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# MULTIPLE SOLUTIONS TO A DIRICHLET EIGENVALUE PROBLEM WITH $p$-LAPLACIAN

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Abstract. The existence of a greatest negative, a smallest positive, and a nodal weak solution to a homogeneous Dirichlet problem with $p$-Laplacian and reaction term depending on a positive parameter is investigated via variational as well as topological methods, besides truncation techniques.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $1 < p < +\infty$, and let $j\colon \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be a Carathéodory function. Consider the homogeneous Dirichlet problem:

$$\begin{cases} -\Delta_p u = j(x, u, \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p$ denotes the $p$-Laplace differential operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. As usual, a function $u \in W_0^{1,p}(\Omega)$ is called a (weak) solution to (1.1) provided

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega j(x, u(x), \lambda) v(x)\, dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

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The literature concerning (1.1) is by now very wide and many existence, multiplicity, or bifurcation-type results are already available. In particular, a meaningful case occurs when

$$j(x,t,\lambda) := \lambda|t|^{q-2}t + |t|^{r-2}t, \quad (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \tag{1.2}$$

with $1 < q < p < r < p^*$. If $p = 2$ then (1.2) reduces to a so-called concave-convex nonlinearity and, after the seminal paper [1], the corresponding problem has been thoroughly investigated. A similar comment can also be made when $p \neq 2$, in which case we cite [2]. The work [6] treats jumping nonlinearities not explicitly depending on $\lambda$, i.e.

$$j(x,t,\lambda) := a(t^+)^{p-1} - b(t^-)^{p-1} + g(x,t) \quad \text{for all } (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \tag{1.3}$$

where $(a,b) \in \mathbb{R}^2$ lies above the Cuesta–de Figueiredo–Gossez [7] curve $\mathcal{C}$ in the Fučik spectrum of $-\Delta_p$ while the Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{t\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly in } x \in \Omega, \tag{1.4}$$

besides some standard growth condition. Under the assumption that a negative sub-solution $\underline{u}$ and a positive super-solution $\overline{u}$ to (1.1) are available, the existence of at least three nontrivial solutions, one negative, another positive, and the third nodal, within the order interval $[\underline{u}, \overline{u}]$ is established. If $a = b = \lambda$ then (1.3) becomes

$$j(x,t,\lambda) := \lambda|t|^{p-2}t + g(x,t). \tag{1.5}$$

The same conclusion as before still holds without requiring sub-super-solutions, provided $\lambda > \lambda_2$, the second eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, while $g$ turns out to be bounded on bounded sets, fulfils (1.4), and

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{p-2}t} = -\infty \quad \text{uniformly in } x \in \Omega; \tag{1.6}$$

see [5, Theorem 4.1]. Finally, [10] investigates the existence of multiple, both constant-sign and nodal, solutions to (1.1) whenever $\lambda$ is small enough, while [13] contains a bifurcation theorem, describing the dependence of positive solutions to (1.1) on the parameter $\lambda > 0$, where the reaction term $j$ takes the form

$$j(x,t,\lambda) := \lambda h(x,t) + g(x,t), \quad (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+,$$

for suitable $g, h\colon \Omega \times \mathbb{R} \to \mathbb{R}$.

Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f(x,t)| \le a_1(1 + |t|^{p-1}) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

$$\limsup_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} \le 0 \quad \text{uniformly in } x \in \Omega, \tag{1.7}$$

and, moreover, there exists $a_2, A_2 > 0$ satisfying

$$a_2 \leq \liminf_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} \leq A_2 \quad \text{uniformly in } x \in \Omega. \tag{1.8}$$

Setting $j(x,t,\lambda) := \lambda f(x,t)$, Problem (1.1) becomes

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In this paper we prove that (1.9) possesses at least three nontrivial solutions, one greatest negative $v_\lambda$, another smallest positive $u_\lambda$, and the third nodal $u_0$, with $v_\lambda \leq u_0 \leq u_\lambda$, provided $\lambda$ is sufficiently large; vide Theorem 5.1 as well as, regarding an explicit estimate of $\lambda$, Remark 4.2. It should be noted that, *for fixed* $\lambda > 0$, the nonlinearity (1.5) fulfils (1.7)–(1.8) once (1.4) and (1.6) hold true, whereas (1.7)–(1.8) do not imply neither (1.4) nor (1.6). As an example, take

$$g(x,t) := \begin{cases} |t|^{p-3}\sin(t|t|) & \text{if } |t| \leq 1, \\ \lambda|t|^{p-2}t(\sin(t|t|) - 2) - \lambda s(t)(\sin(s(t)) - 2) + \sin(s(t)) & \text{otherwise}, \end{cases}$$

where $p > 1$ and $s(t)$ denotes the signum function.

Very recently, in [3], the same conclusion has been achieved supposing $p > N$, the function $f$ independent of $x$, and $\lambda > 0$ small enough. Significantly, no condition at infinity is taken on, but one requires that

$$\lim_{t \to 0} \frac{f(t)}{|t|^{p-2}t} = L \in \mathbb{R}^+, \tag{1.10}$$

besides a suitable condition for $F(z) := \int_0^z f(t)\,dt$ near zero. Obviously, (1.10) forces (1.8).

Our results are obtained via variational and topological methods, as well as truncation arguments. Some of these techniques have already been employed in [5]. Possible extensions to non-smooth settings will be addressed in a future work.

## 2. Basic assumptions and auxiliary results

Let $(X, \|\cdot\|)$ be a real Banach space. Given a set $V \subseteq X$, write $\overline{V}$ for the closure of $V$, $\partial V$ for the boundary of $V$, and $\operatorname{int}(V)$ for the interior of $V$. If $x \in X$ and $\delta > 0$ then

$$B_\delta(x) := \{z \in X : \|z - x\| < \delta\}.$$

The symbol $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of $X$, $\langle \cdot\,, \cdot \rangle$ indicates the duality pairing between $X$ and $X^*$, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in $X$ means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in $X$'.

The next elementary but useful result [13, Proposition 2.1] will be used in Section 3.

Proposition 2.1. *Suppose* $(X, \|\cdot\|)$ *is an ordered Banach space with order cone* $C$. *If* $x_0 \in \operatorname{int}(C)$ *then to every* $z \in X$ *there corresponds* $t_z > 0$ *such that* $t_z x_0 - z \in C$.

A function $\Phi: X \to \mathbb{R}$ fulfilling

$$\lim_{\|x\| \to +\infty} \Phi(x) = +\infty$$

is called coercive. We say that $\Phi$ is weakly sequentially lower semi-continuous when $x_n \rightharpoonup x$ in $X$ implies $\Phi(x) \leq \liminf_{n\to\infty} \Phi(x_n)$. Let $\Phi \in C^1(X)$. The classical Palais–Smale condition for $\Phi$ reads as follows.

(PS) *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{\Phi(x_n)\}$ *is bounded and* $\|\Phi'(x_n)\|_{X^*} \to 0$ *possesses a convergent subsequence.*

Define, for every $c \in \mathbb{R}$,

$$\Phi^c := \{x \in X : \ \Phi(x) \leq c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual, $K(\Phi)$ denotes the critical set of $\Phi$, i.e. $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$.

An operator $A: X \to X^*$ is called of type $(\mathrm{S})_+$ if

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leq 0$$

imply $x_n \to x$. The next simple result is more or less known and will be employed in Section 4.

Proposition 2.2. *Let* $X$ *be reflexive and let* $\Phi \in C^1(X)$ *be coercive. Assume* $\Phi' = A + B$, *where* $A: X \to X^*$ *is of type* $(\mathrm{S})_+$ *while* $B: X \to X^*$ *is compact. Then* $\Phi$ *satisfies* (PS).

Proof. Pick a sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ turns out to be bounded and

$$\lim_{n\to+\infty} \|\Phi'(x_n)\|_{X^*} = 0. \tag{2.1}$$

By the reflexivity of $X$, besides the coercivity of $\Phi$, we may suppose, up to subsequences, $x_n \rightharpoonup x$ in $X$. Since $B$ is compact, using (2.1) and taking a subsequence when necessary, one has

$$\lim_{n\to+\infty} \langle A(x_n), x_n - x\rangle = \lim_{n\to+\infty} (\langle \Phi'(x_n), x_n - x\rangle - \langle B(x_n), x_n - x\rangle) = 0.$$

This forces $x_n \to x$ in $X$, because $A$ is of type $(\mathrm{S})_+$, as desired. □

Throughout the paper, $\Omega$ is a bounded domain of the real Euclidean $N$-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary $\partial\Omega$, $p \in (1, +\infty)$, $p' := p/(p-1)$, $\|\cdot\|_p$ stands for the usual norm of $L^p(\Omega)$, and $W_0^{1,p}(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

Write $p^*$ for the critical exponent of the Sobolev embedding $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding is compact whenever $1 \le q < p^*$.

Define $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. Obviously, $C_0^1(\overline{\Omega})$ turns out to be an ordered Banach space with order cone

$$C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\}.$$

Moreover, one has

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega\right\},$$

where $n(x)$ is the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$; see, for example, [9, Remark 6.2.10].

Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$ and let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative $p$-Laplacian, i.e.

$$\langle A(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega) \tag{2.2}$$

Denote by $\lambda_1$ (respectively, $\lambda_2$) the first (respectively, second) eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. The following properties of $\lambda_1$, $\lambda_2$, and $A$ can be found in [7], [12]; vide also [9, Section 6.2]:

- ($p_1$) $0 < \lambda_1 < \lambda_2$.
- ($p_2$) $\|u\|_p^p \le \|u\|^p/\lambda_1$ *for all* $u \in W_0^{1,p}(\Omega)$.
- ($p_3$) *There exists an eigenfunction* $\phi_1$ *corresponding to* $\lambda_1$ *such that* $\phi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *as well as* $\|\phi_1\|_p = 1$.
- ($p_4$) *If* $S := \{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\}$ *and* $\Gamma_0 := \{\gamma \in C^0([-1,1], S) : \gamma(-1) = -\phi_1,\ \gamma(1) = \phi_1\}$, *then* $\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|u\|^p$.
- ($p_5$) *The operator* $A$ *is maximal monotone and of type* $(S)_+$.

Finally, put, provided $t \in \mathbb{R}$, $t^- := \max\{-t, 0\}$, $t^+ := \max\{t, 0\}$.

If $u, v\colon \Omega \to \mathbb{R}$ belong to a given function space $X$ and $u(x) \le v(x)$ for almost every $x \in \Omega$ then we set

$$[u, v] := \{w \in X : u(x) \le w(x) \le v(x) \text{ a.e. in } \Omega\}.$$

Likewise, $\Omega(u(x) < t) := \{x \in \Omega : u(x) < t\}$, etc. From now on, to avoid unnecessary technicalities, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' and the variable $x$ will be omitted when no confusion can arise. Moreover, we shall write

$$X := W_0^{1,p}(\Omega), \qquad C_+ := C_0^1(\overline{\Omega})_+.$$

Let $\lambda > 0$. If $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(f$_1$) *$f(\,\cdot\,, t)$ is measurable for all $t \in \mathbb{R}$ while $f(x, \,\cdot\,)$ is continuous for every $x \in \Omega$,*

(f$_2$) *there exists $a_1 > 0$ such that $|f(x,t)| \le a_1(1 + |t|^{p-1})$ in $\Omega \times \mathbb{R}$,*

then the functional $\Phi_\lambda\colon X \to \mathbb{R}$ given by

$$\Phi_\lambda(u) := \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(x, u(x))\, dx, \quad u \in X,$$

where, as usual,

$$F(x,\xi) := \int_0^\xi f(x,t)\, dt \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}, \tag{2.3}$$

turns out to be well defined and continuously differentiable. Obviously, critical points of $\Phi_\lambda$ are weak solutions to (1.9), and vice-versa.

We shall assume also that

(f$_3$) $\displaystyle\limsup_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} \le 0$ *uniformly in $x \in \Omega$, and*

(f$_4$) *for suitable $a_2, A_2 > 0$ one has*

$$a_2 \le \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le A_2$$

*uniformly in $x \in \Omega$.*

## 3. Extremal constant-sign solutions

Theorem 3.1. *If* (f$_1$)–(f$_4$) *hold true then, for every $\lambda > 0$ sufficiently large, problem* (1.9) *possesses a smallest positive solution $u_\lambda \in \operatorname{int}(C_+)$ and a greatest negative solution $v_\lambda \in -\operatorname{int}(C_+)$.*

Proof. Put $f_+(x,t) := f(x,t^+)$, $F_+(x,\xi) := \int_0^\xi f_+(x,t)\, dt$, and define, provided $\lambda > 0$, $u \in X$,

$$\Phi_{\lambda,+}(u) := \frac{1}{p}\|u\|^p - \lambda \int_\Omega F_+(x, u(x))\, dx.$$

Since $X$ compactly embeds in $L^p(\Omega)$, the functional $\Phi_{\lambda,+}$ turns out to be weakly sequentially lower semi-continuous. By (f$_3$), for every $\lambda, \varepsilon > 0$ we can find $t_{\lambda,\varepsilon} > 0$ such that

$$f(x,t) < \frac{\lambda_1}{\lambda}\varepsilon t^{p-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R} \text{ with } t \ge t_{\lambda,\varepsilon}.$$

Hence, on account of ($\mathrm{p}_2$),

$$\Phi_{\lambda,+}(u) > \frac{1-\varepsilon}{p}\|u\|^p - a_3(\lambda), \quad u \in X,$$

where $a_3(\lambda) > 0$. Choosing $\varepsilon < 1$ guarantees that $\Phi_{\lambda,+}$ is coercive. Let $\widehat{u} \in X$ satisfy

$$\Phi_{\lambda,+}(\widehat{u}) = \inf_{u\in X} \Phi_{\lambda,+}(u).$$

From $\Phi'_{\lambda,+}(\widehat{u}) = 0$ it follows

$$\langle A(\widehat{u}), v\rangle = \lambda \int_\Omega f_+(x, \widehat{u}(x))v(x)\,dx, \quad v \in X, \tag{3.1}$$

with $A$ as in (2.2). Due to (3.1) written for $v := -\widehat{u}^-$ one has $\|\widehat{u}^-\|^p = 0$. Thus, $\widehat{u} \geq 0$ and, a fortiori, the function $\widehat{u}$ solves (1.9). By ($\mathrm{f}_4$) there exists $\delta > 0$ fulfilling

$$f(x,t) > \frac{a_2}{2}t^{p-1} \quad \text{for all } (x,t) \in \Omega \times (0,\delta). \tag{3.2}$$

Pick $\tau > 0$ so small that $\tau\phi_1(x) < \delta$ in $\Omega$. Through (3.2) and ($\mathrm{p}_3$) we obtain

$$\Phi_{\lambda,+}(\tau\phi_1) < \frac{\tau^p}{p}\left(\lambda_1 - \lambda\frac{a_2}{2}\right) < 0 \tag{3.3}$$

as soon as $\lambda > 2\lambda_1/a_2$. This evidently forces $\widehat{u} \neq 0$. Standard regularity results [8, Theorems 1.5.5–1.5.6] then yield $\widehat{u} \in C_+$. Since, because of (3.2),

$$\Delta_p\widehat{u}(x) = -\lambda f(x, \widehat{u}(x)) \leq 0 \quad \text{in } \Omega(\widehat{u}(x) < \delta),$$

while ($\mathrm{f}_2$) leads to

$$\Delta_p\widehat{u}(x) \leq \lambda\left(\frac{a_1}{\delta^{p-1}} + 1\right)\widehat{u}(x)^{p-1} \quad \text{for every } x \in \Omega(\widehat{u}(x) \geq \delta),$$

Theorem 5 in [15] gives $\widehat{u} \in \operatorname{int}(C_+)$. Now, Proposition 2.1 provides $\varepsilon > 0$ such that $\varepsilon\phi_1 \leq \widehat{u}$. Arguing exactly as in the proofs of [4, Lemma 4.23] and [4, Corollary 4.24], and using [15, Theorem 5] once more, we see that the set

$$S_{\lambda,+} := \{u \in [\varepsilon\phi_1, \widehat{u}] : u \text{ satisfies } (1.9)\}$$

possesses a smallest element, say $u_\varepsilon$. So, in particular, for every sufficiently large $n \in \mathbb{N}$ there exists a least solution

$$u_n \in \operatorname{int}(C_+) \cap [n^{-1}\phi_1, \widehat{u}] \tag{3.4}$$

to (1.9). Consequently,

$$A(u_n) = \lambda f(\,\cdot\,, u_n) \quad \text{in } W^{-1,p'}(\Omega). \tag{3.5}$$

The minimality property of $u_n$ gives

$$u_n \downarrow u_\lambda \quad \text{pointwise in } \Omega, \tag{3.6}$$

where $u_\lambda\colon\Omega\to\mathbb{R}$ complies with $0\le u_\lambda\le\widehat{u}$. We claim that $u_\lambda$ turns out to be a solution of problem (1.9). In fact, by (3.5), ($f_2$), and (3.4), one has

$$\|u_n\|^p=\langle A(u_n),u_n\rangle=\lambda\int_\Omega f(x,u_n(x))u_n(x)\,dx\le\lambda a_1(\|\widehat{u}\|_1+\|\widehat{u}\|_p^p)$$

for all $n\in\mathbb{N}$, i.e. $\{u_n\}\subseteq X$ is bounded. Therefore, up to subsequences, $u_n\rightharpoonup u_\lambda$ in $X$. Gathering ($f_1$), (3.6), ($f_2$), and (3.4) together we next achieve

$$\lim_{n\to+\infty}\langle A(u_n),u_n-u_\lambda\rangle=\lim_{n\to+\infty}\lambda\int_\Omega f(x,u_n(x))(u_n(x)-u_\lambda(x))\,dx=0.$$

Because of ($p_5$) this implies $u_n\to u_\lambda$ in $X$. Now, the assertion follows from (3.5).

If $u_\lambda\equiv0$ then, by (3.6),

$$u_n\downarrow0\quad\text{pointwise in }\Omega. \tag{3.7}$$

Put $v_n:=u_n/\|u_n\|$. Since $\{v_n\}$ is bounded, we may suppose (along a relabelled subsequence, when necessary)

$$v_n\rightharpoonup v\quad\text{in }X,\qquad v_n\to v\quad\text{in }L^p(\Omega), \tag{3.8}$$

as well as

$$|v_n(x)|\le w(x)\quad\text{for all }n\in\mathbb{N},\qquad v_n(x)\to v(x)\quad\text{for almost all }x\in\Omega, \tag{3.9}$$

with $w\in L^p(\Omega)$. Through (3.5) one has

$$\langle A(v_n),v_n-v\rangle=\lambda\int_\Omega\frac{f(x,u_n)}{u_n^{p-1}}\,v_n^{p-1}(v_n-v)\,dx. \tag{3.10}$$

Letting $n\to+\infty$ and using (3.7), ($f_4$), besides (3.9), yields

$$\lim_{n\to+\infty}\langle A(v_n),v_n-v\rangle=0.$$

Hence, as before, $v_n\to v$ in $X$. The choice of $v_n$ forces $v\ne0$. By (3.5) again we next get

$$A(v_n)=\lambda\frac{f(\,\cdot\,,u_n)}{u_n^{p-1}}\,v_n^{p-1}\quad\text{in }W^{-1,p'}(\Omega).$$

Due to (3.7)–(3.9) and ($f_4$), this implies

$$-\Delta_pv(x)=\lambda m_\lambda(x)v(x)^{p-1}\quad\text{for almost every }x\in\Omega,$$

where

$$m_\lambda(x):=\liminf_{n\to+\infty}\frac{f(x,u_n(x))}{u_n(x)^{p-1}}\ge m(x):=\liminf_{t\to0^+}\frac{f(x,t)}{t^{p-1}}. \tag{3.11}$$

So, if $\lambda>\lambda_1(m)$, with $\lambda_1(m)$ being the first eigenvalue of the weighted nonlinear eigenvalue problem

$$-\Delta_pu=\lambda m(x)|u|^{p-2}u\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,$$

then $\lambda > \lambda_1(m_\lambda)$, because (3.11) gives $\lambda_1(m) \geq \lambda_1(m_\lambda)$. Via [9, Proposition 6.2.15] we thus see that $v$ changes sign in $\Omega$, which is impossible. Consequently, $u_\lambda \geq 0$ but $u_\lambda \neq 0$, and Theorem 5 of [15] leads to $u_\lambda \in \mathrm{int}(C_+)$.

Let us finally verify that $u_\lambda$ turns out to be minimal. Suppose $u \in \mathrm{int}(C_+)$ solves (1.9). Through Proposition 2.1 one has $n^{-1}\phi_1 \leq u$ for any sufficiently large $n$. Without loss of generality we may assume that $u \leq \widehat{u}$, otherwise we replace $u$ by a solution $\widetilde{u} \in \mathrm{int}(C_+)$ such that $\widetilde{u} \leq \min\{u, \widehat{u}\}$, whose existence is achieved as in the proof of [4, Corollary 4.24]. Therefore, $u \in [n^{-1}\phi_1, \widehat{u}]$. Since $u_n$ was the least solution of (1.9) belonging to $[n^{-1}\phi_1, \widehat{u}]$, from (3.6) it follows

$$u_\lambda(x) \leq u_n(x) \leq u(x), \quad x \in \Omega,$$

i.e. $u_\lambda \leq u$, which represents the desired conclusion.

Setting

$$\Phi_{\lambda,-}(u) := \frac{1}{p}\,\|u\|^p - \lambda \int_\Omega F_-(x, u(x))\,dx \quad \text{for all } u \in X,$$

where $F_-(x,\xi) := \int_0^\xi f(x, -t^-)\,dt$, analogous arguments produce a greatest negative solution $v_\lambda \in -\mathrm{int}(C_+)$ to problem (1.9). □

Remark 3.2. The preceding proof shows that the conclusion of Theorem 3.1 holds provided $\lambda > \max\{2\lambda_1/a_2, \lambda_1(m)\}$, with $m$ as in (3.11).

## 4. Nodal solutions

Theorem 4.1. *Under assumptions* (f$_1$)–(f$_4$), *for every* $\lambda > 0$ *sufficiently large, problem* (1.9) *possesses a nontrivial sign-changing solution* $u_0 \in C_0^1(\overline{\Omega})$ *such that* $v_\lambda \leq u_0 \leq u_\lambda$, *where* $u_\lambda, v_\lambda$ *are given by Theorem* 3.1.

Proof. Define, provided $x \in \Omega$, $t, \xi \in \mathbb{R}$,

$$\tag{4.1} \begin{aligned} \widehat{f}(x,t) &:= \begin{cases} f(x, v_\lambda(x)) & \text{if } t < v_\lambda(x), \\ f(x,t) & \text{for } v_\lambda(x) \leq t \leq u_\lambda(x), \\ f(x, u_\lambda(x)) & \text{when } t > u_\lambda(x), \end{cases} \\ \widehat{f}_\pm(x,t) &:= \widehat{f}(x, \pm t^\pm) \end{aligned}$$

as well as

$$\widehat{F}(x,\xi) := \int_0^\xi \widehat{f}(x,t)\,dt, \qquad \widehat{F}_\pm(x,\xi) := \int_0^\xi \widehat{f}_\pm(x,t)\,dt.$$

Moreover, put

$$\tag{4.2} \widehat{\Phi}_\lambda(u) := \frac{1}{p}\,\|u\|^p - \lambda \int_\Omega \widehat{F}(x, u(x))\,dx,$$

$$\tag{4.3} \widehat{\Phi}_{\lambda,\pm}(u) := \frac{1}{p}\,\|u\|^p - \lambda \int_\Omega \widehat{F}_\pm(x, u(x))\,dx,$$

for all $u \in X$. The same reasoning made in the proof of Theorem 3.1 ensures here that the functionals $\widehat{\Phi}_\lambda$, $\widehat{\Phi}_{\lambda,\pm}$ are weakly sequentially lower semi-continuous and coercive. Hence, there exists $\overline{u} \in X$ satisfying

$$\widehat{\Phi}_{\lambda,+}(\overline{u}) = \inf_{u\in X} \widehat{\Phi}_{\lambda,+}(u). \tag{4.4}$$

As in the above-mentioned proof we then obtain

$$\overline{u} \in \operatorname{int}(C_+). \tag{4.5}$$

Proposition 2.1 furnishes

$$\tau\phi_1(x) \le \overline{u}(x), \quad x \in \Omega, \tag{4.6}$$

for any $\tau > 0$ small enough. From $\widehat{\Phi}'_{\lambda,+}(\overline{u}) = 0$ it follows

$$\langle A(\overline{u}), v\rangle = \lambda \int_\Omega \widehat{f}_+(x, \overline{u}(x))v(x)\,dx \quad \text{for all } v \in X, \tag{4.7}$$

with $A$ given by (2.2). Due to (4.7), written for $v := (\overline{u} - u_\lambda)^+$, and (4.1) one achieves

$$\langle A(\overline{u}) - A(u_\lambda), (\overline{u} - u_\lambda)^+\rangle = \lambda \int_\Omega [\widehat{f}_+(x, \overline{u}) - f(x, u_\lambda)](\overline{u} - u_\lambda)^+\,dx = 0.$$

On account of ($p_5$) this implies $\overline{u} \le u_\lambda$. So, owing to (4.1) and (4.7) again, the function $\overline{u}$ turns out to be a solution of (1.9). Since $u_\lambda$ was minimal, we must have $\overline{u} = u_\lambda$. Gathering (4.4)–(4.5) together yields that $u_\lambda$ is a $C^1_0(\overline{\Omega})$-local minimum for $\widehat{\Phi}_\lambda$. By [8, Proposition 4.6.10], the function $u_\lambda$ enjoys the same property in the space $X$. Likewise, replacing the functional $\widehat{\Phi}_{\lambda,+}$ with $\widehat{\Phi}_{\lambda,-}$ one realizes that $v_\lambda$ is a local minimizer of $\widehat{\Phi}_\lambda$.

Let $w_0 \in X$ fulfil $\widehat{\Phi}_\lambda(w_0) = \inf_{u\in X} \widehat{\Phi}_\lambda(u)$. Through (4.6) and (3.3) we infer

$$\widehat{\Phi}_\lambda(w_0) \le \widehat{\Phi}_\lambda(\tau\phi_1) = \widehat{\Phi}_{\lambda,+}(\tau\phi_1) = \Phi_{\lambda,+}(\tau\phi_1) < 0,$$

i.e. $w_0 \neq 0$, provided $\lambda > 2\lambda_1/a_2$. Further, $w_0 \in [v_\lambda, u_\lambda]$ because

$$K(\widehat{\Phi}_\lambda) \subseteq [v_\lambda, u_\lambda], \tag{4.8}$$

as a simple computation shows. Thus, $w_0$ turns out to be a nontrivial solution of (1.9). Without loss of generality we may suppose $w_0 = u_\lambda$ or $w_0 = v_\lambda$, otherwise the extremality of $u_\lambda, v_\lambda$ established in Theorem 3.1 would force a changing of sign for $w_0$, which completes the proof. So, let $w_0 = u_\lambda$ (a similar reasoning applies when $w_0 = v_\lambda$). We may assume also that $v_\lambda$ is a strict local minimum of $\widehat{\Phi}_\lambda$. In fact, if this were false then infinitely many nodal solutions to (1.9) might be found via (4.8) besides the extremality of $u_\lambda, v_\lambda$, and the conclusion follows. Pick $\rho \in (0, \|u_\lambda - v_\lambda\|)$ such that

$$\widehat{\Phi}_\lambda(u_\lambda) \le \widehat{\Phi}_\lambda(v_\lambda) < \inf_{u\in\partial B_\rho(v_\lambda)} \widehat{\Phi}_\lambda(u). \tag{4.9}$$

The functional $\widehat{\Phi}_\lambda$ is coercive and one has

$$\langle \widehat{\Phi}'_\lambda(u), v\rangle = \langle A(u), v\rangle + \langle B(u), v\rangle \quad \text{for all } u, v \in X,$$

where

$$\langle B(u), v\rangle := -\lambda \int_\Omega f(x, u(x))v(x)\, dx.$$

By ($p_5$) the operator $A$ turns out to be of type $(S)_+$ while $B\colon X \to X^*$ is compact, because ($f_1$)–($f_2$) hold true and $X$ compactly embeds in $L^p(\Omega)$. So, Proposition 2.2 guarantees that $\widehat{\Phi}_\lambda$ satisfies (PS). Bearing in mind (4.9), the Mountain-Pass Theorem can be applied. Hence, there exists $u_0 \in X$ complying with $\widehat{\Phi}'_\lambda(u_0) = 0$ and

$$\inf_{u\in\partial B_\rho(v_\lambda)} \widehat{\Phi}_\lambda(u) \le \widehat{\Phi}_\lambda(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widehat{\Phi}_\lambda(\gamma(t)), \tag{4.10}$$

where

$$\Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = v_\lambda,\ \gamma(1) = u_\lambda\}.$$

Due to (4.8) and (4.1) the function $u_0$ solves (1.9). By (4.9)–(4.10) one has $u_0 \notin \{u_\lambda, v_\lambda\}$, while standard regularity arguments provide $u_0 \in C^1_0(\overline{\Omega})$. The proof is thus completed once we verify that $u_0 \neq 0$. This immediately comes out from

$$\widehat{\Phi}_\lambda(u_0) < 0, \tag{4.11}$$

which, in view of (4.10), holds whenever we construct a path $\widehat{\gamma} \in \Gamma$ satisfying

$$\widehat{\Phi}_\lambda(\widehat{\gamma}(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{4.12}$$

Owing to ($p_4$), there exists $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\gamma(t)\|^p < \lambda_2 + \frac{a_2}{2^{p+1}}.$$

Define $S_C := S \cap C^1_0(\overline{\Omega})$ and consider on $S_C$ the topology induced by that of $C^1_0(\overline{\Omega})$. Clearly, $S_C$ is a dense subset of $S$. So, we can find $\gamma_0 \in C^0([-1,1], S_C)$ such that $\gamma_0(-1) = -\phi_1$, $\gamma_0(1) = \phi_1$, and

$$\max_{t\in[-1,1]} \|\gamma(t) - \gamma_0(t)\|^p < \frac{a_2}{2^{p+1}}.$$

This evidently forces

$$\max_{t\in[-1,1]} \|\gamma_0(t)\|^p < 2^{p-1}\lambda_2 + \frac{a_2}{2}. \tag{4.13}$$

Assumption ($f_4$) yields

$$F(x,\xi) \ge \frac{a_2}{2p}\, |\xi|^p \quad \text{provided } |\xi| \le \delta, \tag{4.14}$$

where $\delta > 0$. Pick $\varepsilon_0 > 0$ fulfilling

$$\varepsilon_0 \max_{x\in\overline{\Omega}} |u(x)| \le \delta \quad \text{for all } u \in \gamma_0([-1,1]). \tag{4.15}$$

Since $u_\lambda, -v_\lambda \in \operatorname{int}(C_+)$, to every $u \in \gamma_0([-1,1])$ and every bounded neighbourhood $V_u$ of $u$ in $C^1_0(\overline{\Omega})$ there corresponds $\nu_u > 0$ such that

$$u_\lambda - \frac{1}{m}\, v \in \operatorname{int}(C_+), \quad -v_\lambda + \frac{1}{n}\, v \in \operatorname{int}(C_+) \quad \text{whenever } m, n \ge \nu_u,\ v \in V_u\,.$$

Through the compactness of $\gamma_0([-1,1])$ in $C^1_0(\overline{\Omega})$ we thus obtain $\varepsilon_1 > 0$ satisfying

$$v_\lambda(x) \le \varepsilon u(x) \le u_\lambda(x) \quad \text{for all } x \in \Omega,\ u \in \gamma_0([-1,1]),\ \varepsilon \in (0,\varepsilon_1). \tag{4.16}$$

The function $t \mapsto \gamma_0(t)$, $t \in [-1,1]$, is a continuous path in $S_C$ joining $-\phi_1$ with $\phi_1$. Moreover, if $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ then (4.13), (4.16), (4.15), and (4.14) give

$$\begin{aligned} \widehat{\Phi}_\lambda(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}\, \|\gamma_0(t)\|^p - \lambda \int_\Omega \widehat{F}(x, \varepsilon\gamma_0(t)(x))\, dx \\ &\le \frac{\varepsilon^p}{p} \left(2^{p-1}\lambda_2 + \frac{a_2}{2}\right) - \lambda\, \frac{a_2}{2p} \varepsilon^p \int_\Omega |\gamma_0(t)(x)|^p\, dx \\ &= \frac{\varepsilon^p}{p} \left(2^{p-1}\lambda_2 + \frac{(1-\lambda)a_2}{2}\right) < 0, \end{aligned} \tag{4.17}$$

for all $t \in [-1,1]$, whenever $\lambda > (2^p\lambda_2 + a_2)/a_2$.

Now, set $a := \widehat{\Phi}_{\lambda,+}(u_\lambda)$, $b := \widehat{\Phi}_{\lambda,+}(\varepsilon\phi_1)$, and observe that $a < b < 0$. In fact, as the reasoning made below (4.4) actually shows, $u_\lambda$ is the unique global minimizer for $\widehat{\Phi}_{\lambda,+}$. Consequently, $a < b$, while (4.17) written for $t = 1$ yields $b < 0$. Thus, in particular,

$$K_a(\widehat{\Phi}_{\lambda,+}) = \{u_\lambda\}.$$

Since $K(\widehat{\Phi}_{\lambda,+}) \subseteq [0, u_\lambda]$ and, by Theorem 3.1, $u_\lambda$ turns out to be the smallest positive solution of (1.9), no critical value of $\widehat{\Phi}_{\lambda,+}$ lies in $(a,b]$. So, by the second deformation lemma [9, Theorem 5.1.33], there exists a continuous function $h\colon [0,1] \times (\widehat{\Phi}_{\lambda,+})^b \to (\widehat{\Phi}_{\lambda,+})^b$ fulfilling

$$h(0,u) = u, \quad h(1,u) = u_\lambda, \quad \text{and} \quad \widehat{\Phi}_{\lambda,+}(h(t,u)) \le \widehat{\Phi}_{\lambda,+}(u)$$

for all $(t,u) \in [0,1] \times (\widehat{\Phi}_{\lambda,+})^b$. Let $\gamma_+(t) := h(t, \varepsilon\phi_1)^+$, $t \in [0,1]$. Then $\gamma_+(0) = \varepsilon\phi_1$, $\gamma_+(1) = u_\lambda$, as well as

$$\widehat{\Phi}_\lambda(\gamma_+(t)) = \widehat{\Phi}_{\lambda,+}(\gamma_+(t)) \le \widehat{\Phi}_{\lambda,+}(h(t,\varepsilon\phi_1)) \le \widehat{\Phi}_{\lambda,+}(\varepsilon\phi_1) < 0 \quad \text{in } [0,1]. \tag{4.18}$$

In a similar way, but with $\widehat{\Phi}_{\lambda,-}$ in place of $\widehat{\Phi}_{\lambda,+}$, we can construct a continuous function $\gamma_-\colon [0,1] \to X$ such that $\gamma_-(0) = v_\lambda$, $\gamma_-(1) = -\varepsilon\phi_1$, and

$$\widehat{\Phi}_\lambda(\gamma_-(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{4.19}$$

Concatenating $\gamma_-$, $\varepsilon\gamma_0$, and $\gamma_+$ we obtain a path $\widehat{\gamma} \in \Gamma$ which, in view of (4.17)–(4.19), satisfies (4.12). This shows (4.11), whence $u_0 \neq 0$. □

REMARK 4.2. Through Remark 5.3, the above proof, and ($\mathrm{p}_1$) one realizes that the conclusion of Theorem 4.1 holds provided

$$\lambda > \max\left\{\frac{2^p\lambda_2}{a_2} + 1, \lambda_1(m)\right\},$$

with $m$ given by (3.11).

## 5. Existence of multiple solutions

Gathering Theorems 3.1 and 4.1 together directly yields the following result.

THEOREM 5.1. *Assume* ($\mathrm{f}_1$)–($\mathrm{f}_4$) *hold true. Then* (1.9) *has a smallest positive solution* $u_\lambda \in \operatorname{int}(C_+)$, *a biggest negative solution* $v_\lambda \in -\operatorname{int}(C_+)$, *and a sign-changing solution* $u_0 \in C_0^1(\overline{\Omega})$ *such that* $v_\lambda \le u_0 \le u_\lambda$ *for any sufficiently large* $\lambda > 0$.

A meaningful special case occurs when the nonlinearity $(x,t) \mapsto f(x,t)$ is odd in $t$.

THEOREM 5.2. *If* ($\mathrm{f}_1$)–($\mathrm{f}_2$) *are satisfied,* $f(x,\,\cdot\,)$ *turns out to be odd for all* $x \in \Omega$ *and, moreover,*

($\mathrm{f}_3'$) $\displaystyle\limsup_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} \le 0$ *uniformly in* $x \in \Omega$,

($\mathrm{f}_4'$) *there exist* $a_2, A_2 > 0$ *such that*

$$a_2 \le \liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} \le \limsup_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} \le A_2$$

*uniformly in* $x \in \Omega$,

*then the same conclusion of Theorem* 5.1 *holds, with* $v_\lambda = -u_\lambda$.

REMARK 5.3. Unlike most of the multiplicity results for elliptic problems with odd nonlinearities available in the literature (see for instance [11, Section 11.3] and the references therein), due to ($\mathrm{f}_2$), the function $f$ does not fulfil the classical Ambrosetti–Rabinowitz condition:

(AR) There are $\theta > p$, $r > 0$ such that $0 < \theta F(x,\xi) \le \xi f(x,\xi)$ provided $x \in \Omega$ and $|\xi| \ge r$.

Hence, the Symmetric Mountain–Pass Theorem [11, Theorem 11.5] cannot be applied here.

REMARK 5.4. Hypothesis ($\mathrm{f}_4'$) guarantees that $F(x,\xi_0) > 0$ for some $\xi_0 > 0$, with $F$ being as in (2.3).

Theorem 5.2 positively answers under ($f_4'$) the following question, posed to the second author by Prof. B. Ricceri [14]. Let $f_0:\mathbb{R}\to\mathbb{R}$ be an *odd* function. Suppose $f_0$ is continuous and satisfies:

$$\lim_{t\to+\infty}\frac{f_0(t)}{t}=0,\qquad \int_0^{\xi_0} f_0(t)\,dt>0 \quad \text{for some } \xi_0>0.$$

Is there a $\mu>0$ such that, for each $\lambda>\mu$, the problem:

$$-\Delta u=\lambda f_0(u)\quad \text{in } \Omega,\qquad u=0\quad \text{on } \partial\Omega,$$

possesses a sign-changing weak solution?

Finally, to give an idea of possible applications, consider e.g. the case when $p\geq 2$ and

$$f(x,t):=|t|^{p-2}\sin t,\quad (x,t)\in\Omega\times\mathbb{R}.$$

A simple verification shows that ($f_1$)–($f_4$) are fulfilled with $a_1=a_2=1$. Further, $\lambda_1(m)=\lambda_1$ because $m(x)=1$ for all $x\in\Omega$, where $m$ is defined in (3.11). Since $\lambda_2>\lambda_1$ by ($p_1$), Theorem 5.1 and Remark 4.2 assert that the Dirichlet problem:

$$-\Delta_p u=\lambda|u|^{p-2}\sin u\quad \text{in } \Omega,\qquad u=0\quad \text{on } \partial\Omega$$

has two extremal constant-sign solutions and a nodal solution provided $\lambda>2^p\lambda_2+1$.

A similar comment remains true for

$$f(x,t):=|t|^{p-2}((-1)^{[t]}+c)\sin t,\quad (x,t)\in\Omega\times\mathbb{R}.$$

Here $p>2$, the symbol $[t]$ denotes the greatest integer less than or equal to $t$, while $c>1$. It is worth noting that $f(x,\cdot\,)$ does not satisfy (1.10).

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