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LYAPUNOV FUNCTIONS, SHADOWING AND TOPOLOGICAL STABILITY

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Abstract. We use Lyapunov type functions to give new conditions under which a homeomorphism of a compact metric space has the shadowing property. These conditions are applied to establish the topological stability of some homeomorphisms with nonhyperbolic behavior.

1. Introduction

The shadowing property of dynamical systems (diffeomorphisms or flows) is now well-studied (see, for example, the monographs [4], [5] and the recent survey [6]). This property means that, near approximate trajectories (so-called pseudotrajectories), there exist exact trajectories of the system.

Mostly, standard methods allow one to show that the shadowing property follows from hyperbolic behavior of trajectories of the system. It is well known that a structurally stable system has the shadowing property (and this property is Lipschitz), see [5].

One can mention several papers which contain methods of proving the shadowing property for systems with nonhyperbolic behavior (see, for example, [1]).

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In [3], Lewowicz used Lyapunov type functions to study topological stability of dynamical systems (see also [8]). This property is stronger than the shadowing property (and they are equivalent for expansive systems on smooth closed manifolds, see Section 3).

In this paper, we give sufficient conditions of shadowing for a dynamical system generated by a homeomorphism of a compact metric space in terms of existence of a pair of Lyapunov type functions. These conditions are formulated in terms of some sets related to the considered pair of functions.

In fact, our conditions have much in common with the topological conditions used in the classical Ważewski principle in the theory of differential equations [10]. In a sense, a close reasoning has been used by the second author in his joint paper [7] with Plamenevskaya devoted to the C^0-genericity of shadowing.

The structure of the paper is as follows. In Section 2, we formulate and prove our basic shadowing result. Section 3 is devoted to topological stability. We consider an example studied by Lewowicz in [3] and give a comment on the method of [3]. One more example shows that our methods are applicable to homeomorphisms.

2. Lyapunov functions and shadowing

Let f be a homeomorphism of a metric space (X, dist). As usual, we say that a sequence $\{p_k \in X : k \in \mathbb{Z}\}$ is a d-pseudotrajectory of f if

$$\mathrm{dist}(p_{k+1}, f(p_k)) < d, \quad k \in \mathbb{Z}. \tag{2.1}$$

We say that a pseudotrajectory $\{p_k : k \in \mathbb{Z}\}$ is ε-shadowed by a point r if

$$\mathrm{dist}(f^k(r), p_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

We say that f has the standard shadowing property if for any $\varepsilon > 0$ we can find a $d > 0$ such that any d-pseudotrajectory of f is ε-shadowed by some point.

It is well known (see [5]) that to establish the standard shadowing property on a compact phase space it is enough to show that f has the so-called finite shadowing property: For any $\varepsilon > 0$ we can find a $d > 0$ (depending on ε only) such that if $\{p_k : 0 \le k \le m\}$ is a finite d-pseudotrajectory, then there is a point r such that

$$\mathrm{dist}(f^k(r), p_k) < \varepsilon, \quad 0 \le k \le m. \tag{2.2}$$

Our goal is to give sufficient conditions under which a homeomorphism has the finite shadowing property on X. In our conditions, we use analogs of Lyapunov functions.

Let us formulate our main assumptions.

We assume that the space X is compact and there exist two continuous nonnegative functions V and W defined in a closed neighborhood of the diagonal of $X \times X$ such that $V(p,p) = W(p,p) = 0$ for any $p \in X$ and the conditions (C1)–(C9) stated below are satisfied. In what follows, arguments of the functions V and W are assumed to be close enough, so that the functions are defined.

We formulate our conditions not directly in terms of the functions W and V but in terms of some geometric objects defined via these functions. Our main reasoning for the choice of this form of conditions is as follows:

(1) Precisely these conditions are used in the proofs;
(2) it is easy to check conditions of that kind for particular functions W and V (see the examples below).

Fix positive numbers $a, b > 0$ and a point $p \in X$ and let

$$\begin{aligned} P(a,b,p) &= \{q \in X : V(q,p) \le a,\ W(q,p) \le b\}, \\ Q(a,b,p) &= \{q \in P(a,b,p) : V(q,p) = a\}, \\ T(a,b,p) &= \{q \in P(a,b,p) : V(q,p) = 0\}. \end{aligned}$$

Denote by $B(\varepsilon, p)$ the open ε-ball centered at p. Set

$$\begin{aligned} \mathrm{Int}^0 P(a,b,p) &= \{q \in P(a,b,p) : V(q,p) < a,\ W(q,p) < b\}, \\ \partial^0 P(a,b,p) &= Q(a,b,p) \cup \{q \in P(a,b,p) :\ W(q,p) = b\}, \\ \mathrm{Int}^0 Q(a,b,p) &= \{q \in P(a,b,p) : V(q,p) = a,\ W(q,p) < b\}. \end{aligned}$$

Conditions (C1)–(C4) contain our assupmtions on the geometry of the sets introduced above.

(C1) For any $\varepsilon > 0$ there exists a $\Delta_0 = \Delta_0(\varepsilon) > 0$ such that $P(\Delta_0, \Delta_0, p) \subset B(\varepsilon, p)$ for $p \in X$.

There exists a $\Delta_1 > 0$ such that if $p \in X$, $\delta_1, \delta_2, \Delta < \Delta_1$, and $\delta_2 < \Delta$, then there exists a number $\alpha = \alpha(\delta_1, \delta_2, \Delta) > 0$ such that

(C2) $Q(\delta_1, \delta_2, p)$ is not a retract of $P(\delta_1, \delta_2, p)$;
(C3) $Q(\delta_1, \delta_2, p)$ is a retract of $P(\delta_1, \delta_2, p) \setminus T(\delta_1, \delta_2, p)$;
(C4) there exists a retraction $\sigma\colon P(\delta_1, \Delta, p) \to P(\delta_1, \delta_2, p)$ such that

$$V(\sigma(q), p) \ge \alpha V(q,p) \quad \text{for } q \in P(\delta_1, \Delta, p).$$

In the next group of conditions, we state our assumptions on the behavior of the introduced objects and their images under the homeomorphism f.

We assume that for any $\Delta < \Delta_1$ there exist positive numbers $\delta_1, \delta_2 < \Delta$ such that the following relations hold for any $p \in X$:

(C5) $f(P(\delta_1, \delta_2, p)) \subset \mathrm{Int}^0 P(\Delta, \Delta, f(p))$,
$f^{-1}(P(\delta_1, \delta_2, f(p))) \subset \mathrm{Int}^0 P(\Delta, \Delta, p)$;

(C6) $f(T(\delta_1, \delta_2, p)) \subset \operatorname{Int}^0 P(\delta_1, \delta_2, f(p))$;
(C7) $f(T(\delta_1, \Delta, p)) \cap Q(\delta_1, \delta_2, f(p)) = \emptyset$;
(C8) $f(P(\delta_1, \delta_2, p)) \cap \partial^0 P(\delta_1, \delta_2, f(p)) \subset \operatorname{Int}^0 Q(\delta_1, \delta_2, f(p))$;
(C9) $f(S(\delta_1, \Delta, p)) \cap P(\delta_1, \delta_2, f(p)) = \emptyset$, where $S(\delta_1, \Delta, p) = \{q \in P(\Delta, \Delta, p) : V(q, p) \geq \delta_1\}$.

Our main result is as follows.

Theorem 2.1. *Under conditions* (C1)–(C9), *f has the finite shadowing property on the space X.*

In the proof of this statement, we apply the following two lemmas.

First let us formulate one more condition (the letter $\mathcal{W}$ in this condition indicates that, as was mentioned above, this condition has much in common with the classical Ważewski principle in the theory of differential equations).

Let $p, p' \in X$ and $\delta_1, \delta_2 > 0$. We say that condition $\mathcal{W}(\delta_1, \delta_2, p, p')$ holds if

$$f(P) \cap \partial^0 P' \subset Q', \tag{2.3}$$

$$f(Q) \cap P' = \emptyset, \tag{2.4}$$

and Q is a retract of the set $H = H_1 \cup f^{-1}(Q')$, where $P = P(\delta_1, \delta_2, p)$, $Q = Q(\delta_1, \delta_2, p)$, $P' = P(\delta_1, \delta_2, p')$, $Q' = Q(\delta_1, \delta_2, p')$, and $H_1 = P \setminus f^{-1}(\operatorname{Int}^0 P')$.

Lemma 2.2. *Let positive numbers $\delta_1, \delta_2 < \Delta$ satisfy conditions* (C4)–(C9). *Let $\delta = \min(\delta_1, \delta_2)$. There exists a positive $d = d(\delta)$ such that if* $\operatorname{dist}(p', f(p)) < d$, *then condition $\mathcal{W}(\delta_1, \delta_2, p, p')$ holds.*

Proof. In the proof, we several times select a small d (depending on δ) and then take as the required d the minimum of the selected values of d.

Condition (C6), the compactness of the neighborhood of the diagonal of $X \times X$ in which the functions V and W are defined, and the continuity of f imply that there exist positive numbers $c_1 < \delta_1$ and $c_2 < \delta_2$ such that if $q \in f(T(\delta_1, \delta_2, p))$, then $V(q, f(p)) \leq c_1$ and $W(q, f(p)) \leq c_2$. Hence, there exists a $d = d(\delta)$ such that if

$$\operatorname{dist}(p', f(p)) < d, \tag{2.5}$$

then $V(q, p') < \delta_1$ and $W(q, p') < \delta_2$, which means that

$$f(T(\delta_1, \delta_2, p)) \subset \operatorname{Int}^0 P'. \tag{2.6}$$

A similar reasoning based on condition (C5) shows that there exists a $d = d(\delta)$ such that if inequality (2.5) is satisfied, then

$$f(P) \subset P(\Delta, \Delta, p'), \tag{2.7}$$

$$f^{-1}(P') \subset P(\Delta, \Delta, p). \tag{2.8}$$

In particular, inclusion (2.8) implies that

$$f^{-1}(Q') \subset P(\Delta, \Delta, p). \tag{2.9}$$

Let us show that there exists a $d = d(\delta)$ such that if inequality (2.5) is satisfied, then

$$f^{-1}(Q') \subset P(\delta_1, \Delta, p). \tag{2.10}$$

Since the set $S := S(\delta_1, \Delta, p)$ is compact, it follows from condition (C9) that there exists a number $c_3 > 0$ such that if $q \in S$, then

$$\max(V(f(q), f(p)) - \delta_1, W(f(q), f(p)) - \delta_2) \geq c_3.$$

Hence, there exists a $d = d(\delta)$ such that if inequality (2.5) is satisfied, then

$$\max(V(f(q), p') - \delta_1, W(f(q), p') - \delta_2) > 0$$

for $q \in S$, which implies that condition (2.4) is satisfied and $f(S) \cap Q' = \emptyset$. Now inclusion (2.10) follows from inclusion (2.9).

Clearly, condition (C8) (combined with inclusion (2.7)) implies that there exists a $d = d(\delta)$ such that if inequality (2.5) is satisfied, then inclusion (2.3) holds.

Similarly, it follows from condition (C7) that there exists a number $c_4 > 0$ such that if $q \in f^{-1}(Q(\delta_1, \delta_2, f(p)))$, then $V(q, p) \geq 2c_4$. Hence, there exists a $d = d(\delta)$ such that if inequality (2.5) is satisfied, then $V(q, p) \geq c_4$ for $q \in f^{-1}(Q')$.

Apply condition (C4) to find a retraction $\sigma\colon P(\delta_1, \Delta, p) \to P(\delta_1, \delta_2, p)$ such that

$$V(\sigma(q), p) \geq \alpha c_3, \quad q \in f^{-1}(Q'). \tag{2.11}$$

The set $U = \sigma(f^{-1}(Q'))$ is compact, and inequality (2.11) implies that

$$U \cap T(\delta_1, \Delta, p) = \emptyset. \tag{2.12}$$

By condition (C3), there exists a retraction ρ_0 of $P \setminus T(\delta_1, \delta_2, p)$ to Q. Relations (2.6) and (2.12) imply that $H_1 \cup U \subset P \setminus T(\delta_1, \delta_2, p)$. Hence, the restriction of $\rho = \rho_0 \circ \sigma$ to H is the required retraction $H \to Q$. □

Lemma 2.3. *Let $p_0, \dots, p_m$ be points in X such that, for $k = 0, \dots, m-1$, condition $\mathcal{W}(\delta_1, \delta_2, p_k, p_{k+1})$ holds. Then there exists a point $r \in P(\delta_1, \delta_2, p_0)$ such that $f^k(r) \in P(\delta_1, \delta_2, p_k)$ for $k = 1, \dots, m$.*

Proof. Consider the sets

$$A_k = P(\delta_1, \delta_2, p_k) \setminus \bigcap_{l=k+1}^{m} f^{-(l-k)}(\operatorname{Int}^0 P(\delta_1, \delta_2, p_l)), \quad k = 0, \dots, m-1.$$

It follows from equality (2.4) that

$$f(Q(\delta_1,\delta_2,p_k)) \cap P(\delta_1,\delta_2,p_{k+1}) = \emptyset.$$

Hence, $Q(\delta_1,\delta_2,p_k) \subset A_k$.

We claim that there exist retractions $\rho_k\colon A_k \to Q(\delta_1,\delta_2,p_k)$, $k=0,\dots,m-1$. This is enough to prove our lemma since the existence of ρ_0 means that

$$\bigcap_{l=0}^{m} f^{-l}(\operatorname{Int}^0 P(\delta_1,\delta_2,p_l)) \neq \emptyset$$

(otherwise there exists a retraction of $P(\delta_1,\delta_2,p_0)$ to $Q(\delta_1,\delta_2,p_0)$, which is impossible by condition (C3)).

The existence of ρ_{m-1} is obvious since condition $\mathcal{W}(\delta_1,\delta_2,p_{m-1},p_m)$ implies the existence of a retraction

$$A_{m-1} \cup f^{-1}(Q(\delta_1,\delta_2,p_m)) \to Q(\delta_1,\delta_2,p_{m-1}).$$

Let us assume that the existence of retractions $\rho_{k+1},\dots,\rho_{m-1}$ has been proved. Let us prove the existence of ρ_k. For brevity, we denote

$$\begin{aligned} P_k &= P(\delta_1,\delta_2,p_k), & Q_k &= Q(\delta_1,\delta_2,p_k),\\ P_{k+1} &= P(\delta_1,\delta_2,p_{k+1}), & Q_{k+1} &= Q(\delta_1,\delta_2,p_{k+1}). \end{aligned}$$

Note that the definition of the sets A_k implies that

$$A_k \cap f^{-1}(P_{k+1}) \subset f^{-1}(A_{k+1}). \tag{2.13}$$

Define a mapping θ on A_k by setting:

$$\begin{aligned} \theta(q) &= f^{-1}\circ\rho_{k+1}\circ f(q), & q &\in A_k \cap f^{-1}(P_{k+1}),\\ \theta(q) &= q, & q &\in A_k \setminus f^{-1}(P_{k+1}). \end{aligned}$$

Inclusion (2.13) shows that the mapping θ is properly defined.

Let us show that this mapping is continuous. Clearly, it is enough to show that $\rho_{k+1}(r) = r$ for $r \in f(A_k \cap f^{-1}(\partial^0 P_{k+1}))$. For this purpose, we note that

$$f(A_k \cap f^{-1}(\partial^0 P_{k+1})) = f(A_k) \cap \partial^0 P_{k+1} \subset f(P_k) \cap \partial^0 P_{k+1} \subset Q_{k+1}$$

(we refer to inclusion (2.3)) and $\rho_{k+1}(r) = r$ for $r \in Q_{k+1}$.

Clearly, θ maps A_k to the set

$$[P_k \setminus f^{-1}(P_{k+1})] \cup f^{-1}(Q_{k+1}). \tag{2.14}$$

Condition $\mathcal{W}(\delta_1,\delta_2,p_k,p_{k+1})$ implies that there exists a retraction ρ of (2.14) to Q_k. It remains to note that $\theta(q) = q$ for $q \in Q_k$ due to condition (2.4). Thus, $\rho_k = \rho\theta\colon A_k \to Q_k$ is the required retraction. □

To complete the proof of the main theorem, we take an arbitrary $\varepsilon > 0$, apply condition (C1) to find a proper Δ_0 and then find the corresponding numbers δ_1, δ_2, Δ. Lemma 2.2 implies that there exists a $d > 0$ depending on $\delta_1, \delta_2, \Delta$ (i.e. on ε) such that if $p_0, \dots, p_m$ is a finite d-pseudotrajectory of f, then condition $\mathcal{W}(\delta_1, \delta_2, p_k, p_{k+1})$ holds for $k = 0, \dots, m-1$. Now it follows from Lemma 2.3 that f has the finite shadowing property on X. □

3. Topological stability

In this section, we assume, for simplicity of presentation, that X is a smooth closed manifold.

Let $H(X)$ be the space of homeomorphisms of X endowed with the metric

$$\rho(f, g) = \max_{p \in X} \max(\operatorname{dist}(f(p), g(p)), \operatorname{dist}(f^{-1}(p), g^{-1}(p))).$$

It is well known that $H(X)$ is a complete metric space.

A homeomorphism f is called *topologically stable* if for any $\varepsilon > 0$ there exists a neighbourhood Y of f in X such that if $g \in Y$, then there exists a continuous map $h\colon X \to X$ such that $f \circ h = h \circ g$ and

$$\operatorname{dist}(h(p), p) < \varepsilon, \quad p \in X.$$

It is not difficult to show that if a homeomorphism f is topologically stable, then f has the shadowing property (see [5]).

To formulate general sufficient conditions of topological stability, let us recall one more standard definition.

A homeomorphism f is called *expansive* if there exists a positive number a such that if

$$\operatorname{dist}(f^k(p), f^k(q)) \le a, \quad k \in \mathbb{Z},$$

then $p = q$.

Walters proved in [9] the following theorem (see also [5]).

Theorem 3.1. *If a homeomorphism f is expansive and has the shadowing property, then f is topologically stable.*

Remark 3.2. Usually, an expansive homeomorphism having the shadowing property is called topologically Anosov.

It is known that structurally stable diffeomorphisms are topologically stable [2].

We want to show that our shadowing result based on analogs of Lyapunov functions can be applied in the proof of topological stability of dynamical systems that are “far” from the set of structurally stable diffeomorphisms (in particular, such systems may have nonhyperbolic fixed points).

Example 3.2. Consider an example of a diffeomorphism of the 2-torus T^2 studied by Lewowicz in [3]. This diffeomorphism is a perturbation of a hyperbolic automorphism of T^2.

Consider numbers $0 < \alpha < 1 < \beta$ and a small $r > 0$ and define a map $F: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = (\alpha x + \lambda(x)\mu(y), \beta y),$$

where

$$\lambda(x) = \int_0^x ((1-\alpha) - h(s))\, ds,$$

$h: \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $h(0) = 0$, $0 \le h(x) < 1$, and $\lambda(x) = 0$ for $|x| \ge r$; $\mu: \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\mu(0) = 1$, $\mu(y) = \mu(-y)$, μ is not increasing for $y \ge 0$, and $\mu(y) = 0$ for $|y| \ge r$.

Let A be an integer hyperbolic 2×2 matrix with $\det A = 1$. If $0 < \alpha < 1 < \beta$ are the eigenvalues of A and u_1 and u_2 are the corresponding eigenvectors, then

$$A(x, y) = (\alpha x, \beta y)$$

in cooordinates whose axes are parallel to u_1 and u_2.

The lattice Ξ with vertices $\{(n+1/2)u_1, (m+1/2)u_2 : n, m \in \mathbb{Z}\}$ is invariant with respect to the action of the map $v \mapsto Av$. Let $\pi: \mathbb{R}^2 \to \mathbb{R}^2/\Xi$ be the corresponding projection of the plane to the 2-torus.

Define $f: T^2 \to T^2$ by $f(\pi(\xi, \eta)) = \pi \circ F(x, y)$ (of course, we extend F periodically with respect to the above-mentioned lattice).

It is shown in [3] that if r is small enough, then f is an expansive diffeomorphism of the torus. At the same time, f is not Anosov (and is not structurally stable) since the eigenvalues of Df at the zero fixed point are 1 and β.

Consider the functions V and W defined as follows. If $p = (p_x, p_y)$ and $q = (q_x, q_y)$, we set $V(p, q) = |p_y - q_y|$ and $W(p, q) = |p_x - q_x|$ (such functions are properly defined if p and q are close enough).

It is obvious that conditions (C1)–(C5) are satisfied. Let us check condition (C6). We fix $0 < \Delta < \Delta_0$ and a point $p = (p_x, p_y)$. Let $f = (f_x, f_y)$. Take a small $\delta_2 > 0$ and consider points $p' = (p_x + \nu, p_y)$, where $|\nu| \le \delta_2$. If $\nu = 0$, then

$$|f_x(p) - f_x(p')| = 0.$$

If $\nu \ne 0$, then

$$|f_x(p) - f_x(p')| = |\nu(\alpha + \mu(p_y)(1-\alpha)) - \mu(p_y) \int_{p_x}^{p_x+\nu} h(s) ds| < |\nu| \le d_2$$

(we take into account that $\mu(y) \le 1$ and $h(x) > 0$ for $x \ne 0$).

It follows that condition (C6) is satisfied for any (δ_1, δ_2), where $\delta_2 < \Delta$ is small enough. In addition, if δ_2 is small enough, then the "rectangle" $P(\delta_1, \delta_2, p)$

is close to the "segment" $T(\delta_1, \delta_2, p)$, which implies that condition (C8) is satisfied as well.

Since f expands in the y direction, conditions (C7) and (C9) are satisfied automatically.

Thus, we can apply the main theorem to show that f has the shadowing property, which implies the topological stability of f.

REMARK 3.3. Let us make a comment concerning the conditions and proofs in the paper [3]. First, the proof in [3] refers to the smoothness of the system considered (while our proof works for homeomorphisms). Second, the proof in [3] reduces the problem to study of suspension flows, which does not seem natural. Third, in our opinion, the proof in [3] requires stronger assumptions on the regularity of the function V than stated.

As was mentioned, our methods are applicable to homeomorphisms.

EXAMPLE 3.4. Consider a perturbation f of the hyperbolic automorphism of T^2 corresponding to the map

$$F(x, y) = (\mu_1(x), \mu_2(y)),$$

where μ_1 and μ_2 are increasing continuous functions for which there exist numbers $r, \lambda \in (0, 1)$ such that

(1) $|\mu_1(x + \nu) - \mu_1(x)| \leq \lambda|\nu|$ and $\lambda^{-1}|\nu| \leq |\mu_2(y + \nu) - \mu_2(y)|$ for $|\nu| < r$;
(2) $\mu_1(x) = \alpha x$, $|x| \geq r$;
(3) $\mu_2(y) = \beta y$, $|y| \geq r$.

To prove that f is topologically stable, one can apply the same functions V and W and the same reasoning as in Example 3.2 (to show that f is expansive, one can apply the same reasoning as that applied in [3] to Example 3.2 considering the function $\mathbf{V}(p, q) = V(p, q) - W(p, q)$).

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