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DIMENSION OF ATTRACTORS AND INVARIANT SETS OF DAMPED WAVE EQUATIONS IN UNBOUNDED DOMAINS

Martino Prizzi

Abstract. Under fairly general assumptions, we prove that every compact invariant set $\mathcal{I}$ of the semiflow generated by the semilinear damped wave equation

$$\begin{aligned} u_{tt} + \alpha u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\times \partial\Omega, \end{aligned}$$

in $H^1_0(\Omega) \times L^2(\Omega)$ has finite Hausdorff and fractal dimension. Here Ω is a regular, possibly unbounded, domain in $\mathbb{R}^3$ and $f(x,u)$ is a nonlinearity of critical growth. The nonlinearity $f(x,u)$ needs not to satisfy any dissipativeness assumption and the invariant subset $\mathcal{I}$ needs not to be an attractor. If $f(x,u)$ is dissipative and $\mathcal{I}$ is the global attractor, we give an explicit bound on the Hausdorff and fractal dimension of $\mathcal{I}$ in terms of the structure parameters of the equation.

1. Introduction

In this paper we consider the damped wave equation:

$$\begin{aligned} u_{tt} + \alpha u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\times \partial\Omega. \end{aligned} \tag{1.1}$$

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Here Ω is a regular (possibly unbounded) open set in $\mathbb{R}^3$, $\beta(x)$ is a potential such that the operator $-\Delta + \beta(x)$ is positive, and $f(x, u)$ is a nonlinearity of critical growth (i.e. of polynomial growth less than or equal to three). The assumptions on $\beta(x)$ and $f(x, u)$ will be made more precise in Section 2 below. Under such assumptions, equation (1.1) generates a local semiflow Π in the space $H_0^1(\Omega) \times L^2(\Omega)$. Suppose that the semiflow Π admits a compact invariant set $\mathcal{I}$ (i.e. $\Pi(t)\mathcal{I} = \mathcal{I}$ for all $t \geq 0$). We do not make any structure assumption on the nonlinearity $f(x, u)$ and therefore we do not assume that $\mathcal{I}$ is the global attractor of equation (1.1).

Our aim is to prove that $\mathcal{I}$ has finite Hausdorff and fractal dimension and to give an explicit estimate of its dimension. The first results concerning the dimension of invariant sets of dynamical systems are due to Mallet-Paret [14] and Mañé [15]. For a comprehensive study of the subject, see e.g. [3], [12], [21] and [25].

When Ω is a bounded domain and $f(x, u)$ satisfies suitable dissipativeness conditions, the existence of a finite dimensional compact global attractor for (1.1) is a classical achievement (see e.g. [12], [25] and the references therein).

When Ω is unbounded, new difficulties arise due to the lack of compactness of the Sobolev embeddings. These difficulties can be overcome in several ways: by exploiting the *finite speed of propagation property* (e.g. in [7]), by introducing *weighted or uniform spaces* (see e.g. [Z]), by developing suitable *tail-estimates* (see e.g. [18]).

Concerning the finite dimensionality of the attractor, in the *unbounded domain case* very few results are available. In [Z] Zelik proved finite dimensionality of attractors in the context of uniform spaces, assuming that $\beta(x)$ is constant and $f(x, u)$ is independent of x and satisfies $f(u)u \leq 0$, $f'(u) \leq L$ for all $u \in \mathbb{R}$. The technique exploited by Zelik seems not to give explicit bounds for the dimension of the attractor. In [11], Karachalios and Stavrakakis considered an equation of the form:

$$u_{tt} + \alpha u_t + \beta(x)u - g(x)^{-1}\Delta u = f(u) + h(x),$$

where $g(\,\cdot\,)$ is a positive function belonging to $L^\infty \cap L^{3/2}$. In this case the weight $g(x)^{-1}$ "forces" the operator $-g(x)^{-1}\Delta$ to have compact resolvent: the result then is achieved by exploiting directly the technique of *volume tracking* developed by Temam and other authors for bounded domains (see [25]).

In this paper we do not make any structure assumption on the nonlinearity $f(x, u)$. Our only assumption is that $\partial_u f(x, 0)$ is non negative and belongs to $L^r(\Omega)$ for some $r > 3$. The positivity of $\partial_u f(x, 0)$ is not a real restriction, because its negative part can be absorbed in $\beta(x)$. Under this assumption, we shall prove that $\mathcal{I}$ has finite Hausdorff and fractal dimension in the energy space $H_0^1(\Omega) \times L^2(\Omega)$. Also, we give an explicit estimate of the dimension of $\mathcal{I}$, in terms of the main parameters involved in the equation and of the quantity $\sup\{\|(u,v)\| \mid (u,v) \in \mathcal{I}\}$. In order to achieve our result, we shall exploit the technique of *volume tracking*, as expounded in [25]. However, we cannot apply directly the arguments of [25], since the operator $-\Delta + \beta(x)$ does not have compact resolvent. Indeed, in the *bounded domain case* (resp. in the *weighted Laplacian case* considered by Karachalios and Stavrakakis) the key point is that

$$\frac{1}{d}\sum_{j=1}^{d} \lambda_j^{-1} \to 0 \quad \text{as } d \to \infty, \tag{1.2}$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is the sequence of the eigenvalues of $-\Delta$ (resp. of $-g(x)^{-1}\Delta$). In general the operator $-\Delta+\beta(x)$, when Ω is unbounded, does not satisfy such property, since it possesses a nontrivial essential spectrum and its eigenvalues below the bottom of the essential spectrum are finite or form a sequence which accumulate to the bottom the essential spectrum. Yet, a more accurate analysis shows that the numbers λ_j in (1.2) can be replaced by $\check{\lambda}_j$, where $(\check{\lambda}_j)_{j\in\mathbb{N}}$ is the sequence of the eigenvalues of the following *weighted eigenvalue problem*:

$$-\Delta\phi + \beta(x)\phi = \check{\lambda}\partial_u f(x,\overline{u}(x))^2\phi, \tag{1.3}$$

where $\overline{U} = (\overline{u},\overline{v}) \in \mathcal{I}$. It turns out that (1.3) has a pure point spectrum. Moreover, thanks to the Cwickel–Lieb–Rozenblum inequality, it is possible to determine the asymptotics of the sequence $(\check{\lambda}_j)_{j\in\mathbb{N}}$ independently of $\overline{U} \in \mathcal{I}$, and the result will follow.

The paper is organized as follows. In Section 2 we introduce notations, we state the main assumptions and we collect some preliminaries about the semiflow generated by equation (1.1). In Section 3 we recall the definition of Hausdorff and fractal dimension and we prove that any compact invariant set $\mathcal{I}$ of Π has finite Hausdorff and fractal dimension in $H_0^1(\Omega) \times L^2(\Omega)$. In Section 4 we specialize our result to the case of dissipative equations and we show that the dimensions of the attractors of (1.1) remain bounded as $\alpha \to \infty$.

2. Notation, preliminaries and remarks

Let $\sigma \geq 1$. We denote by $L^\sigma_{\mathrm{u}}(\mathbb{R}^N)$ the set of measurable functions $\omega\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$|\omega|_{L^\sigma_{\mathrm{u}}} := \sup_{y\in\mathbb{R}^N}\left(\int_{B(y)} |\omega(x)|^\sigma\,dx\right)^{1/\sigma} < \infty,$$

where, for $y \in \mathbb{R}^N$, $B(y)$ is the open unit cube in $\mathbb{R}^N$ centered at y.

In this paper we assume throughout that $N = 3$, and we fix an open (possibly unbounded) set $\Omega \subset \mathbb{R}^3$.

Proposition 2.1. *Let $\sigma > 3/2$ and let $\omega \in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$. Set $\rho := 3/2\sigma$. Then, for every $\varepsilon > 0$ and for every $u \in H^1_0(\Omega)$,*

$$\int_\Omega |\omega(x)||u(x)|^2\,dx \le |\omega|_{L^\sigma_{\mathrm{u}}}\big(\rho\varepsilon M_B^2|u|^2_{H^1} + (1-\rho)\varepsilon^{-\rho/(1-\rho)}|u|^2_{L^2}\big),$$

where M_B the constant of the Sobolev embedding $H^1(B) \subset L^6(B)$ and B is the open unit cube in $\mathbb{R}^3$. Moreover, for every $u \in H^1_0(\Omega)$,

$$\int_\Omega |\omega(x)||u(x)|^2\,dx \le M_B^{2\rho}|\omega|_{L^\sigma_{\mathrm{u}}}|u|^{2\rho}_{H^1}|u|^{2(1-\rho)}_{L^2}.$$

Proof. See the proof of Lemma 3.3 in [19]. $\square$

Let $\beta \in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$, with $\sigma > 3/2$. Let us consider the following bilinear form defined on the space $H^1_0(\Omega)$:

$$a(u,v) := \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta(x)u(x)v(x)\,dx, \quad u,v \in H^1_0(\Omega).$$

Our first assumption is the following:

Hypothesis 2.2. *There exists $\lambda_1 > 0$ such that*

$$\int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega \beta(x)|u(x)|^2\,dx \ge \lambda_1|u|^2_{L^2}, \quad u \in H^1_0(\Omega). \tag{2.1}$$

Remark 2.3. Conditions on $\beta(x)$ under which Hypothesis 2.2 is satisfied are expounded e.g. in [1], [2].

As a consequence of (2.1) and Proposition 2.1, we have:

Proposition 2.4. *There exist two positive constants λ_0 and Λ_0 such that*

$$\lambda_0|u|^2_{H^1} \le \int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega \beta(x)|u(x)|^2\,dx \le \Lambda_0|u|^2_{H^1}, \quad u \in H^1_0(\Omega).$$

The constants λ_0, Λ_0 can be computed explicitly in terms of λ_1, M_B and $|\beta|_{L^\sigma_{\mathrm{u}}}$.

Proof. Cf. Lemma 4.2 in [18]. $\square$

It follows from Proposition 2.4 that the bilinear form $a(\,\cdot\,,\,\cdot\,)$ defines a scalar product in $H^1_0(\Omega)$, equivalent to the standard one.

Notation 2.5. From now on, we set $\langle\,\cdot\,,\,\cdot\,\rangle_{H^1_0} := a(\,\cdot\,,\,\cdot\,)$ and we denote by $\|\cdot\|_{H^1_0}$ the norm associated with $\langle\,\cdot\,,\,\cdot\,\rangle_{H^1_0}$. Also, we shall use the notation $\|\cdot\|_{L^p}$ to denote the L^p-norm in $L^p(\Omega)$, $1 \le p \le \infty$.

Let $\mathbf{A}$ be the self-adjoint operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta(x)u - \Delta u$. Then $\mathbf{A}$ generates a family X^κ, $\kappa \in \mathbb{R}$, of fractional power spaces with $X^{-\kappa}$ being the dual of X^κ for $\kappa \in\,]0,+\infty[$. For $\kappa \in\,]0,+\infty[$, the space X^κ is a Hilbert space with respect to the scalar product

$$\langle u,v\rangle_{X^\kappa} := \langle \mathbf{A}^\kappa u, \mathbf{A}^\kappa v\rangle_{L^2}, \quad u,v \in X^\kappa.$$

Also, the space $X^{-\kappa}$ is a Hilbert space with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{X^{-\kappa}}$ dual to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{X^{\kappa}}$, i.e.

$$\langle u', v'\rangle_{X^{-\kappa}} = \langle R_\kappa^{-1}u', R_\kappa^{-1}v'\rangle_{X^\kappa}, \quad u, v \in X^{-\kappa},$$

where $R_\kappa\colon X^\kappa \to X^{-\kappa}$ is the Riesz isomorphism $u \mapsto \langle\,\cdot\,, u\rangle_{X^\kappa}$.

We make the following assumption:

HYPOTHESIS 2.6. *The open set Ω is a uniformly C^2 domain in the sense of Browder* [4, p. 36].

As a consequence, by elliptic regularity we have that $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^\infty(\Omega)$. In this situation, the assignment $u \mapsto \beta(x)u$ defines a relatively bounded perturbation of $-\Delta$ and therefore $D(-\Delta + \beta(x)) = H^2(\Omega) \cap H_0^1(\Omega)$. It follows that $X^\kappa \subset L^\infty(\Omega)$ for $\kappa > 3/4$ (see [9, Theorem 1.6.1]).

We write

$$H_\kappa = X^{\kappa/2}, \quad \kappa \in \mathbb{R}.$$

Note that $H_0 = L^2(\Omega)$, $H_1 = H_0^1(\Omega)$, $H_{-1} = H^{-1}(\Omega)$ and $H_2 = H^2(\Omega) \cap H_0^1(\Omega)$.

For $\kappa \in \mathbb{R}$ the operator $\mathbf{A}$ induces a self-adjoint operator $\mathbf{A}_\kappa\colon H_{\kappa+2} \to H_\kappa$. In particular $\mathbf{A} = \mathbf{A}_0$. Moreover,

$$\langle u, v\rangle_{H_0^1} = \langle \mathbf{A}_0 u, v\rangle_{L^2}, \quad u \in D(\mathbf{A}_0),\ v \in H_0^1(\Omega).$$

For $\kappa \in \mathbb{R}$ set $Z_\kappa := H_{\kappa+1} \times H_\kappa$. For $\alpha > 0$ define the linear operator $\mathbf{B}_\kappa\colon Z_{\kappa+1} \to Z_\kappa$ by

$$\mathbf{B}_\kappa(u, v) := (v, -(\alpha v + \mathbf{A}_\kappa u)), \quad (u, v) \in Z_{\kappa+1}.$$

It follows that $\mathbf{B}_\kappa$ is m-dissipative on Z_κ (cf. the proof of Proposition 3.6 in [19]). Therefore, by the Hille–Yosida–Phillips theorem (see e.g. [5]), $\mathbf{B}_\kappa$ is the infinitesimal generator of a C^0-semigroup $\mathbf{T}_\kappa(t)$, $t \in [0, +\infty[$, on Z_κ.

Given a function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\widehat{g}$ the Nemit'skiĭ operator which associates with every function $u\colon \Omega \to \mathbb{R}$ the function $\widehat{g}(u)\colon \Omega \to \mathbb{R}$ defined by

$$\widehat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$

If $I \subset \mathbb{R}$, X is a normed spaces and if $u\colon I \to X$ is a function which is differentiable as a function into X then we denote its X-valued derivative by $(\partial_t \,|\, X)\, u$. Similarly, if X is a Banach space and $u\colon I \to X$ is integrable as a function into X, then we denote its X-valued integral by $\int_I u(t)\,(dt \,|\, X)$. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. If $X = Y$ we write just $\mathcal{L}(X)$.

Hypothesis 2.7.

(a) *$f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is such that, for every $u\in\mathbb{R}$, $f(\,\cdot\,,u)$ is measurable and $f(\,\cdot\,,0)\in L^2(\Omega)$;*

(b) *for almost every $x\in\Omega$, $f(x,\,\cdot\,)$ is of class C^2, $\partial_u f(\,\cdot\,,0)\in L^\infty(\Omega)$ and there exists a constants $C\ge 0$ such that*

$$|\partial_{uu}f(x,u))|\le C(1+|u|),\quad (x,u)\in\Omega\times\mathbb{R}.$$

The main properties of the Nemit'skiĭ operator associated with f are collected in the following Proposition, whose proof is left to the reader.

Proposition 2.8. *Assume Hypothesis 2.7. Then $\widehat{f}\colon H_0^1(\Omega)\to L^2(\Omega)$ is continuously differentiable, $D\widehat{f}(u)[v](x)=\partial_u f(x,u(x))v(x)$ for $u,v\in H_0^1(\Omega)$, and there exists a positive constant $\widetilde{C}>0$ such that the following estimates hold:*

$$\|\widehat{f}(u)\|_{L^2}\le\widetilde{C}(1+\|u\|^3_{H_0^1}),\qquad u\in H_0^1(\Omega), \tag{2.2}$$

$$\|D\widehat{f}(u)\|_{\mathcal{L}(H_0^1,L^2)}\le\widetilde{C}(1+\|u\|^2_{H_0^1}),\qquad u\in H_0^1(\Omega), \tag{2.3}$$

$$\|D\widehat{f}(u_1)-D\widehat{f}(u_2)\|_{\mathcal{L}(H_0^1,L^2)}\le\widetilde{C}(1+\|u_1\|_{H_0^1}+\|u_2\|_{H_0^1})\|u_1-u_2\|_{H_0^1},\quad u_1,u_2\in H_0^1(\Omega). \tag{2.4}$$

If $u\in H_0^1(\Omega)$ and $v\in L^2(\Omega)$, then $\widehat{\partial_u f}(u)\cdot v\in H^{-1}(\Omega)$ and the following estimates hold:

$$\|\widehat{\partial_u f}(u)\|_{\mathcal{L}(L^2,H^{-1})}\le\widetilde{C}(1+\|u\|^2_{H_0^1}),\qquad u\in H_0^1(\Omega), \tag{2.5}$$

$$\|\widehat{\partial_u f}(u_1)-\widehat{\partial_u f}(u_2)\|_{\mathcal{L}(L^2,H^{-1})}\le\widetilde{C}(1+\|u_1\|_{H_0^1}+\|u_2\|_{H_0^1})\|u_1-u_2\|_{H_0^1},\quad u_1,u_2\in H_0^1(\Omega). \tag{2.6}$$

We consider the following semi-linear damped wave equation:

$$\begin{aligned} u_{tt}+\alpha u_t+\beta(x)u-\Delta u &= f(x,u), && (t,x)\in[0,+\infty[\,\times\Omega,\\ u &= 0, && (t,x)\in[0,+\infty[\,\times\partial\Omega, \end{aligned} \tag{2.7}$$

with Cauchy data $u(0)=u_0$, $u_t(0)=v_0$.

We recall the following classical result (see e.g. Theorem II.1.3 in [8]):

Theorem 2.9. *Let X be a Banach space and let $B\colon D(B)\subset X\to X$ be the infinitesimal generator of a C^0-semigroup of linear operators $T(t)$, $t\in\mathbb{R}_+$. Consider the abstract Cauchy problem*

$$\begin{cases} \dot u = Bu(t)+f(t) & \text{for } t\in\mathbb{R}_+,\\ u(0)=u_0. \end{cases} \tag{2.8}$$

Assume that $u_0\in D(B)$ and that either

(a) *$f\in C(\mathbb{R}_+,X)$ takes values in $D(B)$ and $Bf\in C(\mathbb{R}_+,X)$, or*

(b) *$f\in C^1(\mathbb{R}_+,X)$.*

Then (2.8) *has a unique solution* $u \in C^1(\mathbb{R}_+)$ *with values in* $D(B)$. *The solution is given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds. \tag{2.9}$$

Using Theorem 2.9, we rewrite equation (2.7) as an integral evolution equation in the space $Z_0 = H_0^1(\Omega) \times L^2(\Omega)$, namely

$$(u(t), v(t)) = \mathbf{T}_0(t)(u_0, v_0) + \int_0^t \mathbf{T}_0(t-p)(0, \widehat{f}(u(p)))\,(dp \,|\, Z_0). \tag{2.10}$$

Equation (2.10) is called the *mild formulation* of (2.7) and solutions of (2.10) are called *mild solutions* of (2.7). Note that by Proposition 2.1 the nonlinear operator $(u, v) \mapsto (0, \widehat{f}(u))$ is Lipschitz continuous from Z_0 into itself. A classical Picard iteration argument shows that, if $(u_0, v_0) \in Z_0$, then (2.10) possesses a unique continuous maximal solution $(u(\,\cdot\,), v(\,\cdot\,)) \colon [0, t_{\max}[\to Z_0$ (see Theorem 4.3.4 and Proposition 4.3.7 in [5]). We thus obtain a local semiflow on Z_0, which we denote by $\Pi(t)U_0$, $U_0 = (u_0, v_0) \in Z_0$, $t \in [0, t_{\max}(U_0)[$. Notice that the solution $(u(\,\cdot\,), v(\,\cdot\,))$ of (2.10) also satisfies

$$(u(t), v(t)) = \mathbf{T}_{-1}(t)(u_0, v_0) + \int_0^t \mathbf{T}_{-1}(t-p)(0, \widehat{f}(u(p)))\,(dp \mid Z_{-1}). \tag{2.11}$$

Therefore, it follows from Theorem 2.9 that $(u(\,\cdot\,), v(\,\cdot\,))$ is continuously differentiable into Z_{-1} and

$$(\partial_t \,|\, Z_{-1})(u(t), v(t)) = \mathbf{B}_{-1}(u(t), v(t)) + (0, \widehat{f}(u(t))). \tag{2.12}$$

In particular, one has

$$\begin{cases} (\partial_t \,|\, H_0)u(t) = v(t), \\ (\partial_t \,|\, H_{-1})v(t) = -\alpha v(t) - \mathbf{A}_{-1}u(t) + \widehat{f}(u(t)). \end{cases}$$

Definition 2.10. A function $(u(\,\cdot\,), v(\,\cdot\,)) \colon \mathbb{R} \to Z_0$ is called a *full solution* of the semiflow Π generated by (2.10) if and only if, for every s, $t \in \mathbb{R}$, with $s \le t$, one has

$$(u(t), v(t)) = \Pi(t-s)(u(s), v(s))$$

Definition 2.11. A subset $\mathcal{I}$ of Z_0 is called *invariant* for the semiflow generated by (2.10) if for every $(u_0, v_0) \in \mathcal{I}$ there exists a full solution $(u(\,\cdot\,), v(\,\cdot\,))$ of (2.10) with $(u(0), v(0)) = (u_0, v_0)$ and $(u(t), v(t)) \in \mathcal{I}$ for all $t \in \mathbb{R}$.

From now on we assume that $\mathcal{I} \subset Z_0$ is a compact invariant subset of the semiflow Π.

Notation 2.12. If $\mathcal{B}$ is a Banach space such that $\mathcal{I} \subset \mathcal{B}$, we define

$$|\mathcal{I}|_{\mathcal{B}} := \max\{\|u\|_{\mathcal{B}} \mid u \in \mathcal{I}\}.$$

We recall the following result:

Theorem 2.13 (cf. Corollaries 2.10 and 2.13 in [16]). *Assume that Hypotheses* 2.2, 2.6 *and* 2.7 *are satisfied. Let $\mathcal{I} \subset Z_0$ be a compact invariant set of the semiflow generated by* (2.10). *Then $\mathcal{I}$ is a bounded subset of Z_1. Moreover, $|\mathcal{I}|_{Z_1}$ can be explicitly estimated in terms of $|\mathcal{I}|_{Z_0}$ and of the constants in Hypotheses* 2.2 *and* 2.6.

Let $\overline{U}_0 = (\overline{u}_0, \overline{v}_0) \in \mathcal{I}$, and let $\overline{U}(t) = (\overline{u}(t), \overline{v}(t))$, $t \in \mathbb{R}$, be the full bounded solution through $\overline{U}_0$. Given $H_0 = (h_0, k_0) \in Z_0$, let us denote by $\mathcal{U}(\overline{U}_0; t)H_0$ the mild solution of

$$(h(t), k(t)) = \mathbf{T}_0(t)(h_0, k_0) + \int_0^t \mathbf{T}_0(t-p)(0, \widehat{\partial_u f}(\overline{u}(p))h(p))\,(dp \mid Z_0). \tag{2.13}$$

Notice that $\mathcal{U}(\overline{U}_0; t)$ coincides with the restriction to Z_0 of the *evolution family* $\mathbf{U}_{-1}(t,s)$ generated in Z_{-1} by the family $\mathbf{B}_{-1} + \mathbf{C}_{-1}(t)$, $t \in \mathbb{R}$, where $\mathbf{C}_{-1}(t)(h,k) := (0, \widehat{\partial_u f}(\overline{u}(t))h)$ (see [16] and [10]).

A standard computation using Gronwall's inequality and Proposition 2.8 leads to the following:

Proposition 2.14. *For every $t \ge 0$,*

$$\sup_{\overline{U}_0 \in \mathcal{I}} \|\mathcal{U}(\overline{U}_0; t)\|_{\mathcal{L}(Z_0, Z_0)} < +\infty,$$

and

$$\lim_{\varepsilon \to 0} \sup_{\substack{\overline{U}_1, \overline{U}_2 \in \mathcal{I} \\ 0 < \|\overline{U}_1 - \overline{U}_2\|_{Z_0} < \varepsilon}} \frac{\|\Pi(t)(\overline{U}_2) - \Pi(t)(\overline{U}_1) - \mathcal{U}(\overline{U}_1; t)(\overline{U}_2 - \overline{U}_1)\|_{Z_0}}{\|\overline{U}_2 - \overline{U}_1\|_{Z_0}} = 0,$$

where $\overline{U}i = (\overline{u}_i, \overline{v}_i)$, $i = 1, 2, 3$.

3. Dimension of invariant sets

Let $\mathcal{X}$ be a complete metric space and let $\mathcal{K} \subset \mathcal{X}$ be a compact set. For $d \in \mathbb{R}^+$ and $\varepsilon > 0$ one defines

$$\mu_H(\mathcal{K}, d, \varepsilon) := \inf\left\{ \sum_{i \in I} r_i^d \;\middle|\; \mathcal{K} \subset \bigcup_{i \in I} B(x_i, r_i),\ r_i \le \varepsilon \right\},$$

where the infimum is taken over all the finite coverings of $\mathcal{K}$ with balls of radius $r_i \le \varepsilon$. Observe that $\mu_H(\mathcal{K}, d, \varepsilon)$ is a non increasing function of ε and d. The d-dimensional Hausdorff measure of $\mathcal{K}$ is by definition

$$\mu_H(\mathcal{K}, d) := \lim_{\varepsilon \to 0} \mu_H(\mathcal{K}, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(\mathcal{K}, d, \varepsilon).$$

One has:

(1) $\mu_H(\mathcal{K}, d) \in [0, +\infty]$;
(2) if $\mu_H(\mathcal{K}, \overline{d}) < \infty$, then $\mu_H(\mathcal{K}, d) = 0$ for all $d > \overline{d}$;
(3) if $\mu_H(\mathcal{K}, \overline{d}) > 0$, then $\mu_H(\mathcal{K}, d) = +\infty$ for all $d < \overline{d}$.

The Hausdorff dimension of $\mathcal{K}$ is the smallest d for which $\mu_H(\mathcal{K}, d)$ is finite, i.e.

$$\dim_H(\mathcal{K}) := \inf\{d > 0 \mid \mu_H(\mathcal{K}, d) = 0\}.$$

Now let $n_{\mathcal{K}}(\varepsilon)$, $\varepsilon > 0$, denote the minimum number of balls of $\mathcal{X}$ of radius ε which is necessary to cover $\mathcal{K}$. The fractal dimension of $\mathcal{K}$ is the number

$$\dim_F(\mathcal{K}) := \limsup_{\varepsilon \to 0} \frac{\log n_{\mathcal{K}}(\varepsilon)}{\log 1/\varepsilon}.$$

There is a well developed technique to estimate the Hausdorff dimension of an invariant set of a map or a semigroup. We refer the reader e.g. to [25] and [12]. The geometric idea consists in tracking the evolution of a d-dimensional volume under the action of the linearization of the semigroup along solutions lying in the invariant set. One looks then for the smallest d for which any d-dimensional volume contracts asymptotically as $t \to \infty$.

We fix $\delta \in \mathbb{R}$ and we introduce a *change of coordinates* in the space Z_κ, $\kappa \in \mathbb{R}$, by

$$R_\delta \colon Z_\kappa \to Z_\kappa, \quad (u, v) \mapsto (u, v + \delta u).$$

The constant δ is to be fixed later. Clearly the transformation R_δ is linear, bounded and invertible, with inverse $R_\delta^{-1} = R_{-\delta}$. We define the semiflow

$$\Pi_\delta(t) := R_\delta \circ \Pi(t) \circ R_{-\delta}$$

and we set $\mathcal{I}_\delta := R_\delta \mathcal{I}$. Then $\mathcal{I}_\delta$ is a compact invariant set of Π_δ, it is bounded in Z_1, and $\dim \mathcal{I}_\delta = \dim \mathcal{I}$. For $\widetilde{U}_0 \in \mathcal{I}_\delta$ and $t \geq 0$ we set

$$\mathcal{U}_\delta(\widetilde{U}_0; t) := R_\delta \circ \mathcal{U}(R_{-\delta}\widetilde{U}_0; t) \circ R_{-\delta}.$$

Then the conclusions of Proposition 2.14 hold with $\Pi(t)$, $\mathcal{I}$ and $\mathcal{U}(\overline{U}; t)$ replaced by $\Pi_\delta(t)$, $\mathcal{I}_\delta$ and $\mathcal{U}_\delta(\widetilde{U}; t)$.

Let $\widetilde{U}_0 = (\widetilde{u}_0, \widetilde{v}_0) \in \mathcal{I}_\delta$ and let $\widetilde{U}(t) = (\widetilde{u}(t), \widetilde{v}(t)) = \Pi_\delta(t)\widetilde{U}_0$. Let $\Phi_{0,i}$, $i = 1, \ldots, d$, be linearly independent elements of Z_0, $\Phi_{0,i} = (\phi_{0,i}, \psi_{0,i})$. Set $\Phi_i(t) := \mathcal{U}_\delta(\widetilde{U}_0; t)\Phi_{0,i}$. We denote by $G(t)$ the square of the d-dimensional volume delimited by $\Phi_1(t), \ldots, \Phi_\delta(t)$, that is

$$G(t) := \|\Phi_1(t) \wedge \ldots \wedge \Phi_d(t)\|^2_{\wedge^d Z_0} = \det(\langle \Phi_i(t), \Phi_j(t) \rangle_{Z_0})_{ij}.$$

We need to find a differential equation satisfied by $G(t)$.

LEMMA 3.1. *Let i and $j \in \{1, \ldots, d\}$ be fixed. Then the function*

$$t \mapsto \langle \Phi_i(t), \Phi_j(t) \rangle_{Z_0}$$

is continuously differentiable, and

$$\begin{aligned} \frac{d}{dt}\langle \Phi_i, \Phi_j \rangle_{Z_0} = -2\delta \langle \phi_i, \phi_j \rangle_{H_0^1} - 2(\alpha - \delta)\langle \psi_i, \psi_j \rangle_{L^2} + \delta(\alpha - \delta)(\langle \phi_i, \psi_j \rangle_{L^2} \\ + \langle \psi_i, \phi_j \rangle_{L^2}) + (\langle \widehat{\partial_u f}(\widetilde{u}(t))\phi_i, \psi_j \rangle_{L^2} + \langle \psi_i, \widehat{\partial_u f}(\widetilde{u}(t))\phi_j \rangle_{L^2}). \end{aligned}$$

PROOF. First set $\overline{U}_0 := R_{-\delta}\widetilde{U}_0$, $\overline{U}(t) := R_{-\delta}\widetilde{U}(t)$, $\Theta_{0,l} = (\theta_{0,l}, \chi_{0,l}) := R_{-\delta}\Phi_{0,l}$, $l = i, j$, and $\Theta_l(t) = (\theta_l(t), \chi_l(t)) := R_{-\delta}\Phi_l(t)$, $l = i, j$. Notice that $\Theta_l(t) = \mathcal{U}(\overline{U}_0; t)\Theta_{0,l}$, $l = i, j$. It follows that

$$\langle \Phi_i(t), \Phi_j(t) \rangle_{Z_0} = \langle R_\delta \Theta_i(t), R_\delta \Theta_j(t) \rangle_{Z_0}.$$

Now we shall apply Theorem 2.6 in [19]. Set:

(1) $Z := Z_0 \oplus Z_0$;
(2) $T(t) := \mathbf{T}_0(t) \oplus \mathbf{T}_0(t)$;
(3) $B := \mathbf{B}_0 \oplus \mathbf{B}_0$;
(4) $g(s) = (0, \widehat{\partial_u f}(\widetilde{u}(t))\theta_i(t)) \oplus (0, \widehat{\partial_u f}(\widetilde{u}(t))\theta_j(t))$;
(5) $z(t) = \Theta_i(t) \oplus \Theta_j(t)$;
(6) $V(U_1, U_2) := \langle R_\delta U_1, R_\delta U_2 \rangle_{Z_0}$

A standard computation shows that V is Fréchet differentiable in Z; moreover, for $U_i \oplus U_j \in D(B)$ and $H_i \oplus H_j \in Z$,

$$\begin{aligned} &DV(U_i \oplus U_j)[B(U_i \oplus U_j) + H_i \oplus H_j] \\ &\quad = \langle v_i + h_i, u_j \rangle_{H_0^1} + \delta \langle v_i + h_i, \delta u_j + v_j \rangle_{L^2} + \langle -\alpha v_i + k_i, \delta u_j + v_j \rangle_{L^2} \\ &\qquad - \langle \mathbf{A}_0 u_i, \delta u_j + v_j \rangle_{L^2} + \langle u_i, v_j + h_j \rangle_{H_0^1} + \delta \langle \delta u_i + v_i, v_j + h_j \rangle_{L^2} \\ &\qquad + \langle \delta u_i + v_i, -\alpha v_j + k_j \rangle_{L^2} - \langle \delta u_i + v_i, \mathbf{A}_0 u_j \rangle_{L^2} \\ &\quad = -2\delta \langle u_i, u_j \rangle_{H_0^1} + (\langle h_i, u_j \rangle_{H_0^1} + \langle u_i, h_j \rangle_{H_0^1}) \\ &\qquad + (\langle k_i, \delta u_j + v_j \rangle_{L^2} + \langle \delta u_i + v_i, k_j \rangle_{L^2}) \\ &\qquad + (\langle \delta(v_i + h_i) - \alpha v_i, \delta u_j + v_j \rangle_{L^2} + \langle \delta u_i + v_i, \delta(v_j + h_j) - \alpha v_j \rangle_{L^2}) \end{aligned}$$

where $U_l = (u_l, v_l)$ and $H_l = (h_l, k_l)$, $l = i, j$. It follows from Theorem 2.6 in [19] that

$$\begin{aligned} \frac{d}{dt}\langle \Phi_i, \Phi_j \rangle_{Z_0} = \frac{d}{dt} V(\Theta_i, \Theta_j) \\ &= -2\delta \langle \theta_i, \theta_j \rangle_{H_0^1} + (\langle (\delta - \alpha)\chi_i, \delta\theta_j + \chi_j \rangle_{L^2} \\ &\quad + \langle \delta\theta_i + \chi_i, (\delta - \alpha)\chi_j \rangle_{L^2} + (\langle \widehat{\partial_u f}(\widetilde{u}(t))\theta_i, \delta\theta_j + \chi_j \rangle_{L^2} \\ &\quad + \langle \delta\theta_i + \chi_i, \widehat{\partial_u f}(\widetilde{u}(t))\theta_j \rangle_{L^2} \end{aligned}$$

$$
\begin{aligned}
= & -2\delta\langle\phi_i,\phi_j\rangle_{H_0^1} - 2(\alpha-\delta)\langle\psi_i,\psi_j\rangle_{L^2} \\
& + \delta(\alpha-\delta)(\langle\phi_i,\psi_j\rangle_{L^2} + \langle\psi_i,\phi_j\rangle_{L^2}) \\
& + (\langle\widehat{\partial_u f}(\widetilde{u}(t))\phi_i,\psi_j\rangle_{L^2} + \langle\psi_i,\widehat{\partial_u f}(\widetilde{u}(t))\phi_j\rangle_{L^2})
\end{aligned}
$$

and the proof is completed. □

Let $\widetilde{U} = (\widetilde{u},\widetilde{v}) \in \mathcal{I}_\delta$ and let Σ_d be a d-dimensional subspace of Z_0. on Σ_d we define a self-adjoint operator $\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}$ by

$$
\begin{aligned}
\langle\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}(u,v),(\xi,\eta)\rangle_{Z_0} := & -2\delta\langle u,\xi\rangle_{H_0^1} - 2(\alpha-\delta)\langle v,\eta\rangle_{L^2} \\
& + \delta(\alpha-\delta)(\langle u,\eta\rangle_{L^2} + \langle v,\xi\rangle_{L^2}) + (\langle\widehat{\partial_u f}(\widetilde{u})u,\eta\rangle_{L^2} + \langle v,\widehat{\partial_u f}(\widetilde{u})\xi\rangle_{L^2}),
\end{aligned}
$$

for (u,v) and $(\xi,\eta) \in \Sigma_d$.

Now let $\widetilde{U}_0$, $\widetilde{U}(t)$, $\Phi_{0,i}$ and $\Phi_i(t)$, $i = 1,\dots,d$, and $G(t)$ be as above. We set $\Sigma_d(t) := \operatorname{span}(\Phi_1(t),\dots,\Phi_d(t))$ and we define a $(d\times d)$-matrix $(b_{il}(t))_{il}$ such that

$$
\mathbf{B}_{\widetilde{U}(t),\Sigma_d(t),\delta}\Phi_i(t) = \sum_{l=1}^{d} b_{il}(t)\Phi_l(t).
$$

It follows from Lemma 3.1 that

$$
\frac{d}{dt}\langle\Phi_i(t),\Phi_j(t)\rangle_{Z_0} = \langle\mathbf{B}_{\widetilde{U}(t),\Sigma_d(t),\delta}\Phi_i(t),\Phi_j(t)\rangle_{Z_0} = \sum_{l=1}^{d} b_{il}(t)\langle\Phi_l(t),\Phi_j(t)\rangle_{Z_0}.
$$

A straightforward computation now shows that

$$
\frac{d}{dt}G(t) = \left(\sum_{i=1}^{d} b_{ii}(t)\right)G(t) = \operatorname{Tr}(\mathbf{B}_{\widetilde{U}(t),\Sigma_d(t),\delta})G(t).
$$

Therefore we get:

$$
\|\Phi_1(t)\wedge\ldots\wedge\Phi_d(t)\|^2_{\wedge^d Z_0} = \|\Phi_{0,1}\wedge\ldots\wedge\Phi_{0,d}\|^2_{\wedge^d Z_0} \exp\int_0^t \operatorname{Tr}(\mathbf{B}_{\widetilde{U}(s),\Sigma_d(s),\delta})\,ds.
$$

For $j\in\mathbb{N}$, define the quantities

$$
p_j := \sup\{\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_j,\delta}) \mid \widetilde{U}\in\mathcal{I}_\delta,\ \Sigma_j\subset Z_0,\ \dim\Sigma_j = j\}. \tag{3.1}
$$

It follows from the results in [25, Chapter V, pp. 287–291] that if for some d one has $p_d < 0$ then the Hausdorff dimension of $\mathcal{I}_\delta$ in Z_0 is finite and less than or equal to d, and the fractal dimension of $\mathcal{I}_\delta$ in Z_0 is finite and less than or equal to $d \max\limits_{1\le j\le d-1}(1+(p_j)_+/|p_d|)$. Therefore we must choose $\delta>0$ in such a way that we can find d such that $p_d < 0$.

First we observe that, given an orthonormal basis $\check{\Phi}_1, \ldots, \check{\Phi}_d$ of Σ_d, then

$$\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) = \sum_{i=1}^{d} \langle \mathbf{B}_{\widetilde{U},\Sigma_d,\delta}\check{\Phi}_i, \check{\Phi}_i\rangle_{Z_0}$$

$$= \sum_{i=1}^{d}(-2\delta\|\check{\phi}_i\|^2_{H^1_0} - 2(\alpha-\delta)\|\check{\psi}_i\|^2_{L^2} + 2\delta(\alpha-\delta)\langle\check{\phi}_i, \check{\psi}_i\rangle_{L^2} + 2\langle\widehat{\partial_u f}(\widetilde{u})\check{\phi}_i, \check{\psi}_i\rangle_{L^2}),$$

where $\check{\Phi}_i = (\check{\phi}_i, \check{\psi}_i)$, $i = 1, \ldots, d$. Now, following the arguments of [24], we choose $\delta := \lambda_1\alpha/(\alpha^2 + 4\lambda_1)$. With this choice of δ, using Cauchy–Schwartz and Young's inequalities and setting

$$\nu_\alpha := \frac{\lambda_1\alpha}{\sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1})}, \tag{3.2}$$

we get

$$\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) \le -2\nu_\alpha d + \sum_{i=1}^{d}(-\alpha\|\check{\psi}_i\|^2_{L^2} + 2\langle\widehat{\partial_u f}(\widetilde{u})\check{\phi}_i, \check{\psi}_i\rangle_{L^2});$$

using again Cauchy–Schwartz and Young's inequalities, we finally obtain

$$\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) \le -2\nu_\alpha d + \frac{1}{\alpha}\sum_{i=1}^{d}\|\widehat{\partial_u f}(\widetilde{u})\check{\phi}_i\|^2_{L^2}. \tag{3.3}$$

Remark 3.2. Our choice of δ, according to [24], is better than the classical $0 < \delta \le \min\{\alpha/4, \lambda_1/2\alpha\}$ (see e.g. [25]): indeed, when considering attractors of dissipative wave equations, it yields dimensional bounds which are independent of α.

In order to prove finite dimensionality of $\mathcal{I}_\delta$, we have now to find d sufficiently large, so that the right hand side of (3.3) is negative, uniformly with respect to $\widetilde{U}$ and Σ_δ. We introduce the following fundamental Hypothesis:

Hypothesis 3.3.

(a) $\partial_u f(x, 0) \ge 0$ *for almost every* $x \in \Omega$;
(b) *there exists* $r > 3$ *such that* $\partial_u f(\,\cdot\,, 0) \in L^r(\Omega)$.

Notice that property (a) *is not really a restriction, since the negative part of* $\partial_u f(\,\cdot\,, 0)$ *can be absorbed by* $\beta(\,\cdot\,)$.

We observe that, by Hypotheses 2.7 and 3.3, we have:

$$|\partial_u f(x, u)| \le \partial_u f(x, 0) + C(1 + |u|)|u|, \quad (x, u) \in \Omega \times \mathbb{R}.$$

Take $\rho \in \mathcal{S}$ (the Schwartz class) with $\rho(x) > 0$ for all $x \in \mathbb{R}^3$ and, for $\varepsilon \ge 0$, define

$$W_{\widetilde{U}}(x) := \partial_u f(x, 0) + C(1 + |\widetilde{u}|_{L^\infty})|\widetilde{u}(x)|, \quad x \in \Omega, \tag{3.4}$$

and

$$W_{\widetilde{U},\varepsilon}(x) := W_{\widetilde{U}}(x) + \varepsilon\rho(x), \quad x \in \Omega.$$

The reason for introducing the correction $\varepsilon\rho(x)$ will be made clear later. Notice that $W_{\widetilde{U},\varepsilon}(\,\cdot\,) \in L^r(\Omega)$ for $\varepsilon \geq 0$ and $W_{\widetilde{U},\varepsilon} > 0$ for $x \in \Omega$ and $\varepsilon > 0$. Moreover,

$$\|\widehat{\partial_u f}(\widetilde{u})u\|^2_{L^2} \leq \|W_{\widetilde{U}}u\|^2_{L^2} \leq \|W_{\widetilde{U},\varepsilon}u\|^2_{L^2}, \quad u \in H^1_0(\Omega).$$

It follows from Lemma 4.5 in [17] that the assignment $u \mapsto W_{\widetilde{U},\varepsilon}u$ defines a compact linear operator from $H^1_0(\Omega)$ to $L^2(\Omega)$. Let us define the following operator $S_{\widetilde{U},\varepsilon}\colon Z_0 \to Z_0$:

$$S_{\widetilde{U},\varepsilon}(u,v) := (0, W_{\widetilde{U},\varepsilon}u), \quad U = (u,v) \in Z_0.$$

Then $S_{\widetilde{U},\varepsilon}$ is compact, and the same is true for its adjoint $S^*_{\widetilde{U},\varepsilon}$. We have

$$\|W_{\widetilde{U},\varepsilon}u\|^2_{L^2} = \langle S_{\widetilde{U},\varepsilon}U, S_{\widetilde{U},\varepsilon}U\rangle_{Z_0} = \langle S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}U, U\rangle_{Z_0}, \quad U = (u,v) \in Z_0.$$

The operator $S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}$ is compact, self-adjoint and non-negative. It follows that its spectrum is

$$\sigma(S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}) = \{0\} \cup \{\mu_{\widetilde{U},\varepsilon,j} \mid j = 1,2,\dots\},$$

where $(\mu_{\widetilde{U},\varepsilon,j})_{j\in\mathbb{N}}$ is a non-increasing sequence of real numbers tending to 0. The numbers $\mu_{\widetilde{U},\varepsilon,j}$, $j \in \mathbb{N}$, are the eigenvalues of $S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}$, repeated according to their multiplicity. In principle, the sequence $(\mu_{\widetilde{U},\varepsilon,j})_{j\in\mathbb{N}}$ can be ultimately null, but we shall see that this is not the case. Finally, the sequence $(\mu_{\widetilde{U},\varepsilon,j})_{j\in\mathbb{N}}$ is characterized by the min $-$ max formulae:

$$\mu_{\widetilde{U},\varepsilon,j+1} = \min_{\dim E \leq j} \max_{\substack{U \in \mathcal{E}^\perp \\ \|U\|_{Z_0}=1}} \langle S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}U, U\rangle_{Z_0}.$$

Let P_Σ be the Z_0-orthogonal projection onto Σ. Arguing as in the proof of Theorem XIII.3 in [22], we obtain

$$\begin{aligned}\text{(3.5)} \qquad \sum_{i=1}^d \|\widehat{\partial_u f}(\widetilde{u})\check{\phi}_i\|^2_{L^2} &\leq \sum_{i=1}^d \langle S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}\check{\Phi}_i, \check{\Phi}_i\rangle_{Z_0} \\ &= \operatorname{Tr}(P_\Sigma \circ (S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon})|_\Sigma) \leq \sum_{i=1}^d \mu_{\widetilde{U},\varepsilon,i}.\end{aligned}$$

It follows from (3.3) and (3.5) that

$$\text{(3.6)} \qquad \operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) \leq -\frac{d}{\alpha}\bigg(2\nu_\alpha\alpha - \frac{1}{d}\sum_{i=1}^d \mu_{\widetilde{U},\varepsilon,i}\bigg).$$

Now, since $\mu_{\widetilde{U},\varepsilon,i} \to 0$ as $i \to \infty$, then also the *Cesaro means* $(1/d)\sum_{i=1}^d \mu_{\widetilde{U},\varepsilon,i} \to 0$ as $d \to \infty$. Therefore there exists $d = d(\widetilde{U})$ such that the right-hand side of (3.6) is negative. The problem is that $d(\widetilde{U})$ depends on $\widetilde{U}$, so we must perform a more careful inspection of the asymptotic behavior of the sequence $(\mu_{\widetilde{U},\varepsilon,j})_{j\in\mathbb{N}}$.

Let (μ,Φ) be an eigenvalue-eigenvector pair of $S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}$, with $\mu\neq 0$. This is equivalent to say that

$$\langle S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}\Phi,U\rangle_{Z_0}=\mu\langle\Phi,U\rangle_{Z_0}\quad\text{for all } U\in Z_0. \tag{3.7}$$

More explicitly, (3.7) reads

$$\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2\phi u\,dx=\mu\bigg(\int_\Omega\nabla\phi\cdot\nabla u\,dx+\int_\Omega\beta(x)\phi u\,dx+\int_\Omega\psi v\,dx\bigg)$$

for all $U\in Z_0$, where $\Phi=(\phi,\psi)$ and $U=(u,v)$. Choosing first $u=0$ and letting $v\in L^2(\Omega)$ be arbitrary, we get that $\psi=0$. It follows that ϕ must satisfy

$$\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2\phi u\,dx=\mu\bigg(\int_\Omega\nabla\phi\cdot\nabla u\,dx+\int_\Omega\beta(x)\phi u\,dx\bigg)\quad\text{for all } u\in H^1_0.$$

Thus we have obtained that (μ,Φ) is an eigenvalue-eigenvector pair of $S^*_{\widetilde{U},\varepsilon}S_{\widetilde{U},\varepsilon}$ with $\mu\neq 0$ if and only if $\psi=0$ and $(\mu,\phi)=(\lambda^{-1},\phi)$, where (λ,ϕ) is an eigenvalue-eigenvector pair of the *weighted eigenvalue problem*

$$\int_\Omega\nabla\phi\cdot\nabla u\,dx+\int_\Omega\beta(x)\phi u\,dx=\lambda\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2\phi u\,dx\quad\text{for all } u\in H^1_0(\Omega). \tag{3.8}$$

In order to study (3.8) we proceed as in [11]: we denote by $L^2_{\widetilde{W},\varepsilon}(\Omega)$ the closure of $H^1_0(\Omega)$ with respect to the scalar product

$$\langle u_1,u_2\rangle_{L^2_{W_{\widetilde{U},\varepsilon}}}:=\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2u_1u_2\,dx.$$

It turns out that $L^2_{W_{\widetilde{U},\varepsilon}}(\Omega)$ is a separable Hilbert space, and $H^1_0(\Omega)$ is compactly embedded into $L^2_{W_{\widetilde{U},\varepsilon}}(\Omega)$. This is a consequence of the fact that $W^2_{\widetilde{U},\varepsilon}\in L^{r/2}(\Omega)$ with $r>3$ and $W_{\widetilde{U},\varepsilon}(x)>0$ almost everywhere in Ω. The latter observation makes clear the reason for which we introduced the correction $\varepsilon\rho(x)$. It follows from the general theory of self-adjoint operators with compact resolvent (see e.g. [6]) that the eigenvalues of (3.8), counted according to their multiplicity, form a sequence $(\lambda_{\widetilde{U},\varepsilon,j})_{j\in\mathbb{N}}$, with $\lambda_{\widetilde{U},\varepsilon,j}\to+\infty$ as $j\to\infty$. Now let $\widetilde{\lambda}>0$; we want to find an estimate for the number $\mathcal{N}(W_{\widetilde{U},\varepsilon},\widetilde{\lambda})$ of eigenvalues of (3.8) which are strictly smaller than $\widetilde{\lambda}$. To this end, we exploit a trick due to Li and Yau (see [13, Corollary 2]). Namely, we notice that, for $\phi \in H_0^1(\Omega)$,

$$\frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)\phi^2\,dx - \int_\Omega \widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2\phi^2\,dx}{\displaystyle\int_\Omega \phi^2\,dx} \tag{3.9}$$
$$= \frac{\displaystyle\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2\phi^2\,dx}{\displaystyle\int_\Omega \phi^2\,dx}\left(\frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)\phi^2\,dx}{\displaystyle\int_\Omega W_{\widetilde{U},\varepsilon}(x)^2\phi^2\,dx} - \widetilde{\lambda}\right).$$

It follows that, given a finite dimensional subspace E of $H_0^1(\Omega)$, the expression on the left-hand side in (3.9) is negative on E if and only if the expression on the right-hand side (3.9) is negative on E. Now we observe that the mapping $u \mapsto -\widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2 u$ is a relatively compact perturbation of $-\Delta+\beta(x)$. Therefore, by Weyl's Theorem, the essential spectrum of $-\Delta+\beta(x)-\widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2$ is contained in $[\lambda_1, +\infty[$. Then, thanks to the min $-$ max characterization of the eigenvalues of self-adjoint operators (see e.g. [22]), we deduce that

$$\mathcal{N}(W_{\widetilde{U},\varepsilon}, \widetilde{\lambda}) = n(-\Delta + \beta(x) - \widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2),$$

where $n(-\Delta + \beta(x) - \widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2)$ is the number of negative eigenvalues of the operator $-\Delta + \beta(x) - \widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2$. The latter can be estimated by mean of Cwickel-Lieb-Rozenblum inequality in its abstract formulation due to Rozenblum and Solomyak (see [23]). Namely, we have

$$n(-\Delta + \beta(x) - \widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2) \le M_r \int_\Omega (\widetilde{\lambda} W_{\widetilde{U},\varepsilon}(x)^2)^{r/2}\,dx, \tag{3.10}$$

where M_r is an constant depending only on r, λ_1, $|\beta|_{L^\sigma_{\rm u}}$, and on the constant of the embedding $H^2(\Omega) \subset L^\infty(\Omega)$ (see also [17, Section 5] for details; we stress that the constant M_r can be computed explicitly, though the determination of its optimal value seems out of reach). We have thus obtained that

$$\mathcal{N}(W_{\widetilde{U},\varepsilon}, \widetilde{\lambda}) \le \widetilde{\lambda}^{r/2} M_r \int_\Omega W_{\widetilde{U},\varepsilon}(x)^r\,dx.$$

Now fix $j \in \mathbb{N}$. For $\widetilde{\lambda} > \lambda_{\widetilde{U},\varepsilon,j}$ we have

$$j \le N(W_{\widetilde{U},\varepsilon}, \widetilde{\lambda}) \le \widetilde{\lambda}^{r/2} M_r \int_\Omega W_{\widetilde{U},\varepsilon}(x)^r\,dx.$$

By letting $\widetilde{\lambda}$ tend to $\lambda_{\widetilde{U},\varepsilon,j}$ we get

$$j \le \lambda_{\widetilde{U},\varepsilon,j}^{r/2} M_r \int_\Omega W_{\widetilde{U},\varepsilon}(x)^r\,dx.$$

It follows that

$$\lambda_{\widetilde{U},\varepsilon,j} \ge M_r^{-2/r} \|W_{\widetilde{U},\varepsilon}\|_{L^r}^{-2} j^{2/r},$$

whence

$$\mu_{\widetilde{U},\varepsilon,j} \leq M_r^{2/r} \|W_{\widetilde{U},\varepsilon}\|_{L^r}^2 \, j^{-2/r}. \tag{3.11}$$

Putting together (3.6) and (3.11), we get

$$\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) \leq -\frac{d}{\alpha}\bigg(2\nu_\alpha\alpha - \frac{1}{d}\sum_{j=1}^{d} M_r^{2/r}\|W_{\widetilde{U},\varepsilon}\|_{L^r}^2 \, j^{-2/r}\bigg).$$

Letting ε tend to 0 and taking into account (3.4), we finally get

$$\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta}) \leq -\frac{M_r^{2/r}\widetilde{C}(\mathcal{I})^2 d}{\alpha}\bigg(\frac{2\nu_\alpha\alpha}{M_r^{2/r}\widetilde{C}(\mathcal{I})^2} - \frac{1}{d}\sum_{j=1}^{d} j^{-2/r}\bigg), \tag{3.12}$$

where

$$\widetilde{C}(\mathcal{I}) := \|\partial_u f(\,\cdot\,,0)\|_{L^r} + C(1 + \sup_{(u,v)\in\mathcal{I}} \|u\|_{L^\infty}) \sup_{(u,v)\in\mathcal{I}} \|u\|_{L^r}. \tag{3.13}$$

We have thus obtained an estimate for $\operatorname{Tr}(\mathbf{B}_{\widetilde{U},\Sigma_d,\delta})$ which is uniform with respect to $\widetilde{U}$ and Σ_d. Now we are in a position to state and prove the main result of the paper:

Theorem 3.4. *Assume Hypotheses* 2.2, 2.6, 2.7 *and* 3.3 *are satisfied. Let $\mathcal{I} \subset Z_0$ be a compact invariant set of the semiflow $\Pi(t)$ generated by* (2.10). *Let ν_α and $\widetilde{C}(\mathcal{I})$ be defined by* (3.2) *and* (3.13) *respectively, and let M_r be the constant of the Cwickel–Lieb–Rozenblum inequality* (3.10). *Let $d > 0$ be such that*

$$\frac{1}{d}\sum_{j=1}^{d} j^{-2/r} \leq \frac{\nu_\alpha\alpha}{M_r^{2/r}\widetilde{C}(\mathcal{I})^2}. \tag{3.14}$$

Then the Hausdorff (*resp. the fractal*) *dimension of $\mathcal{I}$ in Z_0 is finite, and is less than or equal to d* (*resp.* $2d$).

Proof. Let p_j, $j \in \mathbb{N}$, be the numbers defined by (3.1). If d satisfies condition (3.14), then (3.12) implies that $p_d \leq -\nu_\alpha d$. Moreover, for $j = 1, \ldots, d-1$, one has

$$(p_j)_+ \leq \frac{M_r^{2/r}\widetilde{C}(\mathcal{I})^2}{\alpha}\sum_{i=1}^{j-1} i^{-2r} \leq \frac{M_r^{2/r}\widetilde{C}(\mathcal{I})^2}{\alpha}\sum_{i=1}^{d} i^{-2r} \leq \nu_\alpha d.$$

It follows from Proposition 2.14 and from the results in [25, Chapter V, pp. 287–291] that

$$\dim_H(\mathcal{I}) \leq d \quad \text{and} \quad \dim_F(\mathcal{I}) \leq d \max_{1\leq j\leq d-1}(1 + (p_j)_+/|p_d|) \leq 2d. \qquad \square$$

Remark 3.5. We can give an explicit estimate of d just noticing that

$$\frac{1}{d}\sum_{i=1}^{d} i^{-2r} \leq \frac{1}{d}\int_0^d s^{-2/r}\,ds = \frac{r}{r-2}d^{-2/r}.$$

It follows that

$$\dim_H(\mathcal{I}) \leq \left(\frac{r}{r-2}\frac{M_r^{2/r}\widetilde{C}(\mathcal{I})^2}{\nu_\alpha \alpha}\right)^{r/2} \tag{3.5}$$

and

$$\dim_F(\mathcal{I}) \leq 2\left(\frac{r}{r-2}\frac{M_r^{2/r}\widetilde{C}(\mathcal{I})^2}{\nu_\alpha \alpha}\right)^{r/2}. \tag{3.6}$$

Notice that $\nu_\alpha \alpha \to \lambda_1$ as $\alpha \to \infty$. Therefore, if we have a family $\mathcal{I}_\alpha$ of invariant sets of $\Pi(t) = \Pi_\alpha(t)$ and if we can control $|\mathcal{I}_\alpha|_{Z^1}$ independently of α, we obtain that the dimension of $\mathcal{I}_\alpha$ remains bounded as $\alpha \to \infty$. This is actually the case when the non-linearity f is dissipative and $\mathcal{I}_\alpha$ is the compact global attractor of $\Pi_\alpha(t)$, as we shall see in the next section.

4. Dissipative equations: dimension of the attractor

In this section we consider the equation

$$\begin{aligned} \varepsilon u_{tt} + u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\, \times \Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\, \times \partial\Omega, \end{aligned} \tag{4.1}$$

where $\varepsilon \in\,]0,1]$. Besides Hypotheses 2.2, 2.6, 2.7 and 3.3, we assume:

Hypothesis 4.1. *There exists a positive number μ and a function $c(\,\cdot\,) \in L^1(\Omega)$ such that:*

(a) $f(x,u)u - \mu F(x,u) \leq c(x)$;
(b) $F(x,u) \leq c(x)$.

Here, $F(x,u) := \int_0^u f(x,s)\,ds$, $(x,u) \in \Omega \times \mathbb{R}$.

It was proved in [19] that, under Hypotheses 2.2, 2.7 and 4.1, for every $\varepsilon \in\,]0,1]$ equation (4.1) generates a global semiflow in Z_0, possessing a compact global attractor $\mathcal{A}_\varepsilon$. Moreover, there exists a positive constant R such that

$$\sup_{\varepsilon \in]0,1]} \sup\{\|u\|^2_{H^1_0} + \varepsilon\|v\|^2_{L^2} \mid (u,v) \in \mathcal{A}_\varepsilon\} \leq R.$$

The constant R depends only on the constants in Hypotheses 2.2, 2.7 and 4.1 and on $\|c(\,\cdot\,)\|_{L^1}$, and can be explicitly computed (see [19]). In particular, R is independent of ε. Moreover, it was proved in [16] that there exists a positive constant $\widetilde{R}$ such that

$$\sup_{\varepsilon \in]0,1]} \sup\{\|u\|^2_{H^2 \cap H^1_0} + \|v\|^2_{H^1_0} \mid (u,v) \in \mathcal{A}_\varepsilon\} \leq \widetilde{R}.$$

Also, the constant $\widetilde{R}$ depends only on the constants in Hypotheses 2.2, 2.7 and 4.1 and on $\|c(\,\cdot\,)\|_{L^1}$ and can be explicitly computed (see [16]). In particular, $\widetilde{R}$ is independent of ε. By a time re-scaling ($t = \varepsilon^{1/2} s$) we see that (4.1) is equivalent to

$$\begin{aligned} \varepsilon \check{u}_{ss} + \alpha\ \check{u}_s + \beta(x)\check{u} - \Delta\check{u} &= f(x,\check{u}), &\quad (s,x) &\in [0,+\infty[\,\times\,\Omega, \\ \check{u} &= 0, & (s,x) &\in [0,+\infty[\,\times\,\partial\Omega, \end{aligned} \tag{4.2}$$

where $\alpha := \varepsilon^{-1/2}$. Equation (4.2) possesses a compact global attractor $\check{\mathcal{A}}_\alpha$, such that

$$\check{\mathcal{A}}_\alpha = \{(\check{u},\check{v}) \in Z_0 \mid (\check{u},\alpha\check{v}) \in \mathcal{A}_{\alpha^{-2}}\}.$$

It follows that $|\check{\mathcal{A}}_\alpha|_{Z_0} \le R$ and $|\check{\mathcal{A}}_\alpha|_{Z_1} \le \widetilde{R}$. As a consequence, the constant $\widetilde{C}(\check{\mathcal{A}}_\alpha)$ in (3.5) and (3.6) can be explicitly computed, and in particular it is independent of α. We have then

$$\dim_H(\mathcal{A}_\varepsilon) = \dim_H(\check{\mathcal{A}}_{\varepsilon^{-1/2}}) \le \left(\frac{r}{r-2}\frac{M_r^{2/r}\widetilde{C}(\check{\mathcal{A}}_{\varepsilon^{-1/2}})^2}{\nu_{\varepsilon^{-1/2}}\varepsilon^{-1/2}}\right)^{r/2}$$

and

$$\dim_F(\mathcal{A}_\varepsilon) = \dim_F(\check{\mathcal{A}}_{\varepsilon^{-1/2}}) \le 2\left(\frac{r}{r-2}\frac{M_r^{2/r}\widetilde{C}(\check{\mathcal{A}}_{\varepsilon^{-1/2}})^2}{\nu_{\varepsilon^{-1/2}}\varepsilon^{-1/2}}\right)^{r/2}.$$

Since $\nu_\alpha\alpha \to \lambda_1$ as $\alpha \to \infty$, we obtain that $\dim_H(\mathcal{A}_\varepsilon)$ and $\dim_F(\mathcal{A}_\varepsilon)$ remain bounded as $\varepsilon \to 0$, coherently with the fact that the $\mathcal{A}_\varepsilon$ "converge", as $\varepsilon \to 0$, to the attractor of the parabolic equation

$$\begin{aligned} u_t + \beta(x)u - \Delta u &= f(x,u), &\quad (t,x) &\in [0,+\infty[\,\times\,\Omega, \\ u &= 0, & (t,x) &\in [0,+\infty[\,\times\,\partial\Omega \end{aligned}$$

(see [20] and [16]).

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Martino Prizzi
Università di Trieste
Dipartimento di Matematica e Informatica
Via Valerio 12/1
34127 Trieste, ITALY

E-mail address: mprizzi@units.it