

**INDEX THEORY FOR LINEAR ELLIPTIC EQUATION
AND MULTIPLE SOLUTIONS
FOR ASYMPTOTICALLY LINEAR ELLIPTIC EQUATION
WITH RESONANCE**

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ABSTRACT. In this paper, we consider the existence and multiplicity of solutions to the elliptic equation with resonance. We classify the linear elliptic equation and obtain some new conditions on the existence and multiplicity for asymptotically linear elliptic equation by using critical point theory.

1. Introduction and main results

In this paper, we consider the following problem:

$$(1.1) \quad \Delta u + f(x, u) + f_1(x, u) = 0,$$

$$(1.2) \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain, with smooth boundary $\partial\Omega$, $\Delta u = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u$ and $f: \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a L^1 -Carathéodory function, i.e. $f(x, \cdot)$ is continuous on \mathbb{R} for almost every $x \in \bar{\Omega}$, $f(\cdot, u)$ is measurable on Ω for each $u \in \mathbb{R}$. Moreover, for any constant $r > 0$, there exists some function $\rho_r \in L^1(\Omega)$ such

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that $|f(x, u)| \leq \rho_r(x)$ for almost every $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| \leq r$. And $f_1: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is also a Carathéodory function such that

$$|f_1(x, u)| \leq \tilde{g}(t)$$

for $x \in \mathbb{R}$ and almost every $t \in \Omega$, where $\tilde{g} \in L^2(0, 1)$. We look for the weak solutions of (1.1) and (1.2) which are the same as the critical points of the functional

$$(1.3) \quad I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx - \int_{\Omega} F_1(x, u) dx,$$

for all $u \in H_0^1(\Omega)$, where

$$F(x, u) = \int_0^u f(x, t) dt, \quad F_1(x, u) = \int_0^u f_1(x, t) dt$$

and $H_0^1(\Omega)$ is the Sobolev space with the norm

$$\|u\| = \|\nabla u\|_2 = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

Many results on the existence of solutions for Duffing equation and elliptic equation satisfy asymptotically linear condition. For the non-resonant nonlinearities, we refer to [6], [16]. For the resonant nonlinearities, we refer to [10]–[13], [17], [18]. It is remarkable that there are great deal of approaches for proving these results, including topological degree technique, the critical point theory and the Morse theory. However, there are few results on the multiple solutions to equations with resonance.

We assume that f is asymptotically linear at infinity: there exist functions $a, b \in L^\infty(\Omega)$ such that

$$(f_1) \quad a(x) \leq \frac{f(x, u)}{u} \leq b(x), \quad \text{for all } x \in \Omega, |u| > r > 0.$$

It should be mentioned here that this type condition has been widely considered in Hamiltonian systems. A quantitative way to measure the asymptotically linear condition is the index theory. As in [6], an index for the second order linear Hamiltonian systems was defined. And in [7], an index theory for first order linear Hamiltonian systems was developed. In [4], [14], [15], by Conley, Zehnder and Long, an index theory for symplectic path was defined.

In [8], an index theory was established to investigate the self-adjoint operator equation with finite Morse index which can also be used to study second-order Hamiltonian system and elliptic equation. Since the functional defined in (1.3) is bounded from below, we establish the classification theory for the associated linear elliptic equation of (1.1) following the ideas of [6] and [8]. This paper is largely motivated by the work of [10] in which the existence of solutions to a semilinear elliptic equation with double resonance was proved.

In this paper, we also consider problem (1.1) and (1.2) with resonance under the asymptotically linear condition (f_1) . Obviously, (1.1) does not satisfy the linear growth condition: there exists a constant $C_0 > 0$ such that

$$|f(x, u)| \leq C_0(1 + |u|), \quad \text{for all } u \in \mathbb{R}, x \in \Omega.$$

In order to obtain better results than the earlier works, an index theory for associated linear elliptic equations needs to be developed. Since the functional defined in (1.3) is bounded from below, the index theory for linear elliptic equation will be established by using Morse index. Thanks to this index theory, the existence and multiplicity of critical points of (1.3) were established. And our main results are obtained for the case when I is even.

For any $b(x) \in L^\infty(\Omega)$, we consider the following linear elliptic equation:

$$(1.4) \quad \Delta u + b(x)u = 0, \quad u|_{\partial\Omega} = 0.$$

Set $E = H_0^1(\Omega)$. Define

$$(1.5) \quad q_b(u, v) = \int_{\Omega} (\nabla u(x), \nabla v(x)) dx - \int_{\Omega} (b(x)u(x), v(x)) dx,$$

for all $u, v \in E$.

DEFINITION 1.1 ([8, Definition 2.5.1]). For any $b \in L^\infty(\Omega)$, we define

$$\begin{aligned} \nu(b) &= \dim \ker(q_b) = \dim\{u \mid q_b(u, v) = 0, \text{ for all } v \in E\}, \\ i(b) &= \sum_{\lambda < 0} \nu(b + \lambda). \end{aligned}$$

We call $i(b)$ the index and $\nu(b)$ the nullity of b , respectively.

For any $q \in L^\infty(\Omega)$, let $\lambda_1(q) \leq \lambda_2(q) \leq \dots \leq \lambda_k(q) \leq \dots$ be the eigenvalues of (1.2) and

$$-\Delta u - q(x)u = \lambda u.$$

For any $a, b \in L^\infty(\Omega)$, which satisfy (f_1) , the non-resonant condition

$$\lambda_k(a) < 0, \quad \lambda_{k+1}(b) > 0$$

can be rewritten as

$$i(a) = i(b) = k, \quad \nu(b) = 0,$$

and the double-resonance condition

$$\lambda_k(a) \leq 0, \quad \lambda_{k+1}(b) \geq 0$$

can be rewritten as

$$i(a) + \nu(a) = i(b) = k.$$

We will express this in Proposition 2.2.4 after necessary discussion on the property of the index.

Throughout this paper, for any $a_1, a_2 \in L^\infty(\Omega)$, we write $a_1 \leq a_2$, if $a_2(x) - a_1(x) \geq 0$, for almost every $x \in \Omega$; we write $a_1 < a_2$, if $a_1 \leq a_2$, and $a_2(x) - a_1(x) > 0$ holds on a subset of $[0, 1]$ with nonzero measure. In condition (f_1) , without loss of generality, we assume that $\Omega_0 = \{x \in \Omega \mid a(x) \neq b(x)\}$ is not empty. Thus, by definition, we have $a < b$. Now we use the index to reach our main results.

THEOREM 1.2. *Suppose that condition (f_1) , condition:*

$$(f_2) \quad i(b) = i(a) + \nu(a),$$

and the following generalized Landesman–Lazer conditions:

$$(f_3^+) \quad \lim_{\|v\| \rightarrow \infty} \int_{\Omega} [F(x, v(x)) + F_1(x, v(x)) - \frac{a(x)}{2} v^2(x)] dx = +\infty, \quad v \in \ker(q_a),$$

$$(f_3^-) \quad \lim_{\|v\| \rightarrow \infty} \int_{\Omega} [F(x, v(x)) + F_1(x, v(x)) - \frac{b(x)}{2} v^2(x)] dx = -\infty, \quad v \in \ker(q_b),$$

are satisfied. Then problem (1.1)–(1.2) has at least one weak solution for $n \leq 4$.

COROLLARY 1.3. *Under the assumptions of Theorem 1.2, if $f_1(x, u) = 0$, problem (1.1)–(1.2) has at least one weak solution for any $n \in \mathbb{N}$.*

REMARK 1.4. (a) Due to the Sobolev inequality

$$\|u\|_{L^q} \leq C(n, p) \|u\|_{W^{1,p}},$$

with $q = np/(n-p)$, we restrict $n \leq 4$, when $f_1 \neq 0$.

(b) Note that Corollary 1.3 is similar to Theorem 1.1 in [10], but for different a and b . In this paper we consider a more general form of a , instead of a is a continuous function in [10], we assume that $a \in L^\infty(\Omega)$. In this sense, our result extends the result in [10].

EXAMPLE 1.5. Suppose that $n = 1$, $\Omega = (0, 1)$. Let $t_1 \in (0, 1)$ be fixed,

$$a(x) = \begin{cases} \frac{\pi^2}{4t_1^2} & \text{if } x \in (0, t_1), \\ \frac{\pi^2}{4(1-t_1)^2} & \text{if } x \in (t_1, 1). \end{cases} \quad h(x, u) = \begin{cases} ue^{-u} + \sin x & \text{if } u < 0, \\ \arctan u + \sin x & \text{if } u > 0. \end{cases}$$

The following problem

$$u'' + a(x)u + h(x, u) = 0, \quad u(0) = 0 = u(1)$$

has at least one solution. In fact, let

$$u_0(t) = \sin \frac{\pi}{2t_1} t \quad \text{for } t \in (0, t_1),$$

$$u_0(t) = \sin \left(\frac{\pi}{2(1-t_1)} (t-t_1) + \frac{\pi}{2} \right) \quad \text{for } t \in (t_1, 1).$$

Then $u = u_0(t)$ is a nontrivial solution of

$$u'' + a(x)u = 0, \quad u(0) = 0 = u(1),$$

and by Definition 1.1, we have $i(a) = 0$, $\nu(a) = 1$ and $\ker(q_a) = \{cu_0 \mid c \in \mathbb{R}\}$. All the assumptions of Theorem 1.2 are satisfied. Hence, problem (1.1)–(1.2) has at least one solution. Especially, $a(x)$ is a continuous function if and only if $t_1 = 1/2$.

THEOREM 1.6. *Under the assumptions of Theorem 1.2, for $n < 4$, if further, $f(-x) = -f(x)$, $f(0) = \theta$ and f satisfies following condition:*

(f₄) *there exists $A(x, u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$ such that*

$$\begin{aligned} f(x, u) + f_1(x, u) &= A(x, u)u + o(|u|) \quad \text{as } |u| \rightarrow 0, \\ A(x, u) &\leq a_1(x), \end{aligned}$$

with $i(a) - \nu(a) > i(a_1) + \nu(a_1)$, then (1.1)–(1.2) have at least $i(a) - i(a_1) - \nu(a) - \nu(a_1)$ pairs of weak solutions.

COROLLARY 1.7. *Under the assumptions of Corollary 1.3, if $f_1(x, u) = 0$, for any $n \in \mathbb{N}$, we have that problem (1.1)–(1.2) has at least $i(a) - i(a_1) - \nu(a) - \nu(a_1)$ pairs of weak solutions.*

From these two corollaries, our paper can be regarded as the further work and generalization of [10].

This paper is organized as follows: In Section 2, we introduce some preliminaries including the index theory and some theorems about critical point theory that we need in the proofs. In Section 3, we will prove Theorems 1.2 and 1.6.

2. Index theory for linear elliptic equation

In this section, we first decompose the space E by q_b which is defined in (1.5).

For any $u, v \in E$ if $q_b(u, v) = 0$, we say that u and v are q_b -orthogonal. For any two subspaces E_1 and E_2 of E , if $q_b(u, v) = 0$, for any $u \in E_1, v \in E_2$, we say that E_1 and E_2 are q_b -orthogonal.

PROPOSITION 2.1. *For any $b \in L^\infty(\Omega)$, the space E has a q_b -orthogonal decomposition*

$$E = E^+(b) \oplus E^0(b) \oplus E^-(b),$$

such that q_b is positive definite, zero and negative definite on $E^+(b)$, $E^0(b)$ and $E^-(b)$, respectively. Moreover, $E^0(b)$ and $E^-(b)$ are finitely dimensional.

REMARK 2.2. The decomposition was first used in [6] to investigate second order Hamiltonian system. This was further used in studying self-adjoint operator equation with finite Morse index in [8]. We will present the proof that we needed in the study of the properties of index and the expression of resonant conditions.

PROOF. Let λ_0 be a positive number satisfying $\lambda_0 > b$. The norm $\|\cdot\|_{\lambda_0}$ defined by an product

$$(2.1) \quad (u, v)_{\lambda_0} = \int_{\Omega} (\nabla u(x), \nabla v(x)) dx + \int_{\Omega} ((\lambda_0 - b(x))u(x), v(x)) dx,$$

for all $u, v \in E$, is equivalent to $\|\cdot\|$ and $(E, (\cdot, \cdot)_{\lambda_0})$ is a new Hilbert space. Let $K_{\lambda_0}(b): E \rightarrow E$ be an operator defined by

$$(2.2) \quad \int_{\Omega} (u(x), v(x)) dx = (u, K_{\lambda_0}(b)v)_{\lambda_0}, \quad \text{for all } u, v \in E.$$

Here we use the Riesz representation theorem. Since $E \hookrightarrow L^2$ is compact embedding, K_{λ_0} is self-adjoint and compact. By the spectral theory of self-adjoint and compact operators, there exists a basis $\{e_j\} \subset E$ and $\mu_j(b) \rightarrow 0$ in \mathbb{R} such that

$$(2.3) \quad (e_i, e_j)_{\lambda_0} = \delta_{ij}, \quad K_{\lambda_0}e_j = \mu_j(b)e_j,$$

and $\mu_j(b) = (e_j, K_{\lambda_0}e_j)_{\lambda_0} = \|e_j\|_2^2 > 0$.

For any $x \in \sum_{j=1}^{\infty} c_j e_j$, from (2.1)–(2.3), a simple calculation shows that

$$q_b(u, u) = (u, u)_{\lambda_0} - \lambda_0 \|u\|_2^2 = \sum_{j=1}^{\infty} (1 - \lambda_0 \mu_j(b)) c_j^2 = \sum_{j=1}^{\infty} \left(\frac{1}{\mu_j(b)} - \lambda_0 \right) c_j^2 \|e_j\|_2^2.$$

Set $\lambda_j(b) = 1/\mu_j(b) - \lambda_0$. From $\mu_j \rightarrow 0$ and $\mu_j > 0$, we have that $\lambda_j(b) \rightarrow +\infty$ are positive except a finite number. Without loss of generality, we assume that $\lambda_1(b) \leq \dots \leq \lambda_k(b) \leq \dots$. Here the result holds, if we set

$$\begin{aligned} E^-(b) &:= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } \lambda_j(b) \geq 0 \right\}, \\ E^0(b) &:= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } \lambda_j(b) \neq 0 \right\}, \\ E^+(b) &:= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } \lambda_j(b) \leq 0 \right\}. \quad \square \end{aligned}$$

REMARK 2.3. (a) The above decomposition $E = E^+(b) \oplus E^0(b) \oplus E^-(b)$ is orthogonal with respect to $(\cdot, \cdot)_2$. To show the $(\cdot, \cdot)_2$ orthogonality, observe that, take $e_i \in E^+$, $e_j \in E^-$ for example

$$(e_i, e_j)_2 = (e_i, K_{\lambda_0}(b)e_j)_{\lambda_0} = \lambda_j(e_i, e_j)_{\lambda_0} = 0.$$

(b) The subspace $\text{Ker}(q_a)$ and $\text{Ker}(q_b)$ in condition (f_3^{\pm}) can be expressed as $E^0(a)$ and $E^0(b)$, respectively.

DEFINITION 2.4. For any $b \in L^{\infty}(\Omega)$, we define $i(b) = \dim E^-(b)$, $\nu(b) = \dim E^0(b)$. We call $i(b)$ and $\nu(b)$ the index and the nullity of b , respectively.

Recall that for a bounded symmetric bilinear form ϕ defined on a Hilbert space X , its Morse index and nullity are defined as $m^-(\phi) = \max\{\dim X_1 \mid X_1 \text{ is a subspace of } X \text{ such that } \phi(x, x) < 0, \text{ for any } x \in X_1 \setminus \{\theta\}\}$ and $m^0(\phi) = \dim \ker \phi$ where $\ker \phi = \{x \in X \mid \phi(x, y) = 0, \text{ for all } y \in X\}$.

Applying [8, Proposition 2.1.3, Definition 2.5.1, Proposition 2.5.2], one can prove

PROPOSITION 2.5. *For any $b \in L^\infty(\Omega)$, we have:*

$$(a) \quad i(b) = \sum_{\lambda_i(b) < 0} \dim E_i(b) = \sum_{\lambda < 0} \nu(b + \lambda), \text{ where } \lambda_i(b) \text{ is defined as in}$$

Proposition 2.1. Moreover, $i(b) = \#\{i \mid \lambda_i(b) < 0\}$, $\nu(b) = \#\{i \mid \lambda_i(b) = 0\}$, where $\#S$ denotes the total number of elements in a set S .

(b) $i(b)$ is the Morse index of q_b .

(c) If $b_1 \leq b_2$, then

$$i(b_2) - i(b_1) = \sum_{\lambda \in [0,1]} \nu(b_1 + \lambda(b_2 - b_1)).$$

The summand denoted by $I(b_1, b_2)$ is called the relative Morse index between b_1 and b_2 .

(d) For any $b_1, b_2 \in L^\infty(\Omega)$, if $b_1 \leq b_2$, we have $i(b_1) \leq i(b_2)$; if $b_1 < b_2$, we have $i(b_1) + \nu(b_1) \leq i(b_2)$.

(e) There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have

$$\nu(b + \varepsilon) = 0 = \nu(b - \varepsilon), \quad i(b - \varepsilon) = i(b), \quad i(b + \varepsilon) = i(b) + \nu(b).$$

PROPOSITION 2.6. *For any $a, b \in L^\infty(\Omega)$ and $a < b$:*

(a) $\lambda_k(a) < 0$, and $\lambda_{k+1}(b) > 0$ if and only if $i(a) = i(b) = k$ and $\nu(b) = 0$.

(b) $\lambda_k(a) \leq 0$, and $\lambda_{k+1}(b) \geq 0$ if and only if $i(a) + \nu(a) = i(b) = k$.

(c) $\lambda_k(a) = 0$, and $\lambda_{k+1}(b) = 0$ if and only if $i(a) + \nu(a) = i(b) = k$ and $\nu(b) \neq 0, \nu(a) \neq 0$.

PROOF. (a) From Proposition 2.5(a), we have $i(a) \geq k, i(b) \leq k$. By Proposition 2.5(d), since $a < b$ we have $i(a) \leq i(a) + \nu(a) \leq i(b)$. Hence $i(a) = i(b) = k$ and $\lambda_k(b) < 0$. From the monotonicity of $\lambda_i(b)$ with respect to i , one has $\lambda_i(b) \neq 0$ for all i . So $\nu(b) = 0$.

(b) First, $\lambda_{k+1}(b) \geq 0$, we have $i(b) \leq k$. Since $a < b$, we have $i(a) \leq k$ and $\lambda_{k+1}(a) \geq 0$. If $\lambda_{k+1}(a) = 0$, then $i(a) + \nu(a) \geq k + 1$ and $i(b) \geq k + 1$ and $\lambda_{k+1}(b) < 0$. This is a contradiction. So $\lambda_{k+1}(a) > 0$. Second, $\lambda_k(a) \leq 0$, we have $i(a) + \nu(a) = k$. Hence $i(b) \geq k$ and $\lambda_k(b) < 0$. From the discussion above, we have $i(a) + \nu(a) = k = i(b)$.

(c) *Case 1.* $\lambda_k(a) = 0$, and $\lambda_{k+1}(b) = 0$.

Since $\lambda_k(a) = 0$, then $i(a) + \nu(a) \geq k$. For $a < b$, we have $i(b) \geq i(a) + \nu(a) \geq k$ and $\lambda_k(b) < 0$. Together with $\lambda_{k+1}(b) = 0$, we get $i(b) = k$ and $\nu(b) \neq 0$. Since

$\lambda_{k+1}(b) = 0$, then $i(b) \leq k$, and $i(a) + \nu(a) \leq k$. Hence $\lambda_{k+1}(a) > 0$. Together with $\lambda_k(a) = 0$, we obtain $i(a) + \nu(a) = k$ and $\nu(a) \neq 0$.

Case 2. If $i(a) + \nu(a) = k$ and $\nu(a) \neq 0$, then $\lambda_k(a) = 0$. If $i(b) = k$ and $\nu(b) \neq 0$, then $\lambda_{k+1}(b) = 0$. \square

PROPOSITION 2.7. *Let $a, b \in L^\infty(\Omega)$ satisfy $a < b$ and $i(a) + \nu(a) = i(b)$, $\nu(a) \neq 0$, $\nu(b) \neq 0$. Then the space E has a decomposition:*

$$E = E^-(a) \oplus E^0(a) \oplus E^0(b) \oplus E^+(b).$$

PROOF. First we show that $(E^-(a) \oplus E^0(a)) \cap (E^0(b) \oplus E^+(b)) = \emptyset$.

For any $u \in (E^-(a) \oplus E^0(a)) \cap (E^0(b) \oplus E^+(b))$, we have $q_a(u, u) \leq 0$ and $q_b(u, u) \geq 0$. Furthermore, from $a < b$, we get $q_a(u, u) > q_b(u, u)$. This is a contradiction.

Now we only need to show that for any $u \in E$, $u = u_1 + u_2$ with $u_1 \in E^-(a) \oplus E^0(a)$ and $u_2 \in E^0(b) \oplus E^+(b)$. Let $\{e_1, \dots, e_k\}$ be a basis of $E^-(a) \oplus E^0(a)$ and $k = i(a) + \nu(a) = i(b)$. For each e_i , we have the decomposition $e_i = e_i^- + e_i^*$, $e_i^- \in E^-(b)$, $e_i^* \in E^0(b) \oplus E^+(b)$. We claim that $\{e_i^-\}$ are linear independent.

If not, there exists not all zero constants c_i such that $\sum_i^k c_i e_i^- = 0$. Let

$$u = \sum_i^k c_i e_i = \sum_i^k c_i e_i^* \neq 0.$$

On the one hand,

$$q_b(u, u) = q_b\left(\sum_i^k c_i e_i^*, \sum_i^k c_i e_i^*\right) \geq 0;$$

on the other hand, $q_a(u, u) \leq 0$. This contradicts to $a < b$. Hence, $\{e_1^-, \dots, e_k^-\}$ are linear independent in $E^-(b)$ and $\{e_1^-, \dots, e_k^-\}$ are a basis of $E^-(b)$.

For any $u \in E$, $u = u^- + u^*$ with $u^- \in E^-(b)$, $u^* \in E^0(b) \oplus E^+(b)$,

$$u = u^- + u^* = \sum_{i=1}^k c_i e_i^- + u^* = \sum_{i=1}^k c_i e_i + \left(u^* - \sum_{i=1}^k c_i e_i^*\right).$$

Here

$$\sum_i^k c_i e_i \in E^-(a) \oplus E^0(a) \quad \text{and} \quad u^* - \sum_i^k c_i e_i^* \in E^0(b) \oplus E^+(b).$$

This completes the proof. \square

In order to prove Theorems 1.2 and 1.6, we introduce some lemmas about the existence and multiplicity of critical points in [1].

LEMMA 2.8 ([1, Theorem 2.3]). *Suppose that $f \in C'(X, \mathbb{R})$ satisfies the following properties:*

- (a) f satisfies condition (C) in $[0, +\infty]$,
- (b) there exist a closed subset $S \subset X$ and a Hilbert manifold $Q \subset X$ with boundary ∂Q which fulfill the following properties:
 - (i) There exist two constants $\beta > \alpha \geq 0$ such that

$$f(u) \leq \alpha, \quad \text{for all } u \in \partial Q \quad \text{and} \quad f(u) \geq \beta, \quad \text{for all } u \in S,$$
 - (ii) S and ∂Q link,
 - (iii) $\sup_{u \in Q} f(u) < +\infty$.

Then f possesses a critical value $c \geq \beta$.

Here the condition (C) means: For every sequence $\{u_n\} \subset X$ with

$$(2.4) \quad I(u_n) \text{ being bounded,} \quad (\|u\| + 1)I'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

possesses a convergent subsequence. This weakened condition was introduced by Cerami. This weakening of the Palais–Smale condition was proved to be essential in the study of variational problems in the strong resonance case in [1]. The condition (C) is actually sufficient to get a “deformation theorem”.

LEMMA 2.9 ([1, Lemma 2.4]). *Suppose that $f \in C^1(H, \mathbb{R})$ satisfies the following properties:*

- (a) f satisfies condition (C) in $[0, +\infty]$ and $f(0) \geq 0$,
- (b) there exist two closed subspaces H^+, H^- of H , with $\text{codim } H^+ < +\infty$, and two constants $c_\infty > c_0 > f(0)$ such that
 - (i) $f(u) \geq c_0$ for all $u \in S_\rho \cap H^+$,
 - (ii) $f(u) < c_\infty$ for all $u \in H^-$,
- (c) f is even.

Then, if $\dim H^- \geq \text{codim } H^+$, f possesses at least $m = \dim H^- - \text{codim } H^+$ distinct pairs of critical points whose corresponding critical values belong to $[c_0, c_\infty]$.

3. Proofs of Theorems 1.2 and 1.6

PROPOSITION 3.1. *Under the assumptions of Theorem 1.2, the functional defined in (1.3) satisfies the (C) condition.*

PROOF. We will prove the proposition by several lemmas following the argument in [10] with modifications. Assume that $\{u_n\} \subset E$ satisfies $I(u_n)$ bounded and $(\|u_n\| + 1)I'(u_n) \rightarrow 0$. We first show that u_n is bounded in E . We develop a contradiction argument, assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. By (f_1) , we obtain

$$|f(x, u_n)| \leq c(1 + |u_n|), \quad \text{a.e. on } \Omega.$$

From (1.1),

$$\int_{\Omega} (\Delta u_n, u_n) dx = \int_{\Omega} f(x, u_n) u_n dx + \int_{\Omega} f_1(x, u_n) u_n dx,$$

and

$$\begin{aligned} \|u_n\|^2 &\leq \int_{\Omega} |f(x, u_n)u_n| dx + \int_{\Omega} |\tilde{g}(x)||u_n| dx \\ &\leq c \int_{\Omega} |u_n| + |u_n|^2 dx + \left(\int_{\Omega} |\tilde{g}(x)|^2 \right)^{1/2} \|u_n\|_2 \leq c\|u_n\|_2 + c\|u_n\|_2^2. \end{aligned}$$

Hence, $\|u_n\| \leq c(\|u_n\|_2 + 1)$. Thus, $\|u_n\|_2 \rightarrow \infty$ as $n \rightarrow \infty$. Let $z_n = u_n/\|u_n\|_2$. Since E is compactly embedding in $L^2(\Omega)$, there exist $z \in E$ and a subsequence such that

$$z_n \rightharpoonup z \text{ weakly in } E, \quad \text{and} \quad z_n \rightarrow z \text{ strongly in } L^2, \quad \|z_n\|_2 = 1.$$

By (f₂), let

$$c_n(x) = \frac{f(x, u_n(x))}{u_n(x)} \quad \text{if } |u_n(x)| \geq r, \quad c_n(x) = a(x) \quad \text{if } |u_n(x)| \leq r.$$

There is a $c \in L^\infty(\Omega)$ and $c_n(x) \rightharpoonup c(x)$ weakly in $L^\infty(\Omega)$ as $n \rightarrow \infty$ by going to subsequences if necessary. We have $a(x) \leq c(x) \leq b(x)$, $x \in \Omega$. By (1.3), for any $v \in E$, we have

$$\begin{aligned} (I'(u_n) \setminus \|u_n\|_2, v) &= \int_{\Omega} \nabla z_n \nabla v dx \\ &\quad - \frac{1}{\|u_n\|_2} \int_{\Omega} (f(x, u_n), v) dx - \frac{1}{\|u_n\|_2} \int_{\Omega} (f_1(x, u_n), v) dx. \end{aligned}$$

Let $n \rightarrow \infty$. Standard computations show that

$$\int_{\Omega} \nabla z \nabla v dx = \int_{\Omega} c(x) z v dx,$$

which implies $\nu(c) \neq 0$.

Case 1. $a < c$ and $c < b$.

From Proposition 2.5, we have $i(a) + \nu(a) \leq i(c)$ and $i(c) + \nu(c) \leq i(b)$ which contradicts to $i(a) + \nu(a) = i(b)$.

CLAIM. $c(x) = a(x)$ almost everywhere on Ω .

We write

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} a(x) u^2 dx - \int_{\Omega} G(x, u) dx - \int_{\Omega} F_1(x, u) dx,$$

where $G(x, u) = \int_0^u g(x, t) dt$, $g(x, t) = f(x, t) - a(x)t$. Let $c_n^1(x) = c_n(x) - a(x)$. Then $c_n^1(x) \geq 0$ and $c_n^1 \rightharpoonup 0$ weakly in $L^\infty(\Omega)$. From Proposition 2.1, $E^-(a)$ is finite dimensional, so $(-q_a(u^-, u^-))^{1/2}$ is equivalent to $\|\cdot\|$.

For any $u = \sum_{j=k}^{\infty} c_j e_j \in E^+(a)$, $k = i(a) + \nu(a) + 1$, we have

$$q_a(u, u) = \sum_{j=k}^{\infty} (1 - \lambda_0 \mu_j(a)) c_j^2 \geq (1 - \lambda_0 \mu_k(a)) \sum_{j=k}^{\infty} c_j^2 = (1 - \lambda_0 \mu_k(a)) \|u\|^2.$$

LEMMA 3.2. *There exists a constant $\delta > 0$ such that*

$$\begin{aligned} q_a(u, u) &\leq -\delta\|u\|^2, \quad \text{for all } u \in E^-(a), \\ q_a(u, u) &\geq \delta\|u\|^2, \quad \text{for all } u \in E^+(a). \end{aligned}$$

From Proposition 2.1, u_n has a decomposition:

$$u_n = u_n^+ + u_n^0 + u_n^-, \quad u_n^* \in E^*(a), \quad * = +, 0, -.$$

LEMMA 3.3. *If $c(x) = a(x)$ almost everywhere on Ω , we have $\|u_n^+\|^2 + \|u_n^-\|^2$ is bounded.*

PROOF. Using (2.4), we have

$$(3.1) \quad (\|u_n + 1\|)\langle I'(u_n), u_n^+ - u_n^0 - u_n^- \rangle = o(1)\|u_n^+ - u_n^0 - u_n^-\|.$$

By Lemma 3.2, we have

$$\begin{aligned} (3.2) \quad \langle I'(u_n), u_n^+ - u_n^0 - u_n^- \rangle &= q_a(u_n^+, u_n^+) - q_a(u_n^-, u_n^-) \\ &\quad - \int_{\Omega} g(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx - \int_{\Omega} f_1(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx \\ &\geq \delta(\|u_n^+\|^2 + \|u_n^-\|^2) - \int_{\Omega} g(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx \\ &\quad - \int_{\Omega} f_1(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx. \end{aligned}$$

Together with (3.1), we have

$$(3.3) \quad \delta(\|u_n^+\|^2 + \|u_n^-\|^2) \leq c \frac{\|u_n^+ - u_n^0 - u_n^-\|}{\|u_n\| + 1} + \int_{\Omega} g(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx.$$

Since $u_n^+ - u_n^0 - u_n^- = 2u_n^+ - u_n$, simple computation shows that

$$(3.4) \quad \frac{\|u_n^+ - u_n^0 - u_n^-\|}{\|u_n\| + 1} \leq c\|u_n^+\| + c, \quad \text{for } n \text{ large enough.}$$

Similar to [10, Lemma 2.4], we have

$$(3.5) \quad \int_{\Omega} g(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx \leq c\|u_n^+\| + \frac{\delta}{4}\|u_n^+\|^2 + c,$$

for n large. Now we estimate the last part of (3.2), for any $M_0 > 0$,

$$\begin{aligned} \int_{|u_n| \leq M_0} f_1(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx &\leq \int_{|u_n| \leq M_0} \tilde{g}(x)|2u_n^+ - u_n| dx \\ &\leq \left(\int_{|u_n| \leq M_0} \tilde{g}^2(x) \right)^{1/2} \left(\int_{|u_n| \leq M_0} |2u_n^+ - u_n|^2 \right)^{1/2} \leq c\|u_n^+\| + c. \end{aligned}$$

By the Sobolev inequality for $n \leq 4$, $E \hookrightarrow L_4$,

$$\begin{aligned} \int_{|u_n| \geq M_0} f_1(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx &\leq \int_{|u_n| \geq M_0} \frac{f_1(x, u_n)}{u_n} (u_n^{+2} - (u_n^0 + u_n^-)^2) dx \\ &\leq \frac{1}{M_0} \int_{|u_n| \geq M_0} \tilde{g}(x) u_n^{+2} dx \leq \frac{1}{M_0} c \|u_n^+\|_4^2 \leq \frac{1}{M_0} c \|u_n^+\|^2 \leq \frac{\delta}{4} \|u_n^+\|^2, \end{aligned}$$

for large constant M_0 . Hence,

$$(3.6) \quad \int_{\Omega} f_1(x, u_n)(u_n^+ - u_n^0 - u_n^-) dx \leq c \|u_n^+\| + \frac{\delta}{4} \|u_n^+\|^2 + c.$$

From (3.3)–(3.6), we have $\|u_n^+\|^2 + \|u_n^-\|^2$ is bounded. \square

LEMMA 3.4. *If $\|u_n^+\|^2 + \|u_n^-\|^2$ is bounded, then $I(u_n) \rightarrow -\infty$ as $n \rightarrow \infty$.*

PROOF. Since $\|u_n^+\|^2 + \|u_n^-\|^2$ is bounded, we get

$$\begin{aligned} I(u_n) &= \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\Omega} a(x) u_n^2 dx - \int_{\Omega} G(x, u_n) dx - \int_{\Omega} F_1(x, u_n) dx \\ &\leq c + \int_{\Omega} \left[G\left(x, \frac{u_n^0}{2}\right) - G(x, u_n) \right] dx + \int_{\Omega} \left[F_1\left(x, \frac{u_n^0}{2}\right) - F_1(x, u_n) \right] dx \\ &\quad - \int_{\Omega} G\left(x, \frac{u_n^0}{2}\right) + F_1\left(x, \frac{u_n^0}{2}\right) dx. \end{aligned}$$

From [10, Lemma 2.5], we obtain

$$(3.7) \quad \int_{\Omega} \left[G\left(x, \frac{u_n^0}{2}\right) - G(x, u_n) \right] dx \leq c.$$

In order to prove the boundness of $\int_{\Omega} [F_1(x, u_n^0/2) - F_1(x, u_n)] dx$, we follow the proof of (3.7) closely. By the mean value theorem, we have

$$\int_{\Omega} \left[F_1\left(x, \frac{u_n^0}{2}\right) - F_1(x, u_n) \right] dx = \int_{\Omega} \int_0^1 f_1\left(x, t \frac{u_n^0}{2} + (1-t)u_n\right) \left(\frac{u_n^0}{2} - u_n\right) dt dx.$$

Assume that $h_n = t u_n^0/2 + (1-t)u_n$, by $|f_1(x, u)| \leq \tilde{g}(x)$, one has

$$\begin{aligned} \int_{|h_n| \leq M_0} \left[F_1\left(x, \frac{u_n^0}{2}\right) - F_1(x, u_n) \right] dx &\leq \int_{|h_n| \leq M_0} \tilde{g}(x) \left| \frac{u_n^0}{2} - u_n \right| dt dx \\ &= \int_{|h_n| \leq M_0} \tilde{g}(x) \left| \frac{1}{t-2} h_n + \frac{1}{t-2} (u_n^0 - u_n) \right| dt dx \\ &\leq c \left(\int_{|h_n| \leq M_0} |h_n + (u_n^- + u_n^+)|^2 \right)^{1/2} \leq c. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{|h_n| \geq M_0} \left[F_1 \left(x, \frac{u_n^0}{2} \right) - F_1(x, u_n) \right] dx &= \int_{|h_n| \geq M_0} \int_0^1 f_1(x, h_n) \left(\frac{u_n^0}{2} - u_n \right) dt dx \\ &= \int_{|h_n| \geq M_0} \int_0^1 \frac{f_1(x, h_n)}{h_n} \left[t \left(\frac{u_n^0}{2} - u_n \right)^2 + \left(\frac{u_n^0}{2} - u_n \right) u_n \right] dt dx \\ &\leq \int_{|h_n| \geq M_0} \frac{\tilde{g}(x)}{M_0} \left[\left(\frac{u_n^0}{2} - u_n \right)^2 + \left(\frac{u_n^0}{2} - u_n \right) u_n \right] dx. \end{aligned}$$

By the elementary inequality,

$$\left(\frac{u_n^0}{2} - u_n \right)^2 + \left(\frac{u_n^0}{2} - u_n \right) u_n \leq (u_n^+ + u_n^-)^2.$$

We have

$$\begin{aligned} \int_{|h_n| \geq M_0} \left[F_1 \left(x, \frac{u_n^0}{2} \right) - F_1(x, u_n) \right] dx &\leq \int_{|h_n| \geq M_0} \frac{\tilde{g}(x)}{M_0} (u_n^+ + u_n^-)^2 dx \\ &\leq c \|u_n^+ + u_n^-\|_{L^4}^2 \leq c \|u_n^+ + u_n^-\|^2 \leq c. \end{aligned}$$

This together with the condition (f_3^+) gives $I(u_n) \rightarrow -\infty$, as $n \rightarrow \infty$. \square

Case 3. $c(x) = b(x)$ almost everywhere on Ω .

This case is similar to Case 2. We have the following lemma.

LEMMA 3.5. *If $c(x) = b(x)$ almost everywhere on Ω , then $\|u_n^+\|^2 + \|u_n^-\|^2$ is bounded and $I(u_n) \rightarrow +\infty$ as $n \rightarrow \infty$.*

Lemmas 3.4, 3.5 contradict to condition (2.4), which implies Cases 2 and 3 are also impossible. Hence the proof of Proposition 3.1 is completed. \square

PROOF OF THEOREM 1.2. We apply Lemma 2.11 to the functional I defined by (1.3). Let $X_1 = E^-(a) \oplus E^0(a)$, $X_2 = E^0(b) \oplus E^+(b)$. Then $E = X_1 \oplus X_2$ and X_1 is finite dimensional.

First, we have $I(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$, $u \in X_1$.

If not, there exists a sequence $u_n = u_n^- + u_n^0 \in E^-(a) \oplus E^0(a)$ such that $\|u\| \rightarrow \infty$ and $I(u) \geq c$.

$$\begin{aligned} I(u_n) &= \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\Omega} a(x) u_n^2 dx - \int_{\Omega} G(x, u_n) dx - \int_{\Omega} F_1(x, u_n) dx, \\ &= q_a(u_n^-, u_n^-) - \int_{\Omega} G(x, u_n) dx - \int_{\Omega} -F_1(x, u_n) dx \leq -\delta \|u_n^-\|^2 + c. \end{aligned}$$

Therefore, $\|u_n^-\|^2$ is bounded. And by Lemma 3.4, we obtain $I(u) \rightarrow -\infty$, as $n \rightarrow \infty$. This is a contradiction.

By Lemma 3.5, similar to the discussion above, we get $I(u_n) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$, $u \in X_2$.

Let $Q = B_R \cap X_1$, $S = X_2$. Then ∂Q and S link. Thus it is easy to verify the assumptions of Lemma 2.11. \square

PROOF OF THEOREM 1.6. In order to prove Theorem 1.6, we only need to check that the assumptions of Lemma 2.9 are satisfied. Such work will be done in the following Lemmas 3.6 and 3.7.

LEMMA 3.6. *The assumption (b)(i) of Lemma 2.9 is valid, where f is defined as (1.3) and $\text{codim } H^+ = i(a_1) + \nu(a_1)$.*

PROOF. Let $\varepsilon > 0$ small enough and set $4 < r < 2n/(n-2)$, $n < 4$. Since (f_1) and (f_4) hold, there exist $\delta_2 > \delta_1 > 0$ and $M > 0$ for the above ε such that

$$F(x, u) + F_1(x, u) \leq \frac{1}{2}(a_1(x) + \varepsilon)|u|^2, \quad \text{for all } |u| \leq \delta_1, x \in \bar{\Omega},$$

$$F(x, u) + F_1(x, u) \leq M|u|^2 \leq \frac{M}{\delta_1^{r-2}}|u|^r, \quad \delta_1 \leq |u| \leq \delta_2, x \in \bar{\Omega},$$

$$F(x, u) \leq \frac{1}{2}(b(x) + \varepsilon)|u|^2 \leq \frac{1}{2\delta_2^{r-2}}(\|b(x)\|_\infty + \varepsilon)|u|^r, \quad \text{for all } |u| \geq \delta_2, x \in \bar{\Omega}.$$

Furthermore,

$$\int_{|u| \geq \delta_2} F_1(x, u) dx \leq \left(\int_{|u| \geq \delta_2} \tilde{g}^2(x) dx \right)^{1/2} \left(\int_{|u| \geq \delta_2} |u|^2 dx \right)^{1/2} \leq \frac{c}{\delta_2^{r-2}} \|u\|_{L^r}^{r/2}.$$

Set

$$A = \max \left\{ \frac{1}{2\delta_2^{r-2}}(\|b(x)\|_\infty + \varepsilon), \frac{M}{\delta_1^{r-2}} \right\} > 0.$$

Together with (1.3), we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \frac{1}{2}(a_1(x) + \varepsilon)|u|^2 + A|u|^r dx - \frac{c}{\delta_2^{r-2}} \|u\|_{L^r}^{r/2} \\ &= \frac{1}{2} a_{a_1+\varepsilon}(u, u) - A \|u\|_{L^r}^r - \frac{c}{\delta_2^{r-2}} \|u\|_{L^r}^{r/2}. \end{aligned}$$

Let $H^+ = E^+(a_1 + \varepsilon)$. Then $\text{codim } H^+ = i(a_1 + \varepsilon) + \nu(a_1 + \varepsilon)$. For any $u \in H^+$, we have

$$I(u) \geq \frac{1}{2}(1 - \lambda_0 \mu_k(a_1 + \varepsilon)) \|u\|^2 - Ac \|u\|^r - \frac{c}{\delta_2^{r-2}} \|u\|^{r/2}.$$

Here $\mu_k(a_1 + \varepsilon)$ is defined in Proposition 2.1 and $k = i(a_1 + \varepsilon) + \nu(a_1 + \varepsilon) + 1$. Since $1 - \lambda_0 \mu_k(a_1 + \varepsilon) > 0$, let $\rho > 0$ small enough such that

$$I(u) \geq c_0 = \frac{1}{2}(1 - \lambda_0 \mu_k(a_1 + \varepsilon)) \rho^2 - Ac \rho^r - \frac{c}{\delta_2^{r-2}} \rho^{r/2} > 0.$$

Furthermore, from Proposition 2.5, pick ε small enough, we have

$$i(a_1 + \varepsilon) + \nu(a_1 + \varepsilon) = i(a_1) + \nu(a_1).$$

So we have $I(u) \geq c_0$, for all $u \in S_\rho \cap H^+$ and $\text{codim } H^+ = i(a_1) + \nu(a_1)$. The proof is completed. \square

LEMMA 3.7. *The assumption (b)(ii) of Lemma 2.9 is also satisfied and*

$$\dim H^- = i(a) - \nu(a).$$

PROOF. (f_1) holds. Let

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx.$$

For any $\varepsilon > 0$, there exists a constant M such that

$$(3.8) \quad J(u) \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} (a(x) - \varepsilon)|u|^2 dx + M = \frac{1}{2} q_{a-\varepsilon}(u, u) + M,$$

for all $u \in E$. Recall that $|f_1(x, u)| \leq \tilde{g}(x)$, where $\tilde{g}(x) \in L^2$, and the embedding theorem, we get

$$\left| \int_{\Omega} F_1(x, u) dx \right| \leq \int_{\Omega} \tilde{g}(x)|u| dx \leq c\|u\|.$$

Combining (3.7) and (3.8), we obtain

$$I(u) = J(u) - \int_{\Omega} F_1(x, u) dx \leq \frac{1}{2} q_{a-\varepsilon}(u, u) + c\|u\| + M.$$

Let $H^- = E^-(a - \varepsilon)$. Then $\dim H^- = i(a - \varepsilon)$. From Lemma 3.2, there exists a constant δ_0 , such that

$$\frac{1}{2} q_{a-\varepsilon}(u, u) \leq -\delta_0 \|u\|^2 \quad \text{for any } u \in H^-.$$

Thus, $I(u) \leq -\delta_0 \|u\|^2 + c\|u\| + M < +\infty$. Also from Proposition 2.5, pick ε small enough, we have $i(a - \varepsilon) = i(a) - \nu(a)$. \square

PROOF OF COROLLARIES 1.3 AND 1.7. For $f_1(x, u) = 0$, the proof of Corollary 1.3 and 1.7 can be easily deduced from the proof of Theorems 1.2 and 1.6 for any n , respectively. \square

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