Topological Methods in Nonlinear Analysis
Volume 43, No. 1, 2014, 241–249

RELAXED HALPERN TYPE ITERATION SCHEMES FOR SEQUENCES OF NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. Under weaker conditions on parameters, we prove strong convergence theorems of Halpern type iteration schemes for sequences of nonexpansive mappings in CAT(0) spaces. Since there is no assumption of the AKTT-condition imposed on the involved mappings, the results improve those of the authors with related researches.

1. Introduction and preliminaries

Let (X,d) be a metric space and $x,y\in X$ with $l=d(x,y)$. A *geodesic path* from x to y is an isometry $c\colon[0,l]\to X$ such that $c(0)=x$ and $c(l)=y$. The image of a geodesic path is called a *geodesic segment*, denoted by $[x,y]$ as it is unique. A metric space X is a *(uniquely) geodesic space* if every two points of X are joined by only one geodesic path. A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x_1,x_2,x_3)$ is the triangle $\overline{\triangle}(x_1,x_2,x_3):=\triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i,x_j)=d_{\mathbb{R}^2}(\overline{x}_i,\overline{x}_j)$ for all $i,j=1,2,3$, where $\overline{x}_i$ is called the *comparison vertex* of $x_i, i=1,2,3$.

2010 *Mathematics Subject Classification.* 47H05, 47H10, 47J25.

Key words and phrases. CAT(0) spaces, Halpern's iterations, sequences of nonexpansive mappings, strong convergence.

This work was supported by the National Natural Science Foundation of China (Grant No.11061037).

A geodesic space X is a $CAT(0)$ *space* if for each geodesic triangle $\triangle := \triangle(x_1, x_2, x_3)$ in X and its comparison triangle $\overline{\triangle} := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is satisfied by all $x, y \in \triangle$ and their *comparison points* $\overline{x}, \overline{y} \in \overline{\triangle}$. The meaning of the CAT(0) *inequality* is that a geodesic triangle in X is at least thin as its comparison triangle in the Euclidean plane. A thorough discussion of these spaces and their important role in various branches of mathematics are given in [1], [2]. The complex Hilbert ball with the hyperbolic metric is an example of a CAT(0) space (see [9]).

Fixed point theory in a CAT(0) space was first studied by Kirk (see [12] and [13]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared (see, e.g. [3]–[7], [10], [11], [14], [15], [18]–[20]).

In 2010, Saejung [17] proved strong convergence theorem of a Halpern's iterative sequence for a sequence of nonexpansive mappings in CAT(0) spaces. However, the results were obtained under some stronger assumption conditions, such as the AKTT-condition imposed on the involved mappings and more restrictions on parameters.

Remark 1.1. Let C be a subset of a complete CAT(0) X and let $\{T_n\}$ be a sequence of mappings from C into itself. For a bounded subset B of C, we say that $(\{T_n\}, B)$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{d(T_{n+1}z, T_n z) : z \in B\} < \infty.$$

Inspired and motivated by those studies mentioned above, in this paper, under less restrictions on parameters, we use Halpern type iteration schemes for approximating common fixed points of sequences of nonexpansive mappings and obtain strong convergence theorems in CAT(0) space without stronger assumption imposed on the involved mappings. The results improve and extend those of Saejung.

In this paper, we write $(1-t)x \oplus ty$ for the the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y), \quad d(z, y) = (1 - t)d(x, y), \quad \text{for all } t \in [0, 1].$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is, $[x, y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$. A subset C of a CAT(0) space is convex if $[x, y] \subset C$

for all $x, y \in C$. For elementary facts about CAT(0) spaces, we refer the readers to [1] (or, briefly in [8]).

In the sequel we shall need the following preliminaries.

Let X be a uniquely geodesic space equipped with two operations $\circ$ and $\oplus$, respectively defined by:

Definition 1.2. (a) For any $\alpha \in \mathbb{R}$ and any $x \in X$, $\alpha \circ x$ stands for the unique point $u \in X$ such that

$$\overline{u} = \alpha \overline{x},$$

where $\overline{\cdot}$ is the comparison vertex in the comparison triangle

$$\triangle(\overline{\cdot}, \overline{\theta}, \overline{\cdot}) := \triangle(\overline{\cdot}, \overrightarrow{0}, \overline{\cdot})$$

of $\triangle(\,\cdot\,, \theta, \,\cdot\,)$; and θ denotes a fixed $x_0 \in X$.

(b) For any $x, y \in X$, $x \oplus y$ stands for the unique point $v \in X$ such that

$$\overline{v} = \overline{x} + \overline{y},$$

where $\overline{v}$ is the comparison vertex in the comparison triangles $\triangle(\overline{x}, \overline{\theta}, \overline{v})$ and $\triangle(\overline{y}, \overline{\theta}, \overline{v})$ of $\triangle(x, \theta, v)$ and $\triangle(y, \theta, v)$.

We then have the following conclusion:

Proposition 1.3. *A uniquely geodesic space X equipped with two operations $\circ$ and $\oplus$ forms a vector space whenever its power is no larger than $\aleph$, namely the cardinality of continuum. Such a space is called geodesic vector space.*

This follows from the fact that it is reasonable to define the mappings $x \mapsto \overline{x}$ and $v \mapsto \overline{v}$ as injections, determined respectively by the mappings $\triangle(x, \theta, x) \mapsto \triangle(\overline{x}, \overline{\theta}, \overline{x})$ and $(\triangle(x, \theta, v), \triangle(y, \theta, v)) \mapsto (\triangle(\overline{x}, \overline{\theta}, \overline{v}), \triangle(\overline{y}, \overline{\theta}, \overline{v}))$, since X is equivalent to $\mathbb{R}^2$.

By the uniqueness of the *negative element* of any member of X, denoting the same one in Proposition 1.2, an operation $\ominus$ is defined by

$$x \ominus y = x \oplus (-1 \circ y), \quad \text{for all } x, y \in X.$$

Let X be a geodesic vector space.

Definition 1.4. An analogue of inner product $\langle\,\cdot\,,\,\cdot\,\rangle \colon X \times X \to \mathbb{R}$ is defined by

$$\langle x, y \rangle = \langle \overline{x}, \overline{y} \rangle_{\mathbb{R}^2},$$

where $\overline{x}, \overline{y}$ are the comparison vertices in the comparison triangle $\triangle(\overline{x}, \overline{\theta}, \overline{y})$ of $\triangle(x, \theta, y)$.

It is obvious from the definition of the function $\langle\cdot,\cdot\rangle$ that it has the following properties: for any $x, y, z \in X$ and any $\alpha \in \mathbb{R}$,

(1) $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ if and only if $x = \theta$;

(2) $\langle x, y\rangle = \langle y, x\rangle$;
(3) $\langle \alpha \circ x, y\rangle = \alpha\langle x, y\rangle$;
(4) $\langle x \oplus y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

Then a distance ρ on X can be defined by

$$\rho(x, y) := \sqrt{\langle x \ominus y, x \ominus y\rangle},$$

which coincides with the original distance d on X, since the distance $d_{\mathbb{R}^2}$ on $\mathbb{R}^2$ is just induced by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^2}$ and $d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$.

Next, we define a function $\phi\colon X \times X \to \mathbb{R}^+$ by

$$\phi(x, y) := d^2(x, y),$$

which obviously has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z \ominus y, x \ominus z\rangle, \quad \text{for all } x, y, z \in X. \tag{1.1}$$

In what follows we shall make use of the following lemmas.

Lemma 1.5. *A geodesic space X is a* CAT(0) *space if and only if the following inequality*

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y)$$

is satisfied by all $x, y, z \in X$ and all $t \in [0, 1]$. In particular, if x, y, z are points in a CAT(0) *space and $t \in [0, 1]$, then*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z).$$

Lemma 1.6 ([16]). *Let $\{a_n\}$, $\{\delta_n\}$, and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \textit{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2. Main results

Let X be a CAT(0) space and C a closed convex subset of X. In the sequel, we denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of a mapping T in X. Note that a CAT(0) space X with the cardinality of continuum can be seen as a subset of some geodesic vector space. We then have the following theorem.

THEOREM 2.1. *Let X be a complete* CAT(0) *space with the cardinality of continuum and C a closed convex subset of X. Let $\{T_i\}_{i=1}^{\infty}: C \to C$ be a sequence of nonexpansive mappings with the interior of $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T_n^* x_n, \quad \text{for all } n \geq 1, \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $T_n^* = T_{i_n}$ with i_n being the solution to the positive integer equation: $n = i + (m-1)m/2$ (for $m \geq i, n = 1, 2, \ldots$), that is, for each $n \geq 1$, there exists a unique i_n such that, $i_1 = 1$, $i_2 = 1$, $i_3 = 2$, $i_4 = 1$, $i_5 = 2$, $i_6 = 3$, $i_7 = 1$, $i_8 = 2$, $i_9 = 3$, $i_{10} = 4$, $i_{11} = 1, \ldots$ Then $\{x_n\}$ converges to an $x^* \in F$.

PROOF. We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} d(x_n, q)$ exists forall $q \in F$.

For any $q \in F$, from (2.1), we have that

$$\begin{aligned} d(x_{n+1}, q) &= d(\alpha_n u \oplus (1-\alpha_n) T_n^* x_n, q) \leq \alpha_n d(u,q) + (1-\alpha_n) d(T_n^* x_n, q) \\ &\leq \alpha_n d(u,q) + (1-\alpha_n) d(x_n, q) \leq d(x_n, q) + \mu_n, \end{aligned}$$

where $\mu_n := \alpha_n d(u,q)$, and so $\sum_{n=1}^{\infty} \mu_n < \infty$. So by Lemma 1.6 we conclude that $\lim_{n\to\infty} d(x_n, q)$ exists and hence $\{x_n\}$ and $\{T_n^* x_n\}$ are bounded.

Step 2. $x_n \to x^* \in C$ as $n \to \infty$.

From (2.1) and Lemma 1.5, we also have

$$\begin{aligned} d^2(x_{n+1}, q) &= d^2(\alpha_n u \oplus (1-\alpha_n) T_n^* x_n, q) \\ &\leq \alpha_n d^2(u,q) + (1-\alpha_n) d^2(T_n^* x_n, q) - \alpha_n (1-\alpha_n) d^2(u, T_n^* x_n) \\ &\leq (1-\alpha_n) d^2(x_n, q) + \alpha_n d^2(u,q) \leq d^2(x_n, q) + \nu_n, \end{aligned} \tag{2.2}$$

where $\nu_n := \alpha_n d^2(u,q)$ and $\sum_{n=1}^{\infty} \nu_n < \infty$.

Furthermore, it follows from (1.1) that

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} \ominus p, x_n \ominus x_{n+1} \rangle, \quad \text{for all } p \in X.$$

This implies that

$$\langle x_{n+1} \ominus p, x_n \ominus x_{n+1} \rangle + \frac{1}{2} \phi(x_{n+1}, x_n) = \frac{1}{2} (\phi(p, x_n) - \phi(p, x_{n+1})). \tag{2.3}$$

Moreover, since the interior of F is nonempty, there exists a $p^* \in F$ and $r > 0$ such that $(p^* \oplus r \circ h) \in F$ whenever $\sqrt{\langle h, h \rangle} \leq 1$. Thus, from (2.2) and (2.3) we obtain that

$$0 \leq \langle x_{n+1} \ominus (p^* \oplus r \circ h), x_n \ominus x_{n+1} \rangle + \frac{1}{2} \phi(x_{n+1}, x_n) + \frac{1}{2} \nu_n. \tag{2.4}$$

Then from (2.3) and (2.4) we obtain that

$$\begin{aligned} r\langle h, x_n \ominus x_{n+1}\rangle &\le \langle x_{n+1} \ominus p^*, x_n \ominus x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\nu_n \\ &= \frac{1}{2}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2}\nu_n, \end{aligned}$$

and hence

$$\langle h, x_n \ominus x_{n+1}\rangle \le \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}\nu_n.$$

Since h with $\sqrt{\langle h, h\rangle} \le 1$ is arbitrary, taking $h = (1/d(x_n, x_{n+1})) \circ (x_n \ominus x_{n+1})$, we have

$$d(x_n, x_{n+1}) \le \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}\nu_n. \tag{2.5}$$

So, if $n > m$, then we have that

$$\begin{aligned} d(x_m, x_n) &\le \sum_{j=m}^{n-1} d(x_j, x_{j+1}) \\ &\le \frac{1}{2r}\sum_{j=m}^{n-1}(\phi(p^*, x_j) - \phi(p^*, x_{j+1})) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j \\ &= \frac{1}{2r}(\phi(p^*, x_m) - \phi(p^*, x_n)) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j. \end{aligned} \tag{2.6}$$

But we know that $\{\phi(p^*, x_n)\}$ converges, and $\sum_{n=1}^{\infty} \nu_n < \infty$. Therefore, we obtain from (2.6) that $\{x_n\}$ is a Cauchy sequence. Since X is complete there exists an $x^* \in X$ such that $x_n \to x^* \in X$ as $n \to \infty$. Thus, since $\{x_n\} \subset C$ and C is closed and convex, then $x^* \in C$, that is,

$$x_n \to x^* \in C \quad (n \to \infty). \tag{2.7}$$

Step 3. x^* is a member of F.

It follows from (2.1) and (2.5) that, as $n \to \infty$,

$$d(x_{n+1}, T_n^* x_n) = \alpha_n d(u, T_n^* x_n) \to 0 \quad \text{and} \quad d(x_{n+1}, x_n) \to 0,$$

which implies that, by induction, for any nonnegative integer j,

$$\lim_{n\to\infty} d(x_{n+j}, x_n) = 0.$$

We then have, as $n \to \infty$,

$$d(x_n, T_n^* x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, T_n^* x_n) \to 0. \tag{2.8}$$

Now, set $\mathcal{K}_i = \{k \ge 1 : k = i + (m-1)m/2,\ m \ge i,\ m \in \mathbb{Z}^+\}$ for each $i \ge 1$. Note that $T_k^* = T_{i_k} = T_i$ whenever $k \in \mathcal{K}_i$. For example, by the definition of $\mathcal{K}_1$, we have $\mathcal{K}_1 = \{1, 2, 4, 7, 11, 16, \dots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \dots = 1$. It then follows from (2.8) that

$$\lim_{\mathcal{K}_i \ni k \to \infty} d(T_i x_k, x_k) = 0, \quad \text{for all } i \geq 1. \tag{2.9}$$

Since $\{x_k\}_{k \in \mathcal{K}_i}$ is a subsequence of $\{x_n\}$, (2.7) implies that $x_k \to x^*$ as $\mathcal{K}_i \ni k \to \infty$. It immediately follows from (2.9) and the continuity of T_i that $x^* \in F(T_i)$ for each $i \geq 1$, and hence $x^* \in F$. □

Next, we consider nonself mappings.

LEMMA 2.2 ([17]). *Let X be a complete* CAT(0) *space and C a closed convex subset of X. Then the followings hold true*:

(a) *For each $x \in X$, there exists an element $\pi(x) \in C$ such that*

$$d(x, \pi(x)) = \inf_{y \in C} d(x, y).$$

(b) *$\pi(x) = \pi(x')$ for all $x' \in [x, \pi(x)]$.*
(c) *The mapping $x \mapsto \pi(x)$ is nonexpansive.*

The mapping π in the preceding theorem is called the *metric projection* from X onto C. From this, we have the following result.

LEMMA 2.3 ([17]). *Let X be a complete* CAT(0) *space and C a closed convex subset of X. Let $T: C \to X$ be a nonself nonexpansive mapping with its fixed points set $F(T) \neq \emptyset$ and π the metric projection from X onto C. Then the mapping $\pi \circ T$ is nonexpansive and $F(\pi \circ T) = F(T)$.*

THEOREM 2.4. *Let X be a complete* CAT(0) *space with the cardinality of continuum and C a closed convex subset of X. Let $\{T_i\}_{i=1}^{\infty}: C \to X$ be a sequence of nonself nonexpansive mappings and π be the metric projection from X onto C. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \pi \circ T_n^* x_n, \quad \textit{for all } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $T_n^* = T_{i_n}$ with i_n being the solution to the positive integer equation: $n = i + (m-1)m/2$ ($m \geq i$, $n = 1, 2, \dots$). If the interior of $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, then $\{x_n\}$ converges to an $x^* \in F$.

PROOF. Since $\bigcap_{i=1}^{\infty} F(\pi \circ T_i) = \bigcap_{i=1}^{\infty} F(T_i)$, this conclusion can be immediately obtained from Lemmas 2.2 and 2.3 and Theorem 2.1. □

Acknowledgments. The author is greatly grateful to the referees for their useful suggestions by which the contents of this article are improved. This work was supported by the National Natural Science Foundation of China (Grant No. 11061037).

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Manuscript received Jly 4, 2012

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