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NONLINEAR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we consider initial value problems for a class of nonlinear impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space. We give a natural formula of the solution and some related existence results by applying Mönch's fixed point theorem and the technique of measures of noncompactness.

1. Introduction

The study of impulsive differential equation has seen a rapid development in the last few years and played a very important role in modern applied mathematical models of real processes, especially describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so on. For the basic theory on impulsive differential equations, the reader can refer to the monographs of Deo et al. [7], Bainov et al. [4] and Lakshmikantham et al. [14]. It is remark that Wang et al. [32] apply new method, Picard and weakly Picard operators technique, to restudy impulsive Cauchy problems. Some interesting existence results are obtained.

On the other hand, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. Applications of fractional differential equations to

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different areas were considered by many authors and some basic results on fractional differential equations have been obtained, see for example, Gaul et al. [10], Glockle and Nonnenmacher [11], Hilfer [12], Mainardi [16], Metzler et al. [17] and Podlubny [23]. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details on fractional calculus theory, one can see the monographs of Diethelm [8], Kilbas et al. [13], Lakshmikantham et al. [15], Miller and Ross [19], Michalski [18] and Tarasov [25]. Fractional differential equations involving the Riemann–Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions (see [1]–[3], [6], [22], [26]–[31], [33], [34], [36]–[39]).

To the best of our knowledge, the theory for impulsive fractional differential equations in Banach spaces has not been sufficiently developed. Recently, Fečkan et al. [9] make a counterexample to show that the formula of solutions in previous papers are incorrect and reconsider a class of impulsive fractional differential equations and introduce a correct formula of solutions for a impulsive Cauchy problem with Caputo fractional derivative. Further, some sufficient conditions for existence of the solutions are established by applying fixed point methods.

Motivated by [5], [9], [20], [21], we reconsider the following initial value problem (IVP for short), for fractional differential equations with nonlinear impulsive conditions

$$
\begin{cases} {}^cD^q_{0,t}u(t) := {}^cD^q_t u(t) = f(t,u(t)), & t \in J' := J \setminus \{t_1,\ldots,t_m\}, \\ & J := [0,T], \\ \Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), & k = 1,\ldots,m, \\ u(0) = u_0 \in E. \end{cases} \tag{1.1}
$$

where ${}^cD^q_t$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $f\colon J \times E \to E$ is a given function, $I_k\colon E \to E$ is continuous for $k = 1,\ldots,m$, where E is a Banach space. Impulsive points t_k satisfy $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$. $u(t_k^+) = \lim\limits_{\varepsilon\to 0^+} u(t_k+\varepsilon)$ and $u(t_k^-) = \lim\limits_{\varepsilon\to 0^-} u(t_k+\varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = 1,\ldots,m$.

In order to investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. Compared with the earlier results obtained in [9], there are at least three differences: (i) the work space is not $\mathbb{R}$ but the abstract Banach space E; (ii) f is not necessary jointly continuous and satisfies some weaker assumptions; (iii) technique of measures of noncompactness is used to deal such problem.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In addition, we introduce a suitable definition of solutions for the IVP (1.1) and give a very important equivalent result. In Section 3, the existence of solution for the IVP (1.1) is showed by virtue of fractional calculus, Mönch fixed point theorem and properties of the measure of noncompactness. In Section 4, we discuss existence of the solution for the nonlocal impulsive differential equations. At last, an example is given to demonstrate the application of our main results.

2. Preliminaries

Throughout this paper, let $\mathrm{C}(J,E)$ be the Banach space of all continuous functions from J into E with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in \mathrm{C}(J,E)$. Let $L^1(J,E)$ be the Banach space of measurable functions $u\colon J \to E$ which are Bochner integrable, equipped with the norm

$$\|u\|_{L^1} = \int_J \|u(t)\|\,dt.$$

Denote $\mathrm{PC}(J,E) = \{u\colon J \to E : u \in \mathrm{C}((t_k,t_{k+1}],E),\ k=0,\ldots,m$ and there exist $u(t_k^-)$ and $u(t_k^+)$, $k=1,\ldots,m$, with $u(t_k^-) = u(t_k)\}$, $\mathrm{PC}(J,E)$ is a Banach space with the norm $\|u\|_{\mathrm{PC}} := \sup\{\|u(t)\| : t \in J\}$.

Let us recall the following known definitions and some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1 ([5]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha\colon \Omega_E \to [0,+\infty)$ defined by

$$\alpha(B) = \inf\bigg\{d > 0 : B \subseteq \bigcup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) < d\bigg\}, \quad \text{here } B_i \in \Omega_E.$$

Proposition 2.2 ([5]). *The Kuratowski measure of noncompactness satisfies some properties*:

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact* (B *is relatively compact*).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$.
(d) $\alpha(A+B) \le \alpha(A) + \alpha(B)$.
(e) $\alpha(cB) = |c|\,\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(\operatorname{conv} B) = \alpha(B)$.

Definition 2.3. The fractional integral of order γ with lower limit zero for a function $f \in L^1([0,\infty)) \to \mathbb{R}$ can be written as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\,ds, \quad t > 0,\ \gamma > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\,\cdot\,)$ is the gamma function.

Definition 2.4. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f\colon [0,\infty)\to\mathbb{R}$ can be written as

$$^{L}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, \quad t>0,\ n-1<\gamma<n.$$

Definition 2.5. The Caputo derivative of order γ for $f\colon[0,\infty)\to\mathbb{R}$ can be written as

$$^{c}D_t^{\gamma} f(t) = {}^{L}D_t^{\gamma}\bigg[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\bigg], \quad t>0,\ n-1<\gamma<n.$$

Remark 2.6. (a) If $f(t)\in C^n[0,\infty)$, then

$$^{c}D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I_t^{n-\gamma} f^{(n)}(t), \quad t>0,\ n-1<\gamma<n.$$

(b) The Caputo derivative of a constant is equal to zero.

(c) If f is an abstract function with values in E, then integrals which appear in Definitions 2.3 and 2.4 are taken in Bochner's sense.

Definition 2.7. A map $f\colon J\times E\to E$ is said to Carathéodory if

(a) $t\to f(t,u)$ is measurable for each $u\in E$;
(b) $u\to f(t,u)$ is continuous for almost all $t\in J$.

Theorem 2.8 ([1], [20]). *Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{\operatorname{conv}}\, N(V) \ \text{or}\ V = N(V)\cup\{0\} \Rightarrow \alpha(V)=0$$

hold for every subset V of D, then N has a fixed point.

Lemma 2.9 ([24]). *Let D be a bounded, closed and convex subset of the Banach space $\mathrm{C}(J,E)$, G be a continuous function on $J\times J$ and f a function from $J\times E\to E$ which satisfies the Carathéodory conditions and assume there exists $p\in L^1(J,\mathbb{R}_+)$ such that for each $t\in J$ and each bounded set $B\in E$ we have*

$$\lim_{k\to 0^+} \alpha(f(J_{t,k}\times B)) \le p(t)\alpha(B); \quad \text{here } J_{t,k}=[t-k,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\bigg(\bigg\{\int_J G(s,t) f(s,u(s))\, ds : u\in V\bigg\}\bigg) \le \int_J \|G(s,t)\| p(s)\alpha(V(s))\, ds,$$

where $V(s)=\{v(s): v\in V\}$, $s\in J$.

Now, let us define what we mean by a solution of the IVP (1.1).

Definition 2.10. A function $u \in \mathrm{PC}(J,E)$ is said to be a solution of IVP (1.1) if u satisfies the equation ${}^cD^q u(t) = f(t,u(t))$ on J', and conditions $\Delta u(t_k) = I_k(u(t_k^-))$, $k = 1,\dots,m$, and $u(0) = u_0$.

Motivated by Fečkan et al. [9], one can obtain the following important lemma.

Lemma 2.11. *Let $h\colon J \to E$ be continuous. A function u is a solution of the fractional integral equation*

$$
(2.1)\quad u(t) = \begin{cases}
u_0 + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}h(s)\,ds, & \text{for } t \in [0,t_1],\\[2mm]
u_0 + I_1(u(t_1^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}h(s)\,ds, & \text{for } t \in (t_1,t_2],\\[2mm]
u_0 + I_1(u(t_1^-)) + I_2(u(t_2^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}h(s)\,ds, & \\
& \text{for } t \in (t_2,t_3],\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
u_0 + \displaystyle\sum_{k=1}^{m} I_k(u(t_k^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}h(s)\,ds, & \\
& \text{for } t \in (t_m,T],
\end{cases}
$$

if and only if u is a solution of the following impulsive problem

$$
(2.2)\quad \begin{cases}
{}^cD_t^q u(t) = h(t), & t \in J',\\
\Delta u(t_k) = I_k(u(t_k^-)), & k = 1,\dots,m,\\
u(0) = u_0.
\end{cases}
$$

To end this section, we collect the following PC-type Ascoli–Arzelà theorem.

Theorem 2.12 (Theorem 2.1, [35]). *Let E be a Banach space and $\mathscr{W} \subset \mathrm{PC}(J,E)$. If the following conditions are satisfied*:

(a) *$\mathscr{W}$ is uniformly bounded subset of* $\mathrm{PC}(J,E)$;

(b) *$\mathscr{W}$ is equicontinuous in (t_k,t_{k+1}), $k = 0,\dots,m$, where $t_0 = 0$, $t_{m+1} = T$*;

(c) *$\mathscr{W}(t) = \{u(t) \mid u \in \mathscr{W},\ t \in J \setminus \{t_1,\dots,t_m\}\}$, $\mathscr{W}(t_k^+) = \{u(t_k^+) \mid u \in \mathscr{W}\}$ and $\mathscr{W}(t_k^-) = \{u(t_k^-) \mid u \in \mathscr{W}\}$ are relatively compact subsets of E.*

Then $\mathscr{W}$ is a relatively compact subset of $\mathrm{PC}(J,E)$.

3. Existence of solutions for IVP

This section deals with the existence of solutions for IVP (1.1). Before stating and proving the main results, we introduce the following hypotheses:

(H1) $f\colon J \times E \to E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in L^1(J,\mathbb{R}_+) \cap \mathrm{C}(J,\mathbb{R}_+)$, such that,

$$\|f(t,u)\| \le p(t)\|u\|, \quad \text{for } t \in J \text{ and each } u \in E.$$

(H3) There exists $c > 0$ such that

$$\|I_k(u)\| \le c\|u\|, \quad \text{for each } u \in E.$$

(H4) For each bounded set $B \subset E$, we have

$$\alpha(I_k(B)) \le c\alpha(B), \quad k = 1, \dots, m.$$

(H5) For each $t \in J$ and each bounded set $B \subset E$ we have

$$\lim_{h \to 0^+} \alpha(f(J_{t,h} \times B)) \le p(t)\alpha(B); \quad \text{here } J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.1. *Assume that* (H1)–(H5) *hold. Let* $p^* = \sup_{t \in J} p(t)$. *If*

$$\frac{p^* T^q}{\Gamma(q+1)} + mc < 1, \tag{3.1}$$

then the IVP (1.1) *has at least one solution.*

Proof. We shall reduce the existence of solutions of IVP (1.1) to a fixed point problem. To this end we consider the operator $N\colon \mathrm{PC}(J,E) \to \mathrm{PC}(J,E)$ defined by

$$(Nu)(t) = \begin{cases} u_0 + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s,u(s))\,ds, & \text{for } t \in [0,t_1], \\ u_0 + I_1(u(t_1^-)) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s,u(s))\,ds, & \text{for } t \in (t_1,t_2], \\ u_0 + I_1(u(t_1^-)) + I_2(u(t_2^-)) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s,u(s))\,ds, & \\ & \text{for } t \in (t_2,t_3], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ u_0 + \displaystyle\sum_{k=1}^m I_k(u(t_k^-)) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s,u(s))\,ds, & \\ & \text{for } t \in (t_m,T]. \end{cases}$$

Clearly, the fixed points of the operator N are solution of IVP (1.1).

Choose

$$r_0 \ge \frac{\|u_0\|}{1 - mc - p^* T^q / \Gamma(q+1)}, \tag{3.3}$$

and consider the set $D_{r_0} = \{u \in \mathrm{PC}(J,E) : \|u\|_{\mathrm{PC}} \le r_0\}$.

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.8. The proof will be given in three steps.

Step 1. N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathrm{C}([0,t_1],E)$. Then for each $t \in [0,t_1]$,

$$\|(Nu_n)(t) - (Nu)(t)\| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,u_n(s)) - f(s,u(s))\|\, ds.$$

Note that f is Carathéodory type function, then by the Lebesgue dominated convergence theorem we have

$$\|Nu_n - Nu\|_{\mathrm{C}([0,t_1],E)} \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

For each $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$,

$$\begin{aligned}\|(Nu_n)(t) - (Nu)(t)\| \le {} & \sum_{i=1}^{k} \|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\| \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,u_n(s)) - f(s,u(s))\|\, ds.\end{aligned}$$

Note that I_k is continuous and f is Carathéodory type fucntion, then again by the Lebesgue dominated convergence theorem we have

$$\|Nu_n - Nu\|_{\mathrm{C}((t_k,t_{k+1}],E)} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Combining (3.4) and (3.5), we have $\|Nu_n - Nu\|_{\mathrm{PC}} \to 0$ as $n \to \infty$.

Step 2. N maps D_{r_0} into itself.

It is obvious that N maps D_{r_0} into $\mathrm{PC}(J,E)$. For each $u \in D_{r_0}$, by (H2), (H3) and the condition (3.1), we have for each $t \in J$

$$\begin{aligned}\|(Nu)(t)\| &\le \|u_0\| + \sum_{k=1}^{m} \|I_k(u(t_k^-))\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,u(s))\|\, ds \\ &\le \|u_0\| + mc\|u\|_{\mathrm{PC}} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s) \|u\|_{\mathrm{PC}}\, ds \\ &\le \|u_0\| + mcr_0 + \frac{p^* r_0}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds \\ &\le \|u_0\| + r_0 \left[\frac{(p^* T^q)}{\Gamma(q+1)} + mc \right] \le r_0,\end{aligned}$$

which implies that $\|Nu\|_{\mathrm{PC}} \le r_0$.

Step 3. $N(D_{r_0})$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(D_{r_0}) \subset \mathrm{PC}(J,E)$ is bounded. For arbitrary $s_1, s_2 \in [0,t_1]$, $s_1 < s_2$, and let $u \in D_{r_0}$, then

$$\begin{aligned}
&\|(Nu)(s_2) - (Nu)(s_1)\| \\
&\quad = \frac{1}{\Gamma(q)} \bigg\| \int_0^{s_1} [(s_2-s)^{q-1} - (s_1-s)^{q-1}] f(s,u(s))\,ds \\
&\qquad + \int_{s_1}^{s_2} (s_2-s)^{q-1} f(s,u(s))\,ds \bigg\| \\
&\quad \le \frac{1}{\Gamma(q)} \int_0^{s_1} \|[(s_2-s)^{q-1} - (s_1-s)^{q-1}] f(s,u(s))\|\,ds \\
&\qquad + \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} \|(s_2-s)^{q-1} f(s,u(s))\|\,ds \\
&\quad \le \frac{1}{\Gamma(q)} \int_0^{s_1} [(s_1-s)^{q-1} - (s_2-s)^{q-1}] p(s) \|u\|_{\mathrm{PC}}\,ds \\
&\qquad + \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} (s_2-s)^{q-1} p(s) \|u\|_{\mathrm{PC}}\,ds \\
&\quad \le \frac{p^* r_0}{\Gamma(q)} \int_0^{s_1} [(s_1-s)^{q-1} - (s_2-s)^{q-1}]\,ds + \frac{p^* r_0}{\Gamma(q)} \int_{s_1}^{s_2} (s_2-s)^{q-1}\,ds \\
&\quad \le \frac{p^* r_0}{\Gamma(q+1)} [s_1^q + (s_2-s_1)^q - s_2^q] + \frac{p^* r_0}{\Gamma(q+1)} (s_2-s_1)^q \\
&\quad \le \frac{2p^* r_0}{\Gamma(q+1)} (s_2-s_1)^q.
\end{aligned}$$

As $s_2 \to s_1$, the right-hand side of the above inequality tends to zero. Then Nu is equicontinuous on interval $[0,t_1]$.

In general, for the time interval $(t_k, t_{k+1}]$, one can repeat the above process, we obtain the following inequality

$$\|(Nu)(s_2) - (Nu)(s_1)\| \le \frac{2p^* r_0}{\Gamma(q+1)} (s_2-s_1)^q.$$

This yields that Nu is equicontinuous on $(t_k, t_{k+1}]$ for $k = 1, \ldots, m$.

Now let V be a subset of D_{r_0} such that $V \subset \overline{\mathrm{conv}}(N(V) \cup 0)$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By (H4), (H5), Lemma 2.9 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) &\le \alpha(N(V)(t) \cup \{0\}) \le \alpha(N(V)(t)) \\
&\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s) \alpha((V(s)))\,ds + \sum_{k=1}^m \alpha(I_k(V(s))) \\
&\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s) v(s)\,ds + \sum_{k=1}^m c v(s) \le \|v\|_C \left[\frac{p^* T^q}{\Gamma(q+1)} + mc \right].
\end{aligned}$$

This means that

$$\|v\|_C\left(1-\left[\frac{p^*T^q}{\Gamma(q+1)}+mc\right]\right)\leq 0.$$

By (3.1) it follows that $\|v\|_C = 0$, that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the PC-type Ascoli–Arzelà theorem (Theorem 2.12), V is relatively compact in D_{r_0}. Applying Theorem 2.8, we conclude that N has a fixed point which is a solution of IVP (1.1). □

4. Nonlocal impulsive differential equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence results for the following nonlocal problem

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), & t\in J',\\ \Delta u(t_k) = I_k(u(t_k^-)), & k=1,\dots,m,\\ u(0)+g(u)=u_0, \end{cases} \tag{4.1}$$

where f, I_k are as in Section 3 and $g\colon \mathrm{PC}(J,E)\to E$ is a continuous function.

Let us introduce the following set of conditions:

(H6) There exists a constant $M^*>0$ such that

$$|g(u)|\leq M^* \quad \text{for each } u\in \mathrm{PC}(J,E).$$

(H7) For each bounded set $B\in \mathrm{PC}(J,E)$ we have $\alpha(g(B))\leq M^*\alpha(B)$.

Theorem 4.1. *Assume that* (H1)–(H7) *hold. If*

$$\frac{p^*T^q}{\Gamma(q+1)}+mc+M^*<1,$$

then the nonlocal problem (4.1) *has at least one solution on* J.

Proof. Transform the problem (4.1) into a fixed point problem. Consider the operator $F\colon \mathrm{PC}(J,E)\to \mathrm{PC}(J,E)$ defined by

$$(Fu)(t)=u_0-g(u)+\sum_{0<t_k<t} I_k(u(t_k^-))+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))\,ds.$$

Clearly, the fixed points of the operator F are solution of the problem (4.1). By repeating the same process of Theorem 3.1, we can easily show all the conditions of Theorem 3.1 are satisfied by F. Since the proof is standard, we omit it here. □

5. Examples

In this section we give an example to illustrate the usefulness of our main results.

Let us consider the following fractional impulsive problem:

$$(5.1)\qquad \begin{cases} {}^cD_t^{1/2}u(t) = \dfrac{e^{-t}u(t)}{9+e^t}, & t \in J' = [0,1] \setminus \{t_1\}, \\ \Delta u\left(\dfrac{1}{2}\right) = \dfrac{1}{2}u\left(\dfrac{1}{2}^-\right), & t_1 = \dfrac{1}{2}, \\ u(0) = 0, \end{cases}$$

where $u\colon J \to E := \mathbb{R}^n$. Set

$$f(t,u) = \frac{e^{-t}u}{9+e^t},\ (t,u) \in J \times E, \quad I_k(u) = \frac{1}{2}u.$$

Obviously, for all $u \in E$ and each $t \in [0,1]$,

$$\|f(t,u)\| = \frac{e^{-t}\|u\|}{9+e^t} \leq \frac{1}{10}\|u\|.$$

conditions (H2) and (H3) hold with $p(t) = 1/10$, $c = 1/2$.

We shall check that condition (3.1) is satisfied with $q = 1/2$, $T = 1$, $m = 1$ and $p^* = 1/10$. Indeed,

$$\frac{p^*T^q}{\Gamma(q+1)} + mc = \frac{1/10}{\Gamma(1/2+1)} + \frac{1}{2} \cong 0.61 < 1.$$

Then by Theorem 3.1, the problem (5.1) has at least one solution on $[0,1]$.

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