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NONDECREASING SOLUTIONS OF FRACTIONAL QUADRATIC INTEGRAL EQUATIONS INVOLVING ERDÉLYI–KOBER SINGULAR KERNELS

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Abstract. In this paper, we firstly present the existence of nondecreasing solutions of a fractional quadratic integral equations involving Erdélyi–Kober singular kernels for three provided parameters $\alpha \in (1/2, 1)$, $\beta \in (0, 1]$ and $\gamma \in [\beta(1-\alpha)-1, \infty)$. Moreover, we prove this restriction on α and β can not be improved. Secondly, we obtain the uniqueness and nonuniqueness of the monotonic solutions by utilizing a weakly singular integral inequality and putting $\gamma \in [1/2 - \alpha, \infty)$. Finally, two numerical examples are given to illustrate the obtained results.

1. Introduction

Fractional continuous models are used to describe the real fractal structure of matter and the medium in many physics problems. With the development of fractional calculus, we find that it is better to apply this powerful tools to describe the medium with non-integer mass dimension. In fact, fractional continuous models provide a good method to describe dynamics of fractal media for us. Fractional integrals are also used to derive the generalizations of the equations of balance for the fractal media [31], [32].

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Recently, Banaś and Rzepka [4] studied nondecreasing solutions of the following quadratic integral equation of fractional order:

$$(1.1)\quad x(t)=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s,x(s))\,ds,\quad t\in[0,1],\ \alpha\in(0,1),$$

where $\Gamma(\,\cdot\,)$ is the gamma function. In the paper [4], the authors not only point an error in calculation in [6] but also give a correct proof to obtain the interesting existence theorems of nondecreasing solutions of the equation (1.1).

In general, the term $(t-s)^{\alpha-1}$, $\alpha\in(0,1)$ can be named as Riemann–Liouville singular kernel since it appears in the standard Riemann–Liouville fractional integral of order α of a continuous function y defined by

$$(1.2)\qquad I^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,\quad t>0,\ \alpha>0.$$

In addition to the classical Riemann–Liouville fractional integrals, there are many differ type fractional integrals such as Erdélyi–Kober fractional integrals, Hadamard fractional integrals, Riesz fractional integrals and etc. One can also find more about basic definitions and applications of the fractional calculus in physics, viscoelasticity, electrochemistry and porous media [8]–[10], [13], [24], [27], [33]. In recent years, there has been a significant development in Cauchy problems (boundary value problems, nonlocal problems, impulsive problems) for fractional differential (integral, evolution) equations and related optimal controls, one can see the monographs [7], [19], [22], [28], [29] and the papers [1], [5], [11], [12], [15], [17], [21], [23], [36]–[**?**].

The Erdélyi–Kober fractional integral [30], [20] of a continuous function y is defined by

$$(1.3)\qquad I_{\beta}^{\gamma,\alpha}y(t)=\frac{t^{-\beta(\gamma+\alpha)}}{\Gamma(\alpha)}\int_0^t (t^{\beta}-s^{\beta})^{\alpha-1}y(s)\,ds^{\beta},$$

where α,γ and $\beta>0$. Compared with the term $(t-s)^{\alpha-1}$ in (1.2), the term $(t^{\beta}-s^{\beta})^{\alpha-1}$, $\alpha\in(0,1)$ is more general and can be named as Erdélyi–Kober singular kernel (or named as weakly singular kernel in [25], [26]).

Since Erdélyi–Kober fractional integral can better describe the memory property than Riemann–Liouville fractional integral, quadratic integral equations involving Erdélyi–Kober singular kernels maybe better applicable in the theory of kinetic theory of gases [14] and in the theory of neutron transport [18] than integral equations involving Riemann–Liouville singular kernels. It is remarkable that Wang et al. [35] obtained the existence and uniqueness -results of solutions in a closed ball by Schauder fixed po nt theorem via a weakly singular integral inequality in [26]. Moreover, the authors constructed three certain solutions sets tending to zero at appropriate rate and presented local stability results of solutions. For some other pioneer works on such equations, we refer to [16], [26] and references therein.

Motivated by the above mentioned, we will apply a differ way in [35] to study nondecreasing solutions of a fractional quadratic integral equation involving Erdélyi–Kober singular kernels of the form:

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t^\beta - s^\beta)^{\alpha-1} s^\gamma u(s, x(s))\, ds, \quad t \in [0,1], \tag{1.4}$$

where $\alpha \in (0,1)$, $\beta \in (0,\infty)$ and $\gamma \in (-1,\infty)$ are some suitable positive constants, f and u are two functions will be defined later. Obviously, the equation (1.3) is a particular case of the equation (1.1) when $\beta = 1$ and $\gamma = 0$.

By adopting the methods and techniques of Hausdorff measure of noncompactness in [4], [35], we firstly obtain the existence results of the nondecreasing solutions for the equation (1.4) by restricting the parameters $\alpha \in (1/2, 1)$, $\beta \in (0,1]$ and $\gamma \in [\beta(1-\alpha) - 1, \infty)$. Here, we also prove that the restriction on α and β in Lemma 3.1 can not be improved due to Beta function $\mathbb{B}((\gamma+1)/\beta, 2\alpha - 1)$ (see Remark 3.2). Secondly, we derive the uniqueness and nonuniqueness of the nondecreasing solutions by utilizing a weakly singular integral inequality [26] and imposing additional sublinear continuous condition on u.

Compared with the results in [4], the existence theorem of nondecreasing solutions of the equation (1.1) is included in the equation (1.4). Moreover, the uniqueness and nonuniqueness of the nondecreasing solutions of the equation (1.4) are given in current text. Compared with the results in [35], a more general nonlinear term f is appeared, a differ problem (nondecreasing solutions) is posed, and a differ fixed point theorem (associated with Hausdorff measure of noncompactness) is used to study the equation (1.4) under the suitable conditions on f, u and γ.

2. Preliminaries

Let E be a Banach space with the norm $\|\cdot\|$ and $X \subseteq E$ is bounded, but $X \neq \emptyset$. We denote by $\chi(X)$ the Hausdorff measure of noncompactness of X, which is defined by $\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}$ (see [3]).

We remark that the concept of a measure of noncompactness may be defined in other ways [3], [34] but for our problem the Hausdorff measure of noncompactness will be useful enough. Denote $C([a,b])$ by the Banach space which consisting of all real functions defined and continuous on $[a,b]$ with the general maximum norm.

For a nonempty and bounded subset X of $C([a,b])$ its Hausdorff measure of noncompactness can be expressed by a handy formula which is described below.

We need the following symbols:

$$\omega(X,\varepsilon)=\sup\{\omega(x,\varepsilon):x\in X\},\qquad \omega_0(X)=\lim_{\varepsilon\to 0}\omega(X,\varepsilon),$$

where $\omega(x,\varepsilon)=\sup\{|x(t)-x(s)|:t,s\in[a,b],\ |t-s|\le\varepsilon\}$.

In what follows we recall some basic facts concerning the so-called superposition operator which are drawn from [34].

Let $J\subseteq\mathbb{R}$ but $J\neq\emptyset$ and $g\colon[a,b]\times J\to\mathbb{R}$ be a given function. Denote X_J by all the functions acting from $[a,b]$ into J. For any $x\in X_J$, a function $\mathcal{F}$ is defined by

$$(\mathcal{F}x)(t)=g(t,x(t)),\quad t\in[a,b]. \tag{2.1}$$

Then operator $\mathcal{F}$ defined in (2.1) is called the superposition operator generated by g.

The following simple result is taken from [2] which will be used in the sequel. For more theory concerning the superposition operator, the reader can refer to [2].

Lemma 2.1. *Let $f\colon[a,b]\times J\to\mathbb{R}$ be a continuous function on $[a,b]\times J$. Then the superposition operator $\mathcal{F}$ defined in* (2.1) *maps continuously X_J into $C([a,b])$. In addition, if $t\to g(t,x)$ is nondecreasing on $[a,b]$ for any $x\in J$ and $x\to g(t,x)$ is nondecreasing on J for any $t\in[a,b]$, then the operator $\mathcal{F}$ transforms every nondecreasing function from X_J into a nondecreasing function in $C([a,b])$.*

The following two basic inequalities [30] will be used in the sequel.

Lemma 2.2. *For $0<\sigma\le 1$ and $0\le a<b$, we have*

$$|a^\sigma-b^\sigma|\le(b-a)^\sigma.$$

Lemma 2.3. *Let α, β, γ and p be some suitable positive constants. Then*

$$\int_0^t(t^\alpha-s^\alpha)^{p(\beta-1)}s^{p(\gamma-1)}\,ds=\frac{t^\theta}{\alpha}\mathbb{B}\bigg(\frac{p(\gamma-1)+1}{\alpha},p(\beta-1)+1\bigg),$$

where $t\in\mathbb{R}_+:=[0,+\infty)$ and

$$\mathbb{B}(\xi,\eta)=\int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds,\quad (\mathfrak{Re}(\xi)>0,\ \mathfrak{Re}(\eta)>0)$$

is the well-known Beta function and $\theta=p[\alpha(\beta-1)+\gamma-1]+1$.

To end this section, we collect a Darbox type fixed point theorem [3] which will be used in the sequel.

Lemma 2.4. *Let $Q\subseteq E$ be a nonempty, bounded, closed and convex set and $T\colon Q\to Q$ be a continuous mapping. If there exists a constant $0\le k<1$ such that $\chi(TX)\le k\chi(X)$ for any nonempty subset $X\subseteq Q$, then T has a fixed point in the set Q.*

3. Existence of nondecreasing solutions

In this section, we will apply Lemma 2.4 to study the existence of nondecreasing solutions of the equation (1.4).

For brevity, denote $I = [0, 1]$. We introduce the following assumptions:

(H_1) The function $a\colon I \to \mathbb{R}_+$ is continuous and nondecreasing on I.

(H_2) The function $f\colon I \times J \to \mathbb{R}_+$ is continuous, where $J \subset \mathbb{R}_+$ is an unbounded interval and $a_0 \in J$, where $a_0 = a(0)$.

(H_3) The function $f = f(t, x)$ is nondecreasing with respect to each of both variables t and x separately, i.e. the function $t \to f(t, x)$ is nondecreasing on I for any fixed $x \in J$ and the function $x \to f(t, x)$ is nondecreasing on J for any fixed $t \in I$.

(H_4) There exists a nondecreasing function $k(r) = k\colon [a_0, +\infty) \to \mathbb{R}_+$ such that

$$|f(t, x_1) - f(t, x_2)| \leq k(r)|x_1 - x_2|$$

for any $t \in I$ and all $x_1, x_2 \in [a_0, r]$.

(H_5) The function $u\colon I \times \mathbb{R}_+ \to \mathbb{R}_+$ continuous and $u(t, x)$ is nondecreasing with respect to each variable t and x separately, i.e. the function $t \to u(t, x)$ is nondecreasing on I for any fixed $x \in \mathbb{R}_+$ and the function $x \to u(t, x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $t \in I$.

We need the following additional conditions.

(H_6) There exists a positive solution r_0 of the inequality

$$\|a\| + \frac{\mathbb{B}((\gamma + 1)/\beta, \alpha)\Phi(r)}{\Gamma(\alpha)\beta}(rk(r) + F_1) \leq r, \tag{3.1}$$

where $F_1 = f(1, 0)$. Moreover,

$$\mathbb{B}\bigg(\frac{\gamma + 1}{\beta}, \alpha\bigg)\Phi(r_0)k(r_0) < \Gamma(\alpha)\beta. \tag{3.2}$$

Denote $P = \{x \in C(I) : x(t) \geq a_0,\ t \in I\}$. Clearly, P is a subset of $C(I)$.

Define operators F and U on the set P as follows:

$$(Fx)(t) = f(t, x(t)), \tag{3.3}$$

$$(Ux)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t^\beta - s^\beta)^{\alpha - 1} s^\gamma u(s, x(s))\, ds, \tag{3.4}$$

and operator T on the set P as follows:

$$(Tx)(t) = a(t) + (Fx)(t)(Ux)(t). \tag{3.5}$$

In view of our assumptions on a, f and u, F, U and T are well defined on P.

For the sake of convenience, we will subdivide our main result into several lemmas.

Lemma 3.1. *The operators F and U transform the set P into a subset of the space $C(I)$ consisting of functions being nonnegative on I provided $\alpha \in (1/2,1)$ and $\beta \in (0,1]$.*

Proof. By (H_2) and Lemma 2.1, the operator $F\colon P \to C(I)$ consisting of functions being nonnegative on the interval I.

Next, we prove U has the same property. In fact, for any $x \in P$ and $t_1, t_2 \in I$ with $t_1 < t_2$. Then, by (H_5), Lemmas 2.2 and 2.3 we can derive that

$$
\begin{aligned}
(3.6)\quad &|(Ux)(t_2) - (Ux)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\bigg\{\int_0^{t_1} |(t_2^\beta - s^\beta)^{\alpha-1} - (t_1^\beta - s^\beta)^{\alpha-1}| s^\gamma u(s,x(s))\,ds \\
&\quad + \int_{t_1}^{t_2} (t_2^\beta - s^\beta)^{\alpha-1} s^\gamma u(s,x(s))\,ds\bigg\} \\
&\leq \frac{\Phi(\|x\|)}{\Gamma(\alpha)}\bigg\{\int_0^{t_1}\bigg[\frac{1}{t_1^\beta - s^\beta} - \frac{1}{t_2^\beta - s^\beta}\bigg]^{1-\alpha} s^\gamma\,ds \\
&\quad + \frac{1}{\beta}\int_{t_1}^{t_2} (t_2^\beta - s^\beta)^{\alpha-1} s^\gamma s^{1-\beta}\,ds^\beta\bigg\} \\
&\leq \frac{\Phi(\|x\|)}{\Gamma(\alpha)}\bigg\{\int_0^{t_1}\bigg[\frac{(t_2-t_1)^\beta}{(t_1^\beta - s^\beta)^2}\bigg]^{1-\alpha} s^\gamma\,ds + \frac{1}{\beta}\int_{t_1}^{t_2} (t_2^\beta - s^\beta)^{\alpha-1}\,ds^\beta\bigg\} \\
&= \frac{\Phi(\|x\|)}{\Gamma(\alpha)}\bigg[(t_2-t_1)^{\beta(1-\alpha)}\int_0^{t_1}(t_1^\beta - s^\beta)^{2(\alpha-1)} s^\gamma\,ds + \frac{(t_2^\beta - t_1^\beta)^\alpha}{\alpha\beta}\bigg] \\
&\leq \frac{\Phi(\|x\|)}{\Gamma(\alpha)\beta}\bigg[\mathbb{B}\bigg(\frac{\gamma+1}{\beta}, 2\alpha-1\bigg)(t_2-t_1)^{\beta(1-\alpha)} + \frac{1}{\alpha}(t_2-t_1)^{\alpha\beta}\bigg],
\end{aligned}
$$

which implies that

$$|(Ux)(t_2) - (Ux)(t_1)| \to 0, \quad \text{as } t_2 \to t_1.$$

Thus, we have $Ux \in C(I)$. Clearly, Ux is nonnegative on I. □

Remark 3.2. After reviewing the above result, it is natural to pose such a problem: can we extend the above result from $\alpha \in (1/2,1)$ to $\alpha \in (\delta,1)$ and $\beta \in (0,1]$ to $\beta \in [1,\infty)$ where $0 < \delta < 1/2$? The answer is No. In fact, for $t_1 > 0$, using the fact $0 \leq z^\beta \leq z \leq 1$ for $\beta \in [1,\infty)$ we obtain

$$
\begin{aligned}
&\int_0^{t_1} (t_1^\beta - s^\beta)^{2(\alpha-1)} s^\gamma\,ds \geq \int_{t_1/2}^{t_1} (t_1^\beta - s^\beta)^{2(\alpha-1)} s^\gamma\,ds \\
&\quad \geq \bigg(\frac{t_1}{2}\bigg)^\gamma \int_{t_1/2}^{t_1} (t_1^\beta - s^\beta)^{2(\alpha-1)}\,ds \\
&\quad = \bigg(\frac{t_1}{2}\bigg)^\gamma t_1^{2\beta(\alpha-1)+1}\int_{t_1/2}^{t_1}\bigg[1 - \bigg(\frac{s}{t_1}\bigg)^\beta\bigg]^{2(\alpha-1)} d\bigg(\frac{s}{t_1}\bigg)
\end{aligned}
$$

$$= \left(\frac{t_1}{2}\right)^{\gamma} t_1^{2\beta(\alpha-1)+1} \int_{1/2}^{1} (1-z^{\beta})^{2(\alpha-1)}\, dz$$
$$\geq \left(\frac{t_1}{2}\right)^{\gamma} t_1^{2\beta(\alpha-1)+1} \int_{1/2}^{1} (1-z)^{2(\alpha-1)}\, dz.$$

However, for $0 < \alpha \leq 1/2$, one can show

$$\int_{1/2}^{1} (1-z)^{2(\alpha-1)}\, dz = \infty,$$

so we can not expect some extension on α, β.

Denote $P_{r_0} = \{x \in P : \|x\| \leq r_0\}$. Clearly, the set P_{r_0} is nonempty since $r_0 \geq a_0$.

Lemma 3.3. *The operator T transforms the set P_{r_0} into itself.*

Proof. Note Lemma 3.1 and (H_1) we can know the operator T maps the set P into itself.

Let $\Phi := \Phi(r) = u(1,r)$ be the function defined on $\mathbb{R}_+$. In view of (H_5), the function $\Phi(r)$ is nonnegative and nondecreasing on $\mathbb{R}_+$ and $u(t,x) \leq \Phi(r)$ for $t \in I$ and $x \in [0,r]$. Then, for an arbitrary $x \in P$ and $t \in I$, using Lemma 2.3 again, we have

$$|(Tx)(t)| \leq \|a\| + \frac{k(\|x\|)x(t) + F_1}{\Gamma(\alpha)} \int_0^t (t^{\beta} - s^{\beta})^{\alpha-1} s^{\gamma} \Phi(\|x\|)\, ds$$
$$\leq \|a\| + \frac{\mathbb{B}((\gamma+1)/\beta, \alpha)\Phi(\|x\|)}{\Gamma(\alpha)\beta} (k(\|x\|)\|x\| + F_1).$$

Now, taking into account the condition (3.1) in (H_6) we can deduce that there exists $r_0 > 0$ with

$$\frac{\mathbb{B}((\gamma+1)/\beta, \alpha)\Phi(r_0)k(r_0)}{\Gamma(\alpha)\beta} < 1,$$

which is equal to the condition (3.2) in (H_6).

From the above results, the operator T transforms the set P_{r_0} into itself. $\square$

Consider the subset Q of P_{r_0} consisting of all functions from P_{r_0} which are nondecreasing on I. Clearly, the set Q is nonempty, bounded, closed and convex.

Lemma 3.4. *The operator F transforms each function belonging to Q into a function from $C(I)$ being nondecreasing on I. The same assertion is also valid for the operator U provided that $\beta(\alpha-1)+\gamma+1 \geq 0$.*

Proof. Obviously, the operator F transforms each function belonging to Q into a function from $C(I)$ being nondecreasing on I. Next, we show that the same assertion is also valid for the operator U.

Take an arbitrary function $x \in Q$ and $t_1, t_2 \in I$ with $t_1 < t_2$. Using Lemma 2.3 we have

$$\begin{aligned}(Ux)(t_2) &- (Ux)(t_1)\\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \bigg[\frac{1}{(t_2^\beta - s^\beta)^{1-\alpha}} - \frac{1}{(t_1^\beta - s^\beta)^{1-\alpha}} \bigg] s^\gamma u(s, x(s))\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2^\beta - s^\beta)^{\alpha-1} s^\gamma u(s, x(s))\, ds \\ &\geq \frac{u(t_1, x(t_1))}{\Gamma(\alpha)} \bigg[\int_0^{t_2} (t_2^\beta - s^\beta)^{\alpha-1} s^\gamma\, ds - \int_0^{t_1} (t_1^\beta - s^\beta)^{\alpha-1} s^\gamma\, ds \bigg] \\ &= \frac{u(t_1, x(t_1)) \mathbb{B}((\gamma+1)/\beta, \alpha)}{\Gamma(\alpha)\beta} [t_2^{\beta(\alpha-1)+\gamma+1} - t_1^{\beta(\alpha-1)+\gamma+1}] \geq 0,\end{aligned}$$

which implies that the function Ux is nondecreasing on I. □

Remark 3.5. Clearly, $\alpha \in (1/2, 1)$, $\beta \in (0, 1)$ and $\gamma \in [\beta(1-\alpha) - 1, \infty)$ will imply $\beta(\alpha - 1) + \gamma + 1 \geq 0$.

Lemma 3.6. *The operator T maps Q into itself and T is also continuous on Q.*

Proof. Linking Lemmas 3.1, 3.3 and 3.4, we obtain that the operator T maps the set Q into itself. Next, we show that T is continuous on the set Q. In view of Lemma 2.1 it is sufficient to show that the operator U is continuous on the set Q. Thus, fix an arbitrarily $\varepsilon > 0$ and $x_0 \in Q$. In view of assumption (H_5) and Lemma Lemma 2.1 we can find $\delta > 0$ such that for an arbitrary $x \in Q$ with $\|x - x_0\| \leq \delta$, we have $|u(s, x(s)) - u(s, x_0(s))| \leq \varepsilon$ for $s \in I$. Hence, for any fixed $t \in I$ we can derive that

$$|(Ux)(t) - (Ux_0)(t)| \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t^\beta - s^\beta)^{\alpha-1} s^\gamma\, ds \leq \frac{\mathbb{B}((\gamma+1)/\beta, \alpha)}{\Gamma(\alpha)\beta} \varepsilon.$$

This shows that U is continuous on Q and implies the desired continuity of the operator T on the set Q. □

Now, we are ready to state the main result in this paper.

Theorem 3.7. *Let* (H_1)–(H_5) *be satisfied and* $\alpha \in (1/2, 1)$ *and* $\beta \in (0, 1)$. *Then the equation* (1.4) *has at least one solution* $x \in C(I)$. *Moreover, all the solutions of the equation* (1.4) *must be nonnegative and nondecreasing.*

Proof. By Lemma 3.6, we know T maps $Q \to Q$ is continuous. Now, take a nonempty subset X of the set Q. Fix $\varepsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \varepsilon$. Assume that $t_1 \leq t_2$. Then, using (H_1)–(H_5) and (3.6) we can derive

$$\begin{aligned}
|(Tx)(t_2) &- (Tx)(t_1)| \le |a(t_2) - a(t_1)| \\
&+ (Fx)(t_2)|(Ux)(t_2) - (Ux)(t_1)| + (Ux)(t_1)|(Fx)(t_2) - (Fx)(t_1)| \\
&\le \omega(a,\varepsilon) + [|f(t_2, x(t_2)) - f(t_2, 0)| + |f(t_2, 0)|] \\
&\quad \times \frac{\Phi(r_0)}{\Gamma(\alpha)\beta}\left[\varepsilon^{\beta(1-\alpha)}\mathbb{B}\left(\frac{\gamma+1}{\beta}, 2\alpha - 1\right) + \frac{\varepsilon^{\alpha\beta}}{\alpha}\right] \\
&\quad + \left[\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1^\beta - s^\beta)^{\alpha-1}s^\gamma u(s, x(s))\,ds\right][|f(t_2, x(t_2)) - f(t_2, x(t_1))| \\
&\quad + |f(t_2, x(t_1)) - f(t_1, x(t_1))|] \\
&\le \omega(a,\varepsilon) + (r_0 k(r_0) + F_1)\frac{\Phi(r_0)}{\Gamma(\alpha)\beta}\left[\mathbb{B}\left(\frac{\gamma+1}{\beta}, 2\alpha - 1\right)\varepsilon^{\beta(1-\alpha)} + \frac{1}{\alpha}\varepsilon^{\alpha\beta}\right] \\
&\quad + \frac{\Phi(r_0)}{\Gamma(\alpha)\beta}\mathbb{B}\left(\frac{\gamma+1}{\beta}, \alpha\right)[k(r_0)\omega(x,\varepsilon) + v(f,\varepsilon)],
\end{aligned}$$

where $v(f,\varepsilon) = \sup\{|f(t_2,x) - f(t_1,x)| : t_1, t_2 \in I,\ |t_2 - t_1| \le \varepsilon,\ x \in [a_0, r_0]\}$. Hence, using the uniform continuity of f on $I \times [a_0, r_0]$ we deduce the inequality

$$\omega_0(TX) \le \frac{\mathbb{B}((\gamma+1)/\beta, \alpha)\Phi(r_0)k(r_0)}{\Gamma(\alpha)\beta}\,\omega_0(X). \tag{3.7}$$

Applying the fact $\chi(X) = \omega_0(X)/2$ [3] to the inequality (3.7), we can obtain

$$\chi(TX) \le \frac{\mathbb{B}((\gamma+1)/\beta, \alpha)\Phi(r_0)k(r_0)}{\Gamma(\alpha)\beta}\,\chi(X).$$

Note that (3.2) yields

$$0 \le \frac{\mathbb{B}((\gamma+1)/\beta, \alpha)\Phi(r_0)k(r_0)}{\Gamma(\alpha)\beta} < 1.$$

Thus, all the assumptions in Lemma 2.4 are satisfied. As a result, T has at least a fixed point in the set Q. The desired results are obvious. □

4. Uniqueness and nonuniqueness

In order to obtain uniqueness and nonuniqueness result, we need the following nonlinear integral inequality involving weakly singular kernels [25] which can be widely used in the study of integral equations involving Erdélyi–Kober singular kernels.

Lemma 4.1. *Let $\kappa \in (0,1]$, $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following inequality*

$$x(t) \le \widehat{a}(t) + \widehat{c}(t)\int_0^t (t^{\widehat{\alpha}} - s^{\widehat{\alpha}})^{\widehat{\beta}-1}s^{\widehat{\gamma}-1}\widehat{F}(s)[x(s)]^\kappa\,ds, \quad t \ge 0, \tag{4.1}$$

where $\widehat{a}, \widehat{c} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are not decreasing functions and $\widehat{F} \in C(\mathbb{R}_+, \mathbb{R}_+)$.

If $\widehat{\alpha} \in (0,1]$, $\widehat{\beta} \in (1/2,1)$ *and* $\widehat{\gamma} \geq 3/2 - \widehat{\beta}$, *then*

$$x(t) \leq 2^{\widehat{\beta}}\widehat{a}(t) \exp\bigg\{(1-\widehat{\beta})B(t)\int_0^t [\widehat{F}(s)]^{1/(1-\widehat{\beta})}\,ds\bigg\},$$

for $\kappa = 1$,

$$x(t) \leq \bigg[(2^{\widehat{\beta}}\widehat{a}(t))^{(1-\kappa)/(1-\widehat{\beta})} + (1-\widehat{\beta})B(t)\int_0^t [\widehat{F}(s)]^{1/(1-\widehat{\beta})}\bigg]^{(1-\widehat{\beta})/(1-\kappa)},$$

for $0 < \kappa < 1$, *where*

$$B(t) = \bigg[\frac{2}{\widehat{\alpha}}\mathbb{B}\bigg(\frac{\widehat{\beta}+\widehat{\gamma}-1}{\widehat{\alpha}\widehat{\beta}}, \frac{2\widehat{\beta}-1}{\widehat{\beta}}\bigg)\bigg]^{\widehat{\beta}/(1-\widehat{\beta})} [\widehat{c}(t)]^{1/(1-\widehat{\beta})} t^{(\widehat{\alpha}(\widehat{\beta}-1)+\widehat{\gamma}-1+\widehat{\beta})/(1-\widehat{\beta})}.$$

Remark 4.2. (a) If $\widehat{\alpha} \in (0,1]$, $\widehat{\beta} \in (0,1/2)$ and $\widehat{\gamma} \geq (1-2\widehat{\beta}^2)/(1-\widehat{\beta}^2)$, the authors also give explicit bounds for $\kappa = 1$ and $0 < \kappa < 1$ respectively in [25].

(b) For more general nonlinear integral inequalities involving Erdélyi–Kober singular kernels and their applications, the reader can refer to [26].

Theorem 4.3. *Under the assumptions of Theorem* 3.7 *hold and there exists a positive constant* L *such that*

$$|u(t,x) - u(t,y)| \leq L|x-y|^{\nu}, \quad 0 < \nu \leq 1,$$

for any $t \in I$ *and all* $x_1, x_2 \in [a_0, r]$. *Then for some* $\gamma \in [1/2 - \alpha, \infty)$, *either the equation* (1.4) *has a unique nonnegative and nondecreasing solution for* $\nu = 1$, *or has at least two nonnegative and nondecreasing solutions for* $0 < \nu < 1$.

Proof. Suppose that y be another nonnegative and nondecreasing solution of the equation (1.4). Then y satisfies the following integral equation

$$y(t) = a(t) + \frac{f(t,y(t))}{\Gamma(\alpha)}\int_0^t (t^{\beta} - s^{\beta})^{\alpha-1} s^{\gamma} u(s,y(s))\,ds, \quad t \in I.$$

Hence,

$$\begin{aligned}
|x(t) - y(t)| &\leq \frac{k(r)|x(t)-y(t)|\Phi(\|x\|)}{\Gamma(\alpha)}\int_0^t (t^{\beta}-s^{\beta})^{\alpha-1}s^{\gamma}\,ds \\
&\quad + \frac{L(|f(t,y(t)) - f(t,0)| + |f(t,0)|)}{\Gamma(\alpha)}\int_0^t (t^{\beta}-s^{\beta})^{\alpha-1}s^{\gamma}|x(s)-y(s)|^{\nu}\,ds \\
&\leq \frac{k(r)\Phi(r)}{\Gamma(\alpha)\beta}\mathbb{B}\bigg(\frac{\gamma+1}{\beta}, \alpha\bigg)|x(t)-y(t)| \\
&\quad + \frac{L(k(r)|y(t)| + F_1)}{\Gamma(\alpha)}\int_0^t (t^{\beta}-s^{\beta})^{\alpha-1}s^{\gamma}|x(s)-y(s)|^{\nu}\,ds,
\end{aligned}$$

for any $t \in I$ and all $x_1, x_2 \in [a_0, r]$. This yields that

$$\left[1 - \frac{k(r)\Phi(r)}{\Gamma(\alpha)\beta}\mathbb{B}\left(\frac{\gamma+1}{\beta}, \alpha\right)\right]|x(t) - y(t)| \\ \leq \frac{L(k(r)r + F_1)}{\Gamma(\alpha)} \int_0^t (t^\beta - s^\beta)^{\alpha-1} s^\gamma |x(s) - y(s)|^\nu \, ds.$$

In view of (3.2), we can rewrite the above inequality to

$$|x(t) - y(t)| \leq \widehat{a}(t) + \widehat{c}(t) \int_0^t (t^{\widehat{\beta}} - s^{\widehat{\beta}})^{\widehat{\alpha}-1} s^{\widehat{\gamma}-1} |x(s) - y(s)|^\kappa \, ds, \tag{4.2}$$

where $\widehat{\alpha} := \beta$, $\widehat{\beta} := \alpha$, $\widehat{\gamma} := \gamma + 1$, $\kappa = \nu$, $\widehat{a}(t) := 0$ and

$$\widehat{c}(t) := \frac{L(k(r_0)r_0 + F_1)}{\left[1 - \dfrac{k(r_0)\Phi(r_0)}{\Gamma(\alpha)\beta}\mathbb{B}\left(\dfrac{\gamma+1}{\beta}, \alpha\right)\right]\Gamma(\alpha)}.$$

Note the restriction $\alpha \in (1/2, 1)$, $\beta \in (0, 1]$ and $\gamma \in [1/2 - \alpha, \infty)$, we can apply Lemma 4.1 to the inequality (4.2), one can obtain the desired results immediately. $\square$

5. Examples

In what follows we illustrate the above obtained results by the following two examples.

Example 5.1. Consider a fractional quadratic integral equation involving Erdélyi–Kober singular kernels

$$x(t) = \frac{1}{n}\sin t + \frac{\sqrt[n]{x(t)}}{\Gamma(3/4)} \int_0^t (t^{1/2} - s^{1/2})^{-1/4} s[s^2 + x^2(s)] \, ds, \quad t \in I := [0, 1]. \tag{5.1}$$

Let $\alpha := 3/4 \in (1/2, 1)$, $\beta := 1/2 \in (0, 1]$, $\gamma := 1 \in [-1/4, \infty)$, $n \in \mathbb{N}$, and

$$a(t) := \frac{1}{n}\sin t, \qquad f(t, x) := \sqrt[n]{x}, \qquad u(t, x) := t^2 + x^2,$$
$$F_1 := f(t, 0) = 0, \qquad \Phi(r) := u(1, r) = 1 + r^2.$$

Obviously, the equation (5.1) is a special case of the equation (1.4).

Now, we show that all assumptions of Theorem 4.3 are satisfied for the equation (5.1).

In fact we can observe that:

(1) the function $a(t)$ is continuous, positive and nondecreasing on I with $a_0 = a(0) = 0$ and $\|a\| = (\sin 1)/n$;
(2) the function $f \colon I \times J \to \mathbb{R}_+$ is continuous, where $J := [a_0, \infty) = [0, \infty)$;
(3) f is nondecreasing with respect to each of both variables t and x separately;
(4) f is Lipschitzian on $[a_0, r]$ with the constant $k(r)e^{2(1-1/n)}/n$, for any $r \geq a_0$;

(5) the function $u\colon I \times J \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to each of both variables t and x separately. Moreover, u is also Lipschitzian on $[a_0, r]$

$$|u(t,x) - u(t,y)| = |x+y||x-y| \leq L|x-y|^{\nu}, \quad \nu = 1,$$

with the constant $L := 2r$.

Moreover, the inequality (3.1) has the form

$$\frac{\sin 1}{n} + \frac{\mathbb{B}(4,3/4)(1+r^2)}{\Gamma(3/4)/2} \times \frac{re^{2(1-1/n)}}{n} \leq r,$$

which implies that

$$\sin 1 + \frac{2\mathbb{B}(4,3/4)r(1+r^2)e^{2(1-1/n)}}{\Gamma(3/4)} \leq nr.$$

It is easy to see that $r_0 = 1$ satisfies the above inequality for any $n \geq 10$, where $\sin 1 \simeq 0.8415$, $\mathbb{B}(4,3/4) \simeq 0.4433$, $\Gamma(3/4) \simeq 1.2254$.

Also, the inequality (3.2) has the form

$$0.5364 \simeq \mathbb{B}\left(4, \frac{3}{4}\right)\Phi(1)k(1) < \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \simeq 0.6125, \tag{5.2}$$

where $\Phi(1) = 2$ and $k(1) = 0.60496$.

By Theorem 4.3, the equation (5.1) has a unique positive and nondecreasing solution $x(t) \in I$ for $t \in I$.

The unique positive and nondecreasing solution of the equation (5.1) is displayed in Figure 1.

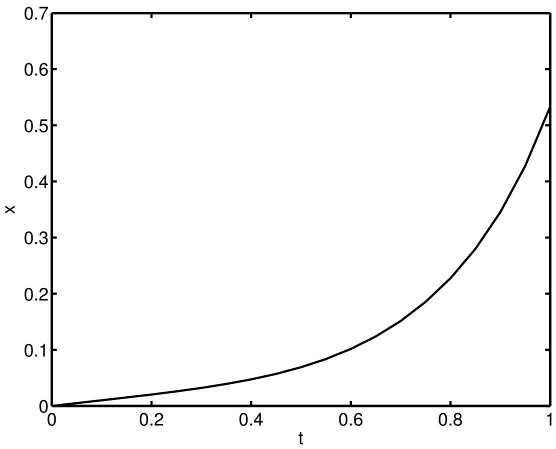

Figure 1. The unique positive and nondecreasing of the equation (5.1).

EXAMPLE 5.2. Consider another fractional quadratic integral equation involving Erdélyi–Kober singular kernels

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t (t^\beta - s^\beta)^{\alpha-1} s^\gamma u(s, x(s))\, ds, \quad t \in I, \tag{5.3}$$

where α, β, γ, I, a, f are the same as in the Example 5.1 and

$$u(t, x) := t + Lx^{1/2}, \quad L := 1,$$

Denote $r_0 := 1$ and $\Phi(r_0) := u(1, r_0) = 1 + \sqrt{r_0}$. Clearly, for all $x, y \in [0, 1]$,

$$|u(t, x) - u(t, y)| \le L|x - y|^{1/2}, \quad \nu = \frac{1}{2}.$$

From the discussion in the Example 5.1, one can see that $r_0 = 1$ satisfies the following inequality

$$\sin 1 + \frac{2\mathbb{B}(4, 3/4) r(1 + \sqrt{r}) e^{2(1-1/n)}}{\Gamma(3/4)} \le nr$$

for any $n \ge 10$. Meanwhile, the inequality (5.2) also holds since $\Phi(1) = 2$.

By Theorem 4.3 again, the equation (5.3) has at least two positive and nondecreasing solutions $x(t) \in I$ for $t \in I$.

Two positive and nondecreasing solutions of the equation (5.3) are displayed in Figure 2.

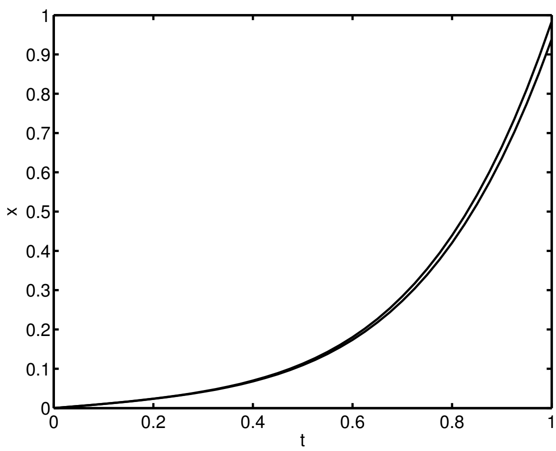

FIGURE 2. Two positive and nondecreasing solutions of the equation (5.3).

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