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PROPERNESS AND TOPOLOGICAL DEGREE FOR NONLOCAL INTEGRO-DIFFERENTIAL SYSTEMS

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Abstract. Reaction-diffusion systems of equations with integral terms are studied. Essential spectrum of the corresponding linear operators is determined and the Fredholm property is studied. Properness of nonlinear operators is proved and topological degree is constructed.

1. Integro-differential systems

Consider the system of integro-differential equations:

$$\left\{\begin{array}{l} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2} + F_1(u_1, \ldots, u_n, \varphi_{11} * u_1, \ldots, \varphi_{1n} * u_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial u_n}{\partial t} = \dfrac{\partial^2 u_n}{\partial x^2} + F_n(u_1, \ldots, u_n, \varphi_{n1} * u_1, \ldots, \varphi_{nn} * u_n), \end{array}\right. \tag{1.1}$$

where $\varphi_{ij}\colon \mathbb{R} \to \mathbb{R}$, $\varphi_{ij} \geq 0$ on $\mathbb{R}$, $\operatorname{supp} \varphi_{ij} = [-N_{ij}, N_{ij}]$ is bounded,

$$\int_{-\infty}^{\infty} \varphi_{ij}(y)\, dy = 1, \quad \text{for } i, j = 1, \ldots, n,$$

while $F_1, \ldots, F_n\colon \mathbb{R}^{2n} \to \mathbb{R}$ are given functions such that $F_i \in C^1(\mathbb{R}^{2n}, \mathbb{R})$. Here $\varphi_{ij} * u_j$ is the convolution product

$$(\varphi_{ij} * u_j)(x) = \int_{-\infty}^{\infty} \varphi_{ij}(x - y) u_j(y)\, dy.$$

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Denote in the sequel by the superscript ^{T} the transposed of any n-dimensional vector or $n \times n$ matrix. Let $u = (u_1, \ldots, u_n)^T$, $F = (F_1, \ldots, F_n)^T$ and

$$\Phi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix}, \qquad \Phi * u = \begin{pmatrix} \varphi_{11} * u_1 & \cdots & \varphi_{1n} * u_n \\ \vdots & \ddots & \vdots \\ \varphi_{n1} * u_1 & \cdots & \varphi_{nn} * u_n \end{pmatrix}.$$

Then the integro-differential system (1.1) can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u, \Phi * u). \tag{1.2}$$

A travelling wave solution of system (1.1) or equivalently, of equation (1.2) is a solution of the form $u(x,t) = w(x - ct)$, where $c \in \mathbb{R}$ is a constant, called the wave speed. If $w = (w_1, \ldots, w_n)^T$, then $u_i(x,t) = w_i(x - ct)$, $i = 1, \ldots, n$ and function w verifies the equation

$$w'' + cw' + F(w, \Phi * w) = 0. \tag{1.3}$$

Nonlocal reaction-diffusion equations of this type arise in population dynamics (see [1]–[3], [6]). The integral term describes nonlocal consumption of resources and intraspecific competition resulting in the emergence of biological species in the process of evolution.

In this paper we study the operator B defined by the left-hand side of (1.3) as acting from the Hölder space $E = (C^{2+\alpha}(\mathbb{R}))^n$ to $E^0 = (C^{\alpha}(\mathbb{R}))^n$, $0 < \alpha < 1$:

$$Bw = w'' + cw' + F(w, \Phi * w).$$

We are interested in the solutions $w = (w_1, \ldots, w_n)^T$ of system (1.3) with some specific limits $w^{\pm} = (w_1^{\pm}, \ldots, w_n^{\pm})^T$ at $\pm\infty$. We are looking for the solutions w_i of (1.3) under the form $w_i = u_i + \psi_i$, where $\psi_i \in C^{\infty}(\mathbb{R})$ are chosen such that $\psi_i(x) = w_i^+$ for $x \geq 1$ and $\psi_i(x) = w_i^-$ for $x \leq -1$. Thus equation (1.3) becomes

$$(u + \psi)'' + c(u + \psi)' + F(u + \psi, \Phi * (u + \psi)) = 0, \tag{1.4}$$

where $\psi = (\psi_1, \ldots, \psi_n)^T$. Denote by A the operator in the left-hand side of (1.4), $A: E \to E^0$,

$$Au = (u + \psi)'' + c(u + \psi)' + F(u + \psi, \Phi * (u + \psi)). \tag{1.5}$$

The linearization of A about a function $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n) \in E$ is the operator

$$Lu = u'' + cu' + \frac{\partial F}{\partial u}(\widetilde{u} + \psi, \Phi * (\widetilde{u} + \psi))u + \frac{\partial F}{\partial U}(\widetilde{u} + \psi, \Phi * (\widetilde{u} + \psi))(\Phi * u), \tag{1.6}$$

where $\frac{\partial F}{\partial u} = \left(\frac{\partial F_i}{\partial u_j}\right)$ and $\frac{\partial F}{\partial U} = \left(\frac{\partial F_i}{\partial U_j}\right)$ are the matrices of the derivatives of $F_1, \ldots, F_n$ with respect to variables $u_1, \ldots, u_n$ and $U_1, \ldots, U_n$, respectively.

In this work we will prove the Fredholm property of the linear operator L and the properness of the nonlinear operator A. These results will be used to define the topological degree, which is a powerful tool to study existence and bifurcations of solutions. We generalize here the previous results obtained for the scalar equation [4] and used to prove the existence of travelling waves [5].

The paper is organized as follows. Section 2 contains some basic definitions and auxiliary results that we need in the sequel. In Section 3 we prove the properness of our nonlinear operator A in weighted Hölder spaces. More generally, this property is studied for a family of operators A^τ associated to A. Section 4 is devoted to the Fredholm property of the linearized operators L^τ. Using all these results, we construct in Section 4 a topological degree for an associated class of homotopies.

2. Definitions and auxiliary results

2.1. Limiting operators. For the linearized operator L, we introduce the limiting operators $L^{\pm}$, that is the operators obtained from L by replacing the coefficients of u, u', u'', $\Phi * u$ with their limits as $x \to \pm\infty$. Since for $\widetilde{w} = \widetilde{u} + \psi$ there exist the limits $\lim_{x\to\pm\infty} \widetilde{w}(x) = w^{\pm}$, it follows that $\Phi * (\widetilde{u} + \psi) \to w^{\pm}$ as $x \to \pm\infty$ and therefore the limiting operators associated to L are

$$L^{\pm}u = u'' + cu' + \frac{\partial F}{\partial u}(w^{\pm}, w^{\pm})u + \frac{\partial F}{\partial U}(w^{\pm}, w^{\pm})(\Phi * u). \tag{2.1}$$

Similar to elliptic problems in unbounded domains [8], [9], limiting operators determine the Fredholm property and properness of the integro-differential operators.

2.2. A priori estimates. In [2] the following estimate of Schauder type has been proved.

Lemma 2.1. *If $\varphi_{ij} \in L^1(\mathbb{R})$, $i,j = 1,\dots,n$, $f \in E^0$ and u is a solution of the equation $Lu = f$, then there exists a constant $C > 0$ independent of u, such that*

$$||u||_E \le C(||Lu||_{E^0} + ||u||_{(C(\mathbb{R}))^n}).$$

We will use this lemma below in the proof of Fredholm property.

2.3. Topological degree. We finish this section by recalling the definition of the topological degree that we need in the last section.

Consider two Banach spaces E_1, E_2, a class Ψ of operators acting from E_1 to E_2 and a class of homotopies

$$H = \{A_\tau(u)\colon E_1 \times [0,1] \to E_2, \text{ such that } A_\tau(u) \in \Psi, \text{ for all } \tau \in [0,1]\}.$$

Let $D \subset E_1$ be an open bounded set and $A \in \Psi$ such that $A(u) \neq 0$, $u \in \partial D$, where ∂D is the boundary of D. Suppose that for such a pair (D, A), there exists an integer $\gamma(A, D)$ with the following properties:

(a) (Homotopy invariance) If $A_\tau(u) \in H$ and $A_\tau(u) \neq 0$, for $u \in \partial D$, $\tau \in [0,1]$, then
$$\gamma(A_0, D) = \gamma(A_1, D).$$

(b) (Aditivity). If $A \in \Psi$, $\overline{D}$ is the closure of D and $D_1, D_2 \subset D$ are open sets, such that $D_1 \cap D_2 = \emptyset$ and $A(u) \neq 0$, for all $u \in \overline{D} \setminus (D_1 \cup D_2)$, then
$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

(c) (Normalization) There exists a bounded linear operator $J: E_1 \to E_2$ with a bounded inverse defined on all E_2 such that, for every bounded set $D \subset E_1$ with $0 \in D$,
$$\gamma(J, D) = 1.$$

The integer $\gamma(A, D)$ is called a *topological degree associated to the operator A and the set D*.

3. Properness in weighted spaces

This section is devoted to the properness of the semilinear operator A as acting between some weighted Hölder spaces. Introduction of weighted spaces is necessary since problems in unbounded domains may not be proper in Hölder spaces without weight [9]. Recall first the definition of properness.

Definition 3.1. If X, Y are Banach spaces, an operator $A: X \to Y$ is called *proper* if for any compact set $D \subset Y$ and any bounded closed set $B \subset X$, the intersection $A^{-1}(D) \cap B$ is a compact set in X.

Remark 3.2. Let us note that the set of solutions of the operator equation $A(u) = 0$ is compact in closed bounded sets if the operator A is proper. This shows importance of this property of nonlinear operators. As it is shown in [4] for a single equation, the operator $A: E \to E^0$ is not proper from E into E^0.

We will prove in this section that A is proper in some weighted spaces defined as below.

Let $\mu: \mathbb{R} \to \mathbb{R}$ be the function given by $\mu(x) = 1 + x^2$, $x \in \mathbb{R}$. Consider $E = (C^{2+\alpha}(\mathbb{R}))^n$, $E^0 = (C^\alpha(\mathbb{R}))^n$ endowed with the usual norms $||\cdot||_E$ and $||\cdot||_{E^0}$ such that for $u = (u_1, \dots, u_n) \in E$, $||u||_E = \sum_{j=1}^n ||u_j||_{C^{2+\alpha}(\mathbb{R})}$ and similarly for $||\cdot||_{E^0}$. We will work in the weighted Hölder spaces E_μ and E^0_μ, which are E and E^0, respectively, with the norms $||u||_\mu = ||\mu u||_E$ and $||u||_{0\mu} = ||\mu u||_{E^0}$.

We begin with the following estimate for the integral term

$$(\varphi_{ij} * u_j)(x) = \int_{-\infty}^{\infty} \varphi_{ij}(x-y)u_j(y)\,dy$$

with $\varphi_{ij}, u_j \in C^\alpha(\mathbb{R})$, $i,j = 1,\dots,n$.

LEMMA 3.2 ([4]). *Suppose that $\mu(x) = 1 + x^2$ and $\varphi_{ij}: \mathbb{R} \to \mathbb{R}$ $(1 \le i,j \le n)$ are given functions such that $\varphi_{ij} \ge 0$ on $\mathbb{R}$, $\operatorname{supp}\varphi_{ij} = [-N,N]$ is bounded, $\int_{-\infty}^{\infty} \varphi_{ij}(y)\,dy = 1$ and $\varphi_{ij} \in C^\alpha(\mathbb{R})$. Then*

$$||\varphi_{ij} * u_j||_{C^\alpha_\mu(\mathbb{R})} \le K||\mu u_j||_{C(\mathbb{R})}, \quad \text{for all } u_j \in C^\alpha(\mathbb{R}), \tag{3.1}$$

for some constant $K > 0$. We have denoted by $C^\alpha_\mu(\mathbb{R})$ the space $C^\alpha(\mathbb{R})$ with the weight μ.

We study the operator A acting from E_μ into E^0_μ. In order to introduce a topological degree in a future section, we prove the properness of A in the more general case when the coefficient c and function F depend also on a parameter $\tau \in [0,1]$. Let $A^\tau: E_\mu \to E^0_\mu$, $\tau \in [0,1]$ be the operator defined as follows

$$A^\tau u = (u+\psi)'' + c(\tau)(u+\psi)' + F^\tau(u+\psi, \Phi * (u+\psi)). \tag{3.2}$$

The operator L^τ linearized about a function $\widetilde{u} \in E_\mu$ is

$$L^\tau u = u'' + c(\tau)u' + \frac{\partial F^\tau}{\partial u}(\widetilde{u}+\psi, \Phi * (\widetilde{u}+\psi))u + \frac{\partial F^\tau}{\partial U}(\widetilde{u}+\psi, \Phi * (\widetilde{u}+\psi))(\Phi * u). \tag{3.3}$$

The associated limiting operators are given by

$$(L^\tau)^\pm u = u'' + c(\tau)u' + \frac{\partial F^\tau}{\partial u}(w^\pm, w^\pm)u + \frac{\partial F^\tau}{\partial U}(w^\pm, w^\pm)(\Phi * u). \tag{3.4}$$

Assume the following hypotheses are satisfied:

(H1) For any $\tau \in [0,1]$, functions $F_i^\tau(u,U)$ and their derivatives with respect to u and U satisfy the Lipschitz condition: there exists $K > 0$ such that

$$|F_i^\tau(u,U) - F_i^\tau(\widehat{u},\widehat{U})| \le K(|u-\widehat{u}| + |U-\widehat{U}|)$$

for any $(u,U), (\widehat{u},\widehat{U}) \in \mathbb{R}^{2n}$. Similarly, for $\partial F_i^\tau/\partial u_j$ and $\partial F_i^\tau/\partial U_j$:

$$\left|\frac{\partial F_i^\tau}{\partial u_j}(u,U) - \frac{\partial F_i^\tau}{\partial u_j}(\widehat{u},\widehat{U})\right| \le K(|u-\widehat{u}| + |U-\widehat{U}|),$$
$$\left|\frac{\partial F_i^\tau}{\partial U_j}(u,U) - \frac{\partial F_i^\tau}{\partial U_j}(\widehat{u},\widehat{U})\right| \le K(|u-\widehat{u}| + |U-\widehat{U}|).$$

(H2) $c(\tau)$, $F_i^\tau(u,U)$ and the derivatives of $F_i^\tau(u,U)$ are Lipschitz continuous in τ, i.e. there exists a constant $K > 0$ such that

$$|c(\tau) - c(\tau_0)| \le K|\tau - \tau_0|, \qquad |F_i^\tau(u,U) - F_i^{\tau_0}(u,U)| \le K|\tau - \tau_0|,$$

$$\left|\frac{\partial F_i^\tau}{\partial u_j}(u,U) - \frac{\partial F_i^{\tau_0}}{\partial u_j}(u,U)\right| \le K|\tau-\tau_0|,$$
$$\left|\frac{\partial F_i^\tau}{\partial U_j}(u,U) - \frac{\partial F_i^{\tau_0}}{\partial U_j}(u,U)\right| \le K|\tau-\tau_0|.$$

for all $\tau,\tau_0 \in [0,1]$ and for all (u,U) from any bounded set in $\mathbb{R}^{2n}$.

(H3) (Condition NS) For any $\tau \in [0,1]$, the limiting equations

$$u'' + c(\tau)u' + \frac{\partial F^\tau}{\partial u}(w^\pm, w^\pm)u + \frac{\partial F^\tau}{\partial U}(w^\pm, w^\pm)(\Phi * u) = 0$$

do not have nonzero solutions in E.

Recall the following auxiliary result from [4].

Lemma 3.4. *Suppose that conditions* (H1)–(H2) *hold. If $\tau_k \to \tau_0$ and $\mu u_i^k \to \mu u_i^0$ in $C(\mathbb{R})$ for a fixed i $(1 \le i \le n)$, then the following convergence holds in $C_\mu^\alpha(\mathbb{R})$:*

$$F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) \to F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi)) \quad \text{in } C_\mu^\alpha(\mathbb{R}).$$

where

$$J_{ij}(u_j)(x) = (\varphi_{ij} * u_j)(x) = \int_{-\infty}^{\infty} \varphi_{ij}(x-y)u_j(y)\,dy$$

and $J_i(u) = (J_{i1}(u_1), \ldots, J_{in}(u_n))$.

Theorem 3.5. *Assume that functions $\varphi_{ij}\colon \mathbb{R} \to \mathbb{R}$ and $F_i\colon \mathbb{R}^{2n} \to \mathbb{R}$ $(1 \le i,j \le n)$ satisfy the conditions from Section* 1 *and hypotheses* (H1)–(H3). *In addition assume that $\varphi_{ij} \in C^\alpha(\mathbb{R})$. Then the operator $A^\tau\colon E_\mu \times [0,1] \to E_\mu^0$ from* (3.2) *is proper on $E_\mu \times [0,1]$ (with respect to both u and τ).*

Proof. Let $f^k = (f_1^k, \ldots, f_n^k) \in E_\mu^0$ be a given sequence such that $f^k \to f^0$ in E_μ^0, where $f^0 = (f_1^0, \ldots, f_n^0)$ and let (u^k, τ_k) be a solution in $E_\mu \times [0,1]$ of the system $A^{\tau_k}(u^k) = f^k$, such that $u^k = (u_1^k, \ldots, u_n^k)$ and

$$||u^k||_\mu \le M, \quad \text{for all } k \ge 1.$$

We can also assume that $\tau_k \to \tau_0$ as $k \to \infty$. Our aim is to show that we can extract a converging subsequence of u^k (in E_μ). The above system can be written component-wise in the form

$$(3.5)\quad (u_i^k+\psi_i)'' + c(\tau_k)(u_i^k+\psi_i)' + F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) = f_i^k, \quad 1 \le i \le n,$$

One multiplies this equation by $\mu(x) = 1+x^2$, $x \in \mathbb{R}$ and denotes $v^k = (v_i^k)$, $g^k = (g_i^k)$, where $v_i^k(x) = \mu(x)u_i^k(x)$ and $g_i^k(x) = \mu(x)f_i^k(x)$, $1 \le i \le n$. We note that $\mu(u_i^k)' = (\mu u_i^k)' - \mu' u_i^k = (v_i^k)' - \mu' v_i^k/\mu$ and $\mu(u_i^k)'' = (\mu u_i^k)'' - \mu'' u_i^k -$

$2\mu'(u_i^k)' = (v_i^k)'' - \mu'' v_i^k/\mu - 2\mu'((v_i^k)'/\mu - \mu' v_i^k/\mu^2)$. Then the above system becomes

$$(3.6)\quad (v_i^k)'' + \left[-2\frac{\mu'}{\mu} + c(\tau_k)\right](v_i^k)' + \left[-\frac{\mu''}{\mu} + 2(\frac{\mu'}{\mu})^2 - c(\tau_k)\frac{\mu'}{\mu}\right] v_i^k \\ + \mu F_i^{\tau_k}(u^k + \psi, J_i(u^k + \psi)) + \mu(\psi_i'' + c(\tau_k)\psi_i') = g_i^k,$$

for $1 \le i \le n$. The boundedness of u^k implies that

$$(3.7)\qquad ||v^k||_E = ||\mu u^k||_E = ||u^k||_\mu \le M, \quad \text{for all } k \ge 1.$$

Then for each bounded interval $I \subset \mathbb{R}$ of x, there exists a subsequence, denoted also v^k, such that $v^k \to v^0$ in $(C^{2+\alpha}(I))^n$, with $v_0 \in (C^{2+\alpha}(I))^n$. The limit v^0 can be extended to $\mathbb{R}$ by a diagonalization process, such that $v^0 \in E$. In view of (3.7) we can get $||v^0||_E \le M$.

If $u^0 = v^0/\mu$, then $v^k = \mu u^k \to v^0 = \mu u^0$ and $u^k \to u^0$ in $(C^{2+\alpha}(I))^n$, for every bounded interval I.

We note that hypotheses (H1) and (H2) lead to inequality

$$||F_i^{\tau_k}(u^k + \psi, J_i(u^k + \psi)) - F_i^{\tau_0}(u^0 + \psi, J_i(u^0 + \psi))||_{C_\mu^\alpha(\mathbb{R})} \\ \le K|\tau_k - \tau_0| + K[|u_1^k - u_1^0| + \ldots + |u_n^k - u_n^0| \\ + |J_{i1}(u_1^k + \psi_1) - J_{i1}(u_1^0 + \psi_1)| + \ldots + |J_{in}(u_n^k + \psi_n) - J_{in}(u_n^0 + \psi_n)|].$$

The right-hand side of this inequality tends to 0 as $k \to \infty$ because $J_{il}(u_l^k + \psi_l) \to J_{il}(u_l^0 + \psi_l)$ in $C_\mu^\alpha(I)$, for every bounded interval I. Here $C_\mu^\alpha(I)$ is the space $C^\alpha(I)$ with the weight μ. Hence, for each $1 \le i \le n$,

$$(3.8)\qquad F_i^{\tau_k}(u^k + \psi, J_i(u^k + \psi)) \to F_i^{\tau_0}(u^0 + \psi, J_i(u^0 + \psi))$$

in $C_\mu^\alpha(I)$ for every I (Lemma 3.4). Passing to the limit as $k \to \infty$ in $C^\alpha(I)$ in (3.5) and (3.6) we get

$$(3.9)\qquad (u_i^0 + \psi_i)'' + c(\tau_0)(u_i^0 + \psi_i)' + F_i^{\tau_0}(u^0 + \psi, J_i(u^0 + \psi)) = f_i^0,$$

$$(3.10)\quad (v_i^0)'' + \left[-2\frac{\mu'}{\mu} + c(\tau_0)\right](v_i^0)' + \left[-\frac{\mu''}{\mu} + 2(\frac{\mu'}{\mu})^2 - c(\tau_0)\frac{\mu'}{\mu}\right] v_i^0 \\ + \mu F_i^{\tau_0}(u^0 + \psi, J_i(u^0 + \psi)) + \mu(\psi_i'' + c(\tau_0)\psi_i') = \mu f_i^0.$$

Subtracting (3.10) from (3.6) and denoting $V_i^k = v_i^k - v_i^0$, we obtain

$$(3.11)\quad (V_i^k)'' + \left[-2\frac{\mu'}{\mu} + c(\tau_k)\right](V_i^k)' + \left[-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau_k)\frac{\mu'}{\mu}\right] V_i^k \\ + \mu[F_i^{\tau_k}(u^k + \psi, J_i(u^k + \psi)) - F_i^{\tau_0}(u^0 + \psi, J_i(u^0 + \psi))] \\ + [c(\tau_k) - c(\tau_0)]\left((v_i^0)' - \frac{\mu'}{\mu}v_i^0 + \mu\psi_i'\right) = \mu f_i^k - \mu f_i^0.$$

Recall that $V_i^k = v_i^k - v_i^0 \to 0$ as $k \to \infty$ in $C^{2+\alpha}(I)$, for any bounded interval I. We show that $V_i^k \to 0$ in $C(\mathbb{R})$. Suppose by contradiction that this is not true. Then, without loss of generality, we can chose a sequence $x_k \to \infty$ such that $|V_i^k(x_k)| \geq \varepsilon > 0$. This means that $|v_i^k(x_k) - v_i^0(x_k)| \geq \varepsilon > 0$. Let

$$\begin{aligned} \widetilde{V}_i^k(x) &= V_i^k(x+x_k) = v_i^k(x+x_k) - v_i^0(x+x_k) \\ &= \mu(x+x_k)[u_i^k(x+x_k) - u_i^0(x+x_k)]. \end{aligned} \tag{3.12}$$

Therefore

$$|\widetilde{V}_i^k(0)| = |V_i^k(x_k)| \geq \varepsilon > 0. \tag{3.13}$$

Writing (3.11) in $x + x_k$, one obtains

$$\begin{aligned} (\widetilde{V}_i^k)''(x) &+ \bigg[-2\frac{\mu'(x+x_k)}{\mu(x+x_k)} + c(\tau_k)\bigg](\widetilde{V}_i^k)'(x) \\ &+ \bigg[-\frac{\mu''(x+x_k)}{\mu(x+x_k)} + 2\bigg(\frac{\mu'(x+x_k)}{\mu(x+x_k)}\bigg)^2 - c(\tau_k)\frac{\mu'(x+x_k)}{\mu(x+x_k)}\bigg]\widetilde{V}_i^k(x) \\ &+ \mu(x+x_k)[F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) \\ &- F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi))](x+x_k) \\ &+ [c(\tau_k) - c(\tau_0)]\bigg((v_i^0)' - \frac{\mu'}{\mu}v_i^0 + \mu\psi_i'\bigg)(x+x_k) \\ =& (\mu f_i^k - \mu f_i^0)(x+x_k). \end{aligned} \tag{3.14}$$

We pass to the limit as $k \to \infty$ in this system. Remark that (3.12) and (3.7) imply that there exists $\widetilde{V}^0 = (\widetilde{V}_1^0, \ldots, \widetilde{V}_n^0) \in E$ such that $\widetilde{V}_i^k \to \widetilde{V}_i^0$ as $k \to \infty$ in $C^{2+\alpha}(I)$, for all bounded intervals I of x $(1 \leq i \leq n)$. Observe that

$$\frac{1}{\mu(x+x_k)} \to 0, \quad \frac{\mu'(x+x_k)}{\mu(x+x_k)} \to 0, \quad \frac{\mu''(x+x_k)}{\mu(x+x_k)} \to 0, \quad k \to \infty,$$

while condition $f^k \to f^0$ in E_μ^0 leads to $(\mu f_i^k - \mu f_i^0)(x+x_k) \to 0$. Inequality (3.7) implies a similar estimate for v^0, so $v^0(x+x_k)$ and $(v^0)'(x+x_k)$ are bounded in E. We also have $\psi_i'(x+x_k) = 0$ for $x + x_k > 1$ and for $x + x_k < -1$ and

$$\begin{aligned} \mu(x+x_k)[F_i^{\tau_k}(u^k &+ \psi, J_i(u^k+\psi)) \\ &- F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi))](x+x_k) \\ =& \mu(x+x_k)[F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) \\ &- F_i^{\tau_0}(u^k+\psi, J_i(u^k+\psi))](x+x_k) \\ &+ \mu(x+x_k)[F_i^{\tau_0}(u^k+\psi, J_i(u^k+\psi)) \\ &- F_i^{\tau_0}(u^0+\psi, J_i(u^k+\psi))](x+x_k) \\ &+ \mu(x+x_k)[F_i^{\tau_0}(u^0+\psi, J_i(u^k+\psi)) \\ &- F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi))](x+x_k). \end{aligned} \tag{3.15}$$

By hypothesis (H2) we get

$$\mu(x+x_k)[F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) - F_i^{\tau_0}(u^k+\psi, J_i(u^k+\psi))](x+x_k) \to 0, \tag{3.16}$$

as $k \to \infty$ in $C^\alpha(I)$, on bounded intervals I of x.

By virtue of (3.12) the second term from (3.15) can be written as

$$\begin{aligned} H_i^k &= \sum_{j=1}^n \widetilde{V}_j^k(x) \frac{[F_i^{\tau_0}(u^k+\psi, J_i(u^k+\psi)) - F_i^{\tau_0}(u^0+\psi, J_i(u^k+\psi))](x+x_k)}{u_j^k(x+x_k) - u_j^0(x+x_k)} \\ &= \sum_{j=1}^n \widetilde{V}_j^k(x) \frac{\partial F_i^{\tau_0}}{\partial u_j}(s_j(u^k+\psi) + (1-s_j)(u^0+\psi), J_i(u^k+\psi))(x+x_k) \\ &= \sum_{j=1}^n \widetilde{V}_j^k(x) \frac{\partial F_i^{\tau_0}}{\partial u_j}(s_j u^k + (1-s_j)u^0 + \psi, J_i(u^k+\psi))(x+x_k), \end{aligned}$$

for some $s_j \in [0,1]$, $1 \le j \le n$. The boundedness of u^k and u^0 leads to

$$|u^k(x+x_k)| \le M/\mu(x+x_k), \qquad |u^0(x+x_k)| \le M/\mu(x+x_k),$$

hence $(s_j u^k + (1-s_j)u^0 + \psi)(x+x_k) \to w^\pm$ and

$$J_{ih}(u_h^k+\psi)(x+x_k) = \int_{-\infty}^\infty \varphi_{ih}(x+x_k-y)u_h^k(y)dy + \int_{-\infty}^\infty \varphi_{ih}(x+x_k-y)\psi_h(y)\,dy,$$

for each $1 \le h \le n$. By the change of variable $x_k - y = -z$, it follows that

$$J_{ih}(u_h^k+\psi)(x+x_k) = \int_{-\infty}^\infty \varphi_{ih}(x-z)u_h^k(x_k+z)\,dz + \int_{-\infty}^\infty \varphi_{ih}(x-z)\psi_h(x_k+z)\,dz \to w^\pm,$$

uniformly on bounded intervals of x. Hypothesis (H1) leads to

$$H_i^k \to \sum_{j=1}^n \frac{\partial F_i^{\tau_0}}{\partial u_j}(w^\pm, w^\pm)\widetilde{V}_j^0, \quad \text{as } k \to \infty \text{ in } C^\alpha(I), \tag{3.17}$$

on every bounded interval I. We now estimate the third term from (3.15), which we denote by K_i^k. We have

$$\begin{aligned} K_i^k &= \mu(x+x_k) \sum_{j=1}^n (J_{ij}(u_j^k+\psi_j) - J_{ij}(u_j^0+\psi_j)) \\ &\quad \times \frac{F_i^{\tau_0}(u^0+\psi, J_i(u^k+\psi)) - F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi))}{J_{ij}(u_j^k+\psi_j) - J_{ij}(u_j^0+\psi_j)}(x+x_k) \\ &= \sum_{j=1}^n I_{ij}^k(x) \cdot \frac{\partial F_i^{\tau_0}}{\partial U_j}(u^0+\psi, s_j J_i(u^k+\psi) + (1-s_j)J_i(u^0+\psi))(x+x_k), \end{aligned}$$

for some $s_j \in [0,1]$, where $I_{ij}^k(x) = \mu(x+x_k)(J_{ij}(u_j^k+\psi_j) - J_{ij}(u_j^0+\psi_j))$. For $\mu(x) = 1+x^2$, $x \in \mathbb{R}$, with the aid of (3.8) we arrive at

$$I_{ij}^k(x) = \int_{-\infty}^{\infty} \frac{\mu(x+x_k)}{\mu(z+x_k)} \varphi_{ij}(x-z)\widetilde{V}_j^k(z)\,dz \to \int_{-\infty}^{\infty} \varphi_{ij}(x-z)\widetilde{V}_j^0(z)\,dz = J_{ij}(\widetilde{V}_j^0),$$

in $C^\alpha(I)$ for arbitrary bounded I.

As above, since $J_i(u^k)(x+x_k) \to 0$, $J_i(u^0)(x+x_k) \to 0$, $J_i(\psi)(x+x_k) \to w^\pm$ uniformly on bounded intervals of x, we deduce that

$$K_i^k \to \sum_{j=1}^n \frac{\partial F_i^{\tau_0}}{\partial U_j}(w^\pm, w^\pm) J_{ij}(\widetilde{V}_j^0), \quad \text{as } k \to \infty \tag{3.18}$$

in $C^\alpha(I)$ for any arbitrary bounded interval I.

Using (3.15)–(3.18) and (H2), we may pass to the limit in (3.14). One obtains

$$(\widetilde{V}_i^0)'' + c(\tau_0)(\widetilde{V}_i^0)' + \sum_{j=1}^n \frac{\partial F_i^{\tau_0}}{\partial u_j}(w^\pm, w^\pm)\widetilde{V}_j^0 + \sum_{j=1}^n \frac{\partial F_i^{\tau_0}}{\partial U_j}(w^\pm, w^\pm) J_{ij}(\widetilde{V}_j^0) = 0,$$

which contradicts (H3). Therefore we have proved that $V_i^k \to 0$ in $C(\mathbb{R})$ for all i, with $1 \le i \le n$.

Now we have to show that $V_i^k \to 0$ in $C^{2+\alpha}(\mathbb{R})$. To this end, we write equation (3.11) in the form $S(V_i^k) = h_i^k$, where

$$S(V_i^k) = (V_i^k)'' + \left[-2\frac{\mu'}{\mu} + c(\tau_k)\right](V_i^k)' + \left[-\frac{\mu''}{\mu} + 2(\frac{\mu'}{\mu})^2 - c(\tau_k)\frac{\mu'}{\mu}\right] V_i^k$$

and

$$\begin{aligned} h_i^k = \mu f_i^k - \mu f_i^0 - \mu[F_i^{\tau_k}(u^k+\psi, J_i(u^k+\psi)) - F_i^{\tau_0}(u^0+\psi, J_i(u^0+\psi))] \\ - [c(\tau_k) - c(\tau_0)]\left((v_i^0)' - \frac{\mu'}{\mu} v_i^0 + \mu\psi_i'\right). \end{aligned}$$

Applying Lemma 2.1 for the linear operator S, we derive that

$$||V_i^k||_{C^{\alpha+2}(\mathbb{R})} \le C(||S(V_i^k)||_{C^\alpha(\mathbb{R})} + ||V_i^k||_{C(\mathbb{R})}).$$

Since $f^k \to f^0$ in E_μ^0, using Lemma 3.4 and hypothesis (H2), one derives that $S(V_i^k) = h_i^k \to 0$ in $C^\alpha(\mathbb{R})$. Since also $V_i^k \to 0$ in $C(\mathbb{R})$, we conclude that $u^k \to u^0$ in E_μ. The theorem is proved. □

4. Normal solvability and Fredholm property

Properness and topological degree for elliptic problems in unbounded domains may not hold in the usual Hölder spaces [9]. This is why we use weighted spaces. We will now prove the Fredholm property in weighted spaces. Together with properness (Section 3) it will allow us to define the topological degree in the next section. We will work with the more general operators A^τ and L^τ that are necessary for introducing the topological degree. First we recall an auxiliary result from [7] that we need in the sequel.

LEMMA 4.1. *If* $\mathcal{L}: E \to E^0$ *is a normally solvable operator with a finite dimensional kernel and the operator* $\mathcal{K}: E_\mu \to E^0$, $\mathcal{K}u = \mu\mathcal{L}u - \mathcal{L}(\mu u)$ *is compact, then* $\mathcal{L}: E_\mu \to E^0_\mu$ *is normally solvable with a finite dimensional kernel.*

THEOREM 4.2. *Under hypothesis* (H3), *the operator* $L^\tau: E_\mu = (C^{2+\alpha}(\mathbb{R}))^n_\mu \to E^0_\mu = (C^\alpha(\mathbb{R}))^n_\mu$ *from* (3.3) *is normally solvable and its kernel* $\ker L^\tau$ *has a finite dimension.*

PROOF. The assertion is true in Hölder spaces without weight μ according to Theorem 2.2 from [2]. To prove it in weighted spaces we make use of Lemma 4.1, i.e. we show that the operator $K: E_\mu \to E^0$, $Ku = \mu L^\tau u - L^\tau(\mu u)$ is compact. To this end, consider a sequence $\{u^k\} \subset E_\mu$, $u^k = (u^k_1, \dots, u^k_n)$ with $||u^k||_\mu \le M$, for all $k \ge 1$. We show that $\{Ku^k\}$ admits a convergent subsequence in E^0. Let $v^k = \mu u^k$, $v^k = (v^k_1, \dots, v^k_n)$ and $Ku = (K_1 u, \dots, K_n u)$ for all $u \in E_\mu$. The sequence v^k is bounded in E ($||v^k||_E = ||u^k||_\mu \le M$ for all $k \ge 1$), so like in the proof of Theorem 3.5 we can show that v^k admits a convergent subsequence locally in C^2, say $v^k \to v^0$. By a diagonalization process we can extend v^0 to $\mathbb{R}$ such that $v^0 \in E$. In addition $||v^0||_E \le M$. If $v^0 = (v^0_1, \dots, v^0_n)$, we denote $u^0 = (u^0_1, \dots, u^0_n)$ and $\zeta^k = (\zeta^k_1, \dots, \zeta^k_n)$ the sequences given by $u^0_i = v^0_i/\mu$ and $\zeta^k_i = v^k_i - v^0_i = \mu(u^k_i - u^0_i)$, $1 \le i \le n$. In such a way we get $||\zeta^k||_E \le 2M$, $\zeta^k_i \to 0$ in $C^{2+\alpha}(I)$ for every bounded interval I and

$$||Ku^k - Ku^0||_{E^0} = \sum_{i=1}^n ||K_i u^k - K_i u^0||_{C^\alpha(\mathbb{R})} = \sum_{i=1}^n \left\| K_i\left(\frac{\zeta^k}{\mu}\right)\right\|_{C^\alpha(\mathbb{R})}.$$

We compute the term

$$\begin{aligned}(4.1)\quad K_i\left(\frac{\zeta^k}{\mu}\right) &= \mu L^\tau_i\left(\frac{\zeta^k}{\mu}\right) - L^\tau_i(\zeta^k)\\ &= \left(-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau)\frac{\mu'}{\mu}\right)\zeta^k_i - 2\,\frac{\mu'}{\mu}(\zeta^k_i)'\\ &\quad + \sum_{j=1}^n \frac{\partial F^\tau_i}{\partial U_j}(\widetilde{u} + \psi, \varphi_{i1} * (\widetilde{u}_1 + \psi_1), \dots, \varphi_{in} * (\widetilde{u}_n + \psi_n))\\ &\quad \times \left[\mu(\varphi_{ij} * \frac{\zeta^k_j}{\mu}) - (\varphi_{ij} * \zeta^k_j)\right].\end{aligned}$$

But

$$\begin{aligned}\mu\left(\varphi_{ij} * \frac{\zeta^k_j}{\mu}\right) - (\varphi_{ij} * \zeta^k_j) &= \int_{-\infty}^\infty \varphi_{ij}(x-y)\zeta^k_j(y)\left[\frac{\mu(x)}{\mu(y)} - 1\right] dy\\ &= \int_{-\infty}^\infty \varphi_{ij}(\xi)\zeta^k_j(x-\xi)\left[\frac{\mu(x)}{\mu(x-\xi)} - 1\right] d\xi.\end{aligned}$$

Since $\mu(x)/\mu(x-\xi)-1\le h(x)$, where $h(x)\to 0$ as $x\to\pm\infty$, $\zeta_j^k(x)$ is uniformly bounded and $\zeta_j^k(x)\to 0$ as $k\to\infty$ locally with respect to x, it follows that

$$\mu\biggl(\varphi_{ij}*\frac{\zeta_j^k}{\mu}\biggr)-(\varphi_{ij}*\zeta_j^k)\to 0$$

as $k\to\infty$, uniformly with respect to x on $\mathbb{R}$. Similarly, ζ_j^k, $(\zeta_j^k)'$ are uniformly bounded, $\zeta_j^k\to 0$, $(\zeta_j^k)'\to 0$ as $k\to\infty$ locally and $\mu''/\mu\to 0$, $\mu'/\mu\to 0$ as $x\to\pm\infty$, so the first two terms from (4.1) tend to zero uniformly with respect to $x\in\mathbb{R}$, as $k\to\infty$. This implies that $K_i(\zeta^k/\mu)\to 0$ as $k\to\infty$ in $C(\mathbb{R})$. Therefore, with the aid of the local convergence $\zeta_j^k\to 0$ in C^2, we conclude that $Ku^k\to Ku^0$ as $k\to\infty$ in $E^0=C^\alpha(\mathbb{R})$. □

We now prove the Fredholm property of the operator $L^\tau\colon E_\mu\to E_\mu^0$. Assume in addition that the condition below holds, where I is the identity operator. Then we will show that the co-dimension of the image is finite. Together with normal solvability and finite dimensional kernel, it determines the Fredholm property.

Condition NS(λ). *For each $\tau\in[0,1]$, the limiting equations $(L^\tau)^\pm u-\lambda u=0$ associated to the operator $L^\tau-\lambda I$ do not have nonzero solutions in E_μ, for any $\lambda\ge 0$.*

Lemma 4.3 ([2]). *Let the operators $L^0,L^1,L^s\colon E_\mu\to E_\mu^0$ defined by $L^0u=L^\tau u-\rho u$, $L^1u=u''-\rho u(\rho\ge 0)$, and $L^s=(1-s)L^0+sL^1$, $s\in[0,1]$. Then there exists $\rho\ge 0$ large enough such that the limiting equations $(L^s)^\pm u=0$ do not have nonzero solutions for any $s\in[0,1]$.*

Theorem 4.4. *If Condition* NS(λ) *is satisfied, then L^τ, regarded as an operator from E_μ to E_μ^0, has the Fredholm property and its index is zero.*

Proof. We define the operators $L^0u=L^\tau u-\lambda u$, $L^1u=u''-\lambda u$ and $L^s=(1-s)L^0+sL^1$, $s\in[0,1]$. Condition NS(λ) for L^τ is the same as Condition NS for $L^0=L^\tau-\lambda I$. Then, Theorem 4.2 ensures that L^0 is normally solvable with a finite dimensional kernel. We also have $\ker L^1=\{0\}$, $\operatorname{Im} L^1=E_\mu^0$. Consequently, L^1 is a Fredholm operator and its index is $\operatorname{ind} L^1=0$.

By Lemma 4.3 applied for L^s, there exists $\lambda\ge 0$ large enough such that Condition NS holds for all L^s, $s\in[0,1]$. Therefore, the operators L^s are normally solvable with a finite dimensional kernel (via Theorem 4.2). Thus the homotopy L^s gives a continuous deformation from the operator L^0 to the operator L^1, in the class of the normally solvable operators with finite dimensional kernels. Such deformation preserves the Fredholm property and the index. This implies that the index of L^s is zero for every $s\in[0,1]$. The claim follows from this assertion taking $s=0$ and $\lambda=0$. □

5. The construction of a topological degree

In this section we apply the construction of a topological degree for Fredholm and proper operators with the zero index to our integro-differential operators. First recall a general result concerning the existence of a topological degree [9].

Let E_1 and E_2 be Banach spaces, $E_1 \subseteq E_2$ algebraically and topologically and let $G \subset E_1$ be an open bounded set. Denote by I is the identity operator and by Λ a class of bounded linear operators $L\colon E_1 \to E_2$ satisfying the following conditions:

(a) The operator $L - \lambda I\colon E_1 \to E_2$ is Fredholm for all $\lambda \geq 0$,

(b) For every operator $L \in \Lambda$, there is $\lambda_0 = \lambda_0(L)$ such that $L - \lambda I$ has a uniformly bounded inverse for all $\lambda > \lambda_0$.

Remark 5.1. Condition (b) can be weakened as follows. Let E_1' and E_2' be two Banach spaces such that $E_i \subset E_i'$, $i = 1, 2$ where the inclusion is understood in the algebraic and topological sense. In the case of the Hölder space $E_1 = C^{k+\alpha}(\mathbb{R})$ with a nonnegative integer k, we can take $E_1' = C^k(\mathbb{R})$. We can also take E_1' an integral spaces of the form $W_\infty^{k,p}(\mathbb{R})$ as in [8]. Instead of hypothesis (b) above, we can impose the following condition (see [9]):

(b′) For every operator $L\colon E_1' \to E_2'$, there is $\lambda_0 = \lambda_0(L)$ such that $L - \lambda I$ has a uniformly bounded inverse for all $\lambda > \lambda_0$.

Denote by $\mathcal{F}$ the class

$$\mathcal{F} = \{B \in C^1(G, E_2),\ B \text{ proper},\ B'(x) \in \Lambda,\ \text{for all } x \in G\},$$

where $B'(x)$ is the Fréchet derivative of the operator B.

Finally, one introduces the class $\mathcal{H}$ of homotopies given by

$$\mathcal{H} = \{B(x,\tau) \in C^1(G \times [0,1], E_2),\ B \text{ proper},\ B(\,\cdot\,,\tau) \in \mathcal{F},\ \text{for all } \tau \in [0,1]\}.$$

Here the properness of B is understood in both variables $x \in G$ and $\tau \in [0,1]$. Combining Section 4.2 from [7] with Remark 5.1 above, we can formulate the following theorem.

Theorem 5.2. *If hypotheses* (a) *and* (b′) *are satisfied, then for every operator $B \in \mathcal{H}$ and every open set D, with $\overline{D} \subset G$, there exists a topological degree $\gamma(B, D)$.*

Now, we apply the above result for our integro-differential operator A from equation (1.5). Consider the weighted spaces E_μ (instead of E_1) and E_μ^0 (instead of E_2), with $\mu(x) = 1 + x^2$, $x \in \mathbb{R}$. Suppose in addition that the following assumptions are satisfied:

(H4) Functions $F_i(u_1, \dots, u_n, U_1, \dots, U_n)$ and their derivatives with respect to u_j and U_j are Lipschitz continuous.

(H5) The limiting equations

$$u'' + cu' + \frac{\partial F}{\partial u}(w^{\pm}, w^{\pm})u + \frac{\partial F}{\partial U}(w^{\pm}, w^{\pm})(\Phi * u) - \lambda u = 0,$$

or in detail,

$$(u_i)'' + c(u_i)' + \sum_{j=1}^{n} \frac{\partial F_i}{\partial u_j}(w^{\pm}, w^{\pm})u_j + \sum_{j=1}^{n} \frac{\partial F_i}{\partial U_j}(w^{\pm}, w^{\pm})(\varphi_{ij} * u_j) - \lambda u_i = 0,$$

for $1 \leq i \leq n$, do not have nonzero solutions in E for all $\lambda \geq 0$.

Under these hypotheses, Theorem 3.5 implies that operator A is proper, while Theorem 4.4 assures that its Fréchet derivative $A' = L$ from (1.6) is a Fredholm operator with the index zero.

Consider $\mathcal{F}$ the class of operators A defined through (1.5) such that (H4)–(H5) are satisfied. Consider also the class $\mathcal{H}$ of homotopies $A^{\tau} : E_{\mu} \to E_{\mu}^{0}$, $\tau \in [0,1]$, of the form (3.2) satisfying conditions (H1)–(H2) and

(H6) For every $\tau \in [0,1]$, the equations

$$u'' + c(\tau)u' + \frac{\partial F^{\tau}}{\partial u}(w^{\pm}, w^{\pm})u + \frac{\partial F^{\tau}}{\partial U}(w^{\pm}, w^{\pm})(\Phi * u) - \lambda u = 0$$

do not have nonzero solutions in E for all $\lambda \geq 0$.

By Theorems 3.5 and 4.4 it follows that the operators $A^{\tau}(u)$ are Fréchet differentiable, proper with respect to (u, τ) and their Fréchet derivatives L^{τ} verify condition (a) above. Condition (b$'$) follows like in [4]. By Theorem 5.2 for the class of operators $\mathcal{F}$ and the class of homotopies $\mathcal{H}$, we obtain the following result.

Theorem 5.3. *Suppose that functions F^{τ} and $c(\tau)$ satisfy conditions* (H1)–(H2) *and* (H4)–(H6). *Then a topological degree can be constructed for the class $\mathcal{F}$ of operators and the class $\mathcal{H}$ of homotopies.*

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