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ON PARAMETRIC EQUILIBRIUM PROBLEMS

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Abstract. The goal of this article is to study a kind of stability property of a sequence of solutions to parametric equilibrium problems. The main result gives sufficient conditions for this purpose, in the presence of the topological pseudomonotonicity in the limit problem.

1. Introduction

One of the most studied topics in nonlinear analysis is the equilibrium problem. Several well known problems in optimization, Nash equilibrium theory, variational inequalities, fixed point theory, etc. can be considered as particular cases. In this paper we consider equilibrium problems of the following type:

(EP) Find an element $a \in X$ such that

$$f(a,b) + \Phi(a,b) \geq \Phi(a,a), \quad \text{for all } b \in X,$$

where (X,σ) is a Hausdorff topological space, while $f(\,\cdot\,,\,\cdot\,)\colon X \times X \to \mathbb{R}$ and $\Phi(\,\cdot\,,\,\cdot\,)\colon X \times X \to \mathbb{R} \cup \{\infty\}$ are given functions.

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As a result of changes in the problem data, the solutions behavior is always of concern. For instance, a sequence of functions may provide a sequence of solutions, therefore we are interested to study a certain stability of this sequence. For this reason, besides this problem, for a given $n \in \mathbb{N}$, we consider the following parametric equilibrium problem:

$(\mathrm{EP})_n$ Find an element $a_n \in X$ such that

$$f_n(a_n, b) + \Phi_n(a_n, b) \geq \Phi_n(a_n, a_n), \quad \text{for all } b \in X,$$

where $f_n(\,\cdot\,,\,\cdot\,)\colon X \times X \to \mathbb{R}$, $\Phi_n(\,\cdot\,,\,\cdot\,) : X \times X \to \mathbb{R} \cup \{\infty\}$ are given functions.

Denote by $S(n)$ the set of the solutions for a fixed n and suppose that, for all $n \in \mathbb{N}$, $S(n) \neq \emptyset$.

Problem (EP) is called the limit problem and can be viewed as the "homogenized problem" of $(\mathrm{EP})_n$. Denote by $S(\infty)$ the set of its solutions.

We pose the following question on the stability of solutions:

(Q) If $a_n \in S(n)$ and $(a_n)_n$ σ-converges to a in X as $n \to \infty$, is it true that $a \in S(\infty)$?

Some answers for the question (Q) have been given for the particular case of $\Phi_n = \Phi = 0$ (see for instance [3], [16]). Such problems also appear by the regularization method in [19], [22]. For problem (EP), in the general case, Lignola and Morgan [18] gave sufficient conditions on f_n, f, Φ_n, Φ so that the answer for (Q) is positive. However, the result from [18] cannot be applied for some important cases, for instance in the case of variational inequalities governed by nonlinear pseudomonotone operators [14], [15], [25], [26]. Our goal is to remove this inconvenient: in this paper we shall give sufficient conditions for the stability of solutions, different from that given in [18]. More, we do not require any linear structure of the space X.

The paper is organized as follows. In Section 2 we prove our main result on the stability of solutions. In Section 3 this result is applied to three particular cases. In Subsection 3.1 we study the convergence of solutions to the so-called quasiequilibrium problems. Subsection 3.2 contains the case of parametric hemivariational inequalities governed by pseudomonotone operators. The last subsection considers the particular case of parametric minimum problems in a relationship with Γ-convergence.

2. Main result

The following notion is a generalization of the topological pseudomonotonicity, introduced by Brézis [5] for variational inequalities and utilized in several monographs (see for instance [24]–[27], [29]) and articles (e.g. [6], [7], [11]–[13], [19], [21], [28]).

Definition 2.1 ([1, p. 410]). A function $f\colon X \times X \to$ is said to be *topologically pseudomonotone with respect to the first variable* if, for each sequence $(a_n)_{n\in\mathbb{N}} \subset X$ with $(a_n)_n$ σ-converging to a in X, $\liminf_n f(a_n, a) \geq 0$ implies

$$\limsup_n f(a_n, b) \leq f(a, b), \quad \text{for all } b \in X.$$

Now, we state our main result.

Theorem 2.2. *Let X be a Hausdorff topological space with σ and τ topologies on X such that $\sigma \subseteq \tau$, i.e. σ is weaker than τ. Suppose that $S(n) \neq \emptyset$, for each $n \in \mathbb{N}$, and the following conditions hold*:

(a) *for any $(a_n)_{n\in\mathbb{N}}$ σ-converging to $a \in X$, with $a_n \in S(n)$, one has*

$$\liminf_n \Phi_n(a_n, a_n) \geq \Phi(a, a);$$

(b) *for any $b \in X$ and any sequence $(a_n)_{n\in\mathbb{N}}$ σ-converging to $a \in X$, with $a_n \in S(n)$, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ τ-converging to b, such that*

$$\limsup_n \Phi_n(a_n, b_n) \leq \Phi(a, b);$$

(c) *the functions $f_n, f\colon X \times X \to \mathbb{R}$ $(n \in \mathbb{N})$ verify condition*:

(**C**) *For each sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n \in S(n)$, $(a_n)_n$ σ-converging to a, and $(b_n)_n$ τ-converging to b, one has*

$$\liminf_n (f(a_n, b) - f_n(a_n, b_n)) \geq 0;$$

(d) *the function $f : X \times X \to \mathbb{R}$ is topologically pseudomonotone w.r.t. the first variable.*

Then, for each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in S(n)$, $(a_n)_n$ σ-converging to a and $\Phi(a, a) < \infty$, imply $a \in S(\infty)$.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n \in S(n)$ and let $(a_n)_n$ be σ-converging to a. From (b), there exists a sequence $(\overline{a}_n)_{n\in\mathbf{N}}$, such that $(\overline{a}_n)_n$ is τ-converging to a and

$$\limsup_n \Phi_n(a_n, \overline{a}_n) \leq \Phi(a, a). \tag{2.1}$$

First use (a), then replacing b with $\overline{a}_n$ in $(\mathrm{EP})_n$ we obtain by (2.1)

$$\begin{aligned}\Phi(a, a) &\leq \liminf_n \Phi_n(a_n, a_n) \leq \liminf_n [f_n(a_n, \overline{a}_n) + \Phi_n(a_n, \overline{a}_n)] \\ &\leq \liminf_n f_n(a_n, \overline{a}_n) + \limsup_n \Phi_n(a_n, \overline{a}_n) \leq \liminf_n f_n(a_n, \overline{a}_n) + \Phi(a, a),\end{aligned}$$

hence $\liminf_n f_n(a_n, \overline{a}_n) \geq 0$. By condition (**C**), for $b_n := \overline{a}_n$, we have

$$\liminf_n f(a_n, a) \geq 0.$$

Now, we apply (d) to get

$$\limsup_n f(a_n, b) \leq f(a, b), \quad \text{for all } b \in X. \tag{2.2}$$

Let $b \in X$ be arbitrary. By (b), there exists $(b_n)_n$ τ-converging to b to have

$$\limsup_n \Phi_n(a_n, b_n) \leq \Phi(a, b).$$

Due to (2.2), condition (**C**), and (a) we obtain

$$\begin{aligned} f(a,b) + \Phi(a,b) &\geq \limsup_n f(a_n, b) + \Phi(a,b) \\ &\geq \limsup_n f_n(a_n, b_n) + \liminf_n (f(a_n, b) - f_n(a_n, b_n)) + \Phi(a,b) \\ &\geq \limsup_n f_n(a_n, b_n) + \limsup_n \Phi_n(a_n, b_n) \\ &\geq \liminf_n [f_n(a_n, b_n) + \Phi_n(a_n, b_n)] \\ &\geq \liminf_n \Phi_n(a_n, a_n) \geq \Phi(a, a), \end{aligned}$$

to conclude $a \in S(\infty)$. □

Remark 2.3. It is easy to see that if $\Phi_n = \Phi = 0$, for all $n \in \mathbb{N}$, then in Theorem 2.2 condition (**C**) can be weakened to:

(**C′**) For each sequence $(a_n)_{n\in\mathbb{N}}$, if $a_n \in S(n)$, $(a_n)_n$ is σ-converging to a, and $b \in X$, then there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that $(b_n)_n$ is τ-converging to b and

$$\liminf_n (f(a_n, b) - f_n(a_n, b_n)) \geq 0.$$

3. Applications

In this section we derive from Theorem 2.2 some results for parametric quasiequilibrium problems, parametric hemivariational inequalities, and parametric minimum problems.

3.1. Parametric quasiequilibrium problems. For a given $n \in \mathbb{N}$ we consider the following parametric quasiequilibrium problem:

(QEP)$_n$ Find an element $a_n \in X$ such that $a_n \in D_n(a_n)$ and

$$f_n(a_n, b) \geq 0, \quad \text{for all } b \in D_n(a_n),$$

where D_n is a non-empty set-valued function from X to X.

Along with the problems above we consider the following one:

(QEP) Find an element $a \in X$ such that $a \in D(a)$ and

$$f(a, b) \geq 0, \quad \text{for all } b \in D(a),$$

where D is a non-empty set-valued function from X to X.

Usually, the term of equilibrium problems is used when D and D_n, $n \in \mathbb{N}$, do not depend on the points a and a_n, respectively. We use the expression "quasiequilibrium" problem inspired by the commonly used "quasivariational" inequality.

As in Section 1, denote by $S(n)$ the set of the solutions of $(\mathrm{QEP})_n$ for a fixed n and by $S(\infty)$ the set of the solutions for problem (QEP).

From Theorem 2.2 we obtain the following result.

Corollary 3.1. *Let X be a Hausdorff topological space with σ and τ topologies on X, $\sigma \subseteq \tau$. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of non-empty set-valued functions from X to X, and D be a set-valued function from X to X such that $D(a)$ is non-empty for any $a \in X$. Suppose that $S(n) \neq \emptyset$, for each $n \in \mathbb{N}$, and the following conditions hold*:

(a′) *for any sequence $(a_n)_{n\in\mathbb{N}}$ σ-converging to $a \in X$, with $a_n \in S(n)$, if there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $a_{n_k} \in D_{n_k}(a_{n_k})$, then $a \in D(a)$;*

(b′) *for any $a \in X$ and any sequence $(a_n)_{n\in\mathbb{N}}$ σ-converging to a with $a_n \in S(n)$, and any $b \in D(a)$, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ τ-converging to b such that $b_n \in D_n(a_n)$;*

(c) *$f_n, f\colon X \times X \to \mathbb{R}$ $(n \in \mathbb{N})$ verify condition* (**C**);

(d) *$f : X \times X \to \mathbb{R}$ is topologically pseudomonotone w.r.t. the first variable.*

Then, for each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in S(n)$, $(a_n)_n$ σ-converging to a implies $a \in S(\infty)$.

Proof. It takes only to consider the functions Φ and Φ_n, $n \in \mathbb{N}$ defined by

$$\Phi(a,b) = \psi_{D(a)}(b), \qquad \Phi_n(a,b) = \psi_{D_n(a)}(b),$$

where ψ_K is, for any subset K of X, the indicator function of K, i.e. the function which takes the value 0 on K and ∞ otherwise. □

Remark 3.2. Following the classical article [20], we say that a sequence of sets $(D_n)_{n\in\mathbb{N}}$ Mosco converges to D if:

(a) for every subsequence $(n_k)_{k\in\mathbb{N}}$, $a_{n_k} \in D_{n_k}$ and $(a_{n_k})_k$ σ-converging to a imply $a \in D$;

(b) for every $b \in D$, there exists $(b_n)_{n\in\mathbb{N}}$, such that $b_n \in D_n$ and $(b_n)_n$ τ-converges to b.

Let us note that, assumptions (a′) and (b′) in Corollary 3.1 amount to saying that for any $a \in X$ and any sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in S(n)$ and $(a_n)_n$ σ-converging to a, the sequence of sets $(D_n(a_n))_{n\in\mathbb{N}}$ Mosco converges to $D(a)$.

3.2. Parametric hemivariational inequalities. For a given $n \in \mathbb{N}$ we consider the following parametric problem:

(HVI)$_n$ Find an element $u_n \in D$ such that, for all $v \in D$,

$$\int_\Omega \bigg\{ \sum_{i=1}^{n} g_i^n(x, u_n(x), \nabla u_n(x)) \cdot \partial_i (v - u_n)(x) \bigg\} \, dx + \int_\Omega g_0^n(x, u_n(x), \nabla u_n(x))(v - u_n)(x) \, dx + \int_\Omega l_n(x; u_n(x), v(x) - u_n(x)) \, dx + \varphi_n(v) \geq \varphi_n(u_n), \tag{3.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, D is a nonempty subset of the Sobolev space $H^1(\Omega)$, $g_i^n \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $l_n \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\varphi_n \colon D \to \mathbb{R} \cup \{\infty\}$ are given functions.

Suppose that the functions g_i^n, $i \in \{0, \ldots, N\}$ have the following properties:

(P1$_n$) $g_i^n(x, \eta, \xi)$ is measurable in $x \in \mathbb{R}^N$ and continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, for each $i \in \{0, \ldots, N\}$;

(P2$_n$) $|g_i^n(x, \eta, \xi)| \leq c(k(x) + |\eta| + \|\xi\|_N)$, for almost every $x \in \mathbb{R}^N$, for all $\eta \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, for each $i \in \{0, \ldots, N\}$, with c a positive constant and k a function in $L^4_{\mathrm{loc}}(\mathbb{R}^N)$.

For every $n \in \mathbb{N}$ we consider the function $h_n \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$h_n(u, w) = \int_\Omega \bigg\{ \sum_{i=1}^{N} g_i^n(x, u(x), \nabla u(x)) \cdot \partial_i w(x) \bigg\} \, dx + \int_\Omega g_0^n(x, u(x), \nabla u(x)) w(x) \, dx.$$

Suppose that $l_n(x; \cdot, \cdot)$ is upper semi-continuous for almost every $x \in \Omega$ and $l_n(\cdot\,; y, z)$ is measurable for all $y, z \in \mathbb{R}$. Further, let $L_n \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be given by

$$L_n(u, v) = \int_\Omega l_n(x; u(x), v(x)) \, dx.$$

Therefore, (3.2) becomes

$$h_n(u_n, v - u_n) + L_n(u_n, v - u_n) + \Phi_n(u_n, v) \geq \Phi_n(u_n, u_n), \quad \text{for all } v \in D,$$

where $\Phi_n(u, v) = \varphi_n(v)$, $n \in \mathbb{N}$.

Let us construct the limit problem. Suppose that the functions $g_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i \in \{0, \ldots, N\}$ have the following properties:

(P1) $g_i(x, \eta, \xi)$ is measurable in $x \in \mathbb{R}^N$ and continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, for each $i \in \{0, \ldots, N\}$;

(P2) $|g_i(x,\eta,\xi)| \le c_N(k_N(x)+|\eta|+\|\xi\|_N)$, for almost every $x\in\mathbb{R}^N$, for all $\eta\in\mathbb{R}$, for all $\xi\in\mathbb{R}^N$, for each $i\in\{0,\dots,N\}$, with c_N a positive constant and k_N a function in $L^4_{\mathrm{loc}}(\mathbb{R}^N)$;

(P3) for almost every $x\in\mathbb{R}^N$, for all $\eta\in\mathbb{R}$, for all $\xi,\widetilde{\xi}\in\mathbb{R}^N$ and $\xi\neq\widetilde{\xi}$

$$\sum_{i=1}^N (g_i(x,\eta,\xi)-g_i(x,\eta,\widetilde{\xi}))(\xi_i-\widetilde{\xi}_i)>0;$$

(P4) for almost every $x\in\mathbb{R}^N$, for all $\eta\in\mathbb{R}$, for all $\xi\in\mathbb{R}^N$, with $c_1,\dots,c_4$ positive constants,

$$\sum_{i=1}^N g_i(x,\eta,\xi)\xi_i \ge c_1\|\xi\|_N^2 - c_2 \quad\text{and}\quad g_0^n(x,\eta,\xi)\eta \ge c_3|\eta|^2-c_4.$$

Define $\Phi(u,v)=\varphi(v)$. Let us consider a function $l\colon\Omega\times\mathbb{R}\times\mathbb{R}$, such that $l(x;\,\cdot\,,\,\cdot\,)$ is upper semi-continuous for almost every $x\in\Omega$ and $l(\,\cdot\,;y,z)$ is measurable for all $y,z\in\mathbb{R}$. We say that l satisfies *the growth condition* if there exist nonnegative functions $d_1,d_2\in L^\infty(\Omega)$ such that

$$|l(x;y,z)|\le(d_1(x)+d_2(x)|y|)|z|,\quad \text{for a.e. } x\in\Omega,\ \text{for all } y,z\in\mathbb{R}.$$

Let $L\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ be given by

$$L(u,v)=\int_\Omega l(x;u(x),v(x))\,dx.$$

If l satisfies the growth condition, then L is weakly upper semi-continuous (see [3]).

Let us consider the function $h\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$,

$$h(u,w)=\int_\Omega\left\{\sum_{i=1}^N g_i(x,u(x),\nabla u(x))\cdot\partial_i w(x)\right\}dx+\int_\Omega g_0(x,u(x),\nabla u(x))w(x)\,dx.$$

Along with the parametric problems above $(\mathrm{HVI})_n$, we consider the limit problem:

(HVI) Find an element $u\in D$ such that

$$h(u,v-u)+L(u,v-u)+\Phi(u,v)\ge\Phi(u,u),\quad \text{for all } v\in D.$$

We have shown in [2] that condition (**C**) applies if one has

$$|g_i^n(x,\eta,\xi)-g_i(x,\eta,\xi)|\le\alpha(1/n)[\|\xi\|_N+|\eta|+\widetilde{k}(x)],$$

for all $n\in\mathbb{N}$, $i\in\{0,\dots,N\}$, $\xi\in\mathbb{R}^N$, $\eta\in\mathbb{R}$, and a.e. $x\in\Omega$. Here $\widetilde{k}$ has nonnegative values and belongs to $L^2(\Omega)$, α is a nonnegative function, continuous at 0 and $\alpha(0)=0$.

When $\varphi\equiv0$ and l is the Clarke directional derivative of a locally Lipschitz function then (HVI) turns out to be a hemivariational inequality. The problem of hemivariational inequalities was initiated by Panagiotopoulos and has been studied by many authors (see [22], [23]). More about some important particular cases of (HVI) can be found in [25] and [26].

For this subsection we have the following result.

Corollary 3.3. *Let D be a nonempty subset of $H^1(\Omega)$. Suppose that* $(\mathrm{P1}_n)$, $(\mathrm{P2}_n)$, *and* (P1)–(P4) *apply and l satisfies the growth condition. Suppose that the following conditions hold:*

(a$''$) *for any sequence $(u_n)_{n\in\mathbb{N}}$ weakly converging to $u \in H^1(\Omega)$, one has*

$$\liminf_n \varphi_n(u_n) \geq \varphi(u);$$

(b$''$) *for any $v \in H^1(\Omega)$ there exists a sequence $(v_n)_{n\in\mathbb{N}}$ strongly converging to v such that*

$$\limsup_n \varphi_n(v_n) \leq \varphi(v);$$

(c) *$f_n = h_n + L_n$, $(n \in \mathbb{N})$ and $f = h + L$ verify condition* (**C**).

Then, for each sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to $(\mathrm{HVI})_n$, *$u_n \rightharpoonup u$ (weakly) in $H^1(\Omega)$ implies that u is a solution to* (HVI).

Proof. By hypotheses (a$''$) and (b$''$), for the functions Φ and Φ_n defined above, hypotheses (a) and (b) from Theorem 2.2 apply.

By the well known result, due to Leray-Lions (Theorem 6.1 in [26]), the function h is topologically pseudomonotone w.r.t. the first variable, therefore $f = h + L$ is also topologically pseudomonotone w.r.t. the first variable. □

Remark 3.4. Theorem 2.2 can be applied for Tikhonov–Browder regularization methods for variational inequalities (see for instance [17] and the references therein). But it is not our intention to further detail it in this paper.

Remark 3.5. By some convenient particularizations of φ_n and φ one can inclose boundary value problems of mixed type (see for example [15]) in our formalism.

The sensitivity of solutions to optimal control problems described by hemivariational inequalities was studied in [9] and [10].

3.4. Parametric minimum problems. Let (X, σ) be a topological space and take $\tau = \sigma$. Let us consider, for any $n \in \mathbb{N}$, the following minimum problems, as particular case of equilibrium problems:

$(\mathrm{M})_n$ Find an element $a_n \in X$ such that $g_n(a_n) \leq g_n(b)$, for all $b \in X$, and

(M) find an element $a \in X$ such that $g(a) \leq g(b)$, for all $b \in X$,

where $g_n: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are given functions.

We take $f(a,b) = g(b) - g(a)$ and $f_n(a,b) = g_n(b) - g_n(a)$.

Condition (d) from Theorem 2.2 automatically applies if g is lower semi-continuous. In this case condition $(\mathbf{C}')$ becomes:

$(\mathbf{C}'')$ For each sequence $(a_n)_{n\in\mathbb{N}}$ of solutions for $(\mathrm{M})_n$, $a_n \to a$ and $b \in X$, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n \to b$ and

$$\liminf_n [g(b) - g_n(b_n) - g(a_n) + g_n(a_n)] \geq 0.$$

By Theorem 2.1 we have:

Corollary 3.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of solutions for* $(\mathrm{M})_n$ *and let $a_n \to a$. Suppose that g is lower semi-continuous at a and the functions g_n, g, $n \in \mathbb{N}$, verify condition* $(\mathbf{C}'')$. *Then, limit a is solution for* (M).

The classical way to study the family of minimum problems $(\mathrm{M})_n$, due to E. De Giorgi, is to use Γ-convergence. Let us remind its definition (see [4], [8]).

Definition 3.7. A sequence of functions $(h_n)_{n\in\mathbb{N}}$, $h_n\colon X \to \mathbb{R}\cup\{\infty\}$ is said to Γ-converge on X to $h\colon X \to \mathbb{R}\cup\{\infty\}$ if, for all $x \in X$ one has:

(a) for each $(x_n)_{n\in\mathbb{N}}$ convergent to x it follows that

$$h(x) \leq \liminf_n h_n(x_n);$$

(b) there exists a sequence $(x'_n)_{n\in\mathbb{N}}$ convergent to x such that

$$h(x) \geq \limsup_n h_n(x'_n).$$

If we define Φ and Φ_n, $n \in \mathbb{N}$, by $\Phi(a,b) = g(b)$ and $\Phi_n(a,b) = g_n(b)$. Therefore, when $(g_n)_{n\in\mathbb{N}}$ is Γ-converging to g we can apply Theorem 2.1 with $f_n = f = 0$.

On the other hand, it is easy to see that if the difference $g_n - g$ Γ-converges to 0, then condition $(\mathbf{C}'')$ applies. It is worth noting that from condition $(\mathbf{C}'')$ the Γ-convergence of the difference $g_n - g$ to 0 does not follow. Indeed, let $g_n(x) = x^2/n + (-1)^n n$, $n \in \mathbb{N}$ and $g(x) = 0$, for all $x \in \mathbb{R}$. In this case, condition $(\mathbf{C}'')$ applies with equality, since $a_n = 0 = a$ is the only possibility. But, $g_n - g$ evidently does not Γ-converge to 0 on $\mathbb{R}$. The same example proves that condition $(\mathbf{C}'')$ does not imply neither Γ-convergence of g_n.

Does Γ-convergence imply condition $(\mathbf{C}'')$ to be verified? The answer is yes, if g is upper semi-continuous at a, taking into account the following inequalities:

$$\begin{aligned}\liminf_n &[g(b) - g_n(b_n) - g(a_n) + g_n(a_n)] \\ &\geq \liminf_n [-g(a_n) + g(a)] + \liminf_n [g_n(a_n) - g(a)] \geq 0.\end{aligned}$$

Nevertheless, without this additional requirement, condition $(\mathbf{C}'')$ is not verified as the following example shows.

Take $g_n(x) = g_1(nx)$, $x \in \mathbb{R}$, where

$$g_1(x) = \begin{cases} 1 & \text{for } x = 1, \\ -1 & \text{for } x = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Example 1.11 in [4], g_n Γ-converges to g on $\mathbb{R}$, where

$$g(x) = \begin{cases} 0 & \text{for } x \neq 0, \\ -1 & \text{for } x = 0. \end{cases}$$

Remark that g is not upper semi-continuous at 0 since

$$\limsup_n g(a_n) = 0 \neq -1 = g(0).$$

Condition $(\mathbf{C}'')$ is not verified. Indeed, $a_n = -1/n$ are solutions for $(\mathrm{M})_n$ and $a = 0$ is the solution for (M). If $b \neq 0$ and $b_n \to b$, then we have $g_n(b_n) = 0$, for n sufficiently large. Therefore, we have

$$\liminf_n [g(b) - g_n(b_n) - g(a_n) + g_n(a_n)] = -1.$$

4. Concluding remarks

Hypotheses (a) and (b) in Theorem 2.2 define a notion of convergence for a sequence of bifunctions that can be seen as an extension of Γ-convergence. By condition $(\mathbf{C})$ one can define another "limit" for the sequence $(f_n)_{n\in\mathbb{N}}$ (of course, not unique), but from Subsection 3.3 it follows that these two convergence notions are independent.

In some applications two extreme situations occur: $f_n = f = 0$ or $\Phi_n = \Phi = 0$. Our Theorem 2.2 insures a treatment if we have none of these two cases. Applications from Section 3 show that the mixed mathematical model is important, as well.

In comparison to the paper of Lignola and Morgan [18], we do not require any convexity assumptions (nor any linear structure) while the hypotheses for Φ and Φ_n are less restrictive in our Theorem 2.2.

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