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EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS FOR FRACTIONAL HYBRID DIFFERENTIAL INCLUSIONS

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Abstract. This paper studies existence results for boundary value problems of nonlinear fractional hybrid differential inclusions of quadratic type, by using a fixed point theorem of Dhage [11].

1. Introduction

Fractional differential equations have aroused great interest, which is caused by both the intensive development of the theory of fractional calculus and the application of physics, mechanics, chemistry and engineering sciences. For some recent development on the topic see [1]–[9] and the references cited therein.

In this paper, we study the following boundary value problem

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right] \in F(t,x(t)) \quad \text{a.e. } t \in (0,1), \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

where D^{α} is the standard Riemann–Liouville fractional derivative of order α, $1 < \alpha \leq 2$, $f \in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $F\colon [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

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The present paper is motivated by a recent paper of Sun *et al.* [18], where it considered problem (1.1) with F single-valued and results on existence of solutions are provided. Here we prove an existence result for the problem (1.1), by using a fixed point theorem in Banach algebras due to Dhage [11], under Lipschitz and Carathéodory conditions. For some recent results on hybrid fractional differential equations we refer to [7], [12], [20] and the references cited therein.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. Preliminaries

2.1. Fractional calculus. Let us recall some basic definitions of fractional calculus [14], [16], [17].

Definition 2.1. The Riemann–Liouville derivative of fractional order q is defined as

$$D^q_{0+}g(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1}g(s)\,ds, \quad n-1<q<n,\ n=[q]+1,$$

provident the integral exists, where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^q_{0+}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q>0,$$

provided the integral exists.

Lemma 2.3. *For $q > 0$, the general solution of the fractional differential equation $D^q_{0+}x(t) = 0$ is given by*

$$x(t) = c_1t^{q-1} + c_2t^{q-2} + \ldots + c_nt^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, \ldots, n$ $(n = [q] + 1)$.

In view of Lemma 2.3, it follows that

$$I^q_{0+}D^q_{0+}x(t) = x(t) + c_1t^{q-1} + c_2t^{q-2} + \ldots + c_nt^{q-n}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$ $(n = [q] + 1)$.

Definition 2.4. A function $x \in AC^1([0,1],\mathbb{R})$ is called a solution of problem (1.1) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$ almost everywhere in $[0,1]$, such that

$$D^\alpha\left[\frac{x(t)}{f(t,x(t))}\right] = v(t), \quad \text{a.e. in } [0,1] \quad \text{and} \quad x(0) = x(1) = 0.$$

LEMMA 2.5 [18]. *Let $1 < \alpha < 2$. Given $y \in C([0,1],\mathbb{R})$, then the unique solution of the problem*

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right] + y(t) = 0, & 0 < t < 1,\ 1 < \alpha \le 2, \\ x(0) = x(1) = 0, \end{cases} \tag{2.2}$$

is given by

$$x(t) = f(t,x(t)) \int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

LEMMA 2.6 [19]. *The functions $G(t,s)$ have the following properties:*

(a) $G(t,s) \ge 0$ *for all* $t,s \in [0,1]$.
(b) $G(t,s) \le (\alpha-1)s(1-s)^{\alpha-1}/\Gamma(\alpha)$ *for all* $t,s \in [0,1]$.

2.2. Multi-valued analysis. Let us recall some basic definitions on multi-valued maps [10], [13].

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{\mathrm{cp,cv}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G\colon X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be compact if $G(X)$ is a relatively compact subset of X. G is said to be totally bounded if $G(\mathbb{B})$ is relatively compact subset of X for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is totally bounded with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G\colon [0;1] \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y,G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x\colon [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|\, dt.$$

Definition 2.7. A multivalued map $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(a) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(b) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

Further a Carathéodory function F is called $L^1_{\mathbb{R}}$−Carathéodory if

(c) there exists a function $g \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le g(t)$$

for all $x \in \mathbb{R}$ and for almost every $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

The following lemma is used in the sequel.

Lemma 2.8 [15]. *Let X be a Banach space. Let $F\colon [0,1]\times\mathbb{R} \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F\colon C([0,1],X) \to \mathcal{P}_{\mathrm{cp,cv}}(C([0,1],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

The following fixed point theorem due to Dhage [11] is fundamental in the proof of our main result.

Theorem 2.9. *Let X be a Banach algebra and let $A\colon X \to X$ be a single valued and $B\colon X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be a multi-valued operator satisfying:*

(a) *A is single-valued Lipschitz with a Lipschitz constant k,*
(b) *B is compact and upper semi-continuous,*
(c) *$2Mk < 1$, where $M = \|B(X)\|$.*

Then either

(i) *the operator inclusion $x \in AxBx$ has a solution, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \mu u \in AuBu,\ \mu > 1\}$ is unbounded.*

Theorem 2.10. *Assume that*:

(H$_1$) *The function* $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ *is continuous and there exists a bounded function* ϕ *such that* $\phi(t)>0$, *for almost every* $t\in[0,1]$ *and*

$$|f(t,x)-f(t,y)|\le\phi(t)|x(t)-y(t)|,$$

for almost every $t\in[0,1]$ *and for all* $x,y\in\mathbb{R}$;

(H$_2$) $F\colon[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is* $L^1_{\mathbb{R}}$*-Carathéodory and has nonempty compact and convex values*;

(H$_3$) $\dfrac{\|\phi\|(\alpha-1)}{\Gamma(\alpha)}\displaystyle\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds<\frac{1}{2}$.

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Set $X=C([0,1],\mathbb{R})$. Transform the problem (1.1) into a fixed point problem. Consider the operator $\mathcal{N}\colon C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}(x)=\left\{h\in C([0,1],\mathbb{R}):h(t)=f(t,x(t))\int_0^1 G(t,s)v(s)\,ds,\ v\in S_{F,x}\right\}.$$

Now we define two operators $\mathcal{A}\colon C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ by

$$\mathcal{A}x(t)=f(t,x(t)),\quad t\in[0,1], \tag{2.4}$$

and $\mathcal{B}\colon C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{B}(x)=\left\{h\in C([0,1],\mathbb{R}):h(t)=\int_0^1 G(t,s)v(s)\,ds,\ v\in S_{F,x}\right\}. \tag{2.5}$$

Then $\mathcal{N}(x)=\mathcal{A}x\mathcal{B}x$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9. For better readability we break the proof into a sequence of steps.

Step 1. We first show that $\mathcal{A}$ *is a Lipschitz on* X, *i.e.* (a) *of Theorem* 2.9 *holds.*

Let $x,y\in X$. Then by (H$_1$) we have

$$|\mathcal{A}x(t)-\mathcal{A}y(t)|=|f(t,x(t))-f(t,y(t))|\le\phi(t)|x(t)-y(t)|\le\|\phi\|\|x-y\|$$

for all $t\in[0,1]$. Taking the supremum over t we obtain

$$\|\mathcal{A}x-\mathcal{A}y\|\le\|\phi\|\|x-y\|$$

for all $x,y\in X$. So $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. Now we show that the multi-valued operator $\mathcal{B}$ *is compact and upper semi-continuous on* X, *i.e.* (b) *of Theorem* 2.9 *holds.*

First we show that $\mathcal{B}$ has convex values.

Let $u_1,u_2\in\mathcal{B}x$. Then there are $v_1,v_2\in S_{F,x}$ such that

$$u_i(t)=\int_0^1 G(t,s)v_i(s)\,ds,\quad t\in[0,1],\ i=1,2.$$

For any $\theta \in [0,1]$, we have

$$\begin{aligned}\theta u_1(t) + (1-\theta)u_2(t) &= \theta \int_0^1 G(t,s)u_1(s)\,ds + (1-\theta)\int_0^1 G(t,s)u_2(s)\,ds \\ &= \int_0^1 G(t,s)[\theta u_1(s) + (1-\theta)u_2(s)]\,ds \\ &= \int_0^1 G(t,s)v(s)\,ds,\end{aligned}$$

where $v(t) = \theta v_1(t) + (1-\theta)v_2(t) \in F(t, x(t))$ for all $t \in [0,1]$. Hence $\theta u_1(t) + (1-\theta)u_2(t) \in \mathcal{B}x$ and consequently $\mathcal{B}x$ is convex for each $x \in X$. As a result $\mathcal{B}$ defines a multi valued operator $\mathcal{B}\colon X \to \mathcal{P}_{\rm cv}(X)$.

Next we show that $\mathcal{B}$ maps X into bounded subset of X. To see this, $x \in X$ be arbitrary. Then for each $h \in \mathcal{B}x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \int_0^1 G(t,s)v(s)\,ds.$$

Then, for each $t \in [0,1]$,

$$|h(t)| \leq \int_0^1 |G(t,s)||v(s)|\,ds \leq \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds.$$

This further implies that

$$\|h\| \leq \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds,$$

and so $\mathcal{B}(X)$ is uniformly bounded.

Next we show that $\mathcal{B}(X)$ is an equicontinuous set. Let $x \in X$ is any point. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \int_0^1 G(t,s)v(s)\,ds.$$

Then for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ we have

$$\begin{aligned}|h(t_2) - h(t_1)| &\leq \int_0^1 |G(t_2,s) - G(t_1,s)||v(s)|\,ds \\ &\leq \int_0^1 |G(t_2,s) - G(t_1,s)|g(s)|\,ds \\ &\leq \int_0^1 |G(t_2,s) - G(t_1,s)|g(s)|\,ds.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in X$ as $t_2 - t_1 \to 0$. Therefore, it follows from the Arzelá–Ascoli theorem that $\mathcal{B}\colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\,\mathbb{R}))$ is compact.

In our next step, we show that $\mathcal{B}$ *has a closed graph.* Let $x_n \to x_*$, $h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}x_*$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \int_0^1 G(t,s)v_n(s)\,ds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^1 G(t,s)v_*(s)\,ds.$$

Let us consider the linear operator $\Theta\colon L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \int_0^1 G(t,s)v(s)\,ds.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \int_0^1 G(t,s)(v_n(s) - v_*(s))\,ds \right\| \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows by Lemma 2.8 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^1 G(t,s)v_*(s)\,ds, \quad \text{for some } v_* \in S_{F,x_*}.$$

As a result we have that the operator $\mathcal{B}$ is compact and upper semicontinuous operator on X.

Step 3. Now we show that $2Mk < 1$, *i.e.* (c) *of Theorem* 2.9 *holds.*

This is obvious by (H_3) since we have

$$M = \|B(X)\| = \sup\{|\mathcal{B}x : x \in X\} \leq \frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} g(s)\,ds$$

and $k = \|\phi\|$.

Thus all the conditions of Theorem 2.9 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $u \in \mathcal{E}$ be arbitrary. Then we have for $\lambda > 1$, $\lambda u \in \mathcal{A}u(t)\mathcal{B}u(t)$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, one has

$$u(t) = \lambda^{-1}[f(t,u(t)] \int_0^t G(t,s)v(s)\,ds$$

for all $t \in [0,1]$. Then we have

$$\begin{aligned}|u(t)| &\le \lambda^{-1}|f(t,u(t)|\int_0^1 |G(t,s)||v(s)|\,ds\\ &\le [|f(t,u(t)-f(t,0)|+|f(t,0)|]\int_0^1 |G(t,s)||v(s)|\,ds\\ &\le [\|\phi\|\|u\|+F_0]\int_0^1 |G(t,s)||v(s)|\,ds\\ &\le [\|\phi\|\|u\|+F_0]\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds\\ &= \|\phi\|\|u\|\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds + F_0\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds,\end{aligned}$$

where we have put $F_0 = \sup_{t\in[0,1]}|f(t,0)|$. Then we have

$$\bigg(1-\frac{\|\phi\|(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds\bigg)\|u\| \le \frac{F_0(\alpha-1)}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds,$$

and consequently

$$\|u\| \le \frac{F_0(\alpha-1)\displaystyle\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds}{\Gamma(\alpha)-\|\phi\|(\alpha-1)\displaystyle\int_0^1 s(1-s)^{\alpha-1}g(s)\,ds}.$$

Thus the condition (ii) of Thorem 2.9 does not hold. Therefore the operator equation $\mathcal{A}x\mathcal{B}x$ and consequently problem (1.1) has a solution on $[0,1]$. This completes the proof. □

Example 2.11. Consider the boundary value problem

$$\begin{cases} D^{3/2}\bigg[\dfrac{x(t)}{\sin x+2}\bigg] \in F(t,x(t)), & 0<t<1,\\ x(0)=x(1)=0, \end{cases} \tag{2.6}$$

where $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$t \to F(t,x) = \bigg[\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{9(|\sin x|+1)}+\frac{8}{9}\bigg].$$

For $u\in F$ we have

$$|u| \le \max\bigg\{\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{9(|\sin x|+1)}+\frac{8}{9}\bigg\} \le 1, \quad x\in\mathbb{R}.$$

Thus

$$\|F(t,x)\| = \sup\{|y| : y\in F(t,x)\} \le 1 = g(t), \quad x\in\mathbb{R}.$$

Here, $f(t,x)=\sin x+2$. Therefore, for any $x,y\in\mathbb{R}$, we have

$$|f(t,x)-f(t,y)| = |\sin x-\sin y| \le |x-y| = \phi(t)|x-y|$$

so that $\phi(t) = 1$ for all $t \in [0,1]$. Finally, we have

$$\frac{\|\phi\|(\alpha-1)}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} g(s)\, ds = \frac{4}{15\sqrt{\pi}} < \frac{1}{2},$$

and thus the boundary value problem (2.6) has a solution.

References

[1] B. Ahmad, A. Alsaedi and B. Alghamdi, *Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions*, Nonlinear Anal. Real World Appl. **9** (2008), 1727–1740.

[2] B. Ahmad and J.J. Nieto, *Existence of solutions for nonlocal boundary value problems of higher order nonlinear fractional differential equations*, Abstr. Appl. Anal., Vol. 2009 (2009), Article ID 494720, 9 pages.

[3] B. Ahmad and S.K. Ntouyas, *Existence results for nonlinear fractional differential equations with four-point nonlocal type integral boundary conditions*, Afr. Diaspora J. Math. **11** (2011), 29–39.

[4] ______ *Some existence results for boundary value problems for fractional differential inclusions with non-separated boundary conditions*, Electron. J. Qual. Theory Differ. Equ., No. **71** (2010), 1–17.

[5] B. Ahmad, S.K. Ntouyas and A. Alsaedi, *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Differential Equations, Vol. 2011, Article ID 107384, 11 pages.

[6] B. Ahmad and S. Sivasundaram, *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl. Math. Comput. **217** (2010), 480–487.

[7] M. Ammi, E. El Kinani and D. Torres, *Existence and uniqueness of solutions to functional integro-differential fractional equations*, Electron. J. Differ. Equ., Vol. 2012 (2012), No. 103, 1–9.

[8] Z.B. Bai, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Anal. **72** (2010), 916–924.

[9] K. Balachandran and J.J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal. **72** (2010) 4587–4593.

[10] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin–New York, 1992.

[11] B.C. Dhage, *Existence results for neutral functional differential inclusions in Banach algebras*, Nonlinear Anal. **64** (2006), 1290–1306.

[12] M. El Borai and M. Abbas, *On some integro-differential equations of fractional orders involving Carathéodory nonlinearities*, Int. J. Mod. Math. **2** (2007), 41–52.

[13] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory* I, Kluwer, Dordrecht, 1997.

[14] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[15] A. Lasota and Z. Opial, *An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[17] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[18] S. Sun, Y. Zhao, Z. Han and Y. Li, *The existence of solutions for boundary value problem of fractional hybrid differential equations*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 4961–4967.

[19] Y. Wang, L. Liu and Y. Wu, *Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity*, Nonlinear Anal. **74** (2011), 6434–6441.

[20] Y. Zhao, S. Sun, Z. Han and Q. Li, *Theory of fractional hybrid differential equations*, Comput. Math. Appl. **62** (2011), 1312–1324.

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