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ROTATION NUMBERS FOR PLANAR ATTRACTORS OF EQUIVARIANT HOMEOMORPHISMS

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Abstract. Given an integer $m > 1$ we consider $\mathbb{Z}_m$-equivariant and orientation preserving homeomorphisms in $\mathbb{R}^2$ with an asymptotically stable fixed point at the origin. We present examples without periodic points and having some complicated dynamical features. The key is a preliminary construction of $\mathbb{Z}_m$-equivariant Denjoy maps of the circle.

1. Introduction

This work is motivated by the study of the global behavior of a planar map having a fixed point which is asymptotically stable but is not a global attractor. For instance, the authors show in [3] that it can happen even when there are no periodic points different from the fixed point. They construct examples of planar dissipative homeomorphisms f such that the set $\mathrm{Rec}(f)\setminus\{p\}$ is a Cantor set with almost automorphic dynamics, being $\mathrm{Rec}(f)$ the set of recurrent points of f and p the asymptotically stable fixed point. Besides, they show that this behaviour is

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strongly related to the fact that these planar attractors have irrational rotation number.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism with a fixed point which is not a global attractor and its basin of attraction is unbounded. In that case, the theory of Prime Ends due to Carathéodory is applied and f induces an orientation preserving homeomorphism f^* in the space of prime ends. Since this space is homeomorphic to the circle, it is possible to associate a rotation number to f being the rotation number of f^*.

Under these condition, the authors in [9] prove that periodic orbits different from the fixed point will appear when the rotation number is rational. However the converse seems to be a very difficult problem with interesting and strong applications. One way to tackle this problem is assuming that f has some symmetry.

Given a Lie group Γ acting on $\mathbb{R}^2$, a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is said to be Γ-*equivariant* (or a Γ-symmetric map) if for all $x \in \mathbb{R}^2$ and $\sigma \in \Gamma$

$$f(\sigma x) = \sigma f(x).$$

When the homeomorphism is equivariant with respect to a subgroup of $\mathrm{O}(2)$, the symmetry forces the existence of a global attractor in all cases except $\mathbb{Z}_m$ and $\mathrm{SO}(2)$ (see [1]). Authors in [2] give a family of $\mathbb{Z}_m$-equivariant homeomorphisms with an asymptotically stable fixed point and rotation number $1/m$. So we might be led to think that the presence of the $\mathbb{Z}_m$-symmetry implies that the rotation number of the homeomorphism should be rational. One consequence would be that the asymptotically stable fixed point is a global attractor if and only if there are no periodic points different from the fixed point.

In this article we give examples showing that this is false. We prove the existence of $\mathbb{Z}_m$-equivariant and dissipative homeomorphisms with an asymptotically stable fixed point such that the induced map in the space of prime ends is conjugated to a Denjoy map, which is also $\mathbb{Z}_m$-equivariant. The idea is to reproduce the construction given in [3] in the context of symmetry. Hence, this paper shows that for $\mathbb{Z}_m$-equivariant homeomorphisms one can not guarantee that the rotation number is rational and proves the existence of $\mathbb{Z}_m$-equivariant homeomorphisms with some complicated and interesting dynamical features.

This work is organized as follows: In Section 2 we explain some notations and results of Denjoy maps in the circle that will be used. In Section 3 we explain the problem in the context of symmetry and construct $\mathbb{Z}_m$-equivariant Denjoy maps in the circle. In Section 4 we prove the existence of homeomorphisms of the plane which induce a symmetric Denjoy map in the space of prime ends with the help of some results in [3] and Section 3.

2. Notation and Denjoy maps in the circle

We introduce the same notation as in [3]:

We consider the quotient space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and points $\overline{\theta} = \theta + \mathbb{Z}$, with $\theta \in \mathbb{R}$. All figures are sketched on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, which is homeomorphic to $\mathbb{T}$.

The *distance between two points* $\overline{\theta}_1, \overline{\theta}_2 \in \mathbb{T}$ is

$$\mathrm{dist}_{\mathbb{T}}(\overline{\theta}_1, \overline{\theta}_2) = \mathrm{dist}_{\mathbb{R}}(\theta_1 - \theta_2, \mathbb{Z})$$

where $\mathrm{dist}_{\mathbb{R}}$ indicates the distance from a point to a set on the real line.

A *closed counter-clockwise arc* in $\mathbb{T}$ from p to q, $p \neq q$, will be denoted by $\alpha = \widehat{pq}$ and by $\dot{\alpha}$ its corresponding *open arc*.

We define the *cyclic order* as follows: Given three different points $p_0, p_1, p_2 \in \mathbb{T}$ we say that $p_0 \prec p_1 \prec p_2$ if $p_1 \in \widehat{p_0 p_2}$.

We define the *cyclic order for arcs* as follows: Given three pairwise-disjoint arcs $\alpha_0, \alpha_1, \alpha_2 \subset \mathbb{T}$, we say that $\alpha_0 \prec \alpha_1 \prec \alpha_2$ if $p_0 \prec p_1 \prec p_2$ for some $p_0 \in \alpha_0$, $p_1 \in \alpha_1$, $p_2 \in \alpha_2$.

A *Cantor set* C is any compact, totally disconnected, perfect subset of $\mathbb{T}$ (see [4]). The set C can be expressed as

$$C = \mathbb{T} \setminus \bigcup_{k=0}^{\infty} \dot{\alpha}_k,$$

where $\{\alpha_k\}_{k \geq 0}$ is a family of pairwise disjoint closed arcs in $\mathbb{T}$. The sets of *accessible* and *inaccessible points* will be denoted by A and I, respectively. The set A is composed by the end points of all α_k, thus

$$I = \mathbb{C} \setminus \bigcup_{k=0}^{\infty} \alpha_k.$$

Using C we define an equivalence relation on $\mathbb{T}$ by putting $\overline{\theta}_1 \sim \overline{\theta}_2 \pmod{C}$ if $\overline{\theta}_1 = \overline{\theta}_2$ or $\overline{\theta}_1, \overline{\theta}_2 \in \alpha_k$ for some $k \geq 0$. Thus, the *Cantor function* associated to C is a continuous function $\mathcal{P}: \mathbb{T} \to \mathbb{T}$ such that

$$\mathcal{P}(\overline{\theta}_1) = \mathcal{P}(\overline{\theta}_2) \Leftrightarrow \overline{\theta}_1 \sim \overline{\theta}_2.$$

The intuitive idea of this type of maps is to collapse every arc α_k into a point in such a way that the cyclic order is preserved. See [3] and [11] for more details.

The Cantor function $\mathcal{P}$ is onto and $\mathcal{P}(A)$ is a countable and dense subset of $\mathbb{T}$. See [8] for more details.

The *rotation* $R_{\overline{\eta}}: \mathbb{T} \to \mathbb{T}$ is defined by $R_{\overline{\eta}}(\overline{\theta}) = \overline{\theta + \eta}$, where $\eta, \theta \in \mathbb{R}$. Given a homeomorphism f of $\mathbb{T}$, the *f-orbit* starting at a point $\overline{\theta} \in \mathbb{T}$ is denoted by $\mathcal{O}(\overline{\theta})$. The *$\omega$-limit* of a point $\overline{\theta} \in \mathbb{T}$ is denoted by $\omega(\overline{\theta})$. It is well known (see [11]) that an orientation preserving homeomorphism f in $\mathbb{T}$ with rational rotation number has periodic points. However, if f has irrational rotation number $\rho(f) = \overline{\tau} \notin \mathbb{Q}$, then:

(a) $\omega(\overline{\theta})$ is independent of $\overline{\theta}$.
(b) f is semi-conjugate to the rigid rotation map $R_{\overline{\tau}}$. The semi-conjugacy takes the orbits of f to orbits of $R_{\overline{\tau}}$, is at most two to one on $\omega(\overline{\theta})$ and preserves orientation.
(c) If $\omega(\overline{\theta}) = \mathbb{T}$, then f is conjugate to $R_{\overline{\tau}}$ and the minimal set of f is the whole circle $\mathbb{T}$.
(d) If $\omega(\overline{\theta}) \neq \mathbb{T}$, then the semi-conjugacy from f to $R_{\overline{\tau}}$ collapses the closure of each open interval in the complement of $\omega(\overline{\theta})$ to a point. Moreover, the only minimal set of f is a Cantor set C in $\mathbb{T}$.

An orientation preserving homeomorphism $f\colon \mathbb{T} \to \mathbb{T}$ is said to be a Denjoy map if f has an irrational rotation number $\overline{\tau}$ and f is not conjugated to any rotation. In that case, f admits a Cantor minimal set C_f that attracts all orbits in the future and in the past — and every point in C_f is a recurrent point of f. We can associate to C_f a Cantor function $\mathcal{P}$ which is unique up to rotations and such that $\mathcal{P} \circ f = R_{\overline{\tau}} \circ \mathcal{P}$. So $\mathcal{P}$ is a semi-conjugacy from f to $R_{\overline{\tau}}$.

The construction in [11] of a Denjoy map with irrational rotation number $\overline{\tau}$ consists of choosing a point $\overline{\theta} \in \mathbb{T}$ and determining a family of pairwise disjoint open arcs in $\mathbb{T}$ with decreasing lengths whose sum is one and the complement of the union of all of them is a Cantor set. Each arc is identified with an element of the orbit of $\overline{\theta}$ via $R_{\overline{\tau}}$ which is always dense in $\mathbb{T}$. They also are put in the same order as the elements of the orbit, that is, preserving the cyclic order. These open intervals correspond to the gaps of the Cantor set and the union of the two extremes of all the intervals is the accessible set A of C_f. Next step is to define f on the union of the intervals and then extend the map to the closure.

However it is also possible to generate a Denjoy map considering the $R_{\overline{\tau}}$-orbit orbit of more than one point. Markley proved in [8] that given an irrational number $\tau \notin \mathbb{Q}$ and a countable set $D \neq \emptyset$ in $\mathbb{T}$ such that $R_{\overline{\tau}}D = D$, there exists a Denjoy map with rotation number $\overline{\tau}$ and minimal Cantor set with Cantor function verifying $\mathcal{P}(A) = D$ and being unique up to rotations. For instance, the construction in [11] corresponds to the countable set $\mathcal{P}(A)$ composed by the orbit of a unique point $\overline{\varphi} \in \mathbb{T}$ by the rotation $R_{\overline{\tau}}$, say

$$\mathcal{P}(A) = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\}$$

Figure 1 illustrates the construction of a Denjoy map considering the orbit of two different points, which is well explained in [3]. The corresponding countable set is

$$\mathcal{P}(A) = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\psi + n\tau} : n \in \mathbb{Z}\}.$$

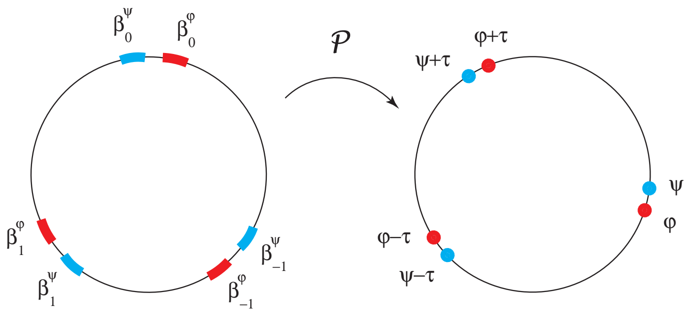

FIGURE 1. Construction of a Cantor set with two orbits

3. $\mathbb{Z}_m$-equivariant Denjoy maps in the circle

Observe that the construction in Figure 1 depends on the points $\overline{\varphi}, \overline{\psi} \in \mathbb{T}$. Since it can be made with every two points, in this section we consider $\overline{\psi} = R_{1/2}\overline{\varphi}$ in order to look for any symmetry of the Denjoy map (see Figure 3). This motivated us to study the more general case when the countable set D is the union of the orbits of points which are the rational rotation $R_{\frac{k}{m}}$ of a given point $\overline{\varphi} \in \mathbb{T}$ for same $k = 0, \dots, m-1$. That is, given a point $\overline{\varphi} \in \mathbb{T}$ and numbers $\tau \notin \mathbb{Q}$, $m \in \mathbb{N}$ we consider the set

$$\mathcal{P}(A) = \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

where $\overline{\varphi}^k = R_{k/m}\overline{\varphi}$, for $k = 0, \dots, m-1$. See Figure 2.

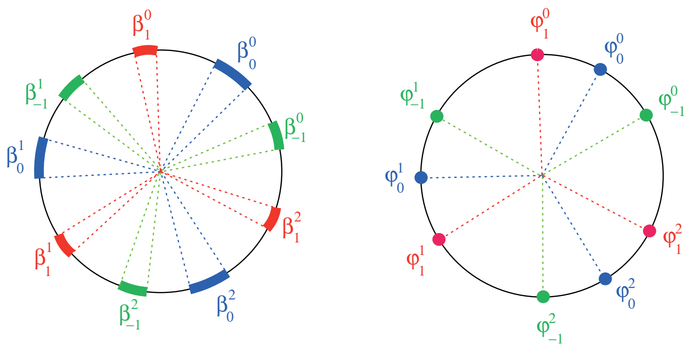

FIGURE 2. Construction of a Cantor set with 3 symmetric points

Since $\mathbb{Z}_m = \{R_{k/m}\}_{k=0}^{m-1}$, given a point $\overline{\varphi} \in \mathbb{T}$ we define the set $\{\overline{\varphi}^k\}_{k=0}^{m-1}$ as the *orbit of the group* $\mathbb{Z}_m$ (or $\mathbb{Z}_m$-orbit) of $\overline{\varphi}$, where $\overline{\varphi}^k = R_{k/m}\overline{\varphi}$.

In addition, $\mathbb{Z}_m$ is a cyclic compact Lie group generated by $R_{1/m}$. So, in order to stay that a map f is $\mathbb{Z}_m$-equivariant, we only need to prove that

$$f\left(\overline{\varphi}+\frac{1}{m}\right)=f(\overline{\varphi})+\frac{1}{m}.$$

See [6] for more details.

In this section we prove the existence of $\mathbb{Z}_m$-equivariant Denjoy maps. For simplicity, details of the proof will be explained only in case $m=2$ because the case $m>2$ is analogous. Firstly we construct a Cantor set which is invariant under the rotation $R_{1/2}$. Secondly we prove the existence of $\mathbb{Z}_2$-equivariant Denjoy maps in the circle with the constructed Cantor set as its minimal set. Finally, we give the keys of the proof in case $m>2$.

LEMMA 3.1 (Herman [7, p. 140]). *Let D_1, D_2 be two dense subsets in $\mathbb{R}$, and $\phi\colon D_1\to D_2$ be a strictly increasing map which is onto. Then ϕ can be extended to a monotone strictly increasing continuous map from $\mathbb{R}$ to $\mathbb{R}$.*

LEMMA 3.2. *Let $\tau\notin\mathbb{Q}$. Given a point $\overline{\varphi}$ in the circle, there exists a Cantor set C such that $R_{1/2}C=C$ and the associated Cantor function $\mathcal{P}\colon\mathbb{T}\to\mathbb{T}$ verifies:*

(a) *$\mathcal{P}(A)=\{\overline{\varphi+n\tau}/n\in\mathbb{Z}\}\cup\{\overline{\varphi'+n\tau}/n\in\mathbb{Z}\}\subset\mathbb{T}$, where A is the accessible set of C and $\overline{\varphi}'=R_{1/2}\overline{\varphi}$*

(b) *$\mathcal{P}$ is $\mathbb{Z}_2$-equivariant.*

PROOF. Given an angle $\tau\notin\mathbb{Q}$ and an orbit $\{\overline{\varphi+n\tau}/n\in\mathbb{Z}\}\subset\mathbb{T}$ we consider the countable dense set

$$\mathcal{D}=\{\overline{\varphi+n\tau}/n\in\mathbb{Z}\}\cup\{\overline{\varphi'+n\tau}/n\in\mathbb{Z}\}\subset\mathbb{T},$$

such that $\overline{\varphi}'=R_{1/2}\overline{\varphi}$.

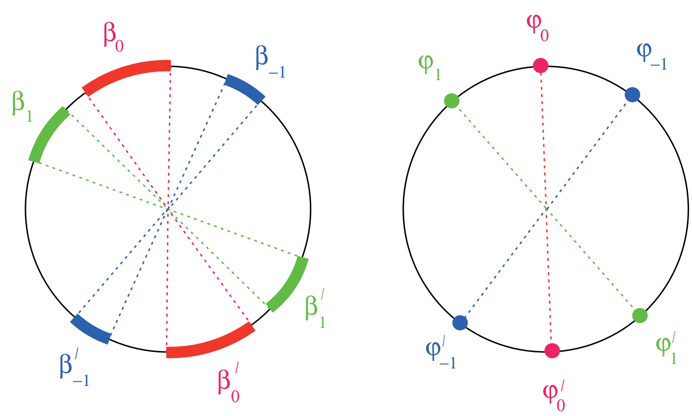

FIGURE 3. Construction of a Cantor set with two symmetric points

We claim that there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{T}$ such that

$$\bigcap_{n\in\mathbb{N}} A_n = \mathbb{T}\setminus \bigcup_{n\in\mathbb{N}}(\dot\beta_n \cup \dot\beta'_n \cup \dot\beta_{-n}\cup \dot\beta'_{-n}),$$

where $\{\beta'_n,\beta_n\}_{n\in\mathbb{Z}}$ is a family of pairwise disjoint closed arcs in $\mathbb{T}$ such that $\beta'_n = R_{\frac{1}{2}}\beta_n$. See Figure 3.

We construct the family $\{A_n\}_{n\in\mathbb{N}}$ by induction on $n\in\mathbb{N}$. It verifies:

(a) $A_n = \bigcup_{i=1}^{2k(n)} \gamma_i^n$, where $\gamma_i^n = \widehat{a_i^n b_i^n}$ is a closed arc in $\mathbb{T}$ with end points a_i^n, b_i^n. In addition $\gamma_i^n \cap \gamma_j^n = \emptyset$ for all $i\neq j$ with $i,j=1,\dots,2k(n)$, where $k(n)=2n-1$. Moreover, $\gamma^n_{k(n)+j} = R_{\frac{1}{2}}\gamma_j^n$ for $j=1,\dots,k(n)$.

(b) $\bigcup_{j=1}^{n}\{a_i^{j-1}, b_i^{j-1}\}_{i=1}^{2k(j-1)} \subset A_n$. That is, every extreme point of each arc γ_i^k of A_k belongs to A_n, for $k=1,\dots,n-1$.

(c) $A_n \subset A_{n-1}$.

(d) $R_{1/2}A_n = A_n$.

(e) The correspondence which associates to each point $\overline{\varphi}_N = \overline{\varphi + N\tau}$ the arc β_N, for all $|N|\le n-1$, preserves the cycle order. Equivalently, if $\overline{\varphi}_{N_1} \prec \overline{\varphi}_{N_2} \prec \overline{\varphi}_{N_3}$, then $\beta_{N_1}\prec\beta_{N_2}\prec\beta_{N_3}$ for all $|N_i|\le n-1$.

Consider $A_0=\mathbb{T}$. We associate to the points $\overline{\varphi}_0 = \overline{\varphi+0\tau}$ and $\overline{\varphi}'_0 = \overline{\varphi'+0\tau}$ two open arcs β_0 and $\beta'_0 = R_{1/2}\beta_0$, respectively, preserving the cyclic order such that $\beta_0\cap\beta'_0=\emptyset$ and $\mu(\mathbb{T}\setminus\beta_0\cup\beta'_0) < \mu(A_0)/2 = 1/2$. We define

$$A_1 = A_0\setminus\beta_0\cup\beta'_0 = \gamma_1^1\cup\gamma_2^1,$$

where γ_1^1,γ_2^1 are closed, $\gamma_1^1\cap\gamma_2^1=\emptyset$ and $\gamma_2^1 = R_{1/2}\gamma_1^1$. Clearly, $R_{1/2}A_1 = A_1$ and $A_1\subset A_0$. See Figure 4.

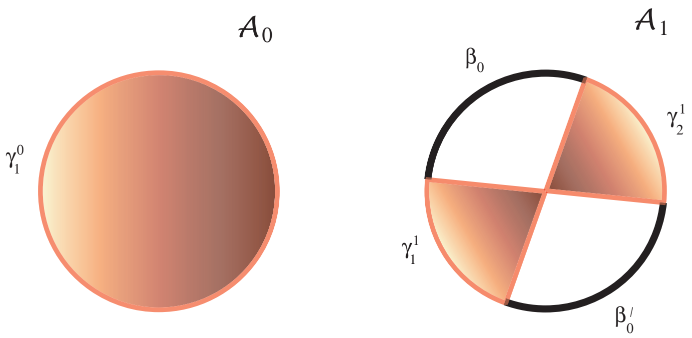

Figure 4. The subsets A_0, A_1

Now we associate to the points $\overline{\varphi}_1 = \overline{\varphi+\tau}$, $\overline{\varphi}_{-1} = \overline{\varphi-\tau}$, $\varphi'_1 = \overline{\varphi'+\tau}$ and $\varphi'_{-1} = \overline{\varphi'-\tau}$ four open arcs in A_1, say β_1, β_{-1}, $\beta'_1 = R_{\frac{1}{2}}\beta_1$ and $\beta'_{-1} = R_{\frac{1}{2}}\beta_{-1}$, respectively, preserving the cycle order and being pairwise-disjoint such that

$$\mu(A_1 \setminus \beta_1 \cup \beta_1' \cup \beta_{-1} \cup \beta_{-1}') \leq \frac{1}{2}\mu(A_1).$$

We define

$$A_2 = A_1 \setminus \beta_1 \cup \beta_1' \cup \beta_{-1} \cup \beta_{-1}' = \gamma_1^2 \cup .. \cup \gamma_6^2,$$

where $\gamma_1^2, \ldots, \gamma_6^2$ are closed, pairwise-disjoint and $\gamma_4^2 = R_{1/2}\gamma_1^2$, $\gamma_5^2 = R_{1/2}\gamma_2^2$ and $\gamma_6^2 = R_{1/2}\gamma_3^2$. Clearly, $R_{1/2}A_2 = A_2$ and $\{a_1^1, b_1^1, a_2^1, b_2^1\} \subset A_2 \subset A_1$. See Figure 5.

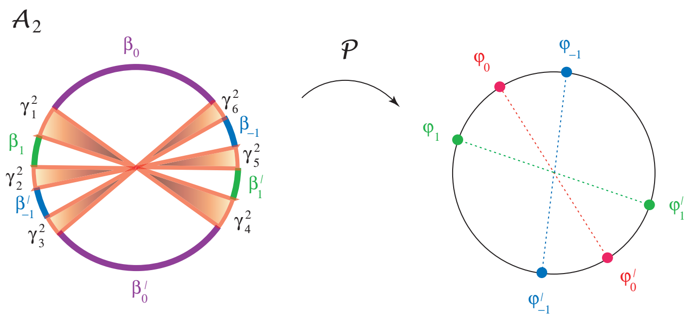

FIGURE 5. The subset A_2

Suppose by induction that there exists A_n verifying assumptions (a)–(e).

As $\tau \notin \mathbb{Q}$, the orbit of φ is dense in $\mathbb{T}$ so we can associate to the points $\overline{\varphi}_n = \overline{\varphi + n\tau}$, $\overline{\varphi}_{-n} = \overline{\varphi - n\tau}$, $\overline{\varphi}_n' = \overline{\varphi' + n\tau}$ and $\overline{\varphi}_{-n}' = \overline{\varphi' - n\tau}$ four open arcs in A_{n-1}, say β_n, β_{-n}, β_n' and β_{-n}', respectively, such that

- they are pairwise-disjoint and preserving the cyclic order,
- $\beta_n' = R_{1/2}\beta_n$ and $\beta_{-n}' = R_{1/2}\beta_{-n}$,
- if $\overline{\varphi}_{N_1} \prec \overline{\varphi}_{N_2} \prec \overline{\varphi}_{N_3}$, then $\beta_{N_1} \prec \beta_{N_2} \prec \beta_{N_3}$ for all $|N_i| \leq n$.

In addition, the arcs verify

$$\mu(\gamma_i^n \setminus \beta) \leq \frac{1}{2}\mu(\gamma_i^n)$$

where β can be one arc β_n, β_{-n}, β_n' and β_{-n}' or the union of two of such arcs. The arc γ_i^n is the component of A_n containing each β.

Figures 6 and 7 show how one component of A_n could be cut by two arcs or by one. But in both cases the obtained new arcs should have less measure than the first one. That is,

Case 1. Suppose the component γ_i^n is cut by only one arc β_n (respectively β_{-n}'), then there appear two new closed arcs $\gamma_j^{n+1}, \gamma_{j+1}^{n+1}$ in γ_i^n such that

$$\mu(\gamma_j^{n+1} \cup \gamma_{j+i}^{n+1}) \leq \frac{1}{2}\mu(\gamma_i^n).$$

Case 2. Suppose now the component γ_i^n is cut by the two arcs β_n and β'_{-n}. Then there appear three closed arcs $\gamma_j^{n+1}, \gamma_{j+1}^{n+1}, \gamma_{j+2}^{n+1}$ in γ_i^n such that

$$\mu(\gamma_j^{n+1} \cup \gamma_{j+i}^{n+1} \cup \gamma_{j+1}^{n+1}) \leq \frac{1}{2}\mu(\gamma_i^n).$$

Observe that in the first case the arc $\gamma_i'^n = R_{1/2}\gamma_i^n$ has also been cut by the arcs β'_n and β_{-n} such a way that the obtained new arcs are $\gamma_j'^{n+1} = R_{1/2}\gamma_j^{n+1}, \gamma_{j+1}'^{n+1} = R_{1/2}\gamma_{j+1}^{n+1}$ and $\gamma_{j+2}'^{n+1} = R_{1/2}\gamma_{j+2}^{n+1}$, all in $\gamma_i'^n$. The case that the arc γ_i^n had been cut only by one arc β is analogous.

We define

$$A_{n+1} = A_n \setminus \beta_n \cup \beta'_n \cup \beta_{-n} \cup \beta'_{-n} = \bigcup_{i=1}^{2k(n+1)} \gamma_i^{n+1},$$

where $k(n) = 2n-1$ and $\{\gamma_i^{n+1}\}_{i=1}^{2k(n+1)}$ is a family of pairwise-disjoint and closed arcs in $\mathbb{T}$ such that $\gamma_{k(m+1)+i}^{n+1} = R_{1/2}\gamma_i^{n+1}$ for all $i = 1, \ldots, k(n+1)$. Clearly, $R_{1/2}A_{n+1} = A_{n+1}$ and

$$\{a_i^1, b_i^1\}_{i=1}^{2k(1)} \cup \ldots \cup \{a_i^n, b_i^n\}_{i=1}^{2k(n)} \subset A_{n+1} \subset A_n \subset \ldots \subset A_1,$$

that is, the two extreme points of each arc γ_i^k of A_k belong to A_{n+1} for all $k = 1, \ldots, n$. Therefore, the family $\{A_n\}$ is well defined and

$$\bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\dot{\beta}_n \cup \dot{\beta}'_n \cup \dot{\beta}_{-n} \cup \dot{\beta}'_{-n})$$

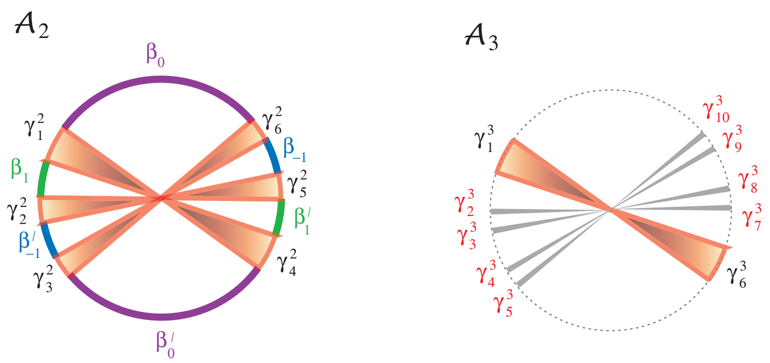

FIGURE 6. The subset A_3. Case 1

Now we claim that $\mu\Big(\bigcap_{n=0}^{\infty} A_n\Big) = 0$, where μ denotes the Lebesgue measure. By construction, for each $m \geq 0$ there exists a natural number $\sigma(m) \geq m$ such that $\mu(A_{\sigma(m)}) < \mu(A_m)/2$. The compact set $A_{\sigma(m)}$ is the $\sigma(m)-$th element of the family $\{A_n\}_{n=0}^{\infty}$ such that every component of A_m have been cut by an arc β_k or β'_k with $k \in \mathbb{Z}$ at least once.

Let us consider the subfamily $\{A_{\sigma_n(m)}\}_{n=0}^{\infty}$ where the compact set $A_{\sigma_{n+1}(m)}$ is an element of $\{A_n\}_{n=0}^{\infty}$ such that every component of $A_{\sigma_n(m)}$ have been cut by an arc β_k or β'_k with $k \in \mathbb{Z}$ at least once. Then,

$$\mu(A_{\sigma_{n+1}(m)}) \leq \frac{1}{2}\mu(A_{\sigma_n(m)}), \quad \text{for all } n \geq 0$$

and $A_{\sigma_{n+1}(m)} \subseteq A_{\sigma_n(m)}$ for all $n \geq 0$. So

$$0 \leq \mu(A_{\sigma_{n+1}(m)}) \leq \frac{1}{2^n}\mu(A_m) < \frac{1}{2^{n+1}}$$

and

$$0 \leq \mu\left(\bigcap_{n=0}^{\infty} A_n\right) \leq \mu\left(\bigcap_{n=0}^{\infty} A_{\sigma_n(m)}\right) = \lim_{n\to\infty} \mu(A_{\sigma_n(m)}) = 0.$$

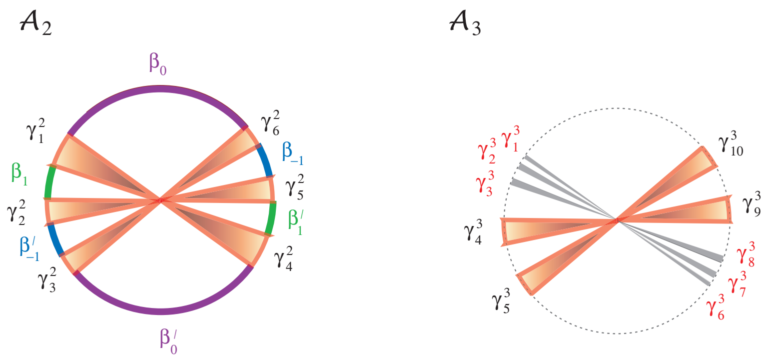

FIGURE 7. The subset A_3. Case 2

Now we claim that

$$C = \bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\dot{\beta}_n \cup \dot{\beta}'_n \cup \dot{\beta}_{-n} \cup \dot{\beta}'_{-n})$$

is a Cantor set. C is compact because it is an intersection of compact sets. Suppose that there exists a connected component of C different from a singleton $\{p\}$, then there exists an arc belonging to C. Therefore $\mu(C) \neq 0$. This is impossible since $\bigcap_{n\in\mathbb{N}} A_n$ has zero measure. So C is totally disconnected. Now take a point $x \in C$, then $x \in A_n$ for all $n \in \mathbb{N}$. In particular, for each $n \in \mathbb{N}$, there exists a $i(n)$-th compact component of A_n such that $x \in \gamma_{i(n)}^n$. Let x_n be an extreme of $\gamma_{i(n)}^n$ such that $x_n \neq x$. By construction, as n increases the component $\gamma_{i(n)}^n$ is being cut appearing other arcs with smaller measure, so the distance between x_n and x is also getting smaller. That is, for all $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ and a component $\gamma_{i(n)}^n$ such that, if x is an extreme of $\gamma_{i(n)}^n$,

$$d(x_n, x) = L(\widehat{x_n x}) = \mu(\gamma_{i(n)}^n)$$

where $L(\widehat{x_n x})$ denotes the length of the arc $\widehat{x_n x}$. Then,

$$d(x_n, x) = \mu(\gamma^n_{i(n)}) < \frac{1}{2}\mu(\gamma^{n-1}_{i(n-1)}) < \ldots < \frac{1}{2^n} < \varepsilon, \quad \text{for all } n > n_0.$$

And if $x \in \dot{\gamma}^n_{i(n)}$, then

$$d(x_n, x) < d(x_n, y) < \frac{1}{2^n} < \varepsilon, \quad \text{for all } n > n_0,$$

where y is the other extreme of $\gamma^n_{i(n)}$. Then C is perfect.

Note that, by construction, $R_{1/2}C = C$ and the set of accessible points of C is the union of the two extremes of all components γ^n_i for all $n \in \mathbb{N}$.

Let now to define the associated Cantor function $\mathcal{P}\colon \mathbb{T} \to \mathbb{T}$ of C. Consider the function

$$\mathcal{P}_*\colon \bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta}'_n) \to \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\varphi' + n\tau} : n \in \mathbb{Z}\}$$

such that $\mathcal{P}_*(\dot{\beta}_n) = \{\overline{\varphi}_n\}$ and $\mathcal{P}_*(\dot{\beta}'_n) = \{\overline{\varphi}'_n\}$, for all $n \in \mathbb{Z}$. It is easy to verify by induction that it is well defined considering the functions

$$\mathcal{P}_n\colon \bigcup_{i=-n}^{i=n} (\dot{\beta}_i \cup \dot{\beta}'_i) \to \{\overline{\varphi + N\tau} : |N| \leq n\} \cup \{\overline{\varphi' + N\tau} : |N| \leq n\}$$

such that $\mathcal{P}_n(\dot{\beta}_N) = \overline{\varphi}_N$, $\mathcal{P}_n(\dot{\beta}'_N) = \overline{\varphi}'_N$, $\mathcal{P}_n(\dot{\beta}_{-N}) = \overline{\varphi}_{-N}$, $\mathcal{P}_n(\dot{\beta}'_{-N}) = \overline{\varphi}'_{-N}$, for all $|N| \leq n$ and a given $n \in \mathbb{N}$.

Observe that $\mathcal{P}_*(\dot{\beta}'_n) = \mathcal{P}_*(\dot{\beta}_n) + 1/2$ for all $n \in \mathbb{Z}$ by construction. This implies that $\mathcal{P}_* \circ R_{1/2} = R_{1/2} \circ \mathcal{P}_*$ and $\mathcal{P}_*$ is $\mathbb{Z}_2$–equivariant.

Now we extend $\mathcal{P}_*$ to the function $\mathcal{P}$ at the points in the closure of $\bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta}'_n)$ applying Lemma 3.1.

Actually, $\Omega = \bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta}'_n)$ is open and $\mu(\mathbb{T} \setminus \Omega) = 0$, so Ω is dense in $\mathbb{T}$. Note that $\mathcal{D} = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\varphi' + n\tau} : n \in \mathbb{Z}\}$ is also dense because $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Let consider $\widehat{\Omega} \subset \mathbb{R}$ and $\widehat{\mathcal{D}} \subset \mathbb{R}$ be the lift of Ω and $\mathcal{D}$, respectively. As $\mathcal{P}_*$ is onto and preserves the cyclic order, the function $\widehat{\mathcal{P}}_*\colon \widehat{\Omega} \to \widehat{\mathcal{D}}$ is onto and strictly increasing. Then, by Lemma 3.1 the function $\widehat{\mathcal{P}}_*$ can be extended to a homeomorphism $\widehat{\mathcal{P}}\colon \mathbb{R} \to \mathbb{R}$ such that $\widehat{\mathcal{P}}(x+1) = \widehat{\mathcal{P}}(x) + 1$, for all $x \in \mathbb{R}$ (see Construction (3.II) in [7, p. 144] for more details).

Therefore $\widehat{\mathcal{P}}$ can be consider as a lift of a cyclic order preserving and continuous function $\mathcal{P}\colon \mathbb{T} \to \mathbb{T}$ such that $\mathcal{P} \circ R_{1/2} = R_{1/2} \circ \mathcal{P}$ and $\mathcal{P}(A) = \mathcal{D}$, where A is the union of the two extremes of every component γ^n_i for all $n \in \mathbb{N}$, the set of accessible points of the Cantor set C. $\square$

Proposition 3.3. *Let $\tau \notin \mathbb{Q}$. Given a point $\overline{\varphi} \in \mathbb{T}$, there exists a Denjoy map $f\colon \mathbb{T} \to \mathbb{T}$ such that:*

(a) *f is $\mathbb{Z}_2-$equivariant.*
(b) *$\rho(f) = \overline{\tau}$.*
(c) *f has a minimal Cantor set as in Lemma* 3.2.

Proof. Let $\tau \notin \mathbb{Q}$ and consider a point $\overline{\varphi} \in \mathbb{T}$. By Lemma 3.2, we can construct a Cantor set C such that $R_{1/2}C = C$ and the associated Cantor function $\mathcal{P}\colon \mathbb{T} \to \mathbb{T}$ is Z_2-equivariant.

Let now to define the Denjoy homeomorphism with minimal set the Cantor set C. We are going to apply Lemma 3.1 in the same way as the construction of $\mathcal{P}$ in Lemma 3.2.

Consider the bijection

$$f_*\colon \bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta'}_n) \to \bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta'}_n)$$

verifying:

- For each $n \in \mathbb{Z}$, $f_*(a_n) = a_{n+1}$, $f_*(b_n) = b_{n+1}$, $f_*(a'_n) = a'_{n+1}$ and $f_*(b'_n) = b'_{n+1}$.
- $f_* \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ f_*$.

Analogously to the definition of the Cantor function associated to C, it is easy to verify that f_* is well defined considering for each $n \in \mathbb{N}$ the bijection

$$f_n\colon \bigcup_{|i|\leq n} (\dot{\beta}_i \cup \dot{\beta'}_i) \to \bigcup_{|i|\leq n+1} (\dot{\beta}_i \cup \dot{\beta'}_i)$$

verifying:

- $f_n(a_i) = a_{i+1}$, $f_n(b_i) = b_{i+1}$, $f_n(a'_i) = a'_{i+1}$ and $f_n(b'_i) = b'_{i+1}$.
- $f_n \circ R_{\frac{1}{2}} = R_{1/2} \circ f_n$.

Let $\Omega = \bigcup_{n\in\mathbb{Z}} (\dot{\beta}_n \cup \dot{\beta'}_n)$. Consider $\widehat{\Omega}$ and $\widehat{f_*}\colon \widehat{\Omega} \to \widehat{\Omega}$ being the lift of Ω and f_*, respectively. As f_* is a bijection and preserves the cyclic order, $\widehat{f_*}$ is a strictly increasing bijection. Then, by Lemma 3.1, it can be extended to a homeomorphism $\widehat{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\widehat{f}(x+1) = \widehat{f}(x) + 1$ for all $x \in \mathbb{R}$ (see Construction (3.II) in [7, p. 144]). Moreover, $\widehat{f} \circ R_{1/2} = R_{1/2} \circ \widehat{f}$. So $\widehat{f}$ can be consider as a lift of an orientation preserving $\mathbb{Z}_2$-equivariant homeomorphism $f\colon \mathbb{T} \to \mathbb{T}$.

By construction, $R_{\overline{\tau}} \circ \mathcal{P} = \mathcal{P} \circ f$. Therefore, f is an orientation preserving $\mathbb{Z}_2$-equivariant Denjoy homeomorphisms of the circle with rotation number $\overline{\tau}$.□

Theorem 3.4. *Let $\tau \notin \mathbb{Q}$ and $m \in \mathbb{N}$. Given a point $\overline{\varphi} \in \mathbb{T}$, there exists a $\mathbb{Z}_m$-equivariant Denjoy map with rotation number $\overline{\tau}$.*

Proof. The case $m = 2$ is given in Proposition 3.3. The case $m \geq 3$ is analogous considering the dense and countable set

$$\mathcal{D} = \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

If $A_0 = \mathbb{T}$, we may associate to $\{\overline{\varphi}_0^k\}_{k=0}^{m-1}$, the orbit of $\overline{\varphi}_0 = \overline{\varphi}$, the family of pairwise-disjoint open arcs $\{\beta_0^k\}_{k=0}^{m-1}$ such that $\mu\Big(\bigcup\limits_{k=0}^{m-1} \beta_0^k\Big) < \mu(A_0)/2 = 1/2$ and define $A_1 = A_0 \setminus \bigcup\limits_{k=0}^{m-1} \beta_0^k$.

Analogously to case $m = 2$ we can define by induction the sets

$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{m-1} \beta_n^k \cup \beta_{-n}^k, \quad \text{for all } n \in \mathbb{N}$$

such that

(a) $A_n = \bigcup\limits_{i=1}^{mk(n)} \gamma_i^n$, where $\gamma_i^n = \widehat{a_i^n b_i^n}$ is closed and $\gamma_i^n \cap \gamma_j^n = \emptyset$ for all $i \neq j$ and $i, j = 1, \ldots, mk(n)$. Moreover, $k(n) = 2n-1$ and $\gamma_{i+mj}^n = R_{j/m}\gamma_i^n$ for $j = 1, \ldots, m-1$ and $i = 1, \ldots, k(n)$.
(b) $\bigcup\limits_{j=1}^{n} \{a_i^{j-1}, b_i^{j-1}\}_{i=1}^{mk(j-1)} \subset A_n$. That is, every extreme point of each arc γ_i^k of A_k belongs to A_n, for $k = 1, \ldots, n-1$.
(c) $A_n \subset A_{n-1}$.
(d) $R_{1/m}A_n = A_n$.
(e) The correspondence which associates to each point $\overline{\varphi}_N = \overline{\varphi + N\tau}$ the arc β_N, for all $|N| \leq n-1$, preserves the cycle order. Equivalently, if $\overline{\varphi}_{N_1} \prec \overline{\varphi}_{N_2} \prec \overline{\varphi}_{N_3}$, then $\beta_{N_1} \prec \beta_{N_2} \prec \beta_{N_3}$ for all $|N_i| \leq n-1$.

Then, the set

$$C = \bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{k=0}^{m-1} (\beta_n^k \cup \beta_{-n}^k)$$

is a Cantor set such that $R_{1/m}C = C$ and the set of accessible points of C is the union of the two extremes of all components γ_i^n for all $n \in \mathbb{N}$.

If we consider the functions

$$\mathcal{P}_*\colon \bigcup_{n\in\mathbb{Z}} \Big(\bigcup_{k=0}^{m-1} \dot{\beta}_n^k \Big) \longrightarrow \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

such that $\mathcal{P}_*(\dot{\beta}_n^k) = \overline{\varphi}_n^k$ for all $k = 0, \ldots, m-1$, $n \in \mathbb{Z}$, where $\overline{\varphi}_n^k = \overline{\varphi^k + n\tau}$. Analogously to case $m = 2$, $\mathcal{P}_*$ is well defined and for all $k = 0, \ldots, m-1$,

$$\mathcal{P}_*(\beta_n^k) = \mathcal{P}_*(\beta_n^0) + \frac{k}{m}.$$

By Lemma 3.1 we can extend $\mathcal{P}_*$ to a function $\mathcal{P}: \mathbb{T} \to \mathbb{T}$ which is the associated Cantor function to C such that $\mathcal{P}(A) = \mathcal{D}$ and it is $\mathbb{Z}_m$-equivariant.

Now consider the bijection

$$f_*: \bigcup_{n\in\mathbb{Z}} \left(\bigcup_{k=0}^{m-1} \dot{\beta}_n^k \right) \longrightarrow \bigcup_{n\in\mathbb{Z}} \left(\bigcup_{k=0}^{m-1} \dot{\beta}_n^k \right)$$

verifying:

- For each $n \in \mathbb{Z}$ and $k = 0, \dots, m-1$, $f_*(a_n^k) = a_{n+1}^k$ and $f_*(b_n^k) = b_{n+1}^k$, where a_n^k, b_n^k and a_{n+1}^k, b_{n+1}^k are the two extremes of β_n^k and β_{n+1}^k, respectively.
- $f_* \circ R_{1/m} = R_{1/m} \circ f_*$.

Analogously as in Proposition 3.3 f_* is well defined. Moreover, by Lemma 3.1, we can extend f_* to a function $f: \mathbb{T} \to \mathbb{T}$ such that:

(a) f is $\mathbb{Z}_m$-equivariant.
(b) $\rho(f) = \overline{\tau}$.
(c) f has a minimal Cantor set. □

4. Main result

In this section we construct $\mathbb{Z}_m$-equivariant homeomorphisms with an asymptotically stable fixed point and irrational rotation number. Consequently, symmetry properties does not give any extra information about global dynamics in the case $\mathbb{Z}_m$.

We say that $h: \mathbb{R}^2 \to \mathbb{R}^2$ is an admissible homeomorphism if h is orientation preserving, dissipative and has an asymptotically stable fixed point with proper and unbounded basin of attraction $U \subset \mathbb{R}^2$. Note that U is non empty, so the proper condition follows when the fixed point is not a global attractor. Since $h(U) = U$, we can obtain automatically the unboundedness condition if we suppose that h is area contracting.

Let $h: \mathbb{R}^2 \to \mathbb{R}^2$ be an admissible homeomorphism and consider $h: \mathbb{S}^2 \to \mathbb{S}^2$ being the compactification of h to the Riemann sphere. Hence $U \subset \mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. A *crosscut* C of U is an arc homeomorphic to the segment $[0,1]$ such that $a, b \notin U$ and $\dot{C} = C \setminus \{a,b\} \subset U$, where a and b are the extrems of C. Every crosscut divides U into two connected components homeomorphic to the open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Let x^* be a point in U. For convenience we will consider only de crosscut such that $x^* \notin C$. We denote by $D(C)$ the component of $U \setminus C$ that does not contain x^*. A *null-chain* is a sequence of pairwise disjoint crosscuts $\{C_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \operatorname{diam}(C_n) = 0 \quad \text{and} \quad D(C_{n+1}) \subset D(C_n),$$

where $\mathrm{diam}(C_n)$ is the diameter of C_n on the Riemann sphere.

Two *null-chains* are *equivalent* $\{C_n\}_{n\in\mathbb{N}} \sim \{C_n^*\}_{n\in\mathbb{N}}$ if given $m \in \mathbb{N}$

$$D(C_n) \subset D(C_m^*) \quad \text{and} \quad D(C_n^*) \subset D(C_m),$$

for n large enough. A *prime end* is defined as a class of equivalence of a null-chain and

$$\mathbb{P} = \mathbb{P}(U) = \mathcal{C}/\sim,$$

where $\mathcal{C}$ is the set of all null-chain of U and $\mathbb{P}$ is the space of prime ends.

The disjoint union $U^* = U \cup \mathbb{P}$ is a topological space homeomorphic to the closed disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ such that its boundary is precisely $\mathbb{P}$.

It is well studied in [10] that the Theory of Prime Ends implies that an admissible homeomorphism h induces an orientation preserving homeomorphism $h^*: \mathbb{P} \to \mathbb{P}$ in the space of prime ends. This topological space is homeomorphic to the circle, that is $\mathbb{P} \simeq \mathbb{T}$, and hence the rotation number of h^* is well defined, say $\overline{\rho} \in \mathbb{T}$. The *rotation number* for an admissible homeomorphisms is defined by $\rho(h) = \overline{\rho}$.

Note that two admissible homeomorphisms h_1, h_2 with the same basin of attraction U verify that

$$(h_1 \circ h_2)^* = h_1^* \circ h_2^*.$$

Let h be an admissible homeomorphisms with basin of attraction U. Suppose h is Z_m-equivariant and U is also invariant by $R_{1/m}$. Hence, the following holds:

$$h^* \circ R_{1/m}^* = R_{1/m}^* \circ h^*.$$

Since $R_{1/m}^*$ is a periodic homeomorphism of $\mathbb{T}^1$ with rotation number $1/m$, $R_{1/m}^*$ is conjugated to the linear rotation $R_{1/m}$ and h^* is said to be $\mathbb{Z}_m$-*equivariant in the space of prime ends.*

In [3] the authors prove that given an irrational number $\tau \notin \mathbb{Q}$ and a Denjoy map $f: \mathbb{T} \to \mathbb{T}$, there exists an admissible homeomorphism $h: \mathbb{R}^2 \to \mathbb{R}^2$ with rotation number $\overline{\tau}$. That motivates us to prove the existence of $\mathbb{Z}_m$-equivariant homeomorphisms of the plane which induce a Denjoy map in the circle of prime ends using the construction given in Section 3.

Proposition 4.1 (Corbato, Ortega and Ruiz del Portal, [3]). *Given a $\tau \in (0,1) \setminus \mathbb{Q}$ and a Denjoy map f, there exists an admissible homeomorphism with rotation number $\overline{\tau}$ and such that h^* is topologically conjugate to f.*

Theorem 4.2. *Given an irrational number $\tau \notin \mathbb{Q}$ and an integer $m > 1$, there exists a $\mathbb{Z}_m$-equivariant and admissible homeomorphism in $\mathbb{R}^2$ with rotation number $\overline{\tau} \in \mathbb{T}$ and such that induces a Denjoy map in the circle of prime ends which is also $\mathbb{Z}_m$-equivariant.*

Proof. Let $\tau \notin \mathbb{Q}$ be an irrational number, let $m > 0$ be an integer and let $\overline{\varphi} \in \mathbb{T}$ be a point in the circle. By Theorem 3.4, it is possible to construct a Denjoy map $f\colon \mathbb{T} \to \mathbb{T}$ which is $\mathbb{Z}_m$-equivariant and has minimal Cantor set C verifying that $R_{2k\pi/m}C = C$, for all $k = 0, \dots, m-1$.

By Proposition 4.1, there exists an admissible homeomorphism with rotation number $\overline{\tau}$ and such that the induced map in the space of prime ends h^* is topologically conjugate to f.

Authors in [3] define the homeomorphism h in polar coordinates by:

$$h: \quad \theta_1 = f(\theta), \quad \rho_1 = R(\theta, \rho)$$

where $R\colon \mathbb{T} \times [0, +\infty) \to [0, +\infty)$ such that

$$R(\overline{\theta}, \rho) = \begin{cases} \dfrac{1}{2}\rho & \text{if } \rho \leq \dfrac{1}{2}, \\ \left(\dfrac{3}{4} - \Pi(\overline{\theta})\right)(2\rho - 1) + \dfrac{1}{4} & \text{if } \dfrac{1}{2} < \rho \leq 1, \\ \dfrac{1}{2}\rho + \dfrac{1}{2} - \Pi(\overline{\theta}) & \text{if } \rho > 1, \end{cases}$$

and $\Pi\colon \mathbb{T} \to \mathbb{R}$ is such that

$$\Pi(\overline{\theta}) = \begin{cases} 0 & \text{if } \overline{\theta} \in C, \\ \dfrac{1}{k(|n|+1)} \dfrac{\operatorname{dist}_{\mathbb{T}}(\overline{\theta}, C)}{\operatorname{length}(\beta_n^k)} & \text{if } \overline{\theta} \in \beta_n^k. \end{cases}$$

We claim that h is $\mathbb{Z}_m$-equivariant. Since f is $\mathbb{Z}_m$-equivariant, we obtain that

$$f\left(\overline{\theta} + \frac{k}{m}\right) = f(\overline{\theta}) + \frac{k}{m}, \quad \text{for all } \overline{\theta} \in \mathbb{T}$$

so we only need to verify that

$$R\left(\overline{\theta} + \frac{k}{m}\right) = R(\overline{\theta})$$

or equivalently,

$$\Pi\left(\overline{\theta} + \frac{k}{m}\right) = \Pi(\overline{\theta}).$$

Indeed, if $\overline{\theta} + k/m \in C$ then $\overline{\theta} \in C$ and $\Pi(\overline{\theta} + k/m) = \Pi(\overline{\theta})$. Otherwise, there exists an arc β_n^j such that $\overline{\theta} + k/m \in \beta_n^j$. In this case, $\overline{\theta} \in \beta_n^{j-k}$ and $\operatorname{dist}_{\mathbb{R}}(\overline{\theta}, C) = \operatorname{dist}_{\mathbb{R}}(\overline{\theta} + k/m, C)$, so $\Pi(\overline{\theta} + k/m) = \Pi(\overline{\theta})$ and h is $\mathbb{Z}_m$-equivariant.

The rest of the proof is the same as the proof of Proposition 2 in [3]. □

Observe that the $\mathbb{Z}_m$-equivariant and admissible map constructed in Theorem 4.1 depends on the initial point $\overline{\varphi}$.

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