Topological Methods in Nonlinear Analysis
Volume 43, No. 1, 2014, 23–52

GENERALIZATIONS OF KRASNOSEL'SKIĬ'S FIXED POINT THEOREM IN CONES AND APPLICATIONS

Sorin Budişan

Abstract. We give some generalizations of Krasnosel'skiĭ's fixed point theorem in cones, replacing norms with functionals. We will apply these theorems to obtain at least one positive solution for the boundary value problems for second-order differential equations. Two positive solution results are also obtained.

1. Introduction

Richard Leggett and Lynn Williams [9] obtained some of the importants results of the type of Krasnosel'skiĭ's fixed point theorem in cones. To present a result from [9] we need to introduce two definitions:

Definition 1.1 ([9]). Let $(E, \|\cdot\|)$ be real Banach space. A closed convex set $K \subset E$ is called a (positive) cone if the following conditions are satisfied:

(a) if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$;
(b) if $x \in K$ and if $-x \in K$, then $x = 0$.

2010 *Mathematics Subject Classification.* 7H10, 34B18.
Key words and phrases. Cone, fixed point, boundary values problem.
This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

DEFINITION 1.2 ([9]). Let K be a cone of the Banach space $(E, \|\cdot\|)$. α is a concave positive functional on K if $\alpha: K \to [0, \infty)$ satisfies

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y), \quad 0 \leq \lambda \leq 1.$$

We observe that if α is a concave positive functional on a cone K, a set of the form

$$S(\alpha, a, b) = \{x \in K : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$$

is closed, bounded and convex in K. Let $K_c := \{x \in K : \|x\| \leq c\}$, where $0 < c < \infty$.

THEOREM 1.3 ([9]). *Suppose $A: K_c \to K$ is completely continuous and suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|(x \in K)$ and numbers $b > a > 0$ $(b \leq c)$ satisfying the following conditions*:

(a) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \phi$ *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b)$;

(b) $Ax \in K_c$ *if* $x \in S(\alpha, a, c)$;

(c) $\alpha(Ax) > a$ *for all* $x \in S(\alpha, a, c)$ *with* $\|Ax\| > b$.

Then A has a fixed point x in $S(\alpha, a, c)$.

Other authors give generalizations of Krasnosel'skiĭ's fixed poit theorem in cones. For example, in [11], is given the following result:

THEOREM 1.4 ([11]). *Let $(X, |\cdot|)$ be a normed linear space, $K_1, K_2 \subset X$ two cones; $K := K_1 \times K_2$; $r, R \in \mathbb{R}^2_+$ with $0 < r < R$ ($r = (r_1, r_2)$, $R = (R_1, R_2)$ and $r < R$ if and only if $r_i < R_i$ for $i \in \{1, 2\}$), and $N: K_{r,R} \to K$, $N = (N_1, N_2)$ a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$, where $u = (u_1, u_2)$:*

(a) $N_i(u) \not< u_i$ *if* $|u_i| = r_i$, *and* $N_i(u) \not> u_i$ *if* $|u_i| = R_i$;

(b) $N_i(u) \not> u_i$ *if* $|u_i| = r_i$, *and* $N_i(u) \not< u_i$ *if* $|u_i| = R_i$.

Then N has a fixed point u in K with $r_i \leq |u_i| \leq R_i$ for $i \in \{1, 2\}$.

In [10] the following theorems are proved($(E, |\cdot|)$ is a normed linear space and $\|\cdot\|$ is an other norm on E. Also $C \subset E$ is a nonempty convex set, not necessarily closed, with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda > 0$. Suppose there exist constants $c_1, c_2 > 0$ such that $c_1|x| \leq \|x\| \leq c_2|x|$ for all $x \in \mathbb{C}$):

THEOREM 1.5 ([10]). *Assume $0 < c_2\rho < R$, $\|\cdot\|$ is increasing with respect to $\mathbb{C}$, that is $\|x+y\| > \|x\|$ for all $x, y \in \mathbb{C}$, and the map $N: \{x \in \mathbb{C} : \|x\| \leq R\} \to \mathbb{C}$ is compact. In addition, assume that the following conditions are satisfied*:

(a) $|N(x)| < |x|$ *for all* $x \in \mathbb{C}$ *with* $|x| = \rho$,

(b) $\|N(x)\| \geq \|x\|$ *for all* $x \in \mathbb{C}$ *with* $\|x\| = R$.

Then N has at least two fixed points $x_1, x_2 \in \mathbb{C}$ with $|x_1| < \rho \leq |x_2|$ and $\|x_2\| \leq R$.

THEOREM 1.6 ([10]). *Assume* $0 < 1/c_1\rho < R$, $|\cdot|$ *is increasing with respect to* $\mathbb{C}$, *and the map* $N\colon \{x \in \mathbb{C} : |x| \le R\} \to \mathbb{C}$ *is compact. In addition, assume that the following conditions are satisfied*:

(a) $\|N(x)\| < \|x\|$ *for all* $x \in \mathbb{C}$ *with* $\|x\| = \rho$,
(b) $|N(x)| \ge |x|$ *for all* $x \in \mathbb{C}$ *with* $|x| = R$.

Then N *has at least two fixed points* $x_1, x_2 \in \mathbb{C}$ *with* $\|x_1\| < \rho \le \|x_2\|$ *and* $|x_2| \le R$.

In [1] the authors give an existence result based on the following properties:

PROPERTY A1 ([1]). Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a contiuous functional $\beta\colon P \to [0,\infty)$ is said to satisfy Property A1 if one of the following conditions hold:

(a) β is convex, $\beta(0) = 0$, $\beta(x) \ne 0$ if $x \ne 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,
(b) β is sublinear, $\beta(0) = 0$, $\beta(x) \ne 0$ if $x \ne 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,
(c) β is concave and unbounded.

PROPERTY A2 ([1]). Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a contiuous functional $\beta\colon P \to [0,\infty)$ is said to satisfy Property A2 if one of the following conditions hold:

(a) β is convex, $\beta(0) = 0$, $\beta(x) \ne 0$ if $x \ne 0$,
(b) β is sublinear, $\beta(0) = 0$, $\beta(x) \ne 0$ if $x \ne 0$,
(c) $\beta(x + y) \ge \beta(x) + \beta(y)$ for all $x, y \in P$, $\beta(0) = 0$, $\beta(x) \ne 0$ if $x \ne 0$.

We present now the existence result.

THEOREM 1.7 ([1]). *Let* Ω_1 *and* Ω_2 *be two bounded open sets in a Banach space* E *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subseteq \Omega_2$ *and* P *is a cone in* E. *Suppose* $A\colon P \cap (\overline{\Omega}_2 - \Omega_1) \to P$ *is completely continuous,* α *and* ψ *are nonnegative continuous functionals on* P, *and one of the two conditions*:

(a) α *satisfies Property* A1 *with* $\alpha(Ax) \ge \alpha(x)$, *for all* $x \in P \cap \partial\Omega_1$, *and* ψ *satisfies Property* A2 *with* $\psi(Ax) \le \psi(x)$, *for all* $x \in P \cap \partial\Omega_2$; *or*
(b) α *satisfies Property* A2 *with* $\alpha(Ax) \le \alpha(x)$, *for all* $x \in P \cap \partial\Omega_1$, *and* ψ *satisfies Property* A1 *with* $\psi(Ax) \ge \psi(x)$, *for all* $x \in P \cap \partial\Omega_2$,

is satisfied. Then A *has at least one fixed point in* $P \cap (\overline{\Omega}_2 - \Omega_1)$.

Theorem 1.7 provide a generalization of some compression-expansion arguments that have utilized the norm or functionals in obtaining the existence of at least one fixed point.

Avery, Henderson and O'Regan proved the result above using the fixed point index.

In our paper we obtain the existence results on a direct way, imposing the appropriate conditions on the functionals that appear. Moreover, we give abstract results that extend and complement previous results from the literature, such as Theorem 1.5 from [1] (At least one of our functionals does not satisfies neither Property A1, nor Property A2).

Also, in [3] the author gives the following result:

THEOREM 1.8 ([3]). *Let* $(X, |\cdot|)$ *be a normed linear space,* $K \subset X$ *a positive cone, "*$\preceq$*" the order relation induced by* K *and "*$\prec$*" the strict order relation induced by* K*. Let be* $K_{r,R} = \{x \in K : r \leq |x| \leq R\}$*, where* $r, R \in \mathbb{R}_+$*,* $0 < r < R$*. We assume that* $N: K_{r,R} \to K$ *is a completely continuous operator and* $\varphi, \psi: K \to \mathbb{R}_+$*. Also, assume that the following conditions are satisfied:*

$$\text{(a)} \quad \begin{cases} \varphi(0) = 0, & \\ \textit{there exists } h \in K \setminus \{0\} \textit{ such that } \varphi(\lambda h) > 0 & \\ & \textit{for all } \lambda \in (0, 1], \\ \varphi(x + y) \geq \varphi(x) + \varphi(y) & \textit{for all } x, y \in K \setminus \{0\}, \end{cases}$$

(b) $\psi(\alpha x) > \psi(x)$ *for all* $\alpha > 1$ *and for all* $x \in K$ *with* $|x| = R$*.*

$$\text{(c)} \quad \begin{cases} \varphi(x) \leq \varphi(Nx) & \textit{if } |x| = r, \\ \psi(x) \geq \psi(Nx) & \textit{if } |x| = R. \end{cases}$$

Then N *has a fixed point in* $K_{r,R}$*.*

For other generalizations of Krasnosel'skiĭ's fixed point theorem in cones see the papers [2] , [8] and [12].

For applications of Krasnosel'skiĭ's fixed point theorem in cones, the reader may see the papers [4]–[6] and [13].

In our paper we are concerned to use conditions of type

$$\varphi(u) \geq \varphi(Nu) \quad \text{if } \varphi(u) = r,$$

instead of condition $|u| \geq |Nu| \quad \text{if } |u| = r$, that is assumed in Krasnosel'skiĭ's fixed point theorem.

2. The main results

In this section we will assume throughout that $(X, |\cdot|)$ is a normed linear space, $K \subset X$ is a positive cone, "$\preceq$" is the order relation induced by K , "$\prec$" the strict order relation induced by K and $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+^* := (0, \infty)$.

Theorem 2.1. *Let* $K_{r,R} = \{x \in K : r \le \varphi(x) \le R\}$ *be nonempty, where* $r, R \in \mathbb{R}_+^*$, $r < R$ *and* $\varphi, \psi\colon K \to \mathbb{R}_+$ *continuous functionals. Assume that* $N\colon K_{r,R} \to K$ *is a completely continuous operator with* $N(K_{r,R})$ *relatively compact,* $\varphi(0) = 0$ *and* φ *is strictly increasing with respect to order relation induced by* K *(in the sense that* $x, y \in K$ *with* $x < y$ *implies* $\varphi(x) < \varphi(y)$*). Also, let* $\varphi^{-1}(0) := \{x \in K : \varphi(x) = 0\}$ *and assume that the following conditions are satisfied*:

(a) $\psi \ge \varphi$ *on* K,
(b) $\varphi(\alpha x) = \alpha\varphi(x)$, *for all* $x \in K$ *and for all* $\alpha \in (0,\infty)$,
(c) $\psi(\alpha x) = \alpha\psi(x)$ *for all* $x \in K$ *and for all* $\alpha \in (0,1)$,
(d) $\psi(\alpha x) \ge \alpha\psi(x)$ *for all* $x \in K$ *with* $\psi(x) \ge R$ *and for all* $\alpha \in (1,\infty)$,
(e) $N(\varphi^{-1}(r))$ *bounded, where* $\varphi^{-1}(r) := \{x \in K : \varphi(x) = r\}$
(f) $\varphi(x) \le \varphi(Nx)$ *if* $\varphi(x) = r$ *and* $\psi(x) \ge \psi(Nx)$ *if* $\psi(x) \ge R$.

Then N *has a fixed point* u^* *in* $K_{r,R}$.

Proof. Let $h \in K \setminus \varphi^{-1}(0)$ and $N_1\colon K \to K$ be defined as

$$N_1(u) = \begin{cases} h & \text{if } \varphi(u) = 0, \\ \left(1 - \dfrac{\varphi(u)}{r}\right)h + \dfrac{\varphi(u)}{r}N\left(\dfrac{r}{\varphi(u)}\,u\right) & \text{if } 0 < \varphi(u) < r, \\ N(u) & \text{if } r \le \varphi(u) \le R, \\ N\left(\dfrac{R}{\varphi(u)}\,u\right) & \text{if } \varphi(u) > R. \end{cases}$$

Since N is completely continuous and φ is continuous, (a)–(c), (e) and (f) imply that N_1 is completely continuous ((b) and (e) imply that

$$\lim_{\varphi(u)\to 0} N_1(u) = \lim_{\varphi(u)\to 0}\left(1 - \frac{\varphi(u)}{r}\right)h + \lim_{\varphi(u)\to 0}\frac{\varphi(u)}{r}N\left(\frac{r}{\varphi(u)}\,u\right) = h = N_1(v),$$

where $\varphi(u) \to 0$ if and only if $u \to v \in \varphi^{-1}(0)$ since φ is continuous). We have that $N_1(K) \subset \operatorname{conv}(\{h\} \cup N(K_{r,R}))$. Since $N(K_{r,R})$ is relatively compact it follows that $\operatorname{conv}(\{h\} \cup N(K_{r,R}))$ is relatively compact by Mazur's lemma. So $N_1(K)$ is a relatively compact set. From our hypothesis we have that K is a convex and closed set and since $N_1(K) \subset K$ is relatively compact, from Schauder' s theorem it follows that there exists $u^* \in K$ with $N_1(u^*) = u^*$. We have to consider three cases.

Case 1. Suppose that $\varphi(u^*) = 0$. We have $u^* = N_1(u^*) = h$, so $\varphi(h) = 0$, $h \in \varphi^{-1}(0)$, a contradiction with $h \in K \setminus \varphi^{-1}(0)$.

Case 2. Suppose that $0 < \varphi(u^*) < r$. We obtain

$$\left(1 - \frac{\varphi(u^*)}{r}\right)h + \frac{\varphi(u^*)}{r} N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = u^*,$$
$$\left(\frac{r}{\varphi(u^*)} - 1\right)h + N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = \frac{r}{\varphi(u^*)}\, u^*.$$

Let $\lambda := r/\varphi(u^*) - 1$, $u_0 := (r/\varphi(u^*))u^*$. Because $\varphi(u^*) < r$ we have that $r/\varphi(u^*) > 1$, so $\lambda > 0$. Also, from (b) we have

$$\varphi\left(\frac{r}{\varphi(u^*)} u^*\right) = \frac{r}{\varphi(u^*)}\, \varphi(u^*) = r,$$

so $\varphi(u_0) = r$. We obtain $\lambda h + N(u_0) = u_0$ so, $\varphi(u_0) = \varphi(N(u_0) + \lambda h)$.

Because $\lambda > 0$ we have $\lambda h > 0$ and $N(u_0) + \lambda h > N(u_0)$. Since φ is strictly increasing we obtain $\varphi(N(u_0) + \lambda h) > \varphi(N(u_0))$, so $\varphi(u_0) > \varphi(N(u_0))$ for $\varphi(u_0) = r$, a contradiction with (f).

Case 3. Suppose that $\varphi(u^*) > R$. It follows that

$$N\left(\frac{R}{\varphi(u^*)}\, u^*\right) = u^*.$$

So we obtain that $\beta := \varphi(u^*)/R > 1$. We have

$$N\left(\frac{R}{\varphi(u^*)} u^*\right) = \left(\frac{R}{\varphi(u^*)}\, u^*\right)\frac{\varphi(u^*)}{R}.$$

Let

$$u_1 := \frac{R}{\varphi(u^*)} u^* = \frac{1}{\beta} u^* \quad \text{with } \frac{1}{\beta} < 1.$$

So

$$N(u_1) = \beta u_1 \tag{2.1}$$

and from (a), (c) we obtain

$$\psi\left(\frac{R}{\varphi(u^*)}\, u^*\right) = \frac{R}{\varphi(u^*)}\, \psi(u^*) \geq R,$$

so $\psi(u_1) \geq R$. From (d) we have

$$\psi(\beta u_1) \geq \beta\psi(u_1) > \psi(u_1). \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$\psi(N(u_1)) > \psi(u_1) \quad \text{for } \psi(u_1) \geq R,$$

a contradiction with (f). So $u^* \in K$ with $r \leq \varphi(u^*) \leq R$. It follows $N(u^*) = u^*$ with $u^* \in K_{r,R}$. □

REMARK 2.2. (a) Let $|\cdot|$, $\|\cdot\|$ be two norms with $|y| < |x|$ if $y < x$ and $|x| \leq \|x\|$ for all $x \in K$. Then $\varphi(x) := |x|$ and $\psi(x) := \|x\|$ are examples of functionals that satisfy the hypothesis of Theorem 2.1.

(b) We note that the conditions on φ from Theorem 2.1 generalize the condition on $|\cdot|$ from Theorem 1.6.

Theorem 2.3. *Let $K_{r,R} = \{x \in K : r \leq \varphi(x) \leq R\}$ be nonempty, where $r, R \in \mathbb{R}_+^*$, $r < R$ and $\varphi, \psi\colon K \to \mathbb{R}_+$ continuous functionals. Assume that $N\colon K_{r,R} \to K$ is a completely continuous operator with $N(K_{r,R})$ relatively compact, $\varphi(0) = 0$. Let $\varphi^{-1}(0) := \{x \in K : \varphi(x) = 0\}$ and assume that there exists $h \in K \setminus \varphi^{-1}(0)$ such that $\varphi(\lambda h) > 0$ for all $\lambda > 0$. Also, assume that the following conditions are satisfied:*

(a) $\psi \geq \varphi$ *on* K,
(b) $\varphi(\alpha x) = \alpha\varphi(x)$ *for all* $x \in K$ *and for all* $\alpha \in (0, \infty)$,
(c) $\varphi(x + y) \geq \varphi(x) + \varphi(y)$, *for all* $x, y \in K$,
(d) $\psi(\alpha x) = \alpha\psi(x)$ *for all* $x \in K$ *and for all* $\alpha > 0$,
(e) $N(\varphi^{-1}(r))$ *bounded, where* $\varphi^{-1}(r) := \{x \in K : \varphi(x) = r\}$,
(f) $\varphi(x) \leq \varphi(Nx)$ *if* $\varphi(x) = r$ *and* $\psi(x) \geq \psi(Nx)$ *if* $\psi(x) \geq R$.

Then N has a fixed point $u^* \in K_{r,R}$.

Proof. Let $N_1\colon K \to K$,

$$N_1(u) = \begin{cases} h & \text{if } \varphi(u) = 0, \\ \left(1 - \dfrac{\varphi(u)}{r}\right)h + \dfrac{\varphi(u)}{r}\, N\left(\dfrac{r}{\varphi(u)}\, u\right) & \text{if } 0 < \varphi(u) < r, \\ N(u) & \text{if } r \leq \varphi(u) \leq R, \\ N\left(\dfrac{R}{\varphi(u)}\, u\right) & \text{if } \varphi(u) > R. \end{cases}$$

Since φ is continuous and N is completely continuous, from (a), (b), (e) we obtain that N_1 is completely continuous. Using the same arguments like in Theorem 2.1, by Schauder's theorem we have that there exists $u^* \in K$ so that $N_1(u^*) = u^*$.We have to analyze three cases.

Case 1. If $\varphi(u^*) = 0$, we have $u^* = N_1(u^*) = h$, so $\varphi(h) = 0$, $h \in \varphi^{-1}(0)$, a contradiction with $h \in K \setminus \varphi^{-1}(0)$.

Case 2. If $0 < \varphi(u^*) < r$. We have

$$\left(1 - \frac{\varphi(u^*)}{r}\right)h + \frac{\varphi(u^*)}{r}\, N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = u^*,$$
$$\left(\frac{r}{\varphi(u^*)} - 1\right)h + N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = \frac{r}{\varphi(u^*)}\, u^*.$$

Let

$$u_0 := \frac{r}{\varphi(u^*)}\, u^* \quad \text{and} \quad \lambda := \frac{r}{\varphi(u^*)} - 1 > 0.$$

From (b) we obtain

$$\varphi(u_0) = \varphi\left(\frac{r}{\varphi(u^*)}\, u^*\right) = \frac{r}{\varphi(u^*)}\, \varphi(u^*) = r.$$

Also $u_0 = \lambda h + N(u_0)$, so

$$\varphi(u_0) = \varphi(\lambda h + N(u_0)). \tag{2.3}$$

From (c) we obtain that

$$\varphi(\lambda h + N(u_0)) \geq \varphi(\lambda h) + \varphi(N(u_0)) > \varphi(N(u_0)). \tag{2.4}$$

From (2.3) and (2.4) we have that$\varphi(u_0) > \varphi(N(u_0))$ for $\varphi(u_0) = r$, a contradiction with (f).

Case 3. If $\varphi(u^*) > R$. We have that $\beta := \varphi(u^*)/R > 1$, and let $u_1 := (R/\varphi(u^*))u^*$. We have that

$$N\left(\frac{R}{\varphi(u^*)}\,u^*\right) = u^* = \left(\frac{R}{\varphi(u^*)}\,u^*\right)\frac{\varphi(u^*)}{R},$$

so $N(u_1) = \beta u_1$. From (d) and (a) we obtain

$$\psi(u_1) = \psi\left(\frac{1}{\beta}\,u^*\right) = \frac{1}{\beta}\,\psi(u^*) = \frac{R}{\varphi(u^*)}\,\psi(u^*) \geq R. \tag{2.5}$$

From (d) we have that

$$\psi(N(u_1)) = \psi(\beta u_1) = \beta\psi(u_1) > \psi(u_1) \tag{2.6}$$

From (2.5) and (2.6) we obtain a contradiction with (f).

It follows that $N(u^*) = u^*$ with $r \leq \varphi(u^*) \leq R$. $\square$

REMARK 2.4. If $X := C([0,1], \mathbb{R}_+)$, $I \subset [0,1]$, $I \neq [0,1]$, $\eta > 0$, $\|x\| := \max_{t\in[0,1]} x(t)$, $K := \{x \in X : x(t) \geq \eta\|x\| \text{ for all } t \in I\}$ then $\varphi(x) := \min_{t\in I} x(t)$ and $\psi(x) := \max_{t\in[0,1]} x(t)$ are functionals that satisfy Theorem 2.3.

THEOREM 2.5. *Let $\varphi, \psi\colon K \to \mathbb{R}_+$ be continuous functionals, $\varphi(0) = 0$, $\varphi^{-1}(0) := \{x \in K : \varphi(x) = 0\}$ and there exists $h \in K \setminus \varphi^{-1}(0)$ such that $\varphi(\lambda h) > 0$ for all $\lambda > 0$. Also assume that $\psi \geq \varphi$ on K and let $K_{r,R} := \{x \in K : r \leq \varphi(x) \leq \psi(x) \leq R\}$ be nonempty, where $r, R \in \mathbb{R}_+^*$, $r < R$. Assume that $N\colon K_{r,R} \to K$ is a completely continuous operator with $N(K_{r,R})$ relatively compact. Also, assume that the following conditions are satisfied*:

(a) *$\varphi(\alpha x) = \alpha\varphi(x)$ for all $x \in K$ and for all $\alpha \in (1, \infty)$,*
(b) *$\varphi(x + y) \geq \varphi(x) + \varphi(y)$, for all $x, y \in K$,*
(c) *$\psi(\alpha x) = \alpha\psi(x)$ for all $x \in K$ and for all $\alpha \geq 0$,*
(d) *$R\varphi(x) \geq r\psi(x)$ for all $x \in K$,*
(e) *$N(\varphi^{-1}(r))$ bounded, where $\varphi^{-1}(r) := \{x \in K : \varphi(x) = r\}$,*
(f) *$\varphi(x) \leq \varphi(Nx)$ if $\varphi(x) = r$ and $\psi(x) \geq \psi(Nx)$ if $\psi(x) = R$.*

Then N has a fixed point $u^* \in K_{r,R}$.

PROOF. Let $N_1: K \to K$,

$$N_1(u) = \begin{cases} h & \text{if } \varphi(u) = 0, \\ \left(1 - \dfrac{\varphi(u)}{r}\right)h + \dfrac{\varphi(u)}{r}\, N\left(\dfrac{r}{\varphi(u)}\, u\right) & \text{if } 0 < \varphi(u) < r \text{ and } \psi(u) \le R, \\ N(u) & \text{if } r \le \varphi(u) \le \psi(u) \le R, \\ N\left(\dfrac{R}{\psi(u)}\, u\right) & \text{if } \psi(u) > R. \end{cases}$$

N_1 is well defined. Indeed, if $0 < \varphi(u) < r$ and $\psi(u) \le R$, from (a) and (c), we have that

$$\varphi\left(\frac{r}{\varphi(u)}\, u\right) = \frac{r}{\varphi(u)}\, \varphi(u) = r \le \psi\left(\frac{r}{\varphi(u)}\, u\right) = \frac{r}{\varphi(u)}\, \psi(u) \le R$$

(from (d)), so $(r/\varphi(u))u \in K_{r,R}$ and $N((r/\varphi(u))u)$ is defined.

Since φ is continuous and N is completely continuous, from (a) and (e), we obtain that N_1 is completely continuous (we use the same argument like in Theorem 2.1). Using the same arguments like in Theorem 2.1, by Schauder's theorem we have that there exists $u^* \in K$ so that $N_1(u^*) = u^*$. We have to analyze three cases.

Case 1. If $\varphi(u^*) = 0$, we have $u^* = N_1(u^*) = h$, so $\varphi(h) = 0$, $h \in \varphi^{-1}(0)$, a contradiction with $h \in K \setminus \varphi^{-1}(0)$.

Case 2. Suppose that $0 < \varphi(u^*) < r$ and $\psi(u^*) \le R$. We obtain

$$\left(1 - \frac{\varphi(u^*)}{r}\right)h + \frac{\varphi(u^*)}{r} N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = u^*,$$

$$\left(\frac{r}{\varphi(u^*)} - 1\right)h + N\left(\frac{r}{\varphi(u^*)}\, u^*\right) = \frac{r}{\varphi(u^*)}\, u^*.$$

Let

$$u_0 := \frac{r}{\varphi(u^*)}\, u^* \quad \text{and} \quad \lambda := \frac{r}{\varphi(u^*)} - 1 > 0,$$

because $\varphi(u^*) < r$. From (a) we obtain

$$\varphi(u_0) = \varphi\left(\frac{r}{\varphi(u^*)}\, u^*\right) = \frac{r}{\varphi(u^*)}\, \varphi(u^*) = r,$$

since $r/\varphi(u^*) > 1$. Also, $u_0 = \lambda h + N(u_0)$, so using (b), we have that

$$\varphi(u_0) = \varphi(\lambda h + N(u_0)) \ge \varphi(N(u_0)) + \varphi(\lambda h) > \varphi(N(u_0)),$$

so

$$\varphi(u_0) > \varphi(N(u_0)) \quad \text{for } \varphi(u_0) = r,$$

a contradiction with (f).

Case 3. Suppose that $\psi(u^*) > R$. Then

$$N\left(\frac{R}{\psi(u^*)}\, u^*\right) = u^*$$

and let $\beta := \psi(u^*)/R > 1$. We have that

$$N\left(\frac{R}{\psi(u^*)}\,u^*\right) = \left(\frac{R}{\psi(u^*)}\,u^*\right)\frac{\psi(u^*)}{R}.$$

Let

$$u_1 := \frac{R}{\psi(u^*)}\,u^* = \frac{1}{\beta}\,u^*.$$

So

$$N(u_1) = \beta u_1 \tag{2.7}$$

and from (c) we obtain

$$\psi(u_1) = \psi\left(\frac{R}{\psi(u^*)}\,u^*\right) = \frac{R}{\psi(u^*)}\,\psi(u^*) = R. \tag{2.8}$$

From (c) we have that

$$\psi(\beta u_1) = \beta\psi(u_1) > \psi(u_1) \tag{2.9}$$

From (2.7)–(2.9) we obtain

$$\psi(N(u_1)) > \psi(u_1) \quad \text{for } \psi(u_1) = R,$$

a contradiction with (f). The conclusion follows. □

THEOREM 2.6. *Let r, R be with $0 < r < R$ and $\varphi, \psi\colon K \to \mathbb{R}_+$ continuous functionals, $\psi(0) = 0$, $\psi^{-1}(0) := \{x \in K : \psi(x) = 0\}$ and there exists $h \in K \setminus \psi^{-1}(0)$ such that $\psi(\lambda h) > 0$ for all $\lambda > 0$. Also, assume that $\psi \geq \varphi$, $R\varphi \geq r\psi$ on K and $c_1 := r/R$, $c_2 := R/r$. We define $K_{r,R} := \{x \in K : r \leq \psi(x),\ \varphi(x) \leq R\}$ and suppose that is a nonempty set. Assume that $N\colon K_{r,R} \to K$ is a completely continuous operator with $N(K_{r,R})$ relatively compact. Also, assume that the following conditions are satisfied:*

(a) *$\psi(\alpha x) = \alpha\psi(x)$ for all $x \in K$ and for all $\alpha \geq 0$,*
(b) *$\psi(x + y) \geq \psi(x) + \psi(y)$ for all $x, y \in K$,*
(c) *$\varphi(\alpha x) = \alpha\varphi(x)$ for all $x \in K$ and for all $\alpha \geq 0$,*
(d) *$N(\psi^{-1}(r))$ bounded, where $\psi^{-1}(r) := \{x \in K : \psi(x) = r\}$,*
(e) *φ and ψ are increasing on K,*
(f) *$c_1 N(x) \leq N(c_1 x)$ for all $x \in K_{r,R}$ with $\psi(x) = R$ and $c_2 N(x) \geq N(c_2 x)$ for all $x \in K_{r,R}$ with $\varphi(x) = r$,*
(g) *$\varphi(x) \geq \varphi(Nx)$ if $\varphi(x) = r$ and $\psi(x) \leq \psi(Nx)$ if $\psi(x) = R$.*

Then N has a fixed point $u_0 \in K_{r,R}$.

PROOF. Let $\varphi', \psi'\colon K \to \mathbb{R}_+$ be continuous functionals

$$\varphi'(u) := \varphi(c_2 u), \qquad \psi'(u) := \psi(c_1 u).$$

Firstly we prove that

$$\psi'(u) \le \varphi'(u), \tag{2.10}$$

which is, using (a), (c) and the definitions of ψ', φ',

$$\frac{r}{R}\psi(u) \le \frac{R}{r}\varphi(u), \qquad \left(\frac{r}{R}\right)^2 \psi(u) \le \varphi(u),$$

which is true since

$$\varphi(u) \ge \frac{r}{R}\psi(u) \ge \left(\frac{r}{R}\right)^2 \psi(u),$$

from our hypothesis.

Now define $K'_{r,R} := \{x \in K : c_1 r \le \psi'(x) \le \varphi'(x) \le c_2 R\}$. We have that $x_0 \in K'_{r,R}$ if and only if

$$c_1 r \le \psi'(x_0) \le \varphi'(x_0) \le c_2 R,$$

which may be written equivalently (basing on (2.10))

$$\begin{array}{ccc} c_1 r \le c_1\psi(x_0) & \text{and} & c_2\varphi(x_0) \le c_2 R, \\ r \le \psi(x_0) & \text{and} & \varphi(x_0) \le R, \end{array}$$

which is $x_0 \in K_{r,R}$. It follows that

$$K'_{r,R} = K_{r,R} \ne \Phi \tag{2.11}$$

We have that $\varphi(u) = r$ if and only if

$$\frac{\varphi'(u)}{c_2} = r, \qquad \frac{r}{R}\,\varphi'(u) = r, \qquad \varphi'(u) = R.$$

So, from (g) we deduce, if $\varphi'(u) = R$, that

$$\varphi(u) \ge \varphi(Nu), \qquad c_2\varphi(u) \ge c_2\varphi(Nu), \qquad \varphi(c_2 u) \ge \varphi(c_2 N u)$$

and from (e) and (f) we have that $\varphi(c_2 N u) \ge \varphi(N(c_2 u))$ and it follows that $\varphi(c_2 u) \ge \varphi(N(c_2 u))$.

Since $\varphi'(c_2 u) = c_2\varphi'(u) = c_2 R$ if and only if $\varphi'(u) = R$, we deduce that

$$\begin{aligned} \varphi(x_1) &\ge \varphi(N(x_1)) && \text{if } \varphi'(x_1) = c_2 R, \\ c_2\varphi(x_1) &\ge c_2\varphi(N(x_1)) && \text{if } \varphi'(x_1) = c_2 R, \\ \varphi(c_2 x_1) &\ge \varphi(c_2 N(x_1)) && \text{if } \varphi'(x_1) = c_2 R, \end{aligned}$$

which is

$$\varphi'(x_1) \ge \varphi'(N(x_1)) \quad \text{if } \varphi'(x_1) = c_2 R. \tag{2.12}$$

We have that $\psi(u) = R$ if and only if

$$\frac{\psi'(u)}{c_1} = R, \qquad \frac{R}{r}\,\psi'(u) = R, \qquad \psi'(u) = r.$$

So, from (g) we deduce, if $\psi'(u) = r$, that

$$\psi(u) \le \psi(Nu), \qquad c_1\psi(u) \le c_1\psi(Nu), \qquad \psi(c_1 u) \le \psi(c_1 Nu),$$

and from (e) and (f) we have that $\psi(c_1 Nu) \le \psi(N(c_1 u))$ it follows that

$$\psi(c_1 u) \le \psi(N(c_1 u)).$$

Since $\psi'(c_1 u) = c_1\psi'(u) = c_1 r$ if and only if $\psi'(u) = r$, we deduce that

$$\begin{aligned} \psi(x_2) &\le \psi(N(x_2)) && \text{if } \psi'(x_2) = c_1 r, \\ c_1\psi(x_2) &\le c_1\psi(N(x_2)) && \text{if } \psi'(x_2) = c_1 r, \\ \psi(c_1 x_2) &\le \psi(c_1 N(x_2)) && \text{if } \psi'(x_2) = c_1 r, \end{aligned}$$

which is

$$\psi'(x_2) \le \psi'(N(x_2)) \quad \text{if } \psi'(x_2) = c_1 r. \tag{2.13}$$

From (a)–(c) and the definitions of φ', ψ' we deduce the following relations:

$$\psi'(\alpha u) = \alpha\psi'(u) \qquad \text{for all } u \in K \text{ and all } \alpha \ge 0, \tag{2.14}$$

$$\psi'(x+y) \ge \psi'(x) + \psi'(y) \qquad \text{for all } x, y \in K, \tag{2.15}$$

$$\varphi'(\alpha u) = \alpha\varphi'(u) \qquad \text{for all } u \in K \text{ and all } \alpha \ge 0, \tag{2.16}$$

$$\psi'(0) = 0 \quad \text{and} \quad \psi'(\lambda h) > 0 \qquad \text{for all } \lambda > 0. \tag{2.17}$$

We have $\psi'(h) \neq 0$ if and only if $\psi(h) \neq 0$. Also, since $\psi(x) = r$ if and only if $\psi'(x) = c_1 r$, from (d) we deduce that

$$N((\psi')^{-1}(c_1 r)) \text{ is bounded, where } (\psi^{'})^{-1}(c_1 r) := \{x \in K : \psi'(x) = c_1 r\}, \tag{2.18}$$

Also, since $\psi \ge \varphi$, we have that $c_2 R c_1 \psi(x) \ge c_1 r c_2 \varphi(x)$ which is

$$(c_2 R)\psi'(x) \ge (c_1 r)\varphi'(x) \tag{2.19}$$

From the relations (2.12)–(2.19) we deduce that φ', ψ', $K'_{r,R}$ and N satisfy the hypothesis of Theorem 2.5 with φ' instead of ψ, ψ' instead of φ and $K'_{r,R}$ instead of $K_{r,R}$. So, from Theorem 2.5 it follows that N has a fixed poit $u_0 \in K'_{r,R}$ and from (2.11) we obtain that $u_0 \in K_{r,R}$ and the proof is completed. $\square$

THEOREM 2.7. *Let $\varphi, \psi, \delta\colon K \to \mathbb{R}_+$ be continuous functionals, $\delta(0) = 0$, $\delta^{-1}(0) := \{x \in K : \delta(x) = 0\}$ and there exists $h \in K\setminus\delta^{-1}(0)$ such that $\varphi(\lambda h) > 0$ for all $\lambda > 0$. Let $K_{r,R} := \{x \in K : r \le \delta(x) \le R\}$ be nonempty, where $r, R \in \mathbb{R}_+^*$, $r < R$. Assume that $N\colon K_{r,R} \to K$ is a completely continuous operator with $N(K_{r,R})$ relatively compact. Also, assume that the following conditions are satisfied:*

(a) $\delta(\alpha x) = \alpha\delta(x)$ *for all* $x \in K$ *and for all* $\alpha \ge 0$,
(b) $\varphi(x + y) \ge \varphi(x) + \varphi(y)$, *for all* $x, y \in K$,
(c) $\psi(\alpha x) \ge \alpha\psi(x)$ *for all* $x \in K$ *with* $\delta(x) = R$ *and for all* $\alpha \in (1, \infty)$,
(d) $\psi(x) > 0$ *for all* $x \in K$ *with* $\delta(x) = R$,
(f) $N(\delta^{-1}(r))$ *bounded, where* $\delta^{-1}(r) := \{x \in K : \delta(x) = r\}$,
(g) $\varphi(x) \le \varphi(Nx)$ *if* $\delta(x) = r$ *and* $\psi(x) \ge \psi(Nx)$ *if* $\delta(x) = R$.

Then N has a fixed point $u^* \in K_{r,R}$.

PROOF. Let $N_1\colon K \to K$, be

$$N_1(u) = \begin{cases} h & \text{if } \delta(u) = 0, \\ \left(1 - \dfrac{\delta(u)}{r}\right) h + \dfrac{\delta(u)}{r} N\left(\dfrac{r}{\delta(u)} u\right) & \text{if } 0 < \delta(u) < r, \\ N(u) & \text{if } r \le \delta(u) \le R, \\ N\left(\dfrac{R}{\delta(u)} u\right) & \text{if } \delta(u) > R. \end{cases}$$

Since $\delta\colon K \to \mathbb{R}_+$ is continuous functional, $\delta(0) = 0$, N is a completely continuous operator and from (a), (e) we have that N_1 is completely continuous. From our hypothesis, using the same arguments like in Theorem 2.1, by Schauder's theorem we have that there exists $u^* \in K$ such that $N_1(u^*) = u^*$.

We have to consider three cases.

Case 1. Suppose that $\delta(u^*) = 0$. We have $u^* = N_1(u^*) = h$, so $\delta(h) = 0$, $h \in \delta^{-1}(0)$, a contradiction with $h \in K \setminus \delta^{-1}(0)$.

Case 2. Suppose that $0 < \delta(u^*) < r$. Then

$$\left(1 - \frac{\delta(u^*)}{r}\right) h + \frac{\delta(u^*)}{r} N\left(\frac{r}{\delta(u^*)} u^*\right) = u^*,$$
$$\left(\frac{r}{\delta(u^*)} - 1\right) h + N\left(\frac{r}{\delta(u^*)} u^*\right) = \frac{r}{\delta(u^*)} u^*.$$

Let

$$u_0 := \frac{r}{\delta(u^*)} u^*, \qquad \lambda := \frac{r}{\delta(u^*)} - 1.$$

We obtain $\lambda > 0$ and

$$\lambda h + N(u_0) = u_0. \tag{2.20}$$

From (a) we have that

$$\delta(u_0) = \delta\left(\frac{r}{\delta(u^*)}\, u^*\right) = \frac{r}{\delta(u^*)}\, \delta(u^*) = r. \tag{2.21}$$

From (2.20) and (b) we obtain

$$\varphi(u_0) = \varphi(\lambda h + N(u_0)) \geq \varphi(\lambda h) + \varphi(N(u_0)) > \varphi(N(u_0)). \tag{2.22}$$

From (2.21) and (2.22) it follows $\varphi(u_0) > \varphi(N(u_0))$ if $\delta(u_0) = r$, a contradiction with (f).

Case 3. Suppose that $\delta(u^*) > R$. Then

$$N\left(\frac{R}{\delta(u^*)}\, u^*\right) = u^*.$$

Let

$$u_1 := \frac{R}{\delta(u^*)}\, u^*, \qquad \beta := \frac{\delta(u^*)}{R} > 1.$$

We obtain that

$$N\left(\frac{R}{\delta(u^*)}\, u^*\right) = \left(\frac{R}{\delta(u^*)}\, u^*\right) \frac{\delta(u^*)}{R},$$

so

$$N(u_1) = \beta u_1 \tag{2.23}$$

From (a) we have that

$$\delta(u_1) = \delta\left(\frac{R}{\delta(u^*)}\, u^*\right) = \frac{R}{\delta(u^*)}\, \delta(u^*) = R. \tag{2.24}$$

Using (c), (d), (2.23) and (2.24) we obtain that

$$\psi(N(u_1)) = \psi(\beta u_1) \geq \beta\psi(u_1) > \psi(u_1). \tag{2.25}$$

From (2.24) and (2.25) it follows $\psi(N(u_1)) > \psi(u_1)$ for $\delta(u_1) = R$, a contradiction with (f). So $u^* \in K_{r,R}$ with $N(u^*) = u^*$, the conclusion. $\square$

THEOREM 2.8. *Let $\varphi, \psi, \delta\colon K \to \mathbb{R}_+$ be continuous functionals, $\delta(0) = 0$, $\delta^{-1}(0) := \{x \in K : \delta(x) = 0\}$ and there exists $h \in K \setminus \delta^{-1}(0)$ such that $\psi(\lambda h) > 0$ for all $\lambda > 0$. Let $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$ be nonempty, where $r, R \in \mathbb{R}_+^*$, $r < R$. Assume that $N\colon K_{r,R} \to K$ is a completely continuous operator with $N(K_{r,R})$ relatively compact. Also, assume that the following conditions are satisfied*:

(a) *$\delta(\alpha x) = \alpha\delta(x)$ for all $x \in K$ and for all $\alpha \geq 0$,*

(b) *$\psi(x + y) \geq \psi(x) + \psi(y)$, for all $x, y \in K$,*

(c) *$\psi\left(\frac{R}{r}\, x\right) = \frac{R}{r}\, \psi(x)$ for all $x \in K$,*

(d) *$\varphi(x) > 0$ for all $x \in K$ with $\delta(x) = R$,*

$$\text{(e)}\quad \begin{cases} \varphi\left(\dfrac{r}{R}\,x\right) = \dfrac{r}{R}\,\varphi(x) & \textit{for all } x \in K, \\ \varphi(\alpha x) \geq \alpha\varphi(x) & \textit{for all } x \in K \textit{ with } \delta(x) = R \\ & \textit{and for all } \alpha \in (1,\infty), \end{cases}$$

(f) *$N(\delta^{-1}(R))$ bounded, where $\delta^{-1}(R) := \{x \in K : \delta(x) = R\}$,*

(g) *$\varphi(x) \geq \varphi(Nx)$ if $\delta(x) = r$ and $\psi(x) \leq \psi(Nx)$ if $\delta(x) = R$.*

Then N has a fixed point $u_0 \in K_{r,R}$.

Proof. Let be $N^*\colon K_{r,R} \to K$ be defined as

$$N^*(u) = \left(\frac{R+r}{\delta(u)} - 1\right)^{-1} N\left(\left(\frac{R+r}{\delta(u)} - 1\right)u\right).$$

Also, let be

$$\lambda := \left(\frac{R+r}{\delta(u)} - 1\right)^{-1}.$$

So we have that

$$N^*(u) = \lambda N\left(\frac{1}{\lambda}u\right).$$

If $\delta(u) = r$ we obtain $\lambda = r/R$ and using (a) we have that

$$\delta\left(\frac{1}{\lambda}\,u\right) = \frac{1}{\lambda}\,\delta(u) = R \tag{2.26}$$

From (2.26) and (g) we deduce that

$$\psi\left(\frac{1}{\lambda}\,u\right) \leq \psi\left(N\left(\frac{1}{\lambda}\,u\right)\right) = \psi\left(\frac{1}{\lambda}\,N^*(u)\right)$$

and from (c) we have that

$$\frac{1}{\lambda}\,\psi(u) \leq \frac{1}{\lambda}\,\psi(N^*(u)),$$

so

$$\psi(u) \leq \psi(N^*(u)) \quad \text{if } \delta(u) = r. \tag{2.27}$$

If $\delta(u) = R$ we have that $\lambda = R/r$ and using (a) we have that

$$\delta\left(\frac{1}{\lambda}\,u\right) = \frac{1}{\lambda}\,\delta(u) = r \tag{2.28}$$

From (2.28) and (g) we deduce that

$$\varphi\left(\frac{1}{\lambda}\,u\right) \geq \varphi\left(N\left(\frac{1}{\lambda}\,u\right)\right) = \varphi\left(\frac{1}{\lambda}\,N^*(u)\right)$$

and from (e) we have that

$$\frac{1}{\lambda}\,\varphi(u) \geq \frac{1}{\lambda}\,\varphi(N^*(u)),$$

so

$$\varphi(u) \geq \varphi(N^*(u)) \quad \text{if } \delta(u) = R. \tag{2.29}$$

Since $N^*(u) = \lambda N(u\lambda)$, from (2.27), (2.29) and (f) we deduce that N^* satisfies the conditions (e)–(f) from Theorem 2.7. Also, the conditions (a)–(e) imply that φ, ψ and δ satisfy the conditions the conditions (a)–(d) and (f) from Theorem 2.7, with φ and ψ changing their places. So, we may apply Theorem 2.7 and we obtain that there exists $u^* \in K_{r,R}$ such that $N^*(u^*) = u^*$, so

$$N\left(\frac{1}{\lambda}\, u^*\right) = \frac{1}{\lambda}\, u^*, \quad \text{where } \lambda = \left(\frac{R+r}{\delta(u^*)} - 1\right)^{-1}.$$

Let be

$$u_0 := \frac{1}{\lambda}\, u^* = \left(\frac{R+r}{\delta(u^*)} - 1\right) u^*.$$

We have that

$$N(u_0) = u_0 \tag{2.30}$$

and from (a) we obtain

$$\delta(u_0) = \delta\left(\left(\frac{R+r}{\delta(u^*)} - 1\right) u^*\right) = \left(\frac{R+r}{\delta(u^*)} - 1\right) \delta(u^*) = R + r - \delta(u^*)$$

and, since $r \leq \delta(u^*) \leq R$, we obtain that

$$r \leq \delta(u_0) \leq R. \tag{2.31}$$

From (2.30) and (2.31) the conclusion follows. □

Remark 2.9. Notice that function ψ from Theorem 2.5 does not satisfies neither Property A1, nor Property A2. Also, φ from Theorem 2.6 does not satisfies neither Property A1, nor Property A2. Thus our theorems clearly extend Theorem 1.7.

3. Applications

In this section we will give some applications of the theorems from previous section. We are concerned with the existence of at least one positive solution for the second order boundary value problem,

$$u''(t) + f(t, u(t)) = 0, \quad 0 \leq t \leq 1, \tag{3.1}$$

$$u(0) = u(1) = 0, \tag{3.2}$$

where $f \colon [0,1] \times \mathbb{R}_+ \to [0,\infty)$ is continuous. We look for solutions $u \in \mathbb{C}^2[0,1]$ of (3.1) with (3.2) which are both nonnegative and concave on $[0,1]$. We will apply theorems from previous section to a completely continuous operator whose kernel $G(t,s)$ is the Green's function for

$$u'' = 0, \tag{3.3}$$

satisfying (3.2). We have that

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \le t \le s \le 1, \\ s(1-t) & \text{if } 0 \le s \le t \le 1. \end{cases} \tag{3.4}$$

$G(t,s)$ has the following properties, which are necessary to our results:

$$\begin{cases} G(t,s) \le G(s,s), & \text{if } 0 \le t, s \le 1, \\ \dfrac{1}{4}\, G(s,s) \le G(t,s) & \text{if } 0 \le s \le 1,\ \dfrac{1}{4} \le t \le \dfrac{3}{4}, \\ \displaystyle\int_0^1 G(s,s)\,ds = \frac{1}{6}, & \displaystyle\int_{1/4}^{3/4} G(s,s)\,ds = \frac{11}{96}. \end{cases} \tag{3.5}$$

Let be

$$\begin{cases} I := \left[\dfrac{1}{4}, \dfrac{3}{4}\right], \\ K := \left\{u \in C[0,1] : u \ge 0 \text{ on } [0,1],\ u(t) \ge \dfrac{1}{4}\|u\| \text{ for all } t \in I\right\}, \\ \varphi, \psi \colon K \to \mathbb{R}_+, \quad \varphi(u) := \min_{t\in I} u(t), \quad \psi(u) := \max_{t\in[0,1]} u(t) = \|u\|. \end{cases} \tag{3.6}$$

This technique with min and max functionals is also used in [1] and many other papers.

It is obviously that K is a cone. Also, $u \in K$ is a solution of (3.1) with (3.2) if and only if

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\,ds. \tag{3.7}$$

Firstly, we will impose conditions on f which ensure the existence of at least one positive solution of (3.1) with (3.2) by applying Theorem 2.5.

Theorem 3.1. *Let be the positive numbers M, r and R such that $0 < r < R$, $r \le M/35$, $64r/11 \le R$ and define $f_1\colon [0,1] \times K \to [0,\infty)$, $f_1(s,x) = f(s, x(s))$. Also, suppose f_1 satisfies the following conditions*:

(a) $f_1(s,x) \le M$ *for all* $s \in [0,1]$, *for all* $x \in K$ *with* $\varphi(x) = r$,
(b) $f_1(s,x) \ge 384r/11$ *for all* $s \in I$, *for all* $x \in K$ *with* $\varphi(x) = r$,
(c) $f_1(s,x) \le 6R$ *for all* $s \in [0,1]$, *for all* $x \in K$ *with* $\psi(x) = R$.

Then (3.1) *with* (3.2) *has a solution* u^* *such that*

$$r \le \min_{t\in[1/4,3/4]} u^*(t) \le \max_{t\in[0,1]} u^*(t) \le R.$$

Proof. Let be $K_{r,R} := \{u \in K : r \leq \varphi(u) \leq \psi(u) \leq R\}$. Define the completely continuous operator N by

$$N(u)(t) = \int_0^1 G(t,s) f(s,u(s))\,ds.$$

Basing on (3.7), we seek a fixed poit of N and we show that N satisfies the conditions of the Theorem 2.5. For $u \in K$ we have, using (3.5), that, for $t \in I$,

$$\begin{aligned} N(u)(t) &= \int_0^1 G(t,s) f(s,u(s))\,ds \geq \frac{1}{4} \int_0^1 G(s,s) f(s,u(s))\,ds \\ &\geq \frac{1}{4} \int_0^1 \max_{t\in[0,1]} G(t,s) f(s,u(s))\,ds \\ &\geq \frac{1}{4} \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))\,ds = \frac{1}{4}\|N(u)\|, \end{aligned}$$

so $N(u) \in K$. It follows that $N\colon K_{r,R} \to K$ is well defined and completely continuous.

For $u \in K_{r,R}$ we have that $\psi(u) = \|u\| \leq R$, so $K_{r,R}$ is bounded, so it follows that $N(K_{r,R})$ is relatively compact, since N is completely continuous.

From (3.6) we deduce that (for $\|u\| > 0$)

$$\frac{\varphi}{\psi} \geq \frac{1}{4} \quad \text{on } K$$

and from our hypothesis we have that $r/R \leq 1/4$, so

$$\frac{\varphi}{\psi} \geq \frac{r}{R} \quad \text{on } K, \qquad R\varphi \geq r\psi \quad \text{on } K,$$

so φ and ψ satisfy the conditions (a)–(d) from Theorem 2.5. Also, the condition (a) of our hypothesis implies that N satisfies the conditions (e) of Theorem 2.5.

From (b) and (3.5) we deduce, for $\varphi(u) = r$, that

$$\begin{aligned} \varphi(Nu) &= \min_{t\in I} \int_0^1 G(t,s) f(s,u(s))\,ds \geq \int_0^1 \min_{t\in I} G(t,s) f(s,u(s))\,ds \\ &\geq \frac{1}{4} \int_{1/4}^{3/4} G(s,s) \frac{384}{11} r\,ds = \left(\frac{96}{11} \int_{1/4}^{3/4} G(s,s)\,ds \right) r = r = \varphi(u). \end{aligned} \tag{3.8}$$

From (c) and (3.5) we deduce, for $\psi(u) = R$, that

$$\begin{aligned} \psi(Nu) &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))\,ds \leq \int_0^1 \max_{t\in[0,1]} G(t,s) f(s,u(s))\,ds \\ &\leq \int_0^1 G(s,s) \max_{s\in[0,1]} f(s,u(s))\,ds = 6R \int_0^1 G(s,s)\,ds = R = \psi(u). \end{aligned} \tag{3.9}$$

Now (3.8) and (3.9) imply the condition (f) from Theorem 2.5. So we may apply Theorem 2.5 and the conclusion follows. □

Example 3.2. Function

$$f(s,x)=\frac{s+1}{s+2}\cdot\frac{x+2}{x+3}$$

is an example of function that satisfies Theorem 3.1. Indeed,

$$\left(\frac{x+p}{x+q}\right)'=\frac{q-p}{(x+q)^2}>0 \quad \text{for } q>p,$$

so $f(\,\cdot\,,\,\cdot\,)$ is increasing in both of its variables. It is obvious that $f(s,x)<1$, for all $s\in[0,1]$ and all $x\in[0,\infty)$, which implies (a).

For $\varphi(x)=\min_{s\in I}x(s)=r$, since f is increasing, we obtain that

$$f(s,x(s))\geq f(s,r) \quad \text{for all } s\in I. \tag{3.10}$$

We are looking for $r>0$ such that

$$f(s,r)\geq\frac{384}{11}r \quad \text{for all } s\in I. \tag{3.11}$$

Since

$$\frac{s+1}{s+2}\geq\frac{1}{2} \quad \text{for all } s\in I,$$

(3.11) is satisfied if we find $r>0$ such that

$$\frac{r+2}{r+3}\cdot\frac{1}{2}\geq\frac{385}{11}r=35r, \qquad 70r^2+209r-2\leq 0.$$

We have that $\Delta=209^2+560=44241$ and $r_{1,2}=(-209\pm\sqrt{\Delta})/140$, so $0<r<\max\{r_1,r_2\}$. For $r:=0.0095$, (3.11) is satisfied and from (3.10) we obtain condition (b) from Theorem 3.1.

For $\psi(x)=\max_{s\in[0,1]}x(s)=R$, since f is increasing, we obtain that

$$f(s,x(s))\leq f(s,R), \quad \text{for all } s\in[0,1]. \tag{3.12}$$

We are looking for $R>64r/11$ such that

$$f(s,R)\leq 6R, \quad \text{for all } s\in[0,1]. \tag{3.13}$$

and since $\frac{s+1}{s+2}\leq 1$ for all $s\in[0,1]$, (3.13) is satisfied if we find $R>\frac{64}{11}r$ such that

$$\frac{R+2}{R+3}\leq 6R, \qquad 0\leq 6R^2+17R-2.$$

We have that $\Delta'=17^2+48$ and $R_{1,2}=(-17\pm\sqrt{\Delta'})/12$, so $R\geq\max\{R_1,R_2\}$. For $R\geq 0.12$, (3.13) is satisfied and from (3.12) we obtain condition (c) from Theorem 3.1.

So f satisfies Theorem 3.1, for $r:=0.0095$ and $R:=0.12$.

Now we give an application of Theorem 2.7.

Theorem 3.3. *Let be the positive numbers M, r and R such that* $0 < r < R$, $0 < M$, $r \leq M/35$ *and define* $f_1 \colon [0,1] \times K \to [0,\infty), f_1(s,x) = f(s,x(s))$. *Suppose* f_1 *satisfies the following conditions*:

(a) $f_1(s,x) \leq M$ *for all* $s \in [0,1]$, *for all* $x \in K$ *with* $\delta(x) = r$,

(b) $f_1(s,x) \geq 384r/11$ *for all* $s \in I$, *for all* $x \in K$ *with* $\delta(x) = r$,

(c) $f_1(s,x) \leq 6R$ *for all* $s \in [0,1]$, *for all* $x \in K$ *with* $\delta(x) = R$, *where* $\delta(x) := (\varphi(x) + \psi(x))/2$.

Then (3.1) *with* (3.2) *has a solution* u^* *such that*

$$r \leq \frac{\min\limits_{t\in[1/4,3/4]} u^*(t) + \max\limits_{t\in[0,1]} u^*(t)}{2} \leq R \tag{3.14}$$

and

$$r \leq \max_{t\in[0,1]} u^*(t) \leq 4R \quad \textit{and} \quad \frac{r}{4} \leq \min_{t\in[1/4,3/4]} u^*(t) \leq R. \tag{3.15}$$

Proof. Let $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$. Define the completely continuous operator N by

$$N(u)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds.$$

From (a) it follows the condition (e) from Theorem 2.7. From φ, ψ and δ definitions we deduce the conditions (a)–(d) from Theorem 2.7.

If $x \in K_{r,R}$ we have that $\delta(x) \leq R$, so $\min\limits_{t\in[1/4,3/4]} x(t) \leq R$ (on the contrary, we obtain $\max\limits_{t\in[0,1]} x(t) \geq \min\limits_{t\in[1/4,3/4]} x(t) > R$ and $\min\limits_{t\in[1/4,3/4]} x(t) + \max\limits_{t\in[0,1]} x(t) > 2R$, $\delta(x) > R$, a contradiction). From (3.6) we obtain $\|x\|/4 \leq R$, so $\|x\| \leq 4R$ and $K_{r,R}$ is bounded. It follows that $N(K_{r,R})$ is relatively compact, since N is completely continuous.

If $\delta(u) = r$ we have $\varphi(u) \leq r$ (on the contrary, we obtain $\varphi(u) := \min\limits_{t\in[1/4,3/4]} u(t) > r$, so $\max\limits_{t\in[0,1]} u(t) \geq \min\limits_{t\in[1/4,3/4]} u(t) > r$ and $\min\limits_{t\in[1/4,3/4]} u(t) + \max\limits_{t\in[0,1]} u(t) > 2r$, $\delta(u) > r$, a contradiction) and from (b) we obtain that

$$\begin{aligned}\varphi(Nu) &= \min_{t\in[1/4,3/4]} \int_0^1 G(t,s) f(s,u(s))\, ds \\ &\geq \int_0^1 \min_{t\in[1/4,3/4]} G(t,s) f(s,u(s))\, ds \geq \frac{1}{4} \int_{1/4}^{3/4} G(s,s) \frac{384}{11} r\, ds = r \geq \varphi(u),\end{aligned}$$

so

$$\varphi(u) \leq \varphi(Nu) \quad \text{if } \delta(u) = r. \tag{3.16}$$

If $\delta(u) = R$ we have $\psi(u) \geq R$ (on the contrary, we obtain $\psi(u) := \max\limits_{t\in[0,1]} u(t) < R$, so $\min\limits_{t\in[1/4,3/4]} u(t) \leq \max\limits_{t\in[k0,1]} u(t) < R$ and $\min\limits_{t\in[1/4,3/4]} u(t) + \max\limits_{t\in[0,1]} u(t) < 2R$,

$\delta(u) < R$, a contradiction) and from (c) we obtain that

$$\begin{aligned}\psi(Nu) &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))\,ds \le \int_0^1 \max_{t\in[0,1]} G(t,s) f(s,u(s))\,ds \\ &\le \int_0^1 G(s,s) \max_{s\in[0,1]} f(s,u(s))\,ds = 6R\int_0^1 G(s,s)\,ds = R \le \psi(u),\end{aligned}$$

so

$$\psi(u) \ge \psi(Nu) \quad \text{if } \delta(u) = R. \tag{3.17}$$

From (3.16) and (3.17) we deduce that N, φ, ψ and δ satisfy (a)–(f) from Theorem 2.7. So it follows that there exists $u^* \in K_{r,R}$ with $N(u^*) = u^*$.

For $u^* \in K_{r,R}$ we obtain that

$$2r \le \min_{t\in[1/4,3/4]} u^*(t) + \max_{t\in[0,1]} u^*(t) \le 2R.$$

Since

$$2 \min_{t\in[1/4,3/4]} u^*(t) \le \min_{t\in[1/4,3/4]} u^*(t) + \max_{t\in[0,1]} u^*(t) \le 2 \max_{t\in[0,1]} u^*(t)$$

and

$$\frac{1}{4} \max_{t\in[0,1]} u^*(t) \le \min_{t\in[1/4,3/4]} u^*(t),$$

we obtain that

$$\frac{r}{4} \le \min_{t\in[1/4,3/4]} u^*(t) \le R \quad \text{and} \quad r \le \max_{t\in[0,1]} u^*(t) \le 4R.$$

The conclusion follows. □

Example 3.4. $f(s,x) := (s+1)/(x+1)$ is an example of funtion that satisfies the hypothesis of Theorem 3.3. Indeed, $f(s,\cdot\,)$ is decreasing and $f(s,x) \le 2$ for all $s \in [0,1]$ and for all $x \ge 0$, so it follows the condition (a) from Theorem 3.3.

If $\delta(x) = r$ it follows that

$$\frac{1}{4}\|x\| \le \min_{t\in I} x(t) \le r, \qquad \|x\| := \max_{t\in[0,1]} x(t) \le 4r$$

and since $f(s,\cdot\,)$ is decreasing it follows that

$$f(s,x(s)) \ge f(s,4r) = \frac{s+1}{4r+1} \ge \frac{1}{4r+1} \tag{3.18}$$

for all $s \in [0,1]$ and for all $x \ge 0$. We search for $r > 0$ such that

$$\frac{1}{4r+1} \ge 35r > \frac{384}{11} r, \qquad 0 \ge 140r^2 + 35r - 1, \tag{3.19}$$

which is true for $r \in (0, (\sqrt{1785} - 35)/280]$. So, from (3.18) and (3.19), it follows that

$$f(s,x(s)) \ge \frac{384}{11} r \quad \text{for all } s \in [0,1] \text{ and for all } x \ge 0, \tag{3.20}$$

If $\delta(x) = R$ we choose $R > 0$ such that

$$2 \leq 6R, \qquad \frac{1}{3} \leq R.$$

Since $f(s, x) \leq 2$ for all $s \in [0, 1]$ and for all $x \geq 0$, we obtain that

$$f(s, x(s)) \leq 6R \quad \text{for all } s \in [0,1] \text{ and for all } x \geq 0. \tag{3.21}$$

From (3.20) and (3.21) we obtain (b) and (c) from Theorem 3.3. So $r = (\sqrt{1785} - 35)/280$, $R = 1/3$ and $f(s, x) := (s + 1)/(x + 1)$ satisfy Theorem 3.3, so (3.1) with (3.2) has a solution u^* with

$$\frac{\sqrt{1785} - 35}{280} \leq \frac{\min\limits_{t \in [1/4, 3/4]} u^*(t) + \max\limits_{t \in [0,1]} u^*(t)}{2} \leq \frac{1}{3},$$

and

$$\frac{\sqrt{1785} - 35}{280} \leq \max_{t \in [0,1]} u^*(t) \leq \frac{4}{3} \quad \text{and} \quad \frac{\sqrt{1785} - 35}{1120} \leq \min_{t \in [1/4, 3/4]} u^*(t) \leq \frac{1}{3}.$$

Let be $\varphi_1, \psi_1 \colon K \to \mathbb{R}_+$ be defined as follows:

$$\psi_1(u) := \min_{t \in I} u(t), \qquad \varphi_1(u) := \max_{t \in [0,1]} u(t) = \|u\|.$$

Now we give an application of Theorem 2.8.

THEOREM 3.5. *Let r and R be positive numbers such that $0 < r < R$ and define $f_1 \colon [0, 1] \times K \to [0, \infty)$, $f_1(s, x) = f(s, x(s))$. Suppose f_1 satisfies the following conditions:*

(a) $f_1(s, x) \leq 6r$ *for all* $s \in [0, 1]$, *for all* $x \in K$ *with* $\delta(x) = r$,

(b) $f_1(s, x) \geq 384R/11$ *for all* $s \in I$, *for all* $x \in K$ *with* $\delta(x) = R$,

where $\delta(x) := (\varphi_1(x) + \psi_1(x))/2$. Then (3.1) with (3.2) has a solution u^* such that

$$r \leq \frac{\min\limits_{t \in [1/4, 3/4]} u^*(t) + \max\limits_{t \in [0,1]} u^*(t)}{2} \leq R \tag{3.22}$$

and

$$r \leq \max_{t \in [0,1]} u^*(t) \leq 4R \quad \text{and} \quad \frac{r}{4} \leq \min_{t \in [1/4, 3/4]} u^*(t) \leq R. \tag{3.23}$$

PROOF. Let $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$. Define the completely continuous operator N by

$$N(u)(t) = \int_0^1 G(t, s) f(s, u(s)) \, ds.$$

The definitions of φ_1, ψ_1 and δ imply conditions (a)–(f) from Theorem 2.8.

If $\delta(u) = R$ we obtain $\psi_1(u) \leq R$ (in the contrary we have $\varphi_1(u) \geq \psi_1(u) > R$ and $\delta(u) > R$, a contradiction), so $||u||/4 \leq \psi_1(u) \leq R$. That is $||u|| =$

$\max_{t\in[0,1]} u(t) \leq 4R$ and $f(s,u(s))$ is bounded since $f(s,\cdot\,)$ is continuous. It follows that $N(u)$ is bounded. Hence condition (g) from Theorem 2.8 is satisfied.

Using the same arguments like in Theorem 3.3 we obtain that $N(K_{r,R})$ is relatively compact.

If $\delta(u) = r$ we have that $\varphi_1(u) := \max_{t\in[0,1]} u(t) \geq r$ (on the contrary, we obtain $\psi_1(u) := \min_{t\in I} u(t) \leq \varphi_1(u) < r$, $(\varphi_1(u)+\psi_1(u))/2 < r$, so $\delta(u) < r$, a contradiction). We deduce, using (a), that

$$\begin{aligned}\varphi_1(Nu) &= \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))\,ds \leq \int_0^1 \max_{t\in[0,1]} G(t,s)f(s,u(s))\,ds \\ &\leq \int_0^1 G(s,s) \max_{s\in[0,1]} f(s,u(s))\,ds \leq 6r\int_0^1 G(s,s)\,ds = r \leq \varphi_1(u),\end{aligned}$$

so

$$\varphi_1(u) \geq \varphi_1(Nu) \quad \text{if } \delta(u) = r. \tag{3.24}$$

If $\delta(u) = R$ we have that $\psi_1(u) := \min_{t\in I} u(t) \leq R$ (on the contrary, we obtain $\varphi_1(u) \geq \psi_1(u) > R$, $(\varphi_1(u)+\psi_1(u))/2 > R$, so $\delta(u) > R$, a contradiction). Using (3.5) and (b), we deduce that

$$\begin{aligned}\psi_1(Nu) &= \min_{t\in I} \int_0^1 G(t,s)f(s,u(s))\,ds \geq \int_0^1 \min_{t\in I} G(t,s)f(s,u(s))\,ds \\ &\geq \frac{1}{4}\int_{1/4}^{3/4} G(s,s)\min_{s\in I} f(s,u(s))\,ds \\ &\geq \frac{1}{4}\cdot\frac{384}{11}\,R\int_{1/4}^{3/4} G(s,s)\,ds = R \geq \psi_1(u),\end{aligned}$$

so

$$\psi_1(u) \leq \psi_1(Nu) \text{ if } \delta(u) = R. \tag{3.25}$$

From (3.24) and (3.25) we obtain (h) from Theorem 2.8. So we may apply Theorem 2.8 and there exists $u^* \in K_{r,R}$ such that $N(u^*) = u^*$.

For $u^* \in K_{r,R}$ we obtain that

$$2r \leq \min_{t\in[1/4,3/4]} u^*(t) + \max_{t\in[0,1]} u^*(t) \leq 2R$$

and since

$$2\min_{t\in[1/4,3/4]} u^*(t) \leq \min_{t\in[1/4,3/4]} u^*(t) + \max_{t\in[0,1]} u^*(t) \leq 2\max_{t\in[0,1]} u^*(t)$$

and

$$\frac{1}{4}\max_{t\in[0,1]} u^*(t) \leq \min_{t\in[1/4,3/4]} u^*(t)$$

we obtain that

$$\frac{r}{4} \leq \min_{t\in[1/4,3/4]} u^*(t) \leq R \quad \text{and} \quad r \leq \max_{t\in[0,1]} u^*(t) \leq 4R.$$

The conclusion follows. $\square$

Remark 3.6. In Theorems 3.3 and 3.4 (the relations (3.15) and (3.23)) we obtain similar localization results with the results from [1]. In this paper the authors obtain the following localization result (see Theorem 4.1):

$$r \leq \max_{t\in[0,1]} u^*(t) \quad \text{and} \quad \min_{t\in[1/2,3/4]} u^*(t) \leq R. \tag{3.26}$$

Now we give an other application of Theorem 2.8, where $f(\cdot,\cdot)$ may be unbounded. In our proof we use here, for the first time in literature, a functional δ that gives a similar result like in (3.26).

Theorem 3.7. *Let be the positive numbers r and R such that $0 < 16r < R$ and suppose f satisfies the following conditions:*

(a) *$f(s,x) \leq 6r$ for all $s \in [0,1]$, for all $x \in [0,16r]$,*
(b) *$f(s,x) \geq 6144R/11$ for all $s \in I$, for all $x \in [R,16R]$.*

Then (3.1) with (3.2) has a solution u_0 with

$$r \leq \min_{t\in[1/4,3/4]} u_0 \leq 4R \quad \textit{and} \quad r \leq \max_{t\in[0,1]} u_0 \leq 16R.$$

Proof. Let be $K_{r,R} := \{x \in K : r \leq \delta(x) \leq R\}$. Define the completely continuous operator N by

$$N(u)(t) = \int_0^1 G(t,s)f(s,u(s))\,ds.$$

Let $\delta\colon K \to \mathbb{R}_+$,

$$\delta(u) = \begin{cases} \dfrac{\left(\min\limits_{t\in I} u(t)\right)^2}{\max\limits_{t\in[0,1]} u((t)} & \text{if } u \text{ is not identically zero,} \\ 0 & \text{if } u \equiv 0. \end{cases}$$

Since it is obviously that

$$0 \leq \frac{\left(\min\limits_{t\in} u(t)\right)^2}{\max\limits_{t\in[0,1]} u((t)} \leq \min_{t\in I} u(t) \quad \text{if } u \text{ is not identically zero,}$$

it follows that $\delta(u) \to 0$ if $u \to 0$, so δ is a continuous functional. Also, if u is not identically zero (on the contrary it is obviously), we have that

$$\begin{aligned}
&\delta(\alpha u) = \frac{\Big(\min\limits_{t\in I} \alpha u(t)\Big)^2}{\max\limits_{t\in[0,1]} \alpha u(t)} = \frac{\alpha^2\Big(\min\limits_{t\in I} u(t)\Big)^2}{\alpha \max\limits_{t\in[0,1]} u(t)} = \alpha\frac{\Big(\min\limits_{t\in I} u(t)\Big)^2}{\max\limits_{t\in[0,1]} u(t)} = \alpha\delta(u) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \alpha > 0, \\
&\delta(\alpha u) = \delta(0) = 0 = \alpha\delta(u), \qquad\qquad\qquad\qquad\qquad\quad \text{for } \alpha = 0.
\end{aligned}$$

So δ satisfies the hypothesis of Theorem 2.8.

If $u \in K_{r,R}$ we obtain

$$\delta(u) \le R, \qquad \Big(\min_{t\in I} u(t)\Big)^2 \le R \max_{t\in[0,1]} u((t) \le 4R \min_{t\in I} u(t), \qquad \min_{t\in I} u(t) \le 4R,$$

and since $\|u\|/4 \le \min\limits_{t\in I} u(t)$, we obtain $\|u\| \le 16R$, so $K_{r,R}$ is bounded. It follows that $N(K_{r,R})$ is relatively compact since N is completely continuous.

For $\delta(u) = r$ we have that $\Big(\min\limits_{t\in I} u(t)\Big)^2 = r\|u\|$ and since $\min\limits_{t\in I} u(t) \ge \|u\|/4$ we obtain

$$r\|u\| \ge \frac{\|u\|^2}{16}, \qquad \|u\| \le 16r. \tag{3.27}$$

Also we have that

$$\Big(\min_{t\in I} u(t)\Big)^2 = r\|u\| \ge r \min_{t\in I} u(t), \qquad \min_{t\in I} u(t) \ge r. \tag{3.28}$$

From (3.27) and (a) we deduce that

$$f(s, u(s)) \le 6r \quad \text{for all } s \in [0,1],$$

so from (3.5), above relation and (3.28) it follows

$$\begin{aligned}
\varphi_1(Nu) = \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))\,ds &\le \int_0^1 G(s,s) f(s,u(s))\,ds \\
&\le 6r \int_0^1 G(s,s)\,ds = r \le \min_{t\in I} u(t) \le \|u\| = \varphi_1(u),
\end{aligned}$$

so

$$\varphi_1(u) \ge \varphi_1(Nu) \quad \text{if } \delta(u) = r. \tag{3.29}$$

For $\delta(u) = R$ we have that, similarly with the relations (3.27) and (3.28), that

$$R \le \min_{t\in I} u(t) \le \|u\| \le 16R,$$

so

$$u(s) \in [R, 16R] \quad \text{for all } s \in I. \tag{3.30}$$

It follows, from (b), that

$$f(s,u(s)) \geq \frac{6144}{11}R \quad \text{for all } s \in I,$$

so from (3.5), above relation and (3.30) we deduce, for $t \in I$, that

$$\begin{aligned}\psi_1(Nu) &= \min_{t\in I}\int_0^1 G(t,s)f(s,u(s))\,ds \geq \int_0^1 \frac{1}{4}G(s,s)\min_{s\in I} f(s,u(s))\,ds\\ &\geq \int_{1/4}^{3/4}\frac{1}{4}G(s,s)\frac{6144}{11}\,R\,ds = 16R \geq u(t) \geq \min_{t\in I} u(t) = \psi_1(u),\end{aligned}$$

so

$$\psi_1(Nu) \geq \psi_1(u) \quad \text{if } \delta(u) = R. \tag{3.31}$$

Since $\delta(u) = R$ implies that $\|u\| \leq 16R$ and $f(\,\cdot\,,\,\cdot\,)$ is continuous, it is obviously that $f(s,u(s))$ and $N(u(t))$ are bounded for $\delta(u) = R$.

It follows (g) from Theorem 2.8. Also, (3.29), (3.31) imply that φ_1 and ψ_1 satisfy (h) from Theorem 2.8 (with φ_1 instead of φ and with ψ_1 instead of ψ). Also, since $\delta(u) = R$ implies (3.30), it follows that $\varphi_1(u) := \max\limits_{t\in[0,1]} u(t) \geq R > 0$ which is (d) from Theorem 2.8.

Applying Theorem 2.8 we obtain that (3.1) with (3.2) has a solution u_0 so that $r \leq \delta(u_0) \leq R$, so

$$r\|u_0\| \leq \Big(\min_{t\in I} u_0(t)\Big)^2 \leq R\|u_0\| \tag{3.32}$$

and since

$$\frac{1}{4}\|u_0\| \leq \min_{t\in I} u_0(t) \leq \|u_0\|, \qquad \frac{1}{16}\|u_0\|^2 \leq \Big(\min_{t\in I} u_0(t)\Big)^2 \leq \|u_0\|^2, \tag{3.33}$$

from (3.32) and (3.33) we deduce that

$$\frac{1}{16}\|u_0\|^2 \leq R\|u_0\|, \quad \|u_0\| \leq 16R \quad \text{and} \quad r\|u_0\| \leq \|u_0\|^2, \quad r \leq \|u_0\|.$$

So

$$r \leq \|u_0\| \leq 16R. \tag{3.34}$$

Also, from (3.32) and (3.33) it follows that

$$r\min_{t\in I} u_0(t) \leq r\|u_0\| \leq \Big(\min_{t\in I} u_0(t)\Big)^2 \leq R\|u_0\| \leq 4R\min_{t\in I} u_0(t),$$

so

$$r \leq \min_{t\in I} u_0(t) \leq 4R. \tag{3.35}$$

From (3.34) and (3.35) the conclusion follows. □

EXAMPLE 3.8. An example of unbounded function that satisfies the conditons of Theorem 3.7 is

$$f(s,x)=\frac{s+1}{s+2}\,x^2.$$

Indeed, for $x\in[0,16r]$ we have that

$$f(s,x)\le(16r)^2\le 6r\quad\text{for } r\le\frac{3}{128}.$$

Also, for $x\in[R,16R]$ and $s\in I:=[1/4,3/4]$ we obtain that

$$f(s,x)\ge\frac{\frac{1}{4}+1}{\frac{1}{4}+2}\,R^2\ge\frac{6144}{11}\,R$$

wich is true for $R\ge 1006$. So for $r\le 3/128$ and $R\ge 1006$, $f(s,x)$ satisfies the conditions of Theorem 3.7.

As an application of Theorem 3.7 we give the following multiplicity result.

COROLLARY 3.9. *Suppose that there exist positive numbers r_1, r_2, R_1 and R_2 such that* $0<16r_1<R_1\le 11r_2/1024$, $16r_2<R_2$ *and f satisfies the following conditions*:

(a) $f(s,x)\le 6r_1$ *for all* $s\in[0,1]$, *for all* $x\in[0,16r_1]$,
(b) $f(s,x)\ge 6144R_1/11$ *for all* $s\in I$, *for all* $x\in[R_1,16R_1]$,
(c) $f(s,x)\le 6r_2$ *for all* $s\in[0,1]$, *for all* $x\in[0,16r_2]$,
(d) $f(s,x)\ge 6144R_2/11$ *for all* $s\in I$, *for all* $x\in[R_2,16R_2]$,

Then (3.1) *with* (3.2) *has at least two solutions u_1 and u_2 with*

$$r_1\le\min_{t\in[1/4,3/4]}u_1\le 4R_1,\qquad r_1\le\max_{t\in[0,1]}u_1\le 16R_1,$$
$$r_2\le\min_{t\in[1/4,3/4]}u_2\le 4R_2,\qquad r_2\le\max_{t\in[0,1]}u_2\le 16R_2.$$

EXAMPLE 3.10. An example of function f that satisfies the conditions from Corollary 3.9 is

$$f(s,x)=\begin{cases} 6r_1\exp(x-16r_1) & \text{if } x\in[0,16r_1],\\ a(x-R_1)\exp(x-16r_1)+\dfrac{6144}{11}R_1\exp(x-R_1) & \\ & \text{if } x\in(16r_1,R_1),\\ \dfrac{6144}{11}\,R_1\exp(x-R_1) & \text{if } x\in[R_1,16R_1],\\ 6cr_2\exp(x-16r_2)+d(x-16r_2) & \text{if } x\in(16R_1,16r_2],\\ \dfrac{6144}{11}\,R_2\exp(x-R_2)+p(x-R_2)\exp(x-16r_2) & \\ & \text{if } x\in(16r_2,R_2),\\ \dfrac{6144}{11}\,R_2\exp(x-R_2) & \text{if } x\in[R_2,\infty), \end{cases}$$

where, beside the conditions from Corollary 3.9, r_1 and R_1 satisfy the conditions

$$\begin{cases} R_1 - 16r_1 \le 1, \\ \dfrac{1024}{11} R_1 \exp(16r_2 - R_1) < r_2 \quad \text{(it is posible for } R_1 \text{ small enough)}, \end{cases}$$

r_2 and R_2 satisfying the conditions from Corollary 3.9. Also,

$$\begin{cases} a := \dfrac{6r_1 - (6144/11)R_1 \exp(16r_1 - R_1)}{16r_1 - R_1}, \\ 1 \ge c > \dfrac{1024}{11} \dfrac{R_1}{r_2} \exp(16r_2 - R_1), \\ p := \dfrac{6cr_2 - (6144/11)R_2 \exp(16r_2 - R_2)}{16r_2 - R_2}, \\ d := \dfrac{(6144/11)R_1 \exp(15R_1) - 6cr_2 \exp(16R_1 - 16r_2)}{16R_1 - 16r_2}. \end{cases}$$

The values of the constants a, c, p, d assure us that f is continuous.

Finally we give an application of Theorem 2.6.

THEOREM 3.11. *Let r, R be positive numbers with $R > 4r > 0$ and $c_1 := r/R$, $c_2 := R/r$. Suppose that $\varphi, \psi : K \to [0, \infty)$, $\varphi(x) := (1/4) \max_{t \in [0,1]} x(t) := \|x\|/4$, $\psi(x) := \min_{t \in I} x(t)$ and $K_{r,R} := \{x \in K : r \le \psi(x), \varphi(x) \le R\}$, $f(s, x) = g(s)h(x)$ where $g : [0, 1] \to [0, \infty)$, $h : [0, \infty) \to [0, \infty)$ are continuous. Define $h_1 : [0, 1] \times K \to [0, \infty)$, $h_1(s, x) = h(x(s))$. Also assume that the following conditions are satisfied:*

(a) *$h_1(s, c_1 x) \ge c_1 h_1(s, x)$ for all $s \in [0, 1]$ and all $x \in K$ with $\psi(x) = R$,*
$h_1(s, c_2 x) \le c_2 h_1(s, x)$ for all $s \in [0, 1]$ and all $x \in K$ with $\varphi(x) = r$,

(b) *$f(s, x) \le 24r$ for all $s \in [0, 1]$ and all $x \in [0, 4r]$,*

(c) *$f(s, x) \ge 384R/11$ for all $s \in I$ and all $x \ge R$.*

Then (3.1) *with* (3.2) *has a solution u_0 with $r \le \min_{t \in I} u_0(t) \le \|u_0\| \le 4R$.*

PROOF. Define the completely continuous operator $N : K_{r,R} \to K$ by

$$N(u)(t) = \int_0^1 G(t, s) f(s, u(s))\, ds.$$

It is obviously that φ and ψ are continuous, there exists $h \in K \setminus \psi^{-1}\{0\}$ such that $\psi(\lambda h) > 0$ for all $\lambda > 0$ and $\psi \ge \varphi$ (from (3.6)). Also, φ and ψ satisfy the conditions (a)–(c) and (e) from Theorem 2.6.

If $u \in K_{r,R}$ we obtain that $\varphi(u) := (1/4) \max_{t \in [0,1]} u(t) \le R$, so $\|u\| = \max_{t \in [0,1]} u(t)$ $\le 4R$, so $K_{r,R}$ is bounded. It follows that $N(K_{r,R})$ is relatively compact, since N is completely continuous.

From $\psi(u) = r$ we obtain $\|u\|/4 \le r$, $\|u\| \le 4r$ and $f(s, \cdot)$ is bounded. It follows that N is bounded and (d) from Theorem 2.6 is satisfied.

From our hyphotesis and (a) it follows (f) from Theorem 2.6. From (3.6) and our hyphotesis $R > 4r$ we have, for all $u \in K$, that

$$R\varphi(u) = \frac{R}{4}\max_{t\in[0,1]} u(t) \geq r \max_{t\in[0,1]} u(t) \geq r \min_{t\in I} u(t) = r\psi(u). \tag{3.36}$$

If $\varphi(u) = r$ we obtain $\|u\| = 4r$, so $u(s) \leq 4r$ for all $s \in [0,1]$. From our hyphothesis (b) we have that

$$\begin{aligned}\varphi(Nu) &= \frac{1}{4}\max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))\,ds \\ &\leq \frac{1}{4}\int_0^1 G(s,s) \max_{s\in[0,1]} f(s,u(s))\,ds \leq \frac{24r}{4}\int_0^1 G(s,s)\,ds = r = \varphi(u),\end{aligned}$$

so

$$\varphi(u) \geq \varphi(Nu) \quad \text{if } \varphi(u) = r. \tag{3.37}$$

If $\psi(u) = R$ we have that $u(s) \geq R$ for all $s \in I$. From our hyphothesis (c) and from (3.5) we have that

$$\begin{aligned}\psi(Nu) &= \min_{t\in I} \int_0^1 G(t,s) f(s,u(s))\,ds \\ &\geq \int_{1/4}^{3/4} \frac{1}{4} G(s,s) \min_{s\in I} f(s,u(s))\,ds \geq \frac{96R}{11}\int_{1/4}^{3/4} G(s,s)\,ds = R = \psi(u),\end{aligned}$$

so

$$\psi(u) \leq \psi(Nu) \quad \text{if } \psi(u) = R. \tag{3.38}$$

From (3.36)–(3.38) it follows that we are in the conditions of Theorem 2.6. So (3.1) with (3.2) has a solution u_0 with

$$r \leq \min_{t\in I} u_0(t) \quad \text{and} \quad \frac{1}{4}\max_{t\in[0,1]} u_0(t) \leq R$$

and the conclusion follows. □

Remark 3.12. Comparing the condition (b) from Theorem 3.11 with the following condition:

$$f(x) \leq 8r \quad \text{for all } x \in [0,r]$$

(the condition (b) from Theorem 4.1 from [1]), we note that Theorem 3.11 extends Theorem 4.1 (from [1]).

References

[1] R. Avery, J. Henderson and D. O'Regan, *Functional compression-expansion fixed point theorem*, Electron. J. Differential Equations **2008** (2008), no. 22, 1–12.

[2] ______, *A dual of the compression-expansion fixed point theorems*, Article ID 90715, Fixed Point Theory Appl. **2007**, 11 pages.

[3] S. Budişan, *Generalizations of Krasnosel'skiĭ's fixed point theorems in cones*, Stud. Univ. Babeş–Bolyai Math. **56** (2011), 165–171.

[4] ______, *Positive solutions of functional differential equations*, Carpathian J. Math. **22** (2006), no. 1-2, 13-19.

[5] ______, *Positive weak radial solutions of nonlinear systems with p-Laplacian*, Differ. Equ. Appl. **3** (2011), 209–224.

[6] S. Budişan and R. Precup, *Positive solutions of functional-differential systems via the vector version of Krasnosel'skiĭ's fixed point theorem in cones*, Carpathian J. Math. **27** (2011), 165–172.

[7] M. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Noordhoff, Gröningen, 1964.

[8] M.K. Kwong, *On Krasnosel'skiĭ's cone fixed point theorem*, Article ID 164537, Fixed Point Theory Appl. **2008**, 18 pages.

[9] R. Leggett and L. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana University Mathematics Journal **28** (1979), no. 4.

[10] D. O'Regan and R. Precup, *Compression-expansion fixed point theorem in two norms and applications*, J. Math. Anal. Appl. **309** (2005), 383–391.

[11] R. Precup, *A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions of nonlinear systems*, J. Fixed Point Theory **2** (2007), 141–151.

[12] ______, *Compression-expansion fixed point theorems in two norms*, Ann. Tiberiu Popoviciu Semin. **3** (2005), 157–163.

[13] H. Wang, *Positive periodic solutions of singular systems with a parameter*, J. Differential Equations **249** (2010), 2986–3002.

Manuscript received November 22, 2011

Sorin Budişan
Babeş–Bolyai University
Department of Applied Mathematics
400084 Cluj-Napoca, ROMANIA

E-mail address: sorinbudisan@yahoo.com