# GENERALIZED STURM–LIOUVILLE
# BOUNDARY CONDITIONS
# FOR FIRST ORDER DIFFERENTIAL SYSTEMS
# IN THE PLANE

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Abstract. We study asymptotically positively homogeneous first order systems in the plane, with boundary conditions which are positively homogeneous, as well. Defining a generalized concept of Fučík spectrum which extends the usual one for the scalar second order equation, we prove existence and multiplicity of solutions. In this way, on one hand we extend to the plane some known results for scalar second order equations (with Dirichlet, Neumann or Sturm–Liouville boundary conditions), while, on the other hand, we investigate some other kinds of boundary value problems, where the boundary points are chosen on a polygonal line, or in a cone. Our proofs rely on the shooting method.

## 1. Introduction

In his pioneering paper [23], Svatopluk Fučík provided, in 1976, a first approach to the study of the Dirichlet problem associated with a second order differential equation with a nonlinearity of asymmetric type. He considered the model

$$(1.1) \qquad \begin{cases} x'' + \mu x^+ - \nu x^- + g(x) = e(t), \\ x(0) = 0 = x(T), \end{cases}$$

where $\mu$, $\nu$ are real parameters, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, and $g\colon \mathbb{R} \to \mathbb{R}$, $e\colon [0, T] \to \mathbb{R}$ are continuous functions. In [23, Lemma 2.8], he defined the set $\Sigma$ which is now known as the *Fučík spectrum*, whose elements are the pairs $(\mu, \nu)$ for which there is a nontrivial solution to the positively homogeneous problem

$$(1.2) \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = 0, \\ x(0) = 0 = x(T). \end{cases}$$

The set $\Sigma$ is well known in this case, and can be explicitly computed. As shown in Figure 1, it is the union of a sequence of curves in the plane.
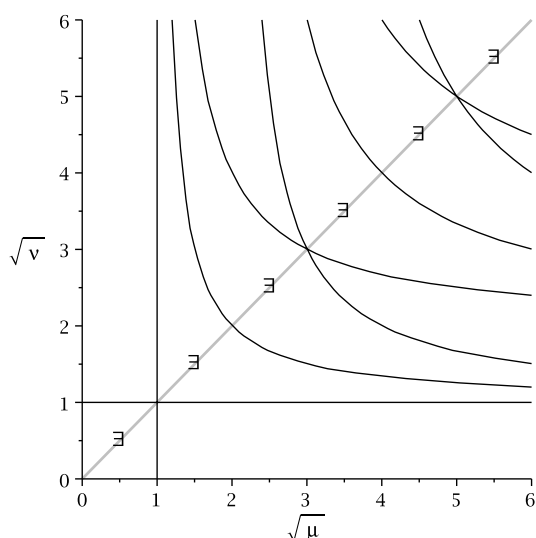


FIGURE 1. The Fučík spectrum for (1.2), with $T = \pi$

Assuming $g(x)$ to have a sublinear growth, Fučík then proved, in [23, Theorem 2.11], an existence result for (1.1), provided that $(\mu, \nu)$ belongs to some "nonresonance regions" determined by the set $\Sigma$. These regions are precisely the connected components of $\mathbb{R}^2 \setminus \Sigma$ which have nonempty intersection with the diagonal (in Figure 1, they are indicated by the symbol $\exists$; not to be misunderstood, the diagonal has been drawn only to show the symmetry of $\Sigma$, and does not belong to it. The same will be true for Figures 3, 4, 7 and 0).

This kind of result has been developed and generalized in several directions. Starting with Dancer [7], different types of boundary conditions were considered, such as the Neumann or the periodic ones (see also, e.g. [13], [15], [18]). Sturm–Liouville boundary conditions of the type

$$ax(0) + bx'(0) = 0 = cx(T) + dx'(T)$$

were considered, as well (we refer, for instance, to [11], [28], [34], [35]). The notion of Fučík spectrum naturally extends to these cases.

There is also a large literature about possible generalizations to partial differential operators (see, for instance, [9], [24]), but, for the sake of briefness, we prefer not entering into this subject.

As observed in [19], when $\mu, \nu > 0$, the scalar equation $x'' + \mu x^+ - \nu x^- = 0$ can be included in the more general framework of planar systems of the type

$$Ju' = \nabla H(u),$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix, and the $C^1$-Hamiltonian function $H \colon \mathbb{R}^2 \to \mathbb{R}$ is positive and positively homogeneous of degree 2, i.e.

(1.3) $\qquad 0 < H(\lambda u) = \lambda^2 H(u), \quad$ for every $\lambda > 0, \ u \in \mathbb{R}^2 \setminus \{0\}$.

For such kind of systems, the origin is an *isochronous center*, i.e. there exists $\tau > 0$ such that all the nontrivial solutions to $Ju' = \nabla H(u)$ wind the origin and are periodic with minimal period $\tau$. In [19], an existence result was given for the $T$-periodic problem associated with the system $Ju' = \nabla H(u) + f(t)$, assuming that $\tau$ is not a submultiple of $T$ (see also [16], [20], [21]). This fact agrees with the usual picture for the scalar periodic problem, when it is sufficient to assume that $(\mu, \nu)$ does not belong to the Fučík spectrum in order to avoid resonance.

The situation for the Dirichlet problem, or, more generally, for Sturm–Liouville boundary value problems, is substantially different from the periodic case. In the Dirichlet case, as already noticed in [8, Proposition 1], if $\mu, \nu > 0$, it is not sufficient that the pair $(\mu, \nu)$ does not belong to the Fučík spectrum to guarantee the existence of a solution to $x'' + \mu x^+ - \nu x^- = e(t)$ satisfying the boundary conditions, but some regions between the Fučík curves must also be avoided (see Section 3 for further details).

In this paper, we consider a planar system of the type

(1.4) $\qquad\qquad\qquad Ju' = \nabla H(u) + R(t, u),$

together with some boundary conditions including the Sturm–Liouville ones. Here, $H(u)$ satisfies (1.3) and $R \colon [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function. As a starting point, we want to generalize to this setting the nonresonance conditions introduced by Fučík in [23], following the scheme proposed in [19] for the periodic problem.

Concerning the boundary conditions, in view of the positive homogeneity of $H(u)$, it seems a natural choice to take the boundary points in some cones. Precisely, we fix a "starting" cone $\mathcal{C}_S$ and an "arrival" cone $\mathcal{C}_A$, and consider (1.4), together with the conditions

$$u(0) \in \mathcal{C}_S, \quad u(T) \in \mathcal{C}_A.$$

To the best of our knowledge, such a general setting has not been dealt with yet in literature, referring to planar positively homogeneous systems. In this way, not only we include in our study the Dirichlet problem (also known as "Bolza problem" in this framework), as well as the Neumann or the Sturm–Liouville ones, but we can also deal with some kind of nonlinear boundary conditions. Incidentally, we remark that boundary conditions of Sturm–Liouville type for planar systems have been considered, e.g. in [36], while, as reviewed in [32], nonlinear boundary conditions for second order scalar equations were already considered in [1], [2], [14], [17], [25], [31], where the upper and lower solutions method was developed.

We briefly summarize the content of the present article. In Section 2, we will state a general existence result under a suitable abstract nonresonance condition, which will be particularized in Section 3, concerning Sturm–Liouville boundary conditions, and in Section 4, for what we call the "polygonal problem". This last situation corresponds to taking the starting and the arrival cones as the gluing of two half-lines generating from the origin, instead of straight lines. Accordingly, we will give some examples for the scalar problem, examining in details the associated Fučík spectra. Contrarily to the case of linear boundary conditions, for which the Fučík spectrum is symmetric (i.e. $(\mu, \nu) \in \Sigma$ if and only if $(\nu, \mu) \in \Sigma$), this symmetry disappears when the boundary conditions are nonlinear (see Figures 7 and 9 in Section 4, where some unusual patterns appear).

The main tool in the proof of our existence results is the well-known shooting method. Since the uniqueness for the Cauchy problems associated with (1.4) is not a priori guaranteed, we need to approximate the continuous function $R(t, u)$ with more regular functions, and then follow a limit procedure.

In the second part of the article, we concentrate on the problem of multiplicity of solutions. Two different situations are analyzed: in Section 5, in the case when the origin is a stationary point, we assume a different asymptotic behavior of the nonlinearity at zero and at infinity. Similar problems have been considered by many authors (for the periodic problem, for instance, see [3], [4], [30]; for the Dirichlet problem, we refer, e.g. to [6], [12], [27], [33]). Under these assumptions, the number of solutions found usually depends on how large is the gap between zero and infinity (see Theorem 5.2 below). In Section 6, on the other hand, we consider a problem in dependence of a real parameter. Our main aim is to generalize a theorem by Hart, Lazer and McKenna [26, Theorem 1], concerning the scalar second order equation. In this setting, we need to consider linear boundary conditions, in order to exploit a technique, based on a suitable change of variable, which allows to reformulate the problem so as to obtain some kind of gap between the rotation numbers of "small" and "large" solutions. Multiple

solutions are then found for large values of the real parameter. For the periodic problem, results in this spirit were obtained, for instance, in [10], [22], [29], [37].

## 2. A semi-abstract result on cones

Let us denote by $\mathcal{P}$ the set of the $C^1$-functions $H:\mathbb{R}^2 \to \mathbb{R}$, with locally Lipschitz continuous gradient, which are positively homogeneous of degree 2 and positive, i.e. satisfying (1.3). As already remarked in the Introduction, if $H \in \mathcal{P}$, then there exists $\tau > 0$ such that all the nontrivial solutions to $Ju' = \nabla H(u)$ are periodic with minimal period $\tau$. The orbits of such a system, moreover, are star-shaped Jordan curves around the origin, on which the motion is performed clockwise. Notice that, setting $\zeta_\theta = \{(\rho\cos\theta, \rho\sin\theta) \mid \rho \geq 0\}$, for any $\theta \in \mathbb{R}$, if $\alpha, \beta$ are real constants such that $\alpha \leq \beta < \alpha + 2\pi$, the time needed by a nontrivial solution to $Ju' = \nabla H(u)$, starting from the ray $\zeta_\beta$, to reach the ray $\zeta_\alpha$, is given by

$$(2.1) \qquad \int_\alpha^\beta \frac{d\theta}{2H(\cos\theta, \sin\theta)}.$$

Thus, such a time is independent of the starting point on the ray $\zeta_\beta$, thanks to the positive homogeneity of $H(u)$, and this will often be exploited throughout the paper.

With these preliminaries, we now deal with a planar system of the kind

$$Ju' = \nabla H(u) + R(t, u),$$

where $H \in \mathcal{P}$ and $R:[0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous. Wishing to consider some kind of nonlinear boundary conditions, it will be natural to choose the boundary points in a cone, in view of the positive homogeneity of $H(u)$.

Thus, let us first specify what we mean by an *admissible cone* in the plane.

DEFINITION 2.1. A nonempty closed subset $\mathcal{C}$ of $\mathbb{R}^2$ is a *cone* if

$$[u \in \mathcal{C} \text{ and } \kappa \geq 0] \Longrightarrow \kappa u \in \mathcal{C}.$$

We say that a cone $\mathcal{C}$ is *admissible* if $\mathbb{R}^2 \setminus \mathcal{C}$ is disconnected.

For every $\overline{u} \in \mathbb{R}^2$, let us denote by $u(\,\cdot\,; \overline{u})$ the solution to

$$(2.2) \qquad \begin{cases} Ju' = \nabla H(u), \\ u(0) = \overline{u}, \end{cases}$$

which is unique and globally defined, since $H \in \mathcal{P}$. We define the continuous function $\mathcal{F}:\mathbb{R}^2 \to \mathbb{R}^2$ by

$$(2.3) \qquad \mathcal{F}(\overline{u}) = u(T; \overline{u}).$$

It is clear that, if $\mathcal{C}$ is a cone, then $\mathcal{F}(\mathcal{C})$ is also a cone, since, by homogeneity, $u(\,\cdot\,;\kappa\overline{u}) = \kappa u(\,\cdot\,;\overline{u})$, for every $\kappa \geq 0$.

We now fix two cones $\mathcal{C}_S$ and $\mathcal{C}_A$ (the "starting" and the "arrival" cones) and consider the boundary value problem

$$(2.4) \qquad \begin{cases} Ju' = \nabla H(u) + R(t,u), \\ u(0) \in \mathcal{C}_S, \ u(T) \in \mathcal{C}_A. \end{cases}$$

Let us state our main abstract result.

THEOREM 2.1. *Let the following assumption hold*:

(A) *The cone $\mathcal{C}_A$ is admissible and $\mathcal{F}(\mathcal{C}_S)$ has a nonempty intersection with at least two different connected components of $\mathbb{R}^2 \setminus \mathcal{C}_A$.*

*If, moreover,*

$$(2.5) \qquad \lim_{|u| \to +\infty} \frac{R(t,u)}{|u|} = 0, \quad \text{uniformly in } t \in [0,T],$$

*then problem (2.4) is solvable.*

PROOF. In view of assumption (A), there exist $\overline{u}_1, \overline{u}_2 \in \mathcal{C}_S \setminus \{0\}$ such that their images under the map $\mathcal{F}$ defined in (2.3) belong to two different connected components $\mathcal{W}_1$, $\mathcal{W}_2$ of $\mathbb{R}^2 \setminus \mathcal{C}_A$. Let $(R_n)_n$ be a sequence of locally Lipschitz continuous functions $R_n \colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $R_n(t,u) \to R(t,u)$ uniformly for $(t,u) \in [0,T] \times \mathbb{R}^2$. We consider, for $\lambda \geq 1$, the Cauchy problems

$$(2.6) \qquad \begin{cases} Ju' = \nabla H(u) + R_n(t,u), \\ u(0) = \lambda\overline{u}_1. \end{cases}$$

Setting $v(t) = u(t)/\lambda$, system (2.6) is equivalent to the following:

$$(2.7) \qquad \begin{cases} Jv' = \nabla H(v) + \dfrac{R_n(t,\lambda v)}{\lambda}, \\ v(0) = \overline{u}_1. \end{cases}$$

We denote by $v_{\lambda,n}(\,\cdot\,;\overline{u}_1)$ the solution to (2.7). By (2.5) and the uniform convergence of $R_n(t,u)$ to $R(t,u)$, we have that

$$\frac{R_n(t,\lambda v)}{\lambda} \to 0, \quad \text{as } \lambda \to +\infty \text{ and } n \to +\infty,$$

uniformly in $t \in [0,T]$ and $v$ in any compact subset of $\mathbb{R}^2$. By continuous dependence, cf. [5], for every fixed $\eta > 0$, there are $\lambda$ and $n$ sufficiently large such that

$$|v_{\lambda,n}(t;\overline{u}_1) - u(t;\overline{u}_1)| \leq \eta,$$

for every $t \in [0,T]$. Since $\mathcal{W}_1$ is open and $u(T;\overline{u}_1) \in \mathcal{W}_1$, taking $\eta$ sufficiently small there exist some sufficiently large $\lambda^* \geq 1$ and $n^* \geq 1$ such that $v_{\lambda,n}(T;\overline{u}_1)$ belongs to $\mathcal{W}_1$ for every $\lambda \geq \lambda^*$ and $n \geq n^*$. Analogously, enlarging $\lambda^*$ and $n^*$

if necessary, we will have that $v_{\lambda,n}(T;\overline{u}_2) \in \mathcal{W}_2$ for every $\lambda \geq \lambda^*$ and $n \geq n^*$, with the analogous convention in the notation.

We now fix $\lambda = \lambda^*$ and use the notation $v_n(t;\overline{u}) = v_{\lambda^*,n}(t,\overline{u})$. We consider the continuous path $\gamma\colon [0,1] \to \mathcal{C}_S$ defined by

$$
\gamma(s) = \begin{cases} (1-2s)\overline{u}_1 & \text{if } s \in [0,1/2], \\ (2s-1)\overline{u}_2 & \text{if } s \in [1/2,1]. \end{cases}
$$

By continuous dependence on the initial data, the map $s \mapsto v_n(T;\gamma(s))$ is continuous, for every $n \geq n^*$; moreover, $v_n(T;\gamma(0)) \in \mathcal{W}_1$ and $v_n(T;\gamma(1)) \in \mathcal{W}_2$. Since $\mathcal{C}_A$ separates $\mathcal{W}_1$ and $\mathcal{W}_2$, by continuity this implies that, for every $n \geq n^*$, there exists $s_n^*$ such that $v_n(T;\gamma(s_n^*)) \in \mathcal{C}_A$.

Let $u_n(t) = \lambda^* v_n(t;\gamma(s_n^*))$. By the above arguments, $(u_n)_n$ is uniformly bounded. Passing to a subsequence, we can assume that $s_n^* \to s^*$ for some $s^* \in [0,1]$. On the other hand, as $u_n(t)$ solves the differential equation in (2.6), we have that $(u_n')_n$ is also uniformly bounded, so that, by Ascoli–Arzelà Theorem, there is a subsequence $(u_{n_k})_k$ which uniformly converges to some continuous function $\widehat{u}(t)$. Then, $\widehat{u}(0) = \lambda^* \gamma(s^*) \in \mathcal{C}_S$ and, passing to the limit in

$$
u_{n_k}(t) = u_{n_k}(0) - J \int_0^t \left( \nabla H(u_{n_k}(r)) + R_{n_k}(r, u_{n_k}(r)) \right) dr,
$$

we have that $\widehat{u}(t)$ is a solution to the differential equation in (2.4). Since $\mathcal{C}_A$ is closed and $u_n(T;\gamma(s_n^*)) \in \mathcal{C}_A$, being

$$
\widehat{u}(T) = \lim_{n \to +\infty} u_n(T;\gamma(s_n^*)),
$$

we have that $\widehat{u}(T) \in \mathcal{C}_A$. The proof is thus completed. $\qquad\square$

As an immediate corollary, under assumption (A) we have existence in the case when the function $R(t,u)$ appearing in (2.4) does not depend on $u$. This fact reminds a classical feature of nonresonance for the forced system

$$
\begin{cases} Ju' = \nabla H(u) + e(t), \\ u(0) \in \mathcal{C}_S, \ u(T) \in \mathcal{C}_A. \end{cases}
$$

REMARK 2.2. The approximation process used above could have been avoided using the shooting approach without uniqueness developed in [6, Section 2].

## 3. The planar Sturm–Liouville boundary value problem

We now want to examine how assumption (A) can be rephrased in a more concrete way, when taking the boundary points on two straight lines in the plane (so that we consider Sturm–Liouville boundary conditions). To this aim, fix two lines passing through the origin, say $l_S$ and $l_A$.

Let us follow a nontrivial solution $u(t)$ to the equation $Ju' = \nabla H(u)$, for which it will be $u(t) \neq 0$ for every $t \in \mathbb{R}$, in view of the uniqueness. Starting from the vertical positive semi-axis and moving clockwise, at some nonnegative time instant $t_0$ such a solution will arrive at a point $u_1$ in $l_S$ (see Figure 2). We denote by $\tau_1$ the least positive time needed by $u(t)$ to arrive at a point $u_2$ in $l_A$, starting from $u_1$, and, correspondingly, we denote by $\theta_1$ the angular width covered in the time $\tau_1$. Continuing in covering the orbit described by $u(t)$, we define $\sigma_1$ as the least nonnegative time needed to encounter again $l_S$, starting from $u_2$, and, accordingly, we denote by $\theta_2$ the angular width spanned in the time $\tau_2$. In the same way, we define $\tau_2$ as the positive time needed to arrive once more on $l_A$ and $\sigma_2$ as the remaining nonnegative time to complete a whole revolution (see Figure 2 to visualize such definitions). In this way, $\tau = \tau_1 + \sigma_1 + \tau_2 + \sigma_2$ (and $2\theta_1 + 2\theta_2 = 2\pi$). It is important to underline that, in view of (2.1), the times $\tau_1, \tau_2, \sigma_1, \sigma_2$ are well-defined and independent of the choice of $u(t)$; moreover, if the lines $l_S$ and $l_A$ coincide, then $\sigma_1 = \sigma_2 = 0$, $\theta_1 = \pi$ and $\theta_2 = 0$.
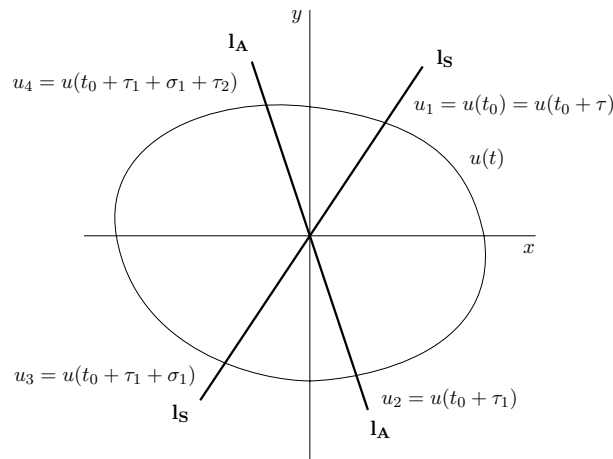


FIGURE 2. Following a solution $u(t)$ to define the times $\tau_1, \sigma_1, \tau_2, \sigma_2$

We are interested in a nonlinear boundary value problem of the kind

$$(3.1) \qquad \begin{cases} Ju' = \nabla H(u) + R(t, u), \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

where $H \in \mathcal{P}$ and $R(t, u)$ is a continuous function. We will prove the following theorem.

THEOREM 3.1. *Let $R(t, u)$ satisfy the sublinear growth condition* (2.5), *and assume that there exists a nonnegative integer $k$ such that one of the following*

*nonresonance assumptions holds: either*

(3.2) $$(k-1)\tau + \tau_1 + \tau_2 + \max\{\sigma_1, \sigma_2\} < T < k\tau + \min\{\tau_1, \tau_2\},$$

*or*

(3.3) $$k\tau + \max\{\tau_1, \tau_2\} < T < k\tau + \tau_1 + \tau_2 + \min\{\sigma_1, \sigma_2\}.$$

*Then, problem* (3.1) *has a solution.*

PROOF. Setting $\mathcal{C}_S = l_S$ and $\mathcal{C}_A = l_A$, we have to prove that assumption (A) holds, so to apply Theorem 2.1.

We focus on the case when condition (3.2) is assumed. Consider a nontrivial solution $u(t) = u(t; \overline{u})$ to the Cauchy problem

$$\begin{cases} Ju' = \nabla H(u), \\ u(0) = \overline{u} \in l_S, \end{cases}$$

and the corresponding function $\mathcal{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined in (2.3). Using polar coordinates, we can write $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, for some continuously differentiable functions $\rho(t) > 0$, $\theta(t) \in \mathbb{R}$. Hence,

$$-\theta' = 2H(\cos\theta, \sin\theta),$$

yielding

$$\int_{\theta(T)}^{\theta(0)} \frac{d\theta}{2H(\cos\theta, \sin\theta)} = T.$$

By (3.2) and the definition of $\theta_1$ and $\theta_2$, we get

$$2(k-1)\pi + \theta_1 + \theta_2 + \theta_1 < \theta(0) - \theta(T) < 2k\pi + \theta_1,$$

so that, since $\theta_1 + \theta_2 = \pi$,

$$\pi < \theta(0) - \theta(T) - 2(k-1)\pi - \theta_1 < 2\pi.$$

Hence, following the solution $u(t)$ when $t$ varies from 0 to $T$, we cover an angular width greater than $(2k-1)\pi + \theta_1$ and smaller than $2k\pi + \theta_1$, where $\theta_1$ has been defined above. It follows that the points $\mathcal{F}(\overline{u})$ and $\mathcal{F}(-\overline{u})$ lie in two different connected components of $\mathbb{R}^2 \setminus l_A$.

In the case when (3.3) is assumed, we analogously get

$$0 < \theta(0) - \theta(T) - 2k\pi - \theta_1 < \pi,$$

giving the conclusion with a similar argument. $\square$

In order to clarify the assumptions of Theorem 3.1, let us make some considerations about the autonomous problem

(3.4) $$\begin{cases} Ju' = \nabla H(u), \\ u(0) \in l_S, \quad u(T) \in l_A, \end{cases}$$

for $H \in \mathcal{P}$. It follows directly from the above definitions that such a problem has a nontrivial solution if and only if, for some nonnegative integer $k$,

$$(3.5) \qquad T - k\tau \in \{\tau_1, \tau_2, \tau_1 + \sigma_1 + \tau_2, \tau_2 + \sigma_2 + \tau_1\}.$$

We thus denote by $\mathcal{S}$ the set of those $H \in \mathcal{P}$ for which there exists a nonnegative integer $k$ such that (3.5) holds. The set $\mathcal{S}$ generalizes to the plane the classical notion of Fučík spectrum, which was originally introduced for the equation

$$(3.6) \qquad x'' + \mu x^+ - \nu x^- = 0,$$

being $\mu, \nu$ positive parameters. In this case,

$$(3.7) \qquad \tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}},$$

and the Fučík spectrum is defined as the set $\Sigma$ of the couples $(\mu, \nu)$ such that the Sturm–Liouville boundary value problem (3.4), with $u = (x, x')$ and

$$(3.8) \qquad H(x, y) = \frac{1}{2}(y^2 + \mu(x^+)^2 + \nu(x^-)^2),$$

has nontrivial solutions.

For instance, in the particular case of the Dirichlet problem, where $l_S = l_A = \{u = (x, y) \in \mathbb{R}^2 \mid x = 0\}$, we have $\sigma_1 = \sigma_2 = 0$, and

$$\tau_1 = \frac{\pi}{\sqrt{\mu}}, \quad \tau_2 = \frac{\pi}{\sqrt{\nu}},$$

so that the Fučík spectrum $\Sigma$ can be easily computed (see [23]), giving rise to the sequence of curves which has been depicted in Figure 1. On the other hand, in the case of the Neumann problem, where $l_S$ and $l_A$ both coincide with the horizontal axis, we have $\sigma_1 = \sigma_2 = 0$, and

$$\tau_1 = \tau_2 = \frac{1}{2}\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) = \frac{\tau}{2},$$

and also in this case the corresponding Fučík spectrum can be easily determined.

These two examples carry a substantial difference: while for Neumann boundary conditions it is enough to assume that $H \in \mathcal{P} \setminus \mathcal{S}$ in order to apply Theorem 3.1, this is not the case for the Dirichlet problem, as shown in [8, Proposition 1]. In particular, for the Dirichlet problem, we need the two stronger conditions (3.2) and (3.3), namely

$$\text{either} \quad k\tau < T < k\tau + \min\{\tau_1, \tau_2\} \quad \text{or} \quad k\tau + \max\{\tau_1, \tau_2\} < T < (k+1)\tau.$$

Notice that these two assumptions also avoid the existence of $T$-periodic solutions to (3.4), case which would give rise to resonance, as well, since $l_S = l_A$. As a consequence of Theorem 3.1, we thus obtain Fučík's original result [23, Theorem 2.11].

We conclude the section with two pictures of the Fučík spectra for the asymmetric equation (3.6), referring, respectively, to the problems

$$\begin{cases} x'' + \mu x^+ - \nu x^- = 0, \\ x(0) = 0, \ x(T) + x'(T) = 0, \end{cases} \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = 0, \\ x(0) + x'(0) = 0, \ x'(T) = 0. \end{cases}$$

Using (2.1), it is readily seen that, in the first case, we have

$$\tau_1 = \frac{\pi}{2\sqrt{\mu}} + \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}, \qquad \sigma_1 = \frac{\pi}{2\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}},$$

$$\tau_2 = \frac{\pi}{2\sqrt{\nu}} + \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}, \qquad \sigma_2 = \frac{\pi}{2\sqrt{\nu}} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}.$$

In the second one, it is

$$\tau_1 = \frac{\tau}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}, \qquad \sigma_1 = \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}},$$

$$\tau_2 = \frac{\tau}{2} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}, \qquad \sigma_2 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}.$$

Thus, in these two cases, condition (3.5) is quite easy to verify.
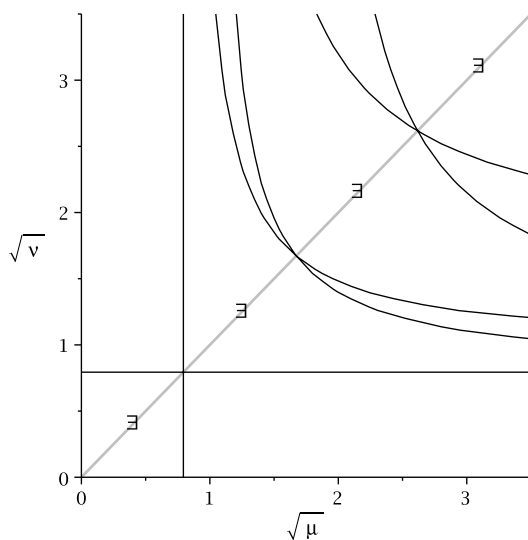


FIGURE 3. The Fučík spectrum for (3.6), with $x(0) = 0, \ x(\pi) + x'(\pi) = 0$.

Moreover, let us observe the following qualitative difference: while in the first case the first curves of the spectrum are two straight lines, parallel to the coordinate axes, like for Dirichlet boundary conditions, in the second one such lines d are not stright any more (see Figure 4). This is due to the fact that any solution of (3.4), with $H(u)$ given by (3.8), has to cross both the half-plane where $x$ is positive and the one where $x$ is negative. As in the classical
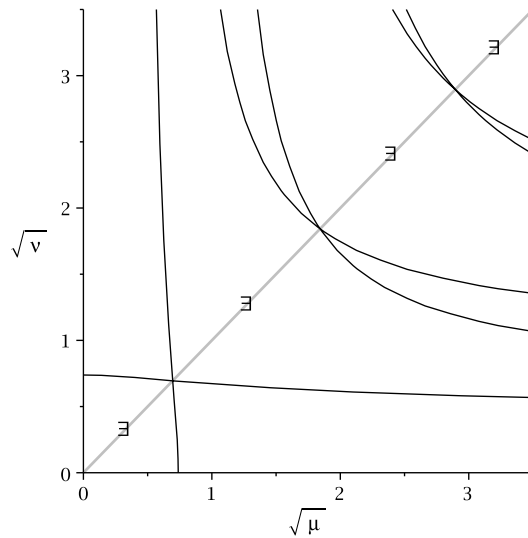
FIGURE 4. The Fučík spectrum for (3.6), with $x(0) + x'(0) = 0$, $x'(\pi) = 0$.

case introduced by Fučík, the regions for which there exists a solution are the connected components of $\mathbb{R}^2 \setminus \Sigma$ which have nonempty intersection with the diagonal.

## 4. The polygonal problem

In this section, we consider more general boundary conditions fitting in the setting of Theorem 2.1. We choose, as the starting and the arrival cones, two polygonal (piecewise linear) lines $p_S$ and $p_A$ which are the union of two half-lines emanating from the origin. For simplicity, we will assume that 0 is the only point of intersection of $p_S$ and $p_A$. Obviously, each of these polygonal lines divides the plane into two connected regions.

We want to study the boundary value problem

(4.1)
$$\begin{cases} Ju' = \nabla H(u) + R(t, u), \\ u(0) \in p_S, \quad u(T) \in p_A. \end{cases}$$

We need to consider two cases, depending on the mutual positions of $p_S$ and $p_A$.

*Case 1.* The polygonal line $p_S$ crosses both the connected regions of the plane separated by $p_A$. The situation is similar to the one in Section 3. As before, we follow a nontrivial solution $u(t)$ to the equation $Ju' = \nabla H(u)$, starting again from the vertical positive semi-axis and moving clockwise. In this way, at some nonnegative time instant $t_0$, $u(t)$ will arrive at a point $u_1$ in $p_S$, and we denote by $\tau_1$ the least time needed by $u(t)$ to arrive at a point $u_2$ in $p_A$, starting from $u_1$. Continuing in covering the orbit described by $u(t)$, we then define $\sigma_1$ as the least

time needed to encounter again $p_S$, starting from $u_2$, and, similarly, we define $\tau_2$ and $\sigma_2$ as the times needed to arrive once more on $p_A$ and $p_S$. The only difference with the Sturm–Liouville boundary value problem lies in the fact that the four angles determined by the intersection of $p_S$ and $p_A$ will all be different, in general (see Figure 5).
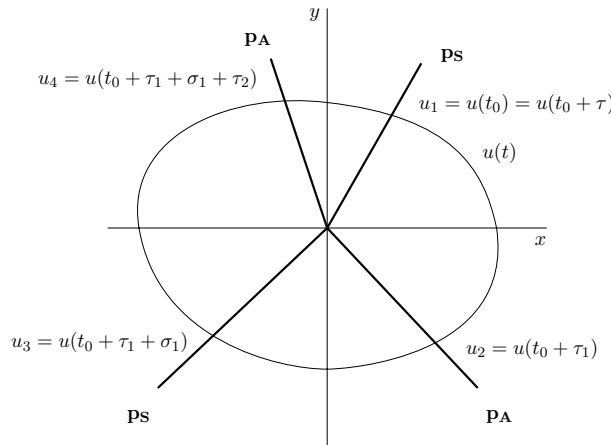


FIGURE 5. The situation described in Case 1

Aside from such a difference, this case can be treated exactly as the previous one, so that we have the following result.

THEOREM 4.1. *In the above configuration, the statement of Theorem 3.1 holds the same for problem* (4.1).

The proof can be done as for Theorem 3.1, except for the fact that, instead of the antipodal points $\overline{u}$ and $-\overline{u}$, one has to take two points on the two different half-lines of $p_S$. As one can expect, the picture concerning the Fučík spectrum, in this case, can be similar to the one of the Sturm–Liouville boundary value problem. However, if the polygonal lines are chosen so as to mix the two situations briefly described at the end of the previous section, some curious phenomena can appear. To give an idea, let us consider the scalar asymmetric equation (3.6), with the boundary conditions:

$$(4.2) \qquad \{x'(0) = 0, \; x(0) \geq 0\} \quad \text{or} \quad \{x(0) + x'(0) = 0, \; x(0) \leq 0\},$$

and

$$(4.3) \quad \{x(T) - x'(T) = 0, \; x'(T) \geq 0\} \quad \text{or} \quad \{x(T) + x'(T) = 0, \; x'(T) \leq 0\}.$$

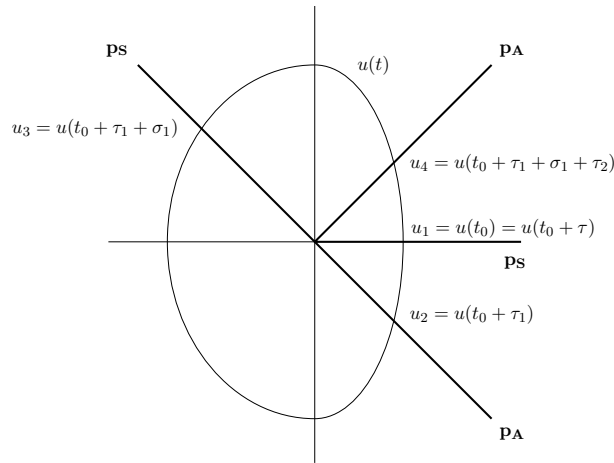We clarify such boundary conditions in Figure 6.

FIGURE 6. A "snapshot" of the boundary conditions (4.2), (4.3) for equation (3.6)

The Fučík spectrum is defined exactly as in the previous section. Recalling (3.7), a direct computation gives, in this case,

$$\tau_1 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}, \qquad \sigma_1 = \frac{\tau}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}},$$

and

$$\tau_2 = \frac{\tau}{2} - \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}} - \frac{1}{\sqrt{\nu}} \arctan \frac{1}{\sqrt{\nu}}, \qquad \sigma_2 = \frac{1}{\sqrt{\mu}} \arctan \frac{1}{\sqrt{\mu}}.$$
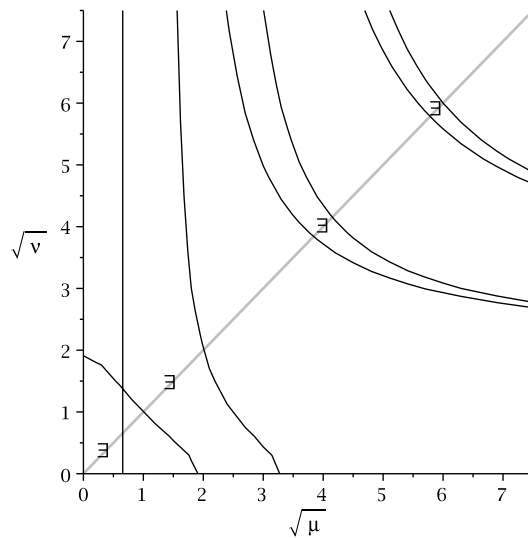


FIGURE 7. The Fučík spectrum for (3.6), with (4.2), (4.3), for $T = \pi/2$

We thus obtain, for the Fučík spectrum $\Sigma$, the curves depicted in Figure 7; notice the asymmetry coming from the fact that the four times $\tau_1, \sigma_1, \tau_2, \sigma_2$ are generally not obtainable one from the other by simply exchanging $\mu$ and $\nu$ (as it is the case for the Sturm–Liouville boundary value problem). The regions for which there exists a solution are not so intuitively clear as in the classical case. Referring to Figure 7, starting from the origin and proceeding along the diagonal, in $\mathbb{R}^2 \setminus \Sigma$ one enters the existence regions alternatively, being them the first, the third, the fifth, . . .

*Case 2.* The polygonal line $p_S$ is all contained (except for the origin) into only one of the two connected regions of the plane separated by $p_A$. Once again, we follow a nontrivial solution $u(t)$ to $Ju' = \nabla H(u)$ but, to simplify the notation, it is convenient to proceed in a slightly different way. We define $\widehat{\tau}_1$ as the least time needed by $u(t)$ to start from $p_S$ and arrive at $p_A$ moving clockwise. Assume that this has been done starting from $u_1 \in p_S$ and arriving at $u_2 \in p_A$, covering an angular width $\widehat{\theta}_1$. From there on, we resume our path along the orbit, defining $\widehat{\tau}_2$ as the positive time to arrive again on $p_A$ (at some point $u_3$) starting from $u_2$; we denote by $\widehat{\theta}_2$ the amplitude of the corresponding angular region. We further define $\widehat{\tau}_3$ as the time to reach $p_S$ in a point $u_4$, starting from $u_3$ (and $\widehat{\theta}_3$ as the angle covered in such a time), and $\widehat{\tau}_4$ as the time to reach again $p_S$, starting from $u_4$ (and $\widehat{\theta}_4$ correspondingly). In view of the mutual positions of $p_S$ and $p_A$, it is guaranteed that $\widehat{\tau}_1 + \widehat{\tau}_2 + \widehat{\tau}_3 + \widehat{\tau}_4 = \tau$; on the other hand, $\widehat{\theta}_1 + \widehat{\theta}_2 + \widehat{\theta}_3 + \widehat{\theta}_4 = 2\pi$. We depict this situation in Figure 8.
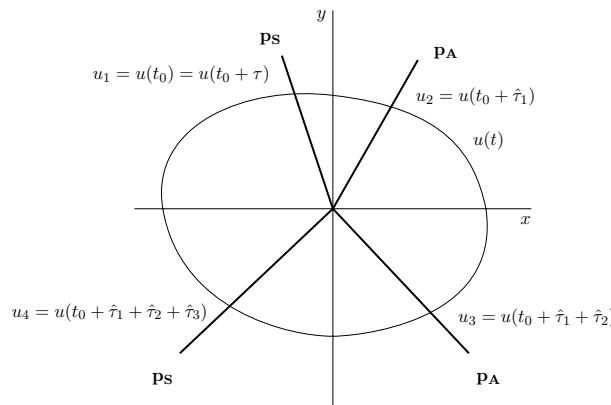


Figure 8. The situation described in Case 2

We now state the following result.

Theorem 4.2. *Let $R(t, u)$ satisfy the sublinear growth condition* (2.5), *and assume that there exists a nonnegative integer $k$ such that one of the following*

*nonresonance assumptions holds:*

$$(4.4) \qquad k\tau + \widehat{\tau}_1 + \max\{\widehat{\tau}_2, \widehat{\tau}_4\} < T < k\tau + \widehat{\tau}_1 + \widehat{\tau}_2 + \widehat{\tau}_4,$$

*or*

$$(4.5) \qquad k\tau + \widehat{\tau}_1 < T < k\tau + \widehat{\tau}_1 + \min\{\widehat{\tau}_2, \widehat{\tau}_4\}.$$

*Then, problem* (4.1) *has a solution.*

PROOF. We consider a nontrivial solution $u(t)$ to the Cauchy problem

$$\begin{cases} Ju' = \nabla H(u), \\ u(0) = \overline{u} \in p_S, \end{cases}$$

and the corresponding function $\mathcal{F} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined in (2.3). Passing to polar coordinates, it is possible to write $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$. Similarly as in the proof of Theorem 3.1, it is possible to see that (4.4) implies

$$\max\{\widehat{\theta}_2, \widehat{\theta}_4\} < \theta(0) - \theta(T) - 2k\pi - \widehat{\theta}_1 < \widehat{\theta}_2 + \widehat{\theta}_4.$$

This means that points belonging to different half-lines of $p_S$ are mapped into different connected components of $\mathbb{R}^2 \setminus p_A$ through the map $\mathcal{F}$. In particular, $\mathcal{F}(u_1)$ (where $u_1$ is as above) will lie in the region – delimited by $p_A$ – which contains $p_S$, while $\mathcal{F}(u_4)$ will belong to the interior of its complementary.

When (4.5) is assumed, we obtain

$$0 < \theta(0) - \theta(T) - 2k\pi - \widehat{\theta}_1 < \min\{\widehat{\theta}_2, \widehat{\theta}_4\},$$

giving rise to the opposite situation.

In both cases, assumption (A) is thus satisfied and we conclude in view of Theorem 2.1. □

We point out that, in this situation, the resonance phenomenon is quite different. In particular, as a counterpart of conditions (4.4) and (4.5), it is readily seen that the problem

$$\begin{cases} Ju' = \nabla H(u), \\ u(0) \in p_S, \ u(T) \in p_A \end{cases}$$

has a (nontrivial) solution if and only if, for some integer $k$,

$$T - k\tau \in \{\widehat{\tau}_1, \widehat{\tau}_1 + \widehat{\tau}_2, \widehat{\tau}_4 + \widehat{\tau}_1, \widehat{\tau}_4 + \widehat{\tau}_1 + \widehat{\tau}_2\}.$$

As an example, let us consider the scalar second order equation (3.6), with the boundary conditions

$$(4.6) \qquad \{x'(0) = 0, \ x(0) \geq 0\} \quad \text{or} \quad \{x(0) = 0, \ x'(0) \geq 0\},$$
$$(4.7) \qquad \{x(T) = 0, \ x'(T) \leq 0\} \quad \text{or} \quad \{x'(T) = 0, \ x(T) \leq 0\}.$$

This means that a solution to such a problem will start on one positive semi-axis and arrive on a negative one, no matter which. In this case, we will have

$$\widehat{\tau}_1 = \widehat{\tau}_4 = \frac{\pi}{2\sqrt{\mu}}, \qquad \widehat{\tau}_2 = \widehat{\tau}_3 = \frac{\pi}{2\sqrt{\nu}},$$

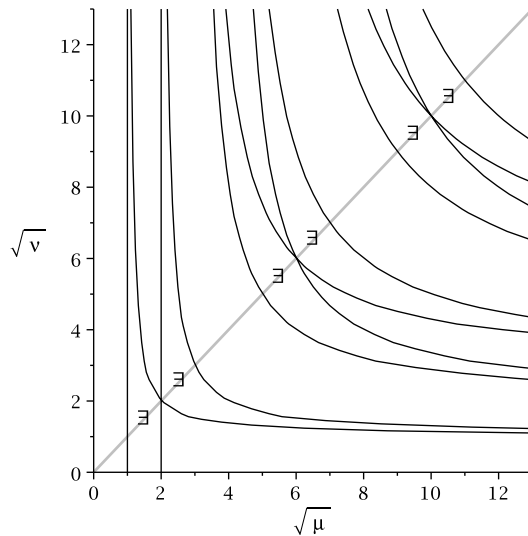and the Fučík curves overlap in a quite curious way, as shown in Figure 9.



FIGURE 9. The Fučík spectrum for (3.6), with (4.6), (4.7), for $T = \pi/2$

Notice that, as already visible in Figure 7, due to the nonlinear boundary conditions, the existence regions do not correspond to those having nonempty intersection with the diagonal.

## 5. Multiplicity of solutions
## in terms of the gap between zero and infinity

In this section, we provide an abstract multiplicity result for systems exhibiting a different behavior at zero and at infinity, and we use it to deduce multiplicity of solutions in some particular cases.

As in Section 2, let us fix two cones $\mathcal{C}_S$ and $\mathcal{C}_A$. We are interested in the following boundary value problem:

$$(5.1) \qquad \begin{cases} Ju' = F(t, u), \\ u(0) \in \mathcal{C}_S, \ u(T) \in \mathcal{C}_A, \end{cases}$$

being $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ continuous, locally Lipschitz continuous in its second variable, and satisfying

$$F(t, 0) \equiv 0.$$

Hence, $u(t) \equiv 0$ trivially satisfies (5.1). Moreover, we will assume that there exist $H_0, H_\infty \in \mathcal{P}$ such that

$$\text{(5.2)} \qquad \lim_{|u| \to 0} \frac{F(t,u) - \nabla H_0(u)}{|u|} = 0,$$

$$\text{(5.3)} \qquad \lim_{|u| \to +\infty} \frac{F(t,u) - \nabla H_\infty(u)}{|u|} = 0.$$

Given $\overline{u} \in \mathbb{R}^2$, let us denote by $u_0(t; \overline{u})$ and $u_\infty(t; \overline{u})$, respectively, the solutions to the Cauchy problems

$$\text{(5.4)} \qquad \begin{cases} Ju' = \nabla H_0(u), \\ u(0) = \overline{u}, \end{cases} \qquad \begin{cases} Ju' = \nabla H_\infty(u), \\ u(0) = \overline{u}. \end{cases}$$

We will write the starting and the arrival cones as union of half-lines: precisely,

$$\mathcal{C}_S = \bigcup_{\alpha \in \mathcal{I}_S} \eta_S^\alpha, \qquad \mathcal{C}_A = \bigcup_{\beta \in \mathcal{I}_A} \eta_A^\beta,$$

where $\mathcal{I}_S$ and $\mathcal{I}_A$ are sets of indexes, possibly infinite and uncountable, $\eta_S^\alpha$ and $\eta_A^\beta$ are half-lines emanating from the origin and the above unions are disjoint (except for the origin). Moreover, we define the nonnegative integers $n_0^{\alpha,\beta}, n_\infty^{\alpha,\beta}$ as follows: denoting by $\widehat{u}^\alpha$ the only point in $\eta_S^\alpha$ with $|\widehat{u}^\alpha| = 1$,

$$\text{(5.5)} \qquad n_0^{\alpha,\beta} = \#\{t \in \,]0, T[ \mid u_0(t, \widehat{u}^\alpha) \in \eta_A^\beta\},$$

$$\text{(5.6)} \qquad n_\infty^{\alpha,\beta} = \#\{t \in \,]0, T[ \mid u_\infty(t, \widehat{u}^\alpha) \in \eta_A^\beta\}.$$

The numbers $n_0^{\alpha,\beta}, n_\infty^{\alpha,\beta}$ just count the intersections of the solutions to the autonomous systems (5.4), starting on the half-line $\eta_S^\alpha$, with the half-line $\eta_A^\beta$. We can now state the following.

LEMMA 5.1. *For $\alpha \in \mathcal{I}_S$, $\beta \in \mathcal{I}_A$ fixed, there exist at least $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ nontrivial solutions to*

$$\text{(5.7)} \qquad \begin{cases} Ju' = F(t,u), \\ u(0) \in \eta_S^\alpha, \ u(T) \in \eta_A^\beta. \end{cases}$$

PROOF. We consider the Cauchy problem

$$\begin{cases} Ju' = F(t,u), \\ u(0) = \overline{u}^\alpha, \end{cases}$$

with $\overline{u}^\alpha \in \mathbb{R}^2 \setminus \{0\}$, and denote by $u(t; \overline{u}^\alpha)$ its solution. Since $F(t, 0) \equiv 0$, in view of the uniqueness it is possible to write

$$u(t; \overline{u}^\alpha) = \rho(t; \overline{u}^\alpha)(\cos \theta(t; \overline{u}^\alpha), \sin \theta(t; \overline{u}^\alpha)),$$

from which, in view of (1.3) and Euler's identity, we have the equation for the angle $\theta = \theta(t; \overline{u}^\alpha)$:

$$-\theta' = \frac{\langle F(t, u) | u \rangle}{|u|^2} = 2H_0(\cos\theta, \sin\theta) + \frac{\langle R_0(t, u) | u \rangle}{|u|^2},$$

where $R_0(t, u) = F(t, u) - \nabla H_0(u)$. By continuous dependence, from (5.2) we deduce that, for $|\overline{u}^\alpha|$ small, the last term gives a negligible contribution, so that we can infer that the number of intersections of $u(t; \overline{u}^\alpha)$ with $\eta_A^\beta$ for $t \in {]0, T[}$ is equal to $n_0^{\alpha,\beta}$. Similarly, by the "elastic property", since $F(t, u)$ has an at most linear growth, we have that if $|\overline{u}^\alpha|$ is sufficiently large, then $u(t; \overline{u}^\alpha)$ remains sufficiently far from the origin for every $t \in [0, T]$, and the number of intersections of $u(t; \overline{u}^\alpha)$ with $\eta_A^\beta$ for $t \in {]0, T[}$ is equal to $n_\infty^{\alpha,\beta}$.

We now exploit the continuous dependence of $\theta(t; \overline{u}^\alpha)$ on the initial datum $\overline{u}^\alpha$, to infer that, moving $\overline{u}^\alpha$ on $\eta_S^\alpha$, we will find $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ points $\overline{u}_i^\alpha \in \eta_S^\alpha$ such that $u(T; \overline{u}_i^\alpha) \in \eta_A^\beta$, giving the desired conclusion. $\square$

With these preliminaries, we can now state the following result.

THEOREM 5.2. *Problem* (5.1) *has at least*

$$(5.8) \qquad \sum_{\alpha \in \mathcal{I}_S} \sum_{\beta \in \mathcal{I}_A} |n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$$

*nontrivial solutions.*

PROOF. It suffices to repeat the reasoning in the proof of Lemma 5.1 for every couple of half-lines $\eta_S^\alpha$, $\eta_A^\beta$. Since, for every $\alpha \in \mathcal{I}_S$, $\beta \in \mathcal{I}_A$, we find $|n_\infty^{\alpha,\beta} - n_0^{\alpha,\beta}|$ initial conditions in $\eta_S^\alpha$ yielding a solution to problem (5.7), the thesis follows. $\square$

Observe that the sum appearing in (5.8) is well defined, since it is a sum of positive integers. Notice moreover that the number of solutions found through Theorem 5.2 could be infinite.

REMARK 5.3. The statement of Theorem 5.2 holds the same if we weaken conditions (5.2) and (5.3) into the following ones, respectively:

$$\lim_{|u| \to 0} \left[ \frac{\langle F(t, u) | u \rangle}{|u|^2} - 2H_0\left(\frac{u}{|u|}\right) \right] = 0,$$

$$\lim_{|u| \to +\infty} \left[ \frac{\langle F(t, u) | u \rangle}{|u|^2} - 2H_\infty\left(\frac{u}{|u|}\right) \right] = 0,$$

up to requiring that $F(t, u)$ has an at most linear growth in the variable $u$.

We now apply Theorem 5.2 to give a few corollaries concerning the problems treated in the previous sections.

Corollary 5.4. *Let $l_S$, $l_A$ be two lines through the origin. Moreover, let $k > m$ be two positive integers, and $H_0, H_\infty \in \mathcal{P}$. Denoting by $\tau^0, \tau_1^0, \sigma_1^0, \tau_2^0, \sigma_2^0$ the times defined in Section 3 for the Sturm–Liouville boundary value problem, relative to the system $Ju' = \nabla H_0(u)$, and using the same convention for $H_\infty$, assume that*

$$(k-1)\tau^0 + \tau_1^0 + \tau_2^0 + \max\{\sigma_1^0, \sigma_2^0\} < T < k\tau^0 + \min\{\tau_1^0, \tau_2^0\},$$
$$m\tau^\infty + \max\{\tau_1^\infty, \tau_2^\infty\} < T < m\tau^\infty + \tau_1^\infty + \tau_2^\infty + \min\{\sigma_1^\infty, \sigma_2^\infty\}.$$

*Assume that $F(t,u)$ satisfies (5.2) and (5.3), and $F(t,0) \equiv 0$. Then, the problem*

$$(5.9) \qquad \begin{cases} Ju' = F(t,u), \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

*has at least $4(k-m) - 2$ nontrivial solutions.*

Proof. It suffices to notice that, writing $l_S = \eta_S^1 \cup \eta_S^2$, $l_A = \eta_A^1 \cup \eta_A^2$, where $\eta_S^i, \eta_A^i$, $i = 1, 2$, are half-lines emanating from the origin, in such a way that the first half-line encountered starting on $\eta_S^1$ and moving clockwise is $\eta_A^1$, one has

$$n_0^{1,1} = n_0^{1,2} = n_0^{2,1} = n_0^{2,2} = k, \qquad n_\infty^{1,1} = n_\infty^{2,2} = m+1, \qquad n_\infty^{1,2} = n_\infty^{2,1} = m.$$

The conclusion follows.                                                      □

It is clear that several combinations of the conditions of the previous sections are possible, giving various different results of multiplicity. For example, we could have the following.

Corollary 5.5. *Let $l_S, l_A$ be two lines through the origin. Moreover, let $k > m$ be two positive integers, and $H_0, H_\infty \in \mathcal{P}$. Using the same notation as in Corollary 5.4, assume that*

$$k\tau^0 + \max\{\tau_1^0, \tau_2^0\} < T < k\tau^0 + \tau_1^0 + \tau_2^0 + \min\{\sigma_1^0, \sigma_2^0\},$$
$$m\tau^\infty + \max\{\tau_1^\infty, \tau_2^\infty\} < T < m\tau^\infty + \tau_1^\infty + \tau_2^\infty + \min\{\sigma_1^\infty, \sigma_2^\infty\}.$$

*Assume that $F(t,u)$ satisfies (5.2) and (5.3), and $F(t,0) \equiv 0$. Then, problem (5.9) has at least $4(k-m)$ nontrivial solutions.*

Concerning the polygonal problem, we will limit our attention to the case when the mutual positions of the two polygonal lines $p_S$ and $p_A$ are as in Case 2 of Section 4.

Corollary 5.6. *Let $p_S$, $p_A$ be two polygonal lines through the origin as in Section 4, Case 2. Moreover, let $k > m$ be two positive integers, and $H_0, H_\infty \in \mathcal{P}$.*

*With an analogous convention for the notation as in the previous corollaries, suppose that*

$$k\tau^0 + \widehat{\tau}_1^0 + \max\{\widehat{\tau}_2^0, \widehat{\tau}_4^0\} < T < k\tau^0 + \widehat{\tau}_1^0 + \widehat{\tau}_2^0 + \widehat{\tau}_4^0,$$
$$m\tau^\infty + \widehat{\tau}_1^\infty < T < m\tau^\infty + \widehat{\tau}_1^\infty + \min\{\widehat{\tau}_2^\infty, \widehat{\tau}_4^\infty\}.$$

*Assume that $F(t,u)$ satisfies (5.2) and (5.3), and $F(t,0) \equiv 0$. Then, the problem*

$$\begin{cases} Ju' = F(t,u), \\ u(0) \in p_S, \ u(T) \in p_A, \end{cases}$$

*has at least $4(k-m)+2$ nontrivial solutions.*

PROOF. In this case, setting $p_S = p_S^1 \cup p_S^2$, where $p_S^1$ is the half-line of $p_S$ which is closer to $p_A$ with respect to the clockwise motion, and $p_A = p_A^1 \cup p_A^2$, where $p_A^1$ is the first half-line of $p_A$ which is encountered by $p_S^1$ after a clockwise rotation, one has

$$n_0^{1,1} = n_0^{1,2} = n_0^{2,1} = k+1, \quad n_0^{2,2} = k, \quad n_\infty^{1,1} = m+1, \quad n_\infty^{1,2} = n_\infty^{2,1} = n_\infty^{2,2} = m,$$

yielding the desired conclusion.                                    □

In the above corollaries, we always assumed conditions of nonresonance type. Clearly, other types of corollaries of Theorem 5.2 can be easily obtained, without this restriction, at the expense of finding a lower number of solutions (see, e.g. [4], [6]).

REMARK 5.7. We could extend Theorem 5.2 assuming a more general control on $F(t,u)$, namely

$$F(t,u) = \gamma_0(t,u)\nabla H_0^{(1)}(u) + (1 - \gamma_0(t,u))\nabla H_0^{(2)}(u) + R_0(t,u)$$

in a neighbourhood of 0, and

$$F(t,u) = \gamma_\infty(t,u)\nabla H_\infty^{(1)}(u) + (1 - \gamma_\infty(t,u))\nabla H_\infty^{(2)}(u) + R_\infty(t,u)$$

at infinity, where $H_0^{(i)}, H_\infty^{(i)} \in \mathcal{P}$, $i = 1, 2$, satisfy $H_0^{(1)} \leq H_0^{(2)}$, $H_\infty^{(1)} \leq H_\infty^{(2)}$, the functions $\gamma_0(t,u)$, $\gamma_\infty(t,u)$ are continuous, taking values between 0 and 1, and the functions $R_0(t,u)$, $R_\infty(t,u)$ are negligible. The multiplicity result then comes from similar estimates on the gap of the angular speeds at zero and infinity (see, e.g. [4], [21]). For briefness, we do not enter into the details.

In the same spirit, other kinds of controls on the angular speed could be considered. For instance, in the case of a problem like

(5.10)
$$\begin{cases} x'' + f(t,x) = 0, \\ x(0) = 0 = x(T), \end{cases}$$

following [6] we could assume that $f(t, 0) \equiv 0$,

$$a_0(t) \leq \liminf_{|x| \to 0} \frac{f(t, x)}{x} \leq \limsup_{|x| \to 0} \frac{f(t, x)}{x} \leq b_0(t),$$

$$a_\infty(t) \leq \liminf_{|x| \to +\infty} \frac{f(t, x)}{x} \leq \limsup_{|x| \to +\infty} \frac{f(t, x)}{x} \leq b_\infty(t),$$

for suitable functions $a_0(t), a_\infty(t), b_0(t), b_\infty(t)$. Denoting by $\lambda_n(\gamma)$ the $n$-th eigenvalue of the Dirichlet problem

$$\begin{cases} x'' + \lambda\gamma(t)x = 0, \\ x(0) = 0 = x(T), \end{cases}$$

it was proved in [6, Theorem 1.1] that, if there exist two integers $m \leq n$ such that

$$(5.11) \qquad\qquad \lambda_n(a_0) < 1 < \lambda_m(b_\infty),$$

then there are $2(n - m + 1)$ solutions to (5.10).

Comparing with our previous results, in the case when the above functions $a_0, a_\infty, b_0, b_\infty$ are constant, let

$$H_0^{(1)}(x, y) = \frac{1}{2}(y^2 + \widehat{\lambda}_n x^2), \qquad H_\infty^{(2)}(x, y) = \frac{1}{2}(y^2 + \widehat{\lambda}_m x^2),$$

where $\widehat{\lambda}_1 < \widehat{\lambda}_2 < \dots$ are the usual eigenvalues of the Dirichlet problem on $[0, T]$. We observe that (5.11) is then equivalent to

$$(5.12) \qquad\qquad b_\infty < \widehat{\lambda}_m \leq \widehat{\lambda}_n < a_0.$$

Let us denote by $\eta_S^1 = \eta_A^1$ the positive vertical semi-axis, and by $\eta_S^2 = \eta_A^2$ the negative one. Using the same notation as in (5.5), (5.6), from (5.12) we deduce that

$$n_0^{1,1} \geq \left\lfloor \frac{n}{2} \right\rfloor, \quad n_0^{1,2} \geq \left\lceil \frac{n}{2} \right\rceil, \quad n_0^{2,1} \geq \left\lceil \frac{n}{2} \right\rceil, \quad n_0^{2,2} \geq \left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$n_\infty^{1,1} \leq \left\lfloor \frac{m-1}{2} \right\rfloor, \quad n_\infty^{1,2} \leq \left\lceil \frac{m-1}{2} \right\rceil, \quad n_\infty^{2,1} \leq \left\lceil \frac{m-1}{2} \right\rceil, \quad n_\infty^{2,2} \leq \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Here, for a real number $a$, the symbol $\lfloor a \rfloor$ denotes the largest integer less than or equal to $a$, while $\lceil a \rceil$ denotes the least integer greater than or equal to $a$. Arguing as in the proofs of Lemma 5.1 and Theorem 5.2, we then find at least

$$2\left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{m-1}{2} \right\rceil \right)$$

nontrivial solutions. This number can be checked to be exactly equal to $2(n - m + 1)$, thus agreeing with [6, Theorem 1.1].

We conclude this section by observing that, as in [6], we could characterize the nontrivial solutions obtained by their nodal properties. For briefness, we will not enter into details.

## 6. Multiplicity in dependence of a real parameter

In this section, we consider the issue of giving multiplicity results for a Sturm–Liouville boundary value problem, depending on a real parameter. Let $l_S$, $l_A$ be two fixed lines passing through the origin. We will deal with the problem

$$(6.1) \qquad \begin{cases} Ju' = \nabla H(u) + R(t, u) + sv^*(t), \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

with $s$ a positive parameter and $v^*(t)$ a fixed continuous function. Moreover, we will suppose that $R(t, u)$ fulfills the sublinear growth assumption (2.5), and $H(u)$ satisfies some nonresonance hypothesis.

We will still denote by $\tau$, $\tau_i$, $\sigma_i$, with $i = 1, 2$, the times introduced in Section 3. Moreover, $\tau_{\mathbb{A}}$, $\tau_{i,\mathbb{A}}$, $\sigma_{i,\mathbb{A}}$, with $i = 1, 2$, will refer to the times, defined as in Section 3, associated with the linear problem

$$\begin{cases} Ju' = \mathbb{A}u, \\ u(0) \in l_S, \ u(T) \in l_A, \end{cases}$$

where $\mathbb{A}$ is a symmetric $2 \times 2$ matrix. We can now state the main result of this section.

THEOREM 6.1. *Let $H \in \mathcal{P}$ satisfy, for a suitable nonnegative integer $k$,*

$$(6.2) \qquad (k-1)\tau + \tau_1 + \tau_2 + \max\{\sigma_1, \sigma_2\} < T < k\tau + \min\{\tau_1, \tau_2\}.$$

*Moreover, assume that there exist a function $w^*: [0, T] \to \mathbb{R}^2$ solving*

$$\begin{cases} Jw' = \nabla H(w) + v^*(t), \\ w(0) \in l_S, \ w(T) \in l_A, \end{cases}$$

*such that $0 \notin \mathcal{W}^* = \{w^*(t) : t \in [0, T]\}$, and two positive definite symmetric matrices $\mathbb{A} \leq \mathbb{B}$ satisfying*

$$(6.3) \qquad \langle \mathbb{A}(u - v)|u - v \rangle \leq \langle \nabla H(u) - \nabla H(v)|u - v \rangle \leq \langle \mathbb{B}(u - v)|u - v \rangle,$$

*for every $u, v \in \mathcal{W}^*$. Assume also that $\mathbb{A}$, $\mathbb{B}$ fulfill, for a suitable nonnegative integer $m$,*

$$(6.4) \qquad m\tau_{\mathbb{A}} + \max\{\tau_{1,\mathbb{A}}, \tau_{2,\mathbb{A}}\} < T < m\tau_{\mathbb{B}} + \tau_{1,\mathbb{B}} + \tau_{2,\mathbb{B}} + \min\{\sigma_{1,\mathbb{B}}, \sigma_{2,\mathbb{B}}\}.$$

*Lastly, suppose that $R(t, u)$ satisfies the sublinear growth condition (2.5).*

*Then, there exists $s^* > 0$ such that, for every $s \geq s^*$, problem (6.1) has at least $2|2(m-k)+1|+1$ solutions.*

The proof is similar to the one for the $T$-periodic problem given in [22, Theorem 1.1]. First, we change variables in (6.1), setting

$$\lambda = \frac{1}{s}, \quad y = \lambda u - w^*.$$

In this way, for $\lambda \in \,]0, +\infty[$, problem (6.1) is equivalent to

$$(6.5) \qquad \begin{cases} Jy' = \nabla H(y + w^*(t)) - \nabla H(w^*(t)) + f(t, y; \lambda), \\ y(0) \in l_S, \ y(T) \in l_A, \end{cases}$$

where

$$f(t, y; \lambda) = \begin{cases} \lambda R\left(t, \dfrac{1}{\lambda}(y + w^*(t))\right) & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

In view of (2.5), we have

$$(6.6) \qquad \lim_{\lambda \to 0^+} f(t, y; \lambda) = 0,$$

uniformly in $t \in [0, T]$ and $y \in B(0, r)$, being $B(0, r)$ the open ball centered in 0 with radius $r > 0$. Thus, $f(t, y; \lambda)$ is continuous up to $\lambda = 0$.

The next two statements are crucial in order to find a first solution to (6.1) for $\lambda$ small enough, via topological methods. In the following, we will denote by $B_\infty(0, r)$ the open ball in $L^\infty(0, T)$, centered in 0 and having radius $r > 0$, and by $\overline{B}_\infty(0, r)$ its closure.

LEMMA 6.2. *There exists $r^* > 0$ such that, if $y(t)$ solves*

$$(6.7) \qquad \begin{cases} Jy' = \nabla H(y + w^*(t)) - \nabla H(w^*(t)), \\ y(0) \in l_S, \ y(T) \in l_A, \end{cases}$$

*and $y \in \overline{B}_\infty(0, r^*)$, then $y(t) \equiv 0$.*

PROOF. If there were a sequence $(y_n)_n \in \overline{B}_\infty(0, 1/n)$ of nontrivial solutions to (6.7), by uniqueness it would be $y_n(t) \neq 0$ for every $t \in [0, T]$, and we could write

$$y_n(t) = \rho_n(t)(\cos \theta_n(t), \sin \theta_n(t)),$$

so that

$$(6.8) \qquad -\theta_n'(t) = \frac{\langle \nabla H(y_n + w^*(t)) - \nabla H(w^*(t))|y_n \rangle}{|y_n|^2}.$$

Fix $\overline{n}$ sufficiently large; in view of the strict inequalities in (6.4), we are able to find two matrices $\widehat{\mathbb{A}}, \widehat{\mathbb{B}}$, with $0 < \widehat{\mathbb{A}} \leq \mathbb{A} \leq \mathbb{B} \leq \widehat{\mathbb{B}}$, such that, replacing $\mathbb{A}$ with $\widehat{\mathbb{A}}$ and $\mathbb{B}$ with $\widehat{\mathbb{B}}$, (6.4) is still satisfied and (6.3) holds for every $u, v \in \{w^*(t) + x :$

$t \in [0, T]$, $|x| \leq 1/\overline{n}\}$. Therefore, since $y_n \in B_\infty(0, 1/\overline{n})$ for $n \geq \overline{n}$, from (6.8) we deduce

$$\int_{\theta_n(0)}^{\theta_n(T)} \frac{d\theta}{\langle \widehat{\mathbb{B}}(\cos\theta, \sin\theta)|(\cos\theta, \sin\theta)\rangle} \leq T \leq \int_{\theta_n(0)}^{\theta_n(T)} \frac{d\theta}{\langle \widehat{\mathbb{A}}(\cos\theta, \sin\theta)|(\cos\theta, \sin\theta)\rangle}.$$

Hence, in view of (6.4), it follows that

$$(6.9) \qquad 2m\pi + \theta_1 < \theta_n(0) - \theta_n(T) < (2m+1)\pi + \theta_1,$$

where $\theta_1$ is as in Section 3. This implies that it is not possible that $y_{\overline{n}}(0) \in l_S$ and $y_{\overline{n}}(T) \in l_A$ at the same time, a contradiction. $\qquad \square$

LEMMA 6.3. *For every $\delta > 0$, there exists $\lambda^* = \lambda^*(\delta)$ such that, for every $\lambda \in [0, \lambda^*(\delta)]$, there is a solution $y_\lambda$ to (6.5), satisfying $\|y_\lambda\|_\infty \leq \delta$.*

PROOF. In view of Lemma 6.2, it turns out that, for $\lambda = 0$, $y_0(t) \equiv 0$. We would like to continue such a solution in a neighbourhood of $\lambda = 0$. Let $L: D(L) \subset C^0([0, T]) \to C^0([0, T])$, with $D(L) = \{u \in C^1([0, T]) \mid u(0) \in l_S, u(T) \in l_A\}$, be defined by $Lu = Ju'$, and let $N_\lambda$ be the Nemytzkiĭ operator associated with the right-hand side of the equation in (6.5). Here, $C^k([0, T]) = C^k([0, T], \mathbb{R}^2)$ . If $\alpha$ does not belong to the spectrum of $L$, we can define $\Phi: C([0, T]) \times [0, 1] \to C([0, T])$ by

$$\Phi(y, \lambda) = (L - \alpha I)^{-1}(N_\lambda y - \alpha y).$$

In this way, (6.5) is equivalent to the fixed point problem

$$\Phi(y, \lambda) = y.$$

Moreover, in view of (6.6), we have that

$$\lim_{\lambda \to 0^+} \Phi(y; \lambda) = \Phi(y; 0),$$

uniformly in $y \in \overline{B}_\infty(0, r^*)$, where $r^* > 0$ is as above.

We are now going to compute the Leray–Schauder degree

$$\deg(\Phi(\,\cdot\,; \lambda) - I, B_\infty(0, r^*)),$$

showing that it is different from 0. To this aim, we first notice that, with the same proof as in [22, Lemma 2.2], we can deduce that there exists $\lambda^* > 0$ such that there are no solutions to (6.5) on the boundary of $B_\infty(0, r^*)$, for $\lambda \in [0, \lambda^*]$. We then pass to consider the problem

$$\begin{cases} Jy' = \sigma(\nabla H(y + w^*(t)) - \nabla H(w^*(t))) + \dfrac{(1-\sigma)}{2}(\mathbb{A} + \mathbb{B})y, \\ y(0) \in l_S, \ y(T) \in l_A. \end{cases}$$

Since $\mathbb{A} \leq \mathbb{B}$, we can use the same argument as the one to obtain (6.9), to deduce that this problem has only the trivial solution in $\overline{B}_\infty(0, r^*)$. By the homotopy

invariance of the topological degree and the previous considerations, it follows that

$$\deg\left(\Phi(\,\cdot\,;\lambda) - I, B_\infty(0, r^*)\right) = \deg\left(\Phi(\,\cdot\,;0) - I, B_\infty(0, r^*)\right)$$
$$= \deg\left((L - \alpha I)^{-1}\left(\frac{\mathbb{A} + \mathbb{B}}{2} - \alpha I\right) - I, B_\infty(0, r^*)\right) \neq 0,$$

since the operator involved in the last degree is linear and invertible.

We are now ready to conclude the proof of the lemma. So far, for every $\lambda \in [0, \lambda^*]$, we have found a solution $y_\lambda$ to (6.5), belonging to $B_\infty(0, r^*)$. We want to prove that

$$(6.10) \qquad \lim_{\lambda \to 0^+} \|y_\lambda\|_\infty = 0.$$

By contradiction, assume that there exist $\varepsilon > 0$, $(t_n)_n$ in $[0, T]$ and $(\lambda_n)_n$ in $[0, 1]$, with $\lambda_n \to 0$, such that, for every $n$,

$$|y_{\lambda_n}(t_n)| \geq \varepsilon.$$

Since $(y_{\lambda_n})_n$ is bounded in $L^\infty(0, T)$, being $y_{\lambda_n} \in B_\infty(0, r^*)$, the sequence $y_{\lambda_n}(t_n)$ is bounded, so there exists $\overline{y}$ such that, up to subsequences,

$$(6.11) \qquad y_{\lambda_n}(t_n) \to \overline{y};$$

obviously, $|\overline{y}| \geq \varepsilon$. Moreover, we can assume, for a subsequence, that $t_n \to \overline{t} \in [0, T]$.

We now consider, for every $n$, the Cauchy problem

$$(6.12) \qquad \begin{cases} Jy' = \nabla H(y + w^*(t)) - \nabla H(w^*(t)) + f(t, y; \lambda), \\ y(t_n) = y_{\lambda_n}(t_n). \end{cases}$$

By uniqueness, (6.12) is solved by $y_{\lambda_n}(t)$; moreover, in view of (6.11), we can infer, by continuous dependence, that

$$\lim_{n \to +\infty} y_{\lambda_n}(t) = \widehat{y}(t),$$

uniformly in $t \in [0, T]$, where $\widehat{y}(t)$ solves

$$\begin{cases} Jy' = \nabla H(y + w^*(t)) - \nabla H(w^*(t)), \\ y(\overline{t}) = \overline{y}. \end{cases}$$

It follows that $\widehat{y} \in \overline{B}_\infty(0, r^*)$, $\widehat{y}(\overline{t}) = \overline{y} \neq 0$, $\widehat{y}(0) \in l_S$, $\widehat{y}(T) \in l_A$, so that $\widehat{y}$ is a nontrivial solution to (6.7). This contradicts Lemma 6.2. $\qquad \square$

We now change variables, by setting $z = y - y_\lambda$, which transforms problem (6.5) into the following one:

$$(6.13) \qquad \begin{cases} Jz' = g(t, z; \lambda), \\ z(0) \in l_S, \ z(T) \in l_A, \end{cases}$$

where

$$g(t, z; \lambda) = \nabla H(z + y_\lambda(t) + w^*(t)) - \nabla H(y_\lambda(t) + w^*(t))$$
$$+ f(t, z + y_\lambda(t); \lambda) - f(t, y_\lambda(t); \lambda).$$

With the goal of applying a shooting method to prove Theorem 6.1, we are now going to consider the Cauchy problem

(6.14)
$$\begin{cases} Jz' = g(t, z; \lambda), \\ z(0) = \overline{z} \in l_S. \end{cases}$$

We will denote by $z(t, \overline{z}; \lambda)$ the solution to such a problem. We will count, similarly as in Section 5, the number of intersections of such a solution with the arrival line $l_A$; precisely, we define

$$n(z(t, \overline{z}; \lambda)) = \#\{t \in \ ]0, T[ \ | \ z(t, \overline{z}; \lambda) \in l_A\}.$$

We first state a lemma concerning the limit case $\lambda = 0$. In view of (6.6), (6.10) and the Lipschitz continuity of $\nabla H(u)$, we have

(6.15)
$$\lim_{\lambda \to 0^+} g(t, z; \lambda) = g(t, z; 0) = \nabla H(z + w^*(t)) - \nabla H(w^*(t)),$$

uniformly for every $t \in [0, T]$ and every $z \in B(0, r)$, with $r > 0$.

LEMMA 6.4. *Let $r^* > 0$ be as in Lemma 6.2. There exist two positive constants $\widehat{r}, \overline{r}$, with $4\widehat{r} < \overline{r} < r^*/4$, such that, if $\overline{z} \in \mathbb{R}^2$ satisfies $|\overline{z}| = \overline{r}$, then the solution $z(t)$ to the Cauchy problem*

(6.16)
$$\begin{cases} Jz' = \nabla H(z + w^*(t)) - \nabla H(w^*(t)), \\ z(0) = \overline{z}, \end{cases}$$

*satisfies, for every $t \in [0, T]$, $4\widehat{r} \leq |z(t)| \leq r^*/4$.*

The proof can be found in [22, Lemma 2.4] and is essentially based on Gronwall's Lemma, which can be used thanks to the Lipschitz continuity of $\nabla H(u)$.

We are now going to display the gap between "small" and "large" solutions to (6.14), in order to apply the shooting method and find multiple solutions to the original problem (6.1). The following lemma gives an estimate for the number of intersections of "small" solutions to (6.14) with the line $l_A$.

LEMMA 6.5. *Let $\overline{r} > 0$ be as in the previous lemma. Then, there exists $\lambda_1 \in \ ]0, \lambda^*]$ such that, if $|\overline{z}| = \overline{r}$, every solution $z(t, \overline{z}; \lambda)$ to (6.14), with $\lambda \in [0, \lambda_1]$, satisfies $n(z(t, \overline{z}; \lambda)) = 2m + 1$.*

PROOF. We first focus on (6.16), to show that $n(z(t, \overline{z}; 0)) = 2m + 1$. In view of Lemma 6.4, if $|\overline{z}| = \overline{r}$, then $z(\cdot, \overline{z}; 0)$ belongs to $\overline{B}_\infty(0, r^*/4)$, so that, reasoning on (6.8) as in Lemma 6.2, we can use (6.4) to argue that $z(t, \overline{z}; 0)$ meets $l_A$ exactly $2m + 1$ times in the time interval $]0, T[$, in view of (6.9).

We now turn our attention to the solution $z(t, \overline{z}; \lambda)$ to (6.14), with a *fixed* $\overline{z}$ such that $|\overline{z}| = \overline{r}$. By continuous dependence, in view of (6.15), we have that $z(t, \overline{z}; \lambda)$ will stay near $z(t, \overline{z}; 0)$, for $\lambda > 0$ sufficiently small.

Using Lemma 6.4, there exists $\lambda_0 > 0$ such that, for every $t \in [0, T]$ and $\lambda \in \,]0, \lambda_0]$, we have $|z(t, \overline{z}; \lambda)| \leq r^*/2$.

Moreover, by (6.10), we can assume that $\|y_\lambda\|_\infty \leq r^*/2$ for every $\lambda \in \,]0, \lambda_0]$. Thus, we can control the angle as in the proof of Lemma 6.2 and, since the inequalities in (6.9) are strict, we deduce $n(z(t, \overline{z}; \lambda)) = 2m + 1$, for every $\lambda \in \,]0, \lambda_0]$. Finally, in view of the continuous dependence and the compactness of $\partial B(0, \overline{r}) \subset \mathbb{R}^2$, we can find $\lambda_1 > 0$ as in the statement. $\qquad\square$

We *now fix* $\lambda \in \,]0, \lambda_1]$. The following lemma gives an estimate for the number of intersections of "large" solutions to (6.14) with the line $l_A$.

LEMMA 6.6. *There exists $\overline{R}_\lambda > \overline{r}$ such that, if $|\overline{z}| = \overline{R}_\lambda$, the solution $z(t, \overline{z}; \lambda)$ to (6.14) satisfies $n(z(t, \overline{z}; \lambda)) = 2k$.*

PROOF. Let us take $\overline{z} \in \mathbb{R}^2$ sufficiently far from the origin. By uniqueness, the usual system of polar coordinates is well defined for $z(t, \overline{z}; \lambda)$. Writing $z(t, \overline{z}; \lambda) = \rho(t, \overline{z}; \lambda)(\cos\theta(t, \overline{z}; \lambda), \sin\theta(t, \overline{z}; \lambda))$, we are led to the usual equation for $\theta'(t) = \theta'(t, \overline{z}; \lambda)$:

$$(6.17) \quad -\theta'(t) = \frac{\langle \nabla H(z + y_\lambda(t) + w^*(t)) | z \rangle}{|z|^2} - \frac{\langle \nabla H(y_\lambda(t) + w^*(t)) | z \rangle}{|z|^2}$$
$$+ \frac{\langle f(t, z + y_\lambda(t); \lambda) | z \rangle}{|z|^2} - \frac{\langle f(t, y_\lambda(t); \lambda) | z \rangle}{|z|^2}.$$

We notice that, for $|z| \to +\infty$, the second and the fourth term in the right-hand side vanish, since $\|y_\lambda + w^*\|_\infty$ is bounded. For what concerns the third summand, writing explicitly its expression we have

$$\frac{\langle f(t, z + y_\lambda(t); \lambda) | z \rangle}{|z|^2} = \lambda \frac{\langle R(t, \frac{1}{\lambda}(z + y_\lambda(t) + w^*(t))) | z \rangle}{|z|^2}.$$

In view of (2.5), fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|R(t, u)| \leq C_\varepsilon + \varepsilon|u|$ for every $t \in [0, T]$, and every $u \in \mathbb{R}^2$. Hence, the third term in (6.17) goes to 0 when $|z| \to +\infty$, as well. To estimate the remaining part, we write it as

$$(6.18) \quad \frac{\langle \nabla H(z + y_\lambda(t) + w^*(t)) | z \rangle}{|z|^2}$$
$$= \frac{\langle \nabla H(z + y_\lambda(t) + w^*(t)) - \nabla H(z) | z \rangle}{|z|^2} + \frac{\langle \nabla H(z) | z \rangle}{|z|^2}$$

and observe that the Lipschitz continuity of $\nabla H(u)$ gives

$$|\nabla H(z + y_\lambda(t) + w^*(t)) - \nabla H(z)| \leq L|y_\lambda(t) + w^*(t)|,$$

for a suitable constant $L > 0$, so that the first term of the right-hand side in (6.18) vanishes for $|z| \to +\infty$. By Euler's identity, this implies that

$$(6.19) \qquad -\theta'(t) = 2H(\cos\theta(t), \sin\theta(t)) + h(t, z(t)),$$

where $h$ is a function satisfying $h(t, z(t)) \to 0$, uniformly in $t \in [0, T]$, when $\min_{[0,T]} |z(t)| \to +\infty$. As a consequence, there exists a number $M > 0$ such that, if $|z(t, \overline{z}; \lambda)| > M$ for every $t \in [0, T]$, then, in view of (6.19) and the strict inequalities in (6.2), $z(t, \overline{z}; \lambda)$ encounters exactly $2k$ times the line $l_A$. It is now possible to find $\overline{R}_\lambda > M$ through the "elastic property", which ensures that, if we start with $|\overline{z}| = \overline{R}_\lambda$, it will be $|z(t, \overline{z}; \lambda)| > M$ for every $t \in [0, T]$. $\qquad\square$

We are now ready to conclude the proof of Theorem 6.1.

Let $l_S^1$ be one of the two half-lines of $l_S$, starting from the origin. In view of Lemmas 6.5 and 6.6 and the continuous dependence on the initial datum, there will be $|2(m-k)+1|$ distinct points $\overline{z}_{i,\lambda} \in l_S^1$ such that the solution $z(t, \overline{z}_{i,\lambda}; \lambda)$ to (6.14) satisfies $z(T, \overline{z}_{i,\lambda}; \lambda) \in l_A$, thus solving (6.13). Notice that, for $\lambda = 0$, the points $\overline{z}_{i,0}$ do not coincide with the origin. Returning to $u$ through the inverse change of variable

$$z(t) = \lambda u(t) - y_\lambda(t) - w^*(t),$$

we find the corresponding (all distinct) starting points $\overline{u}_{i,\lambda} = (\overline{z}_{i,\lambda} + y_\lambda(0) + w^*(0))/\lambda$ yielding to a solution to (6.1). In particular, since $z(0) \in l_S^1, z(T) \in l_A$, it will be $u(0, \overline{u}_{i,\lambda}; \lambda) \in l_S$, $u(T, \overline{u}_{i,\lambda}; \lambda) \in l_A$. The same reasoning could be done on the other half-line of $l_S$. Taking into account the further solution $y_\lambda(t)$ found in Lemma 6.3 which, we recall, for $\lambda = 0$ is identically 0, we see that, for $\lambda$ sufficiently small, $y_\lambda(t)$ does not coincide with any of the other solutions found. The proof of Theorem 6.1 is thus complete. $\qquad\square$

Remark 6.7. It has been essential to deal with linear boundary conditions, otherwise problems (6.5) and (6.14) would not have been equivalent to our original boundary value problem.

Remark 6.8. We have chosen to present a particular result of multiplicity, relying on conditions (6.2) and (6.4). At the end of the section, we will briefly pass through other combinations of the nonresonance conditions introduced in Section 3.

Remark 6.9. We briefly compare Theorem 6.1 with a result by Hart, Lazer and McKenna [26, Theorem 1], concerning the scalar Dirichlet problem

$$(6.20) \qquad \begin{cases} x'' + g(x) = h(t) + s\sin(t), \\ x(0) = 0 = x(\pi), \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function. Using the notation therein, we set $a = \lim_{\xi \to -\infty} g'(\xi)$ and $b = \lim_{\xi \to +\infty} g'(\xi)$, and assume $a < b$. Moreover, we fix $l =$

$l_S = l_A = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$. Writing the equation as a first order system, we have that (6.20) is equivalent to the problem

$$\begin{cases} Ju' = \nabla H(u) + R(t, u) + sv^*(t), \\ u(0) \in l, \ u(\pi) \in l, \end{cases}$$

where

$$u = (x, x'), \qquad H(u) = \frac{1}{2}(b(x^+)^2 + a(x^-)^2 + (x')^2), \qquad v^*(t) = (-\sin t, 0)$$

and $R(t, u)$ is a bounded function which can be computed explicitly. Thus, it turns out that the choice

$$\mathbb{A} = \mathbb{B} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$$

makes the control (6.3) true in the whole set $\{w^*(t) \mid t \in [0, \pi]\}$, where

$$w^*(t) = \frac{1}{b-1}(\sin t, \cos t)$$

(notice that $\sin(t)/(b-1)$ solves the equation $x'' + bx^+ - ax^- = \sin t$, for $t \in [0, \pi]$). We are going to show that Theorem 6.1 agrees with [26, Theorem 1]; not to confuse the notations, we will write the integer numbers appearing in that theorem with their original letters, but in Gothic style. Thus, $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) will be the number of zeros of a solution to $x'' + bx^+ - ax^- = 0$, with $x(0) = 0$, $x'(0) > 0$ (resp. $x(0) = 0$, $x'(0) < 0$), in $]0, \pi[$. Moreover, it is assumed in [26] that there exists a positive integer $\mathfrak{n}$ such that $\mathfrak{n}^2 < b < (\mathfrak{n}+1)^2$, $\mathfrak{n} \in \mathbb{N}$.

Observe now that assumption (6.2) becomes here $k\tau < \pi < k\tau + \min\{\tau_1, \tau_2\}$, so that $\mathfrak{m}^+ = \mathfrak{m}^- = 2k$. On the other hand, a comparison with condition (6.4) yields $\mathfrak{n} = 2m + 1$. Thus, since the assumption $a < b$ implies $2\mathfrak{n} > \mathfrak{m}^+ + \mathfrak{m}^-$ and thus $m \geq k$, we find the same number of solutions than in [26], i.e.

$$2\mathfrak{n} - (\mathfrak{m}^+ + \mathfrak{m}^-) + 1 = 2(2m+1) - 4k + 1 = 4(m-k) + 3.$$

Finally, notice that, in [26], also the case of a negative parameter $s$ has been considered. This situation can be recovered by the change of variable $\widetilde{x}(t) = -x(t)$ in (6.20).

For the sake of completeness, we now combine the nonresonance conditions given in Section 3 in different ways, and state the corresponding multiplicity results.

THEOREM 6.10. *Assume that $H \in \mathcal{P}$ satisfies, for a suitable nonnegative integer $k$,*

(6.21)         $k\tau + \max\{\tau_1, \tau_2\} < T < k\tau + \tau_1 + \tau_2 + \min\{\sigma_1, \sigma_2\}$

*instead of* (6.2). *Then, under all the other assumptions of Theorem* 6.1*, problem* (6.1) *has at least* $4|m - k| + 1$ *solutions.*

It is interesting to observe that, if $m = k$, Theorem 6.1 provides at least three solutions, while in Theorem 6.10 we only find a single solution, the one given by the topological degree argument. This can be explained by the fact that, roughly speaking, assuming together (6.2) and (6.4) implies that a gap between "small" and "large" solutions is already present even if $k, m$ are equal, since "large" solutions starting on a fixed half-line of $l_S$ intersect the arrival line $l_A$ a number of times equal to $2k$, while "small" ones intersect it $2m + 1$ times. In this last theorem, on the contrary, the number of intersections of "large" solutions starting on a fixed half-line of $l_S$, with $l_A$, is equal to $2k + 1$.

Acting on condition (6.4), on the other hand, we have the following counterparts of Theorems 6.1 and 6.10.

THEOREM 6.11. *Assume that* $H \in \mathcal{P}$ *satisfies* (6.2)*, and that* $\mathbb{A}$, $\mathbb{B}$ *fulfill, for a nonnegative integer* $m$,

$$(6.22) \quad (m - 1)\tau_{\mathbb{A}} + \tau_{1,\mathbb{A}} + \tau_{2,\mathbb{A}} + \max\{\sigma_{1,\mathbb{A}}, \sigma_{2,\mathbb{A}}\} < T < m\tau_{\mathbb{B}} + \min\{\tau_{1,\mathbb{B}}, \tau_{2,\mathbb{B}}\},$$

*instead of* (6.4). *Then, under all the other assumptions of Theorem* 6.1*, problem* (6.1) *has at least* $4|m - k| + 1$ *solutions.*

THEOREM 6.12. *Assume that* $H \in \mathcal{P}$ *satisfies* (6.21) *and* $\mathbb{A}, \mathbb{B}$ *fulfill* (6.22). *Then, under all the other assumptions of Theorem* 6.1*, problem* (6.1) *has at least* $2|2(m - k) - 1| + 1$ *solutions.*

REMARK 6.13. Comparing with the periodic boundary value problem, the number of solutions found, e.g. in [4], [10], [22], [29], is given in terms of the gap between the behavior at 0 and at $+\infty$, similarly as in Corollary 5.5, or with a similar interpretation, after a change of variables involving a real parameter, as shown in the proof of Theorem 6.1. Indeed, every complete turn around the origin makes the number $n(z(t, \overline{z}; \lambda))$ defined above increase of two unities, so that the final number of solutions found, e.g. in Theorems 6.10 and 6.11 corresponds to the gap between the rotation numbers of "small" and "large" solutions.

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