \text{ (or } g(t, x) \leq -rx + c \text{ for } t \in \mathbb{R}, x \leq -d).$$

COROLLARY 3.3. *Assume that conditions (f₁) and (g₁) hold. Moreover, g satisfies the sublinear condition as follows:*

$$(3.5) \quad \lim_{x \rightarrow +\infty} \frac{g(t, x)}{x} = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} \frac{g(t, x)}{x} = 0$$

uniformly for $t \in [0, 2\pi]$. Then equation (1.7) has at least one 2π periodic solution provided that the inequality $2\pi\rho < 1$ holds.

PROOF. Since $2\pi\rho < 1$, there exists a constant $\varepsilon > 0$ such that $2\pi(\rho + \varepsilon\pi) < 1$. From (3.5) we know that there exists a constant $d_\varepsilon > 0$ such that

$$(3.6) \quad g(t, x) \leq \varepsilon x \quad \text{for } t \in [0, 2\pi], \quad x \geq d_\varepsilon$$

or

$$(3.7) \quad g(t, x) \geq \varepsilon x \quad \text{for } t \in [0, 2\pi], \quad x \leq -d_\varepsilon.$$

If (3.6) holds, then we know from Theorem 3.1 that equation (1.7) has at least one 2π periodic solution. If (3.7) holds, then we know from Theorem 3.1' that equation (1.7) has at least one 2π periodic solution. \square

In the following, we shall deal with the existence of periodic solutions of equation (1.7) when the condition (g_2) (or (g'_2)) is replaced by one double sided condition.

THEOREM 3.4. Assume that there exist constants $\rho \geq 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:

- (f₁) $|f(t, x)| \leq \rho$ for $(t, x) \in \mathbb{R}^2$;
- (g₁) $\text{sgn}(x)g(t, x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;
- (g₃) $|g(t, x)| \leq r|x| + c$ for $t \in \mathbb{R}$, $|x| \geq d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $\rho + \pi r < 1$ holds.

PROOF. Consider the auxiliary equation (3.1). Let $x(t)$ be any 2π periodic solution of equation (3.1). Multiplying both sides of (3.1) by $x''(t)$ and integrating on the interval $[0, 2\pi]$, we have

$$\begin{aligned} \int_0^{2\pi} x''(t)^2 dt &= -\lambda \int_0^{2\pi} f(t, x(t - \sigma(t)))x'(t)x''(t) dt \\ &\quad - \lambda \int_0^{2\pi} g(t, x(t - \tau(t)))x''(t) dt + \lambda \int_0^{2\pi} p(t)x''(t) dt \\ &\leq \int_0^{2\pi} |f(t, x(t - \sigma(t)))x'(t)x''(t)| dt \\ &\quad + \int_0^{2\pi} |g(t, x(t - \tau(t)))x''(t)| dt + \int_0^{2\pi} |p(t)x''(t)| dt. \end{aligned}$$

According to the conditions (f₁) and (g₃), we get

$$\begin{aligned} \int_0^{2\pi} x''(t)^2 dt &\leq \rho \int_0^{2\pi} |x'(t)x''(t)| dt \\ &\quad + r \int_0^{2\pi} |x(t - \tau(t))x''(t)| dt + (c + l) \int_0^{2\pi} |x''(t)| dt \end{aligned}$$

$$\begin{aligned} &\leq \varrho \left(\int_0^{2\pi} x'(t)^2 dt \right)^{1/2} \left(\int_0^{2\pi} x''(t)^2 dt \right)^{1/2} \\ &\quad + r \|x\|_\infty \int_0^{2\pi} |x''(t)| dt + (c+l) \int_0^{2\pi} |x''(t)| dt. \end{aligned}$$

From (2.1), (3.2) and Lemma 2.3, we have

$$\begin{aligned} \int_0^{2\pi} x''(t)^2 dt &\leq \varrho \int_0^{2\pi} x''(t)^2 dt \\ &\quad + r \left(d + \frac{1}{2} \int_0^{2\pi} |x'(t)| dt \right) \int_0^{2\pi} |x''(t)| dt + (c+l) \int_0^{2\pi} |x''(t)| dt \\ &\leq \varrho \int_0^{2\pi} x''(t)^2 dt + r\pi \left(\int_0^{2\pi} x'(t)^2 dt \right)^{1/2} \left(\int_0^{2\pi} x''(t)^2 dt \right)^{1/2} \\ &\quad + (rd+c+l) \int_0^{2\pi} |x''(t)| dt \\ &\leq (\varrho+r\pi) \int_0^{2\pi} x''(t)^2 dt + (rd+c+l)\sqrt{2\pi} \left(\int_0^{2\pi} x''(t)^2 dt \right)^{1/2}. \end{aligned}$$

Since $\varrho+r\pi < 1$, there exists a constant $M_1 > 0$ such that $\|x''\|_2 \leq M_1$. Furthermore, we have

$$(3.8) \quad \|x'\|_\infty \leq \frac{1}{2} \int_0^{2\pi} |x''(t)| dt \leq \frac{\sqrt{2\pi}M_1}{2} =: M_2.$$

From (3.8) and Lemma 2.2 we get $\|x\|_\infty \leq d + \|x'\|_\infty \leq d + M_2$. □

COROLLARY 3.5. *Assume that conditions (f₁) and (g₁) hold. Moreover, g satisfies the sublinear condition as follows:*

$$(3.9) \quad \lim_{|x| \rightarrow +\infty} \frac{g(t,x)}{x} = 0$$

uniformly for $t \in [0, 2\pi]$. Then equation (1.7) has at least one 2π periodic solution provided that the inequality $\varrho < 1$ holds.

PROOF. Since $\varrho < 1$, there exists a constant $\varepsilon > 0$ such that $\varrho + \varepsilon\pi < 1$. If (3.9) holds, then there exists a constant $d_\varepsilon > 0$ such that $|g(t,x)| \leq \varepsilon|x|$ for $t \in [0, 2\pi]$, $|x| \geq d_\varepsilon$. According to Theorem 3.4, equation (1.7) has at least one 2π periodic solution. □

In the case when f is unbounded, we can obtain the following result.

THEOREM 3.6. *Assume that there exist constants $\varrho > 0$, $d > 0$, $r \geq 0$ and $c > 0$ such that the following conditions hold:*

- (f₂) $|f(t,x)| \geq \varrho$ for $(t,x) \in \mathbb{R}^2$;
- (g₁) $\text{sgn}(x)g(t,x) > l$ for $t \in \mathbb{R}$, $|x| > d$, where $l = \max\{|p(t)| : t \in [0, 2\pi]\}$;
- (g₃) $|g(t,x)| \leq r|x| + c$ for $t \in \mathbb{R}$, $|x| \geq d$.

Then equation (1.7) has at least one 2π periodic solution provided that the inequality $r\pi < \varrho$ holds.

PROOF. We only deal with the case $f(t, x) \geq \varrho$ for $(t, x) \in \mathbb{R}^2$. Another case $f(t, x) \leq -\varrho$ for $(t, x) \in \mathbb{R}^2$ can be handled similarly. Consider the auxiliary equation (3.1). Let $x(t)$ be any 2π periodic solution of equation (3.1). Multiplying both sides of (3.1) by $x'(t)$ and integrating on the interval $[0, 2\pi]$, we get

$$\int_0^{2\pi} f(t, x(t - \sigma(t)))x'(t)^2 dt + \int_0^{2\pi} g(t, x(t - \tau(t)))x'(t) dt = \int_0^{2\pi} p(t)x'(t) dt.$$

According to (f₂), we have

$$\begin{aligned} \varrho \int_0^{2\pi} x'(t)^2 dt &\leq \int_0^{2\pi} f(t, x(t - \sigma(t)))x'(t)^2 dt \\ &= - \int_0^{2\pi} g(t, x(t - \tau(t)))x'(t) dt + \int_0^{2\pi} p(t)x'(t) dt. \end{aligned}$$

Hence,

$$\varrho \int_0^{2\pi} x'(t)^2 dt \leq \int_0^{2\pi} |g(t, x(t - \tau(t)))||x'(t)| dt + \int_0^{2\pi} |p(t)||x'(t)| dt.$$

From condition (g₃) we obtain

$$\varrho \int_0^{2\pi} x'(t)^2 dt \leq r\|x\|_\infty \int_0^{2\pi} |x'(t)| dt + (l + c) \int_0^{2\pi} |x'(t)| dt,$$

which, together with (2.1) and (3.2), implies

$$\begin{aligned} \varrho \int_0^{2\pi} x'(t)^2 dt &\leq r \left(d + \frac{1}{2} \int_0^{2\pi} |x'(t)| dt \right) \int_0^{2\pi} |x'(t)| dt + (l + c) \int_0^{2\pi} |x'(t)| dt \\ &\leq r\pi \int_0^{2\pi} x'(t)^2 dt + (rd + l + c)\sqrt{2\pi} \left(\int_0^{2\pi} x'(t)^2 dt \right)^{1/2}. \end{aligned}$$

Since $r\pi < \varrho$, there exists a constant $M_1 > 0$ such that

$$(3.10) \quad \|x'\|_2 \leq M_1.$$

Furthermore, we get

$$\|x\|_\infty \leq d + \frac{1}{2} \int_0^{2\pi} |x'(t)| dt \leq d + \frac{\sqrt{2\pi}}{2} \left(\int_0^{2\pi} x'(t)^2 dt \right)^{1/2} \leq d + \frac{\sqrt{2\pi}}{2} M_1 =: M_2.$$

On the other hand, we have

$$\begin{aligned} (3.11) \quad \|x'\|_\infty &\leq \frac{1}{2} \int_0^{2\pi} |x''(t)| dt \leq \frac{1}{2} \left(\int_0^{2\pi} |f(t, x(t - \sigma(t)))||x'(t)| dt \right. \\ &\quad \left. + \int_0^{2\pi} |g(t, x(t - \tau(t)))| dt \right) + \frac{1}{2} \int_0^{2\pi} |p(t)| dt. \end{aligned}$$

Set

$$(3.12) \quad c_1 = \max\{|f(t, x)| : 0 \leq t \leq 2\pi, |x| \leq M_2\},$$

$$(3.13) \quad c_2 = \max\{|g(t, x)| : 0 \leq t \leq 2\pi, |x| \leq M_2\}.$$

From (3.10)–(3.13) we obtain

$$\|x'\|_\infty \leq \frac{1}{2}c_1 \int_0^{2\pi} |x'(t)|dt + (c_2 + l)\pi \leq \frac{\sqrt{2\pi}}{2}c_1M_1 + (c_2 + l)\pi. \quad \square$$

REMARK 3.7. The conclusion of Theorem 3.4 (or Theorem 3.6) still holds if the condition (g_1) is replaced by the condition (g'_1) .

COROLLARY 3.8. *Assume that the conditions (f_2) and (g_1) hold. Moreover, g satisfies the sublinear condition (3.9). Then equation (1.7) has at least one 2π periodic solution.*

PROOF. Since $\varrho > 0$, there exists a constant $\varepsilon > 0$ such that $\varepsilon\pi < \varrho$. If (3.9) holds, then there exists a constant $d_\varepsilon > 0$ such that $|g(t, x)| \leq \varepsilon|x|$, for $t \in [0, 2\pi]$ and $|x| \geq d_\varepsilon$. According to Theorem 3.6, equation (1.7) has at least one 2π periodic solution. \square

From Corollary 3.8 we can derive the following result.

COROLLARY 3.9. *Assume that the conditions (f_2) , (g_1) hold and g is bounded on \mathbb{R} . Then equation (1.7) has at least one 2π periodic solution.*

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