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FIXED POINTS AND NON-CONVEX SETS IN CAT(0) SPACES

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Abstract. Dropping the condition of convexity on the domain of a nonexpansive mapping is a difficult and unusual task in metric fixed point theory. Hilbert geometry has been one of the most fruitful at which authors have succeeded to drop such condition. In this work we revisit some of the results in that direction to study their validity in CAT(0) spaces (geodesic spaces of global nonpositive curvature in the sense of Gromov). We show that, although the geometry of CAT(0) spaces resembles at certain points that one of Hilbert spaces, much more than the CAT(0) condition is required in order to obtain counterparts of fixed point results for non-convex sets in Hilbert spaces. We provide significant examples showing this fact and give positive results for spaces of constant negative curvature as well as R-trees.

1. Introduction

Metric fixed point theory [7], [13] studies the existence of fixed points in Banach and metric spaces for mappings satisfying metric conditions such as being a contraction (Banach fixed point theorem) or a nonexpansive mapping

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(i.e. $d(Tx, Ty) \leq d(x, y)$), among others. This field has been extensively developed over the last fifty years, especially after Browder–Göhde–Kirk fixed point theorem for nonexpansive mappings in 1965. From that moment the standard situation has been to study the existence of fixed points for a nonexpansive self-mapping defined on a convex and bounded set. Most techniques developed in the last fifty years work especially well under convexity and boundedness of the domain of the mapping while very little has been said under the lack of any of them. In this work we take up the question of studying the existence of fixed points for nonexpansive and a class of asymptotically nonexpansive mappings with non-convex domain in CAT(0) spaces. Works motivating ours are mainly [20], [21] and the pioneering one [6].

It is well-known (Browder–Göhde–Kirk fixed point theorem) that any nonexpansive mapping defined from a nonempty bounded closed and convex subset D of a Hilbert space to itself has a fixed point. Actually, Browder–Göhde–Kirk fixed point theorem holds for more general Banach spaces [7], [13]. Goebel and Schöneberg [6] proved that if H is a Hilbert space and D is not necessarily convex but it is Chebyshev with respect to its convex hull (i.e. for any y in the closed convex hull of D there is a unique $x \in D$ such that $\|y - x\| = \inf_{z \in D} \|y - z\|$) then every nonexpansive self-mapping on D has a fixed point. This result was obtained via Kirszbraun-Valentine [14], [23] results on extension of nonexpansive mappings. It is very-well known that these extension results are very intrinsic to the Euclidean geometry within the class of normed spaces. Goebel–Schöneberg's result was extended in [21] by Rouhani to different classes of asymptotically nonexpansive mappings by following a completely different approach than that one from [6]. In fact, Kirszbraun–Valentine results are no longer useful for this new class of mappings. In this work we focus on these same problems for CAT(0) spaces which, as it is well-known, contain the class of Hilbert spaces. Our results will require of a different approach than that one from [20], [21], in fact our approach will very much rely on the hyperbolic geometry of the spaces we consider and will take us to a number of technical and geometrical results which will be needed to reach our goals.

Fixed points on CAT(0) spaces, or spaces of globally nonpositive curvature in the sense of Gromov, have been extensively studied in the last years by different authors (see for instance [3], [4], [12] as well as the seminar papers [9], [10] and references therein). [2] provides a very comprehensive exposition on CAT(0) spaces. CAT(0) spaces share a number of good properties with Hilbert spaces and so it is not that surprising that certain results originally obtained for Hilbert spaces find counterparts in CAT(0) spaces. We will show that results obtained by Rouhani in [20], [21] require a very strong symmetry of the metric of the space and so they are far to hold on any CAT(0) spaces. We give two examples of CAT(0) spaces where such results do not hold. One of them will link this problem with property (Q_4) recently studied in [4], [12]. The second example happens to be a CAT(0) tangent cone. The singularity about CAT(0) spaces which are 0-cones is that they contain many flat subsets (isometric to subsets of the 2-dimensional Euclidean space, see [2], [17], [18]), and so they could be considered as good candidates for positive answers. Our main results give extensions of Rouhani's results to CAT(0)-spaces of constant curvature and $\mathbb{R}$-trees.

This work is organized as follows. In Section 2 we introduce the required terminology and preliminary results for a better understanding of our work. In Section 3 we focus on the problem of absolute fixed points. Thanks to a recent result on extension of nonexpansive mappings on spaces of bounded curvature [17] (see also [16]) we can obtain counterparts of Rouhani's results from [20] in spaces of constant curvature and $\mathbb{R}$-trees. We also provide our first example of a CAT(0) space which does not verify Rouhani's results on existence of fixed points for non-convex subsets. In Section 4 we give an example of a 0-cone failing Rouhani's result. Then we present a collection of results which will ultimately lead us to our main result: counterparts of Goebel–Schöneberg and Rouhani's results for spaces of negative constant curvature. In our last section, Section 5, we take up the same problem for $\mathbb{R}$-trees (which can be regarded as spaces of $-\infty$ constant curvature). We will see that in this case the situation is much simpler and we will even be able to provide a multi-valued version of our result. We close our work with an appendix where we study the same problems for nonlinear contraction semigroups. In this work we do not consider the case of positive constant curvature.

2. Preliminaries and definitions

A *geodesic path* joining $x \in X$ to $y \in Y$ (or, more briefly, a *geodesic* from x to y) is a map $c\colon [0,l] \subseteq \mathbb{R} \to X$ such that

$$c(0) = x, \quad c(l) = y \quad \text{and} \quad d(c(t), c(t')) = |t - t'| \quad \text{for all } t, t' \in [0,l].$$

In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (*or metric*) *segment* joining x and y. When it is unique this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, and is said to have *the geodesic extension property* if each geodesic segment is contained in a geodesic which is isometric to the real line. Let $Y \subset X$, we denote by $G_1(Y)$ the union of all geodesic segments in X with endpoints in Y. Then Y is said to be *convex* if $G_1(Y) = Y$ or, equivalently, if every segment connecting two points $x, y \in Y$ is contained in Y. For $n \geq 2$ we inductively define $G_n(Y) = G_1(G_{n-1}(Y))$; then

$$\operatorname{conv}(Y) = \bigcup_{n=1}^{\infty} G_n(Y)$$

is the *convex hull* of Y.

Next we describe the model spaces of negative and zero curvature, the reader may find a much more thorough description of them in [2, Chapter I.2].

Let $\mathbb{E}^n$ stand for the metric space obtained by equipping the vector space $\mathbb{R}^n$ with the metric associated to the norm arising from the Euclidean scalar product $(x \mid y) = \sum_{i=1}^{n} x_i y_i$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, i.e. $\mathbb{R}^n$ endowed with the usual Euclidean distance.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \ldots, u_n, u_{n+1})$ and $v = (v_1, \ldots, v_n, v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u \mid v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Then the *real hyperbolic n-space* $\mathbb{H}^n$ is

$$\{u \in \mathbb{E}^{n,1} : \langle u \mid u\rangle = -1,\ u_{n+1} \geq 1\}.$$

PROPOSITION 2.1. *Let $d: \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the function that assigns to each pair $(A, B) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique non-negative number $d(A, B)$ such that*

$$\cosh d(A, B) = -\langle A \mid B\rangle.$$

Then $(\mathbb{H}^n, d)$ is a uniquely geodesic metric space.

The Model Spaces M_κ^2 for $\kappa \leq 0$ are defined as follows.

DEFINITION 2.2. Given $\kappa \in (-\infty, 0]$, we denote by M_κ^2 the following metric spaces:

(a) if $\kappa = 0$ then M_0^2 is the Euclidean space $\mathbb{E}^2$;
(b) if $\kappa < 0$ then M_κ^2 is obtained from the hyperbolic space $\mathbb{H}^2$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

Let (X, d) be a geodesic metric space. a *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in X consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in M_κ^2 such that $d_{M_\kappa^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. If $\kappa \leq 0$ then such a comparison triangle always exists in M_κ^2 and it is unique up to isometries.

A geodesic triangle $\triangle$ in X is said to satisfy the CAT(κ) *inequality* if, given $\overline{\triangle}$ a comparison triangle in M^2_κ for $\triangle$, for all $x, y \in \triangle$

$$d(x,y) \leq d_{M^2_\kappa}(\overline{x},\overline{y}),$$

where $\overline{x},\overline{y} \in \overline{\triangle}$ are the respective *comparison points* of x, y, i.e. if $x \in [x_i, x_j]$ is such that $d(x,x_i) = \lambda d(x_i,x_j)$ and $d(x,x_j) = (1-\lambda)d(x_i,x_j)$ then $\overline{x} \in [\overline{x}_i,\overline{x}_j]$ is such that $d_{M^2_\kappa}(\overline{x},\overline{x}_i) = \lambda d_{M^2_\kappa}(\overline{x}_i,\overline{x}_j)$ and $d_{M^2_\kappa}(\overline{x},\overline{x}_j) = (1-\lambda)d_{M^2_\kappa}(\overline{x}_i,\overline{x}_j)$.

Definition 2.3. If $\kappa \leq 0$, then X is called a CAT(κ) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

Definition 2.4. We will say that a space is of constant curvature $\kappa \in (-\infty, 0]$ if it is geodesic and all its triangles are isometric to their comparing ones in M^2_κ.

We will also need the notion of Alexandrov's angle. Let (p,q,z) be a triple in a CAT(0) space X and $(\overline{p},\overline{q},\overline{z})$ a comparison triple in M^2_κ. Assume $p, q \neq z$. Then the (κ-)*comparison angle* $\angle^\kappa_z(p,q) \in [0,\pi]$ is the (Riemannian) angle at $\overline{z}$ subtended by the segments $[\overline{z},\overline{p}]$, $[\overline{z},\overline{q}]$ in M^2_κ. Now let x, y be points in X and let $\sigma\colon [0, d(z,x)] \to X$ and $\tau\colon [0, d(z,y)] \to X$ be the geodesics from z to x and y respectively. It can then be shown (see [2]) that the limit

$$\angle_z(x,y) := \lim_{s',t' \to 0} \angle^\kappa_z(\sigma(s'),\tau(t'))$$

exists.

The Hyperbolic Cosine Law for the model space of curvature -1 (see [2, p. 24]) will be heavily used in this work.

Proposition 2.5. *Consider a triangle with side lengths a, b and c in M^2_{-1}, then the Hyperbolic Law of Cosines holds*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma,$$

where γ stands for the hyperbolic angle (which coincides with the Alexandrov angle) opposite to side of length c.

We summarize next some of the properties of CAT(0) spaces which can be found in [2, Chapter II] and will be needed in our work.

Proposition 2.6. *Let (X,d) be a* CAT(0) *space, then the following properties hold*:

(a) *(X,d) is a uniquely geodesic space.*
(b) *If $\Delta{=}\Delta(A,B,C)$ is a triangle in (X,d) and $\overline{\Delta} = \Delta(\overline{A},\overline{B},\overline{C})$ is its Euclidean comparison triangle, then for any vertex of Δ, let us say A,*

$$\gamma = \angle_A(B,C) \leq \angle_{\overline{A}}(\overline{B},\overline{C}).$$

(c) (Law of cosines) *If γ is as above and $a = d(B,C)$, $b = d(A,C)$ and $c = d(A,B)$ then*

$$a^2 \geq b^2 + c^2 - 2bc\cos\gamma.$$

In particular, if $\gamma \geq \pi/2$ then the largest side of Δ is the opposite to γ.

Another important feature from CAT(0) spaces is the behavior of the metric projection. This behavior resembles that of the same projections in Hilbert spaces.

Definition 2.7. Given a metric space X and a nonempty subset K of X, the *metric projection* (or nearest point map) from X onto K is denoted as P_K and defined by

$$P_K(x) = \{y \in K : d(x,y) = \operatorname{dist}(x,K)\}$$

where $\operatorname{dist}(x,K) = \inf_{y\in K} d(x,y)$.

The next proposition, which summarizes the properties of the metric projection onto closed and convex subsets of CAT(0) spaces, can be found in [2].

Proposition 2.8. *Let X be a complete* CAT(0) *space and $K \subseteq X$ nonempty, closed and convex. Then the metric projection onto K is well-defined (single-valued) and nonexpansive. Moreover, if $x \notin K$ and $y \in K$ with $y \neq P_K(x)$ then*

$$\angle_{P_K(x)}(x,y) \geq \frac{\pi}{2}.$$

$\mathbb{R}$-trees are a particular class of CAT(0) spaces with many applications in different fields. Since they are CAT(κ) spaces for any κ they are also referred to as spaces of $-\infty$ constant curvature (see [2, p. 167] for more details). The interested reader may check [1], [5], [11], [19] for recent advances on $\mathbb{R}$-trees and fixed points.

Definition 2.9. An $\mathbb{R}$-tree is a metric space M such that:

(a) it is a uniquely geodesic metric space;
(b) if x, y and $z \in M$ are such that $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

In this work we will deal with nonexpansive mappings and a class of mild asymptotically nonexpansive mappings which is defined next.

Definition 2.10. Let X be a metric space and $D \subseteq X$ nonempty. A mapping $T\colon D \to D$ is said to be

(a) *nonexpansive* if $d(Tx,Ty) \leq d(x,y)$ for every $x, y \in D$.
(b) *asymptotically nonexpansive in the intermediate sense* if

$$\limsup_{n\to\infty} \sup_{x,y\in D} (d(T^n x, T^n y) - d(x,y)) \leq 0.$$

It is immediate to see that nonexpansive mappings are asymptotically nonexpansive in the intermediate sense.

Asymptotic elements are very useful in metric fixed point theory. Consider (x_n) a bounded sequence in X. For $x \in X$ set

$$r(x,(x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(x_n)$ of (x_n) is given by

$$r(x_n) = \inf\{r(x,(x_n)) : x \in X\},$$

and the *asymptotic center* of (x_n), which we will generically denote as $A(x_n)$, is given by the set

$$A(x_n) = \{x \in X : r(x,(x_n)) = r(x_n)\}.$$

The following fundamental property has been proved in [12].

Proposition 2.11. *Let (x_n) be a bounded sequence in a* CAT(0) *space. Then the asymptotic center of (x_n) is a singleton. Moreover,*

$$A(x_n) \in \overline{\mathrm{conv}}\{x_n : n \in \mathbb{N}\}.$$

Next we introduce the notion of absolute fixed point given by Rouhani in [20].

Definition 2.12. Let X be a metric space, $D \subseteq X$ nonempty and $T\colon D \to D$ a nonexpansive mapping. We say that $x \in X$ is an absolute fixed point of T if the extension $\widetilde{T}$ of T from $D \cup \{x\}$ to $D \cup \{x\}$ such that $\widetilde{T}x = x$ is nonexpansive and if x is a fixed point for any nonexpansive extension of T to the union of D and a subset of X containing x.

Remark 2.13. Notice that a fixed point of T is trivially an absolute fixed point after the above definition.

Rouhani obtained the following result in [20] (see also [6]).

Theorem 2.14. *Let D be a nonempty subset of a real Hilbert space H, and $T\colon D \to D$ a nonexpansive mapping. Then T has an absolute fixed point in H if and only if the sequence $(T^n x)$ is bounded for some $x \in D$ (and hence for any point in D). In this case, for any $x \in D$, the asymptotic center of $(T^n x)$ is an absolute fixed point for T. Moreover, the mapping sending each $x \in D$ to the asymptotic center of $(T^n x)$ is nonexpansive.*

Later on, Rouhani proved some results on Hilbert spaces in [21] which extended pioneering fixed point results on non-convex sets from [6].

Theorem 2.15. *Let T be an asymptotically nonexpansive in the intermediate sense mapping defined on a nonempty subset D of a Hilbert space H. Assume T^N is nonexpansive for some integer $N \geq 1$. Then T has a fixed point* *in D if and only if there exists $x \in D$ with bounded orbit and such that for any $y \in \overline{\mathrm{conv}}\{T^n x : n \geq 0\}$ there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$.*

Remark 2.16. We would like to notice here that actually a bit more than that is proved in Rouhani's paper. In fact the condition "there exists $x \in D$ such that for any $y \in \overline{\mathrm{conv}}\{T^n x : n \geq 0\}$ there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$" can be replaced with "there exists $x \in D$ such that there is a unique $p \in D$ in such a way that

$$\|A(T^n x) - p\| = \inf_{z \in D} \|A(T^n x) - z\|,$$

where, as usual, $A(T^n x)$ stands for the asymptotic center of the sequence of iterates".

Remark 2.17. Theorem 2.15 was proved in [21] for the so-called class of asymptotically nonexpansive type self-mappings which is a slightly more general class than that of asymptotically nonexpansive in the intermediate sense mappings.

3. Absolute fixed points in CAT(0) spaces

One of the main keys for Rouhani's results from [20], [21] is Lemma 3.1 in [20] which states that if H is a Hilbert space, $D \subseteq H$ nonempty and $T\colon D \to D$ nonexpansive with bounded orbits, then the sequence $(\|T^n y - c\|)$, where $c = A(T^n x)$, is decreasing for any $y \in D$. We show next that this is no longer true in general CAT(0) spaces.

Example 3.1. Let X be defined by

$$X := \{(x, y) \in \mathbb{R}^2 : (x < 0 \wedge y = 0) \vee x \geq 0\}$$

endowed with the induced length metric from $\mathbb{E}^2$. This space is CAT(0) because it can be seen as gluing of two CAT(0) spaces (see [2, p. 347]). Consider now the set $D = \{X_1, X_2, Y_1, Y_2\}$ where

$$X_1 = (0, -2), \quad Y_1 = (-1, 0), \quad X_2 = (0, 2), \quad Y_2 = (\sqrt{5}, 0).$$

It is enough to define $T\colon D \to D$ as $T(X_i) = X_j$, $T(Y_i) = Y_j$ for $i, j \in \{1, 2\}$ with $i \neq j$. Then T is an isometry (and so nonexpansive). However, for $x = X_1$ and $y = Y_1$ we obtain that $A(T^n x) = (0, 0) =: 0$ while, as it is easy to see, $(d(T^n y, 0))$ is not decreasing.

As a consequence of this example we obtain the next corollary.

COROLLARY 3.2. *Remark* 2.16 *does not hold on general* CAT(0) *spaces even for isometries.*

PROOF. Just notice that the mapping from the example is fixed point free and that $\operatorname{dist}(0, D)$ is reached only at Y_1. $\square$

Another interesting thing from the previous example is that it fails the recently introduced [12] property (Q_4).

DEFINITION 3.3. A CAT(0) space X is said to enjoy property (Q_4) if for points x, y, p, q in X

$$\left.\begin{array}{l} d(x,p) < d(x,q) \\ d(y,p) < d(y,q) \end{array}\right\} \Rightarrow d(m,p) \leq d(m,q)$$

for any point m on the segment $[x, y]$.

Property (Q_4) was further studied in [4]. (Q_4) has not been checked yet in any other space which is not of constant curvature and we think that the lack of condition (Q_4) is very close to the failure of Theorem 2.15. If we restrict ourselves to spaces of constant curvature then we can obtain a counterpart of Rouhani's Lemma 3.1 in [20]. It is worth to say that Rouhani's proof is constructive while, as the same author indicates in his paper, a more direct proof could be derived from Kirszbraun–Valentine [23] results on extension of nonexpansive mappings. We would like to point out that actually Kirszbraun–Valentine's result would be applied when extending from a set D into another set $D \cup \{p\}$ and so it would also provide a constructive proof. Kirszbraun–Valentine's results have been extended recently to spaces of bounded curvature in [17] (see also [16]). We next state a particular case of one of the main results from [17].

THEOREM 3.4. *Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$. Let $D \subseteq X$ nonempty and $T\colon D \to X$ nonexpansive. Then, given $p \notin D$, there exists a nonexpansive extension of T to $D \cup \{p\}$.*

We will need the second part of the next lemma for our absolute fixed point result. The same lemma will be needed at its full in our last section.

LEMMA 3.5. *Let X be a complete* CAT(0) *space and $(x_n) \subseteq X$ a bounded sequence. Then for any $y \in X \setminus \{A(x_n)\}$ there is a subsequence $(x_{q(n)})$ of (x_n) such that*

$$\lim_{n\to\infty} d(x_{q(n)}, A(x_n)) = \limsup_{n\to\infty} d(x_n, A(x_n)) =: r(x_n)$$

and

$$\lim_{n\to\infty} P_{[A(x_n),y]}(x_{q(n)}) = A(x_n).$$

Proof. For an easier exposition make $A = A(x_n)$. Suppose the lemma is not true, then there is a positive number ε and $N \in \mathbb{N}$ such that

$$d(x_n, A) > r(x_n) - \varepsilon \quad \Longrightarrow \quad d(P_{[A,y]}(x_n), A) > \varepsilon$$

for all $n > N$. Let us choose $p \in [A, y]$ such that $d(A, p) = \dfrac{\varepsilon}{2}$. Then, as a consequence of Proposition 2.8, $\angle_p(A, x_n) \geq \pi/2$ for all x_n such that $d(x_n, A) > r(x_n) - \varepsilon$ and $n > N$. Therefore, by (c) of Proposition 2.6,

$$d^2(A, x_n) - d^2(A, p) \geq d^2(p, x_n),$$

still for the same x_n, and finally

$$\limsup d^2(p, x_n) \leq r^2(x_n) - \left(\frac{\varepsilon}{2}\right)^2 < r^2(x_n)$$

what is a contradiction with the definition of asymptotic center. □

The next result (counterpart of Theorem 2.14 for hyperbolic spaces) follows, in part, as a corollary of Theorem 3.4.

Theorem 3.6. *Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subseteq X$ nonempty and $T\colon D \to D$ nonexpansive. Assume that $(T^n x)$ is bounded for some $x \in D$. Then $A(T^n x)$ is an absolute fixed point of T. Moreover, the sequence $(d(T^n y, A(T^m x)))_{n\in\mathbb{N}}$ is non increasing and the mapping U from D into the set of absolute fixed points of T given by $U(x) = A(T^n x)$ is nonexpansive.*

Proof. Notice first that if there is x such that $(T^n x)$ is bounded then, from the nonexpansivity of T, the same is true for any $y \in D$. Denote (x_n) the sequence of iterates of $x \in D$. If it is the case that $A(x_n) \in D$ then it is immediate to check that $T(A(x_n))$ is also an asymptotic center of (x_n) and, since asymptotic centers are unique in CAT(0) spaces (Proposition 2.11), there is a nonexpansive extension of T to $D \cup \{A(x_n)\}$ and, by same arguments as above, $T(A(x_n)) = A(x_n)$. Therefore, $A(x_n)$ is an absolute fixed point of T either way. Also, the nonexpansivity of T or its extension, implies that $(d(T^n y, A(x_m)))_{n\in\mathbb{N}}$ is non increasing for any y in D.

Next we want to show that U is nonexpansive, that is, if (x_n) and (y_n) stand for the sequences of iterates of x and y respectively, then $d(A(x_n), A(y_n)) \leq d(x, y)$.

Let us assume that $d(A(x_n), A(y_n)) > d(x, y) > 0$. There is no loss of generality if we assume that $\kappa = -1$. The case $\kappa = 0$ would also follow after the same reasoning although this case was studied in [20] from a different approach. From Lemma 3.5, we know that $A(x_n)$ is an accumulation point of $(P_{[A(x_n),A(y_n)]}(x_n))$. Let (x_{n_k}) be such that

$$P_{[A(x_n),A(y_n)]}(x_{n_k}) \to A(x_n) \quad \text{as } k \to \infty.$$

At the same time, we may fix (y_{m_k}) such that

$$P_{[A(x_n),A(y_n)]}(y_{m_k}) \to A(y_n) \quad \text{as } k \to \infty.$$

Without loss of generality we may further assume that $n_1 \leq m_1 \leq n_2 \leq m_2 \leq \dots$

Make $l = [A(x_n), A(y_n)]$. Now, the nonexpansivity of the metric projection and T implies that $d(P_l(x_{n_k}), P_l(y_{n_k})) \leq d(x, y)$ for any k and so, since $d(A(x_n), A(y_n)) > d(x, y) > 0$, we can assume that there exists $\delta > 0$ such that

$$d(A(y_n), P_l(y_{n_k})) \geq \delta \quad \text{for any } k.$$

For simplicity we will assume that $P_l(y_{n_k}) = p$ for any k and that $d(p, A(y_n)) = \delta$. Otherwise, since we can assume that $(P_l(y_{n_k}))$ converges to such a p, a continuity reasoning would work.

For $k = 1$ let $d_1 = d(A(x_n), y_{n_1})$ and consider $B(A(x_n), d_1)$. Remember that $m_1 \geq n_1$ and then, since the sequence $(d(y_n, A(x_n)))$ is non increasing, it must also be the case that

$$y_{m_1} \in B(A(x_n), d_1).$$

Let now $a = d(A(x_n), p)$ and $b_1 = d(p, y_{n_1})$. By isometries if needed, we can consider that $[A(x_n), A(y_n)]$ and y_{n_1} are in $\mathbb{H}^2$. Consider the orthogonal line l' to $[A(x_n), A(y_n)]$ thought $A(y_n)$ in $\mathbb{H}^2$. Take $z \in l'$ different from $A(y_n)$, then, from the Hyperbolic Cosine Law, we have that if $b_z = d(A(y_n), z)$

$$\cosh d(A(x_n), z) = \cosh(a + \delta) \cosh b_z$$

and, by applying again the law of cosines, it results that $z \in B(A(x_n), d_1)$ if and only if $b_z \leq b_2$ where b_2 is given by

$$\cosh b_2 = \frac{\cosh a}{\cosh(a + \delta)} \cosh b_1.$$

Since this can be reproduce for any such points, this is a property that holds on X. Now, back to X, consider y_{m_1}. Since $P_l(y_{m_k}) \to A(y_n)$ we can assume that $\angle_{A(y_n)}(p, y_{m_k}) \to \alpha \geq \pi/2$ as $k \to \infty$. For simplicity, we will assume that $\angle_{A(y_n)}(p, y_{m_k}) \geq \pi/2$ for any k since otherwise a continuity argument would work. We know that $y_{m_1} \in B(A(x_n), d_1)$ therefore, as an application of the Hyperbolic Cosine Law, we obtain that $d(A(y_n), y_{m_1}) \leq b_2$.

Next, since $d(A(y_m), y_n)$ is decreasing with n, $y_{n_2} \in B(A(y_n), b_2)$ and so $d(y_{n_2}, p) \leq b_2$. By repeating the same argument we obtain b_3 such that

$$\cosh b_3 = \frac{\cosh a}{\cosh(a + \delta)} \cosh b_2 = \left(\frac{\cosh a}{\cosh(a + \delta)}\right)^2 \cosh b_1$$

and $y_{m_2} \in B(A(y_n), b_3)$ which, after a finite number of steps, leads to a contradiction. □

As a corollary we obtain the following counterparts to the main result from [6].

Corollary 3.7. *Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subseteq X$ nonempty and $T\colon D \to D$ nonexpansive. Then T has a fixed point if and only if there is $x \in D$ for which the sequence of its iterates $(T^n x)$ is bounded and there is a unique $y \in D$ such that*

$$d(A(T^n x), y) = \operatorname{dist}(A(T^n x), D).$$

Corollary 3.8. *Let X be a complete space of constant curvature $\kappa \in (-\infty, 0]$, $D \subseteq X$ nonempty and $T\colon D \to D$ nonexpansive. Then T has a fixed point in D if and only if there is $x \in D$ for which the sequence of its iterates $(T^n x)$ is bounded and for any $y \in \overline{\operatorname{conv}}\{T^n x\colon n \geq 0\}$ there is a unique $p \in D$ such that* $\operatorname{dist}(y, D) = d(y, p)$.

The same results hold true for $\mathbb{R}$-trees.

Theorem 3.9. *Let X be a complete $\mathbb{R}$-tree, $D \subseteq X$ nonempty and $T\colon D \to D$ nonexpansive. Assume that $(T^n x)$ is bounded for some $x \in D$. Then $A(T^n x)$ is an absolute fixed point of T. Moreover, the sequence $(d(T^n y, A(T^m x)))$ is non increasing and the mapping U from D into the set of absolute fixed points of T given by $U(x) = A(T^n x)$ is nonexpansive.*

Proof. The proof for $\mathbb{R}$-trees follows the same scheme as the previous one. Extending nonexpansive mappings in a nonexpansive way when defined into an $\mathbb{R}$-tree is a very well-known fact since complete $\mathbb{R}$-trees are hyperconvex (see [8], [18]). The nonexpansivity of U can be proved by paralleling the reasoning in the previous proof although now it follows in an easier way. $\square$

Equivalent corollaries to Corollaries 3.7 and 3.8 follow for $\mathbb{R}$-trees.

4. Fixed point on non convex sets

Example 3.1 already showed that Theorem 2.15 does not hold on general CAT(0) spaces. That example basically shows that Theorem 2.15 is hard to hold on CAT(0) spaces which are obtained as gluings of other CAT(0) spaces (see [2] for details about gluings). The next example provides a similar example on a 0-cone. The significance of 0-cones is that they can be regarded as quite regular class of CAT(0) spaces since they contain many two-dimensional Euclidean subsets.

Definition 4.1. Given a metric space Y, consider X the quotient of $[0, \infty) \times Y$ given by the equivalence relation: $(t, y) \sim (t', y')$ if $t = t' = 0$ or $t = t' > 0$ and $y = y'$ otherwise. For $x = ty$ and $x' = t'y'$ in X, define

$$d^2(x, x') = t^2 + t'^2 - 2tt' \cos(d_\pi(y, y'))$$

where $d_\pi(y, y') := \min\{\pi, d(y, y')\}$. Then (X, d) is the 0-cone of Y.

A more detailed treatment of 0-cones can be found in [2, p. 59] or [17], [18].

Example 4.2. Let M be the segment $[0, 3\pi)$ with distance d defined by

$$d(x, y) = \min\{|x - y|, 3\pi - |x - y|, \pi\}.$$

It is easy to check that M is a CAT(1) space.

Let X be the 0-cone of M. It is shown in [2, p. 188] that the 0-cone of a CAT(1) space is a CAT(0) space, therefore X is a CAT(0) space. Moreover, X is not a tangent cone of M but M is a space of directions of X at the point 0, so X is a self tangent cone.

Now let us consider $D \subseteq X$ made of the points:

$$X_1 = 2 \cdot \pi, \quad Y_1 = 1 \cdot 0, \quad X_2 = 2 \cdot 2\pi, \quad Y_2 = \sqrt{5} \cdot \frac{3\pi}{2}.$$

If we define $T \colon D \to D$ as $T(X_i) = X_j$, $T(Y_i) = Y_j$ for $i, j \in \{1, 2\}$ with $i \neq j$, then T is an isometry (so nonexpansive) and assumptions from Remark 2.16 hold for $x = X_1$ and $y = Y_1$ because $A(T^n x) = 0$. Notice also that T is fixed point free.

Our positive result in this direction will be on spaces of constant curvature. For simplicity we will consider, however, the spaces $\mathbb{H}^\infty$ of constant curvature -1 in our proofs. In Lemma 4.7 we will apply the geodesic extension property of $\mathbb{H}^\infty$, but since the convex hull of finite sets of spaces of curvature -1 are isometric, we may generalize our results for spaces of constant curvature without the geodesic extension property.

For constant curvature $\kappa \in (-\infty, 0)$ the result would follow the same way. The unique modification we have to apply is that in the definition of barycenters (see Lemma 4.4) distances should be multiplied by $\sqrt{-\kappa}$. The fact that we work on $\mathbb{H}^\infty$ instead of a finite dimensional model space of -1 curvature is only to show that compactness arguments do not play any role in our reasonings. In fact, our proofs work the same with no regards on the dimension of the space.

For our first result we will need the CN inequality of CAT(0) spaces ([2, p. 163]).

Proposition 4.3. *Let X be a CAT(0) space, $p, q, r \in X$ and m the middle point of $[q, r]$, then*

$$d^2(p, q) + d^2(p, r) \geq 2d^2(m, p) + \frac{1}{2} d^2(q, r).$$

A main tool in our results will be $\cosh d$-barycenters in $\mathbb{H}^\infty$. We define such barycenters in the following lemma motivated by Proposition 1.7 in [22].

Lemma 4.4. *Let X be a complete* CAT(0) *space and consider $\{x_1, \dots, x_n\} \subseteq X$. Then there is a unique minimizer $z \in X$ for the function*

$$\phi(y) = \sum_{k=1}^{n} \cosh d(y, x_k).$$

This minimizer will be called the $\cosh d$*-barycenter, or just barycenter for simplicity, of the set $\{x_1, \dots, x_n\}$ and will be denoted as $s(x_1, \dots, x_n)$. Furthermore, the barycenter always belongs to the closed convex hull of $\{x_1, \dots, x_n\}$.*

Proof. The function ϕ is positive and so it has an infimum. Let $\{z_m\}$ be a minimizing sequence, that is,

$$\sum_{k=1}^{n} \cosh d(z_m, x_k) \to \inf_{x \in X} \sum_{k=1}^{n} \cosh d(x, x_k)$$

as $m \to \infty$. Take $k \in \{1, \dots, n\}$ and for $p, q \in \mathbb{N}$ let z_{pq} the middle point between z_p and z_q. Then

$$\cosh d(x_k, z_{pq}) = 1 + \frac{d^2(x_k, z_{pq})}{2!} + \frac{d^4(x_k, z_{pq})}{4!} + \dots$$

(from Proposition 4.3)

$$\leq 1 + \frac{\frac{1}{2}\, d^2(x_k, z_p) + \frac{1}{2}\, d^2(x_k, z_q) - \frac{1}{4}\, d^2(z_p, z_q)}{2!} + \frac{d^4(x_k, z_{pq})}{4!} + \dots \tag{4.1}$$

Now, by recalling that the metric is convex,

$$\begin{aligned} d(x_k, z_{pq}) \ &\leq \frac{1}{2} d(x_k, z_p) + \frac{1}{2} d(x_k, z_q), \\ d^{2j}(x_k, z_{pq}) \ &\leq \left(\frac{1}{2} d(x_k, z_p) + \frac{1}{2} d(x_k, z_q)\right)^{2j} \leq \frac{1}{2} d^{2j}(x_k, z_p) + \frac{1}{2} d^{2j}(x_k, z_q). \end{aligned}$$

Then

$$\begin{aligned} (4.1) \leq 1 &+ \frac{\frac{1}{2}\, d^2(x_k, z_p) + \frac{1}{2}\, d^2(x_k, z_q) - \frac{1}{4}\, d^2(z_p, z_q)}{2!} \\ &+ \frac{\frac{1}{2}\, d^4(x_k, z_p) + \frac{1}{2}\, d^4(x_k, z_q)}{4!} + \dots \\ = &\frac{1}{2}\left(1 + \frac{d^2(x_k, z_p)}{2!} + \frac{d^4(x_k, z_p)}{4!} + \dots\right) \\ &+ \frac{1}{2}\left(1 + \frac{d^2(x_k, z_q)}{2!} + \frac{d^4(x_k, z_q)}{4!} + \dots\right) - \frac{1}{8} d^2(z_p, z_q) \\ = &\frac{1}{2} \cosh d(x_k, z_p) + \frac{1}{2} \cosh d(x_k, z_q) - \frac{1}{8}\, d^2(z_p, z_q), \end{aligned}$$

and therefore

$$\sum_{k=1}^{n} \cosh d(x_k, z_{pq}) \le \frac{1}{2}\sum_{k=1}^{n} \cosh d(x_k, z_p) + \frac{1}{2}\sum_{k=1}^{n} \cosh d(x_k, z_q) - \frac{n}{8} d^2(z_p, z_q),$$

from where the conclusion follows by taking limit when p, q go to ∞.

The fact that the minimizer must be in the closed convex hull of $\{x_1, \dots, x_n\}$ directly follows from the nonexpansivity of the metric projection onto closed and convex subsets of CAT(0) spaces. □

Definition 4.5. Given a sequence (x_n) in a CAT(0) space we will denote by $(s(x_1, \dots, x_n))$ the sequence of its baryncenters and, when it exists, by $A(s(x_1, \dots, x_n))$ the asymptotic center of the sequence of barycenters.

Next we state our main fixed point result for non-convex sets in spaces of negative constant curvature with the geodesic extension property.

Theorem 4.6. *Let D be a nonempty subset of $\mathbb{H}^\infty$ and $T\colon D \to D$ an asymptotically nonexpansive in the intermediate sense mapping. Moreover, suppose that there is N such that T^N is nonexpansive. Then T has at least one fixed point if and only if there is $x \in D$ for which the sequence of its iterates (x_n), where $x_n = T^n(x)$, is bounded and there is a unique $y \in D$ such that*

$$d(A(s(x_1, \dots, x_n)), y) = \operatorname{dist}(A(s(x_1, \dots, x_n)), D).$$

Before proving this theorem we will show a collection of technical lemmas which will be required in its proof.

Lemma 4.7. *Let $\{p, a_1, \dots, a_n\} \subset \mathbb{H}^\infty$ then*

$$\sum_{k=1}^{n} \cosh d(p, a_k) = \cosh d(p, s(a_1, \dots, a_n)) \sum_{k=1}^{n} \cosh d(s(a_1, \dots, a_n), a_k).$$

Proof. Let $s := s(a_1, \dots, a_n)$. On account of the Hyperbolic Cosine Law for each $k \in \{1, \dots, n\}$ and $p \in \mathbb{H}^\infty$ we have

$$\cosh d(a_k, p) = \cosh d(a_k, s) \cosh d(p, s) - \sinh d(a_k, s) \sinh d(p, s) \cos \angle_s(p, a_k).$$

Let $K := \sum_{k=1}^{n} \sinh d(a_k, s) \sinh d(s, p) \cos \angle_s(p, a_k)$. It is sufficient to show that $K = 0$.

Let us consider K positive. Let s' be a point in the metric segment $[s, p]$. Then for each $k \in \{1, \dots, n\}$ an angle $\angle_{s'}(p, a_k)$ is not smaller than $\angle_s(p, a_k)$ and, since $\angle_{s'}(s, a_k) + \angle_{s'}(p, a_k) = \pi$,

$$\begin{aligned}\cosh d(a_k, s) = \cosh d(a_k, s') \cosh d(s', s)& \\ + \sinh d(a_k, s') \sinh d(s, s') \cos \angle_{s'}(p, a_k)&.\end{aligned}$$

Summing up this equality from $k = 1$ to n we get

$$\sum_{k=1}^{n} \cosh d(a_k, s) = \sum_{k=1}^{n} \cosh d(a_k, s') \cosh d(s, s') + \sum_{k=1}^{n} \sinh d(a_k, s') \sinh d(s, s') \cos \angle_{s'}(p, a_k).$$

If $d(s', s)$ is small enough, then the last sum gets as close to $K \cdot \frac{d(s',s)}{d(p,s)}$ as wished and therefore we can assume it is positive. Hence

$$\sum_{k=1}^{n} \cosh d(a_k, s) \geq \sum_{k=1}^{n} \cosh d(a_k, s') \cosh d(s, s') > \sum_{k=1}^{n} \cosh d(a_k, s'),$$

a contradiction.

Now let us suppose that K is negative. Let s' be a point of a ray containing $[s, p]$ such that $s \in (p, s')$. Then for each $k \in \{1, \ldots, n\}$, since s is now the interior point in the segment, the angle $\angle_{s'}(p, a_k)$ is not greater than $\angle_s(p, a_k)$ and

$$\cosh d(a_k, s) = \cosh d(a_k, s') \cosh d(s', s) - \sinh d(a_k, s') \sinh d(s, s') \cos \angle_{s'}(p, a_k).$$

Taking $d(s', s)$ small enough and reasoning as above, we get

$$\sum_{k=1}^{n} \cosh d(a_k, s) \geq \sum_{k=1}^{n} \cosh d(a_k, s') \cosh d(s, s') > \sum_{k=1}^{n} \cosh d(a_k, s'),$$

which is a contradiction. □

Lemma 4.8. *Let (a_n) be a bounded sequence in $\mathbb{H}^\infty$, then*

$$\lim_{n\to\infty} d(s(a_1, \ldots, a_n), s(a_2, \ldots, a_{n+1})) = 0.$$

Proof. Let us fix $n > 2$ and take $s_n := s(a_2, \ldots, a_n)$, $s'_n := s(a_1, \ldots, a_n)$. For simplicity, we will denote $s = s_n$ and $s' = s'_n$. From the definition of s' we have that

$$\sum_{i=1}^{n} \cosh d(a_i, s') \leq \cosh d(a_1, s) + \sum_{i=2}^{n} \cosh d(a_i, s)$$

(from Lemma 4.7)

$$= \cosh d(a_1, s) + \sum_{i=2}^{n} \frac{\cosh d(a_i, s')}{\cosh d(s, s')}.$$

Therefore,

$$\begin{aligned}\cosh d(a_1, s') - \cosh d(a_1, s) &\leq \sum_{i=2}^{n} \frac{\cosh d(a_i, s')}{\cosh d(s, s')} - \sum_{i=2}^{n} \cosh d(a_i, s') \\ &= \sum_{i=2}^{n} \cosh d(a_i, s') \left(\frac{1}{\cosh d(s, s')} - 1 \right).\end{aligned}$$

So, bearing in mind that $\cosh(x) \geq 1$,

$$\begin{aligned}(n-1)\left(1 - \frac{1}{\cosh d(s, s')}\right) &\leq \sum_{i=2}^{n} \cosh d(a_i, s') \left(1 - \frac{1}{\cosh d(s, s')}\right) \\ &\leq \cosh d(a_1, s) - \cosh d(a_1, s'). \qquad (4.2)\end{aligned}$$

Now, since $s, s' \in \overline{\text{conv}}\{a_1, \dots, a_n\} \subseteq \overline{\text{conv}}\{a_1, \dots, a_m, \dots\}$ for any $n \in \mathbb{N}$,

$$(4.2) \leq \cosh M,$$

where M stands for the diameter of (a_n). Hence it must be the case that $d(s, s') \to 0$ as $n \to \infty$. $\square$

Lemma 4.9. *Let $\{z, y, a_1, a_2\} \subseteq \mathbb{H}^\infty$ with diameter M and such that $d(y, a_i) \leq d(z, a_i) \leq d(y, a_i) + \delta$ for $i \in \{1, 2\}$ and for a certain $\delta > 0$, then*

$$d(z, u) \leq d(y, u) + \operatorname{arc\,cosh}(1 + \delta \cdot \sinh M) \qquad (4.3)$$

for any $u \in [a_1, a_2]$.

Proof. Let us consider the triangles $\triangle(a_1, a_2, y)$ and $\triangle(a_1, a_2, z)$ and $u \in (a_1, a_2)$. Apply now the law of cosines to the triangles $\triangle(a_1, a_2, y)$ and $\triangle(a_1, u, y)$ at the angle at a_1. Therefore we obtain two expressions for $\cos \angle_{a_1}(y, a_2)$ which allows us to replace the first one into the second one to obtain

$$\begin{aligned}\cosh d(y, u) &= \cosh d(y, a_1) \cosh d(a_1, u) \\ &\quad - \frac{\sinh d(a_1, u)}{\sinh d(a_1, a_2)} (\cosh d(y, a_1) \cosh d(a_1, a_2) - \cosh d(y, a_2)).\end{aligned}$$

A similar expression is obtained for $\cosh d(z, u)$ by working with triangles $\triangle(a_1, a_2, z)$ and $\triangle(a_1, u, z)$.

From where, applying the formula for the $\sinh(d(a_1, a_2) - d(a_1, u))$ in both cases, we obtain that

$$\cosh d(y, u) = \frac{\sinh d(u, a_2)}{\sinh d(a_1, a_2)} \cosh d(y, a_1) + \frac{\sinh d(a_1, u)}{\sinh d(a_1, a_2)} \cosh d(y, a_2) \qquad (4.4)$$

and

$$\cosh d(z, u) = \frac{\sinh d(u, a_2)}{\sinh d(a_1, a_2)} \cosh d(z, a_1) + \frac{\sinh d(a_1, u)}{\sinh d(a_1, a_2)} \cosh d(z, a_2), \qquad (4.5)$$

Since $\sinh(x)$ is superadditive in $[0,\infty)$, we further obtain

$$
\begin{aligned}
(4.6)\qquad \cosh d(z,u) - \cosh d(y,u) \\
&= \frac{\sinh d(u,a_2)}{\sinh d(a_1,a_2)}(\cosh d(z,a_1)-\cosh d(y,a_1)) \\
&\quad + \frac{\sinh d(a_1,u)}{\sinh d(a_1,a_2)}(\cosh d(z,a_2)-\cosh d(y,a_2)) \\
&\le \max\{\cosh d(z,a_1)-\cosh d(y,a_1), \cosh d(z,a_2)-\cosh d(y,a_2)\}.
\end{aligned}
$$

Now we go into some elementary analysis to estimate $\cosh A - \cosh B$ for $A > B > 0$ (the case $0 < A = B$ follows in a trivial way). By expressing B and $A - B$ as convex combination of A and 0, considering the convexity of $\cosh(x)$ and substracting, we obtain that

$$\cosh(A-B) \le 1 + \cosh A - \cosh B.$$

From Lagrange's theorem, there exists $C \in (B,A)$ such that $\cosh A - \cosh B = (A-B)\sinh C$, and so

$$1 + \cosh A - \cosh B \le 1 + \delta \sinh M,$$

where $\delta = A - B$ and $M \ge C$. Now, it only requires to apply all this with $A = d(z,u)$, $B = d(y,u)$ and δ and M as above to deduce (4.3). $\square$

Notice that from this lemma we obtain an expectable strong version of condition (Q_4) for spaces of constant curvature.

Corollary 4.10. *Let z, y, a_1, a_2 be four points in $\mathbb{H}^\infty$ such that $d(y,a_i) < d(z,a_i)$ for $i \in \{1,2\}$, then $d(y,u) < d(z,u)$ for each $u \in [a_1,a_2]$.*

Proof. This immediately follows from equations (4.4) and (4.5). $\square$

Theorem 4.11. *Let D be a nonempty subset of $\mathbb{H}^\infty$ and $T\colon D \to D$ an asymptotically nonexpansive mapping in the intermediate sense. Suppose that there exists $x \in D$ such that the sequence of its iterates (x_n) is bounded. Then for any $y \in D$ the following equality holds*

$$(4.7)\qquad \lim_{n\to\infty} d(A(s(x_1,\dots,x_m)),y_n) = \inf_{n\in\mathbb{N}} d(A(s(x_1,\dots,x_m)),y_n),$$

where, as usual, $y_n = T^n y$. Moreover, if there is $N \in \mathbb{N}$ such that T^N is nonexpansive then the sequence $(d(y_{nN}, A(s(x_1,\dots,x_m))))_{n\in\mathbb{N}}$ is non increasing.

Proof. Take $y \in D$, the metric conditions on T imply that the sequence of its iterates (y_n) is bounded. Take M as the diameter of $\{x_n, y_n : n \in \mathbb{N}\}$. From the asymptotic character of T and the fact that orbits are bounded, we choose $\varepsilon > 0$ and N_1 such that

$$(4.8)\qquad \cosh d(x_{i+k}, T^k(y)) < \cosh d(x_i,y) + \varepsilon/2 \quad \text{for each } k > N_1.$$

Fix $k > N_1$ and denote $s = s(x_1, \dots, x_n)$, $s' = s(x_{1+k}, \dots, x_{n+k})$. On account of Lemma 4.7,

$$\sum_{i=1}^{n} \cosh d(x_i, y) = \cosh d(s, y) \sum_{i=1}^{n} \cosh d(x_i, s)$$

and

$$\sum_{i=1+k}^{n+k} \cosh d(x_i, T^k(y)) = \cosh d(s', T^k(y)) \sum_{i=1+k}^{n+k} \cosh d(x_i, s').$$

By (4.8) it follows that

$$\sum_{i=1+k}^{n+k} \cosh d(x_i, T^k(y)) \le \sum_{i=1}^{n} \cosh d(x_i, y) + n\varepsilon/2$$

and so

$$\sum_{i=1+k}^{n+k} \cosh d(x_i, s') \cosh d(s', T^k(y)) \le \sum_{i=1}^{n} \cosh d(x_i, s) \cosh d(s, y) + n\varepsilon/2.$$

Therefore,

$$\begin{aligned}
\cosh d(s, T^k(y)) &= \cosh d(s, T^k(y)) - \cosh d(s', T^k(y)) + \cosh d(s', T^k(y)) \\
&\le \cosh d(s, T^k(y)) - \cosh d(s', T^k(y)) \\
&\quad + \frac{\cosh d(s, y) \sum_{i=1}^{n} \cosh d(x_i, s) + n\varepsilon/2}{\sum_{i=1+k}^{n+k} \cosh d(x_i, s')} \\
&\le \cosh d(s, y) + d(s, s') \sinh M \\
&\quad + \cosh d(s, y) \frac{\sum_{i=1}^{k} (\cosh d(x_i, s) - \cosh d(x_{n+i}, s))}{\sum_{i=1+k}^{n+k} \cosh d(x_i, s')} \\
&\quad + \frac{\cosh d(s, y) \sum_{i=k+1}^{n+k} (\cosh d(x_i, s) - \cosh d(x_i, s')) + n\varepsilon/2}{\sum_{i=1+k}^{n+k} \cosh d(x_i, s')} \\
&\le \cosh d(s, y) + d(s, s') \sinh M \\
&\quad + \cosh M \frac{k \cosh M + n \sinh M \; d(s, s')}{n} + \frac{\varepsilon}{2}.
\end{aligned}$$

Now, as a consequence of Lemma 4.8, we can assume that there exists $N(k) \in \mathbb{N}$ such that

$$\cosh d(T^k(y), s(x_1, \dots , x_n)) < \cosh d(y, s(x_1, \dots , x_n)) + \varepsilon, \quad \text{for } n > N(k).$$

We claim next that

$$\begin{aligned}\cosh d(T^k(y), u) &- \cosh d(y, u) \\ &\le \max\{0, \sup\{\cosh d(T^k(y), s(x_1, \dots , x_n)) - \cosh d(y, s(x_1, \dots , x_n)) \colon \\ &\qquad\qquad n > N(k)\}\}.\end{aligned}$$

for an $u \in \overline{\mathrm{conv}}\{s(x_1, \dots , x_n) \colon n > N(k)\}$. Indeed,

$$\mathrm{conv}\{s(x_1, \dots , x_n) \colon n > N(k)\} = \bigcup_{j=1}^{\infty} G_j(s(x_1, \dots , x_n) \colon n > N(k)).$$

First, consider $u \in G_1$. So let $u \in [s(x_1, \dots , x_i), s(x_1, \dots , x_j)]$ for certain i and j. If $\cosh d(T^k(y), s) \le \cosh d(y, s)$ for $s \in \{s(x_1, \dots , x_i), s(x_1, \dots , x_j)\}$, on account of (4.4)–(4.5) the same holds true for u. Otherwise, the conclusion follows as a direct application of (4.6). Now, take $u \in G_2$. Then there exist $v_i \in (s(x_1, \dots , x_n))_{n \ge N(k)+1}$ for $i \in \{1, 2, 3, 4\}$, not necessarily different, such that u is in a segment with endpoints w_1 and w_3, respectively in $[v_1, v_2]$ and $[v_3, v_4]$.

Now we may repeat the above consideration for w_1, w_3 and finally u, to obtain our claim.

Now we only need to recall that $A(s(x_1, \dots , x_n)) \in \overline{\mathrm{conv}}\{s(x_1, \dots , x_n) : n \ge N(k) + 1\}$ (e.g. [4, Proposition 4.5]) to deduce that

$$\cosh d(T^k(y), A(s(x_1, \dots , x_n))) - \cosh d(y, A(s(x_1, \dots , x_n))) \le \varepsilon$$

is still true for all $k > N_1$.

Since ε is arbitrary, we have

$$\limsup_{n\to\infty} \cosh d(T^n(y), A(s(x_1, \dots , x_m))) \le \cosh d(y, A(s(x_1, \dots , x_m))),$$

and repeating our consideration for each $i \in \mathbb{N}$ we obtain that

$$\limsup_{n\to\infty} \cosh d(T^n(y), A(s(x_1, \dots , x_m))) \le \inf_{i\in\mathbb{N}} \cosh d(T^i(y), A(s(x_1, \dots , x_m))),$$

what on account of monotonicity of cosh function completes the proof of (4.7). In a similar way one may prove that

$$d(T^{N(k+1)}(y), A(s(x_1, \dots , x_m))) \le d(T^{Nk}(y), A(s(x_1, \dots , x_m))),$$

if T^N is nonexpansive, which completes our proof. $\square$

Next we prove our main fixed point theorem. After the previous technical results, the proof of this theorem is similar to that of Theorem 3.1 in [21], we show details for completeness.

Proof of Theorem 4.6. Necessity is obvious. Assume now that (x_n) is bounded and let $A(s(x_1, \dots, x_n))$ be the asymptotic center of the sequence of barycenters of (x_n).

From Theorem 4.11 we know that $(d(A(s(x_1, \dots, x_n)), T^{kN}y))$ is decreasing. Hence,

$$\begin{aligned} d(A(s(x_1, \dots, x_n)), y) &= \inf_{z \in D} d(A(s(x_1, \dots, x_n)), z) \\ &\le d(A(s(x_1, \dots, x_n)), T^N y) \le d(A(s(x_1, \dots, x_n)), y) \end{aligned}$$

and so $T^N y = y$. Therefore, applying again Theorem 4.11, for $\varepsilon > 0$ there exists n_0 such that

$$\begin{aligned} &d(A(s(x_1, \dots, x_n)), y) \\ &\quad = \inf_{z \in D} d(A(s(x_1, \dots, x_n)), z) \le d(A(s(x_1, \dots, x_n)), Ty) \\ &\quad = d(A(s(x_1, \dots, x_n)), T^{kN+1}y) \le d(A(s(x_1, \dots, x_n)), y) + \varepsilon \end{aligned}$$

for $k \ge n_0$. Now, the arbitrariness of ε implies that

$$d(A(s(x_1, \dots, x_n)), Ty) = d(A(s(x_1, \dots, x_n)), y)$$

and so it suffices to recall the uniqueness of y to finish the proof. $\square$

Along all this section we have replaced the asymptotic centers of the sequences of iterates $A(x_n)$ by the asymptotic center of the sequence of barycenters $A(s(x_1, \dots, x_n))$. We have not been able to establish a direct relation between them even for spaces of constant negative curvature. However, we think that the following two questions should find a positive answer at least in these spaces.

Question 4.12. Under conditions of Theorem 4.6, is $A(s(x_1, \dots, x_n)) = A(x_n)$?

Rouhani proved in [20], [21] that the sequence $((x_1 + \dots + x_n)/n)$ where x_n is as in Theorem 2.15 is weak convergent to $A(x_n)$. Δ-convergence of sequences, which coincides with weak convergence in Hilbert spaces, has been recently associated to weak convergence in CAT(0) spaces in [4], [12].

Question 4.13. Is the sequence of barycenters $s(x_1, \dots, x_n)$ Δ-convergent to its asymptotic center?

After Question 4.12 we do not know whether the following corollary provides a different information than Corollary 3.7. In any case, Corollary 3.8 would also follow from this one.

Corollary 4.14. *Let D be a nonempty subset of $\mathbb{H}^\infty$ and $T\colon D \to D$ a nonexpansive mapping. Then T has a fixed point if and only if there is $x \in D$ for which the sequence of its iterates (x_n) is bounded and there is a unique $y \in D$ such that*

$$d(A(s(x_1, \ldots , x_n)), y) = \operatorname{dist}(A(s(x_1, \ldots , x_n)), D).$$

5. $\mathbb{R}$-trees

In our last section we will study the same problem as in Section 4 for $\mathbb{R}$-trees. The very special geometry of $\mathbb{R}$-trees will not only provide us with a simpler approach to the problem but it will also allow us for a multi-valued version of our results. We begin studying the single-valued case.

Theorem 5.1. *Let (M, d) be a complete $\mathbb{R}$-tree, D a nonempty subset of M and $T\colon D \to D$ an asymptotically nonexpansive in the intermediate sense mapping. Moreover, suppose that there is N such that T^N is nonexpansive. Then T has at least one fixed point if and only if there is $x \in D$ such that $\{T^n(x) : n \in \mathbb{N}\}$ is bounded and there exists a unique $y \in D$ such that $d(A(T^n(x)), y) = \operatorname{dist}(A(T^n(x)), D)$.*

Proof. Necessity is obvious, let us prove the sufficiency. Suppose that the sequence (x_n), where $x_n = T^n(x)$, is bounded and let us choose an arbitrary $y \in D$. Denote $A(x_n)$ by p and $T^n(y)$ by y_n for $n \in \mathbb{N}$. The proof follows the same patterns as those Theorems 4.6 and 4.11. In this proof we will only show that the sequence $(d(p, y_n))$ satisfies

$$\lim_{n\to\infty} d(p, y_n) = \inf_{n\in\mathbb{N}} d(p, y_n),$$

the rest follows after applying similar arguments.

Let us choose $i \in \mathbb{N}$ and fix $\varepsilon > 0$. By definition it follows that $d(y_{i+m}, x_{n+m}) \leq d(y_i, x_n) + \varepsilon$ for all n and for each $m > M$ for a certain M. We claim that $d(y_{i+m}, p) \leq d(y_i, p) + \varepsilon$ for m as above. Suppose that it is not. Hence $d(y_{i+m}, p) = d(y_i, p) + \varepsilon + \delta$, where $\delta > 0$.

On account of Lemma 3.5 there is a subsequence $(x_{q(n)})$ of (x_n) such that

$$\lim_{n\to\infty} d(x_{q(n)}, p) = \limsup_{n\to\infty} d(x_n, p) =: r(x_n) \quad \text{and} \quad \lim_{n\to\infty} P_{[y_{i+m},p]}(x_{q(n)}) = p.$$

Clearly

$$d(y_{i+m}, x_{q(n)}) \to d(y_{i+m}, p) + r(x_n) = d(y_i, p) + \varepsilon + r(x_n) + \delta.$$

Since T is asymptotically nonexpansive in the intermediate sense, we get $d(x_{q(n)-m}, y_i) \geq d(x_{q(n)}, y_{i+m}) - \varepsilon$, what yields to

$$\limsup_{n\to\infty} d(x_{q(n)-m}, y_i) \geq d(y_i, p) + r(x_n) + \delta$$

and finally

$$\limsup_{n\to\infty} d(x_{q(n)-m},p) \geq \limsup_{n\to\infty} d(x_{q(n)-m},y_i) - d(y_i,p) \geq r(x_n) + \delta,$$

a contradiction. Hence $\limsup_{m\to\infty} d(y_{i+m},p) \leq d(y_i,p)+\varepsilon$ and finally $\limsup_{n\to\infty} d(y_n,p) \leq d(y_i,p)$ for all $i \in \mathbb{N}$. $\square$

The next example shows that the nonexpansivity condition on T^N is needed in Theorem 5.1.

Example 5.2. Suppose that $M = [0,1]$ and $D = \{1, 2^{-1}, 3^{-1}, \dots, 0\}$. Let us define $T\colon D \to D$ as

$$T(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{n+1} & \text{if } x = \dfrac{1}{n}. \end{cases}$$

Then a case by case study shows that $\sup_{x,y\in D} (|T^N(x) - T^N(y)| - |x-y|) \leq N^{-1}$ for each $N \geq 1$ and so T is asymptotically nonexpansive in the intermediate sense. T is fixed point free. All sequences of iterates are convergent to $0 \in D$. However, $T^N(0) - T^N(n^{-1}) = \frac{n}{N(N+n)}$, $N \geq 1$ and so there is no nonexpansive iterate of T.

We end our work with a multi-valued version of Theorem 5.1. Remember first that for D a nonempty subset of a metric space M, a mapping $T\colon D \to 2^M$ is said to be *nonexpansive* if $H(Tx,Ty) \leq d(x,y)$ for each $x,y \in D$, where $H(\,\cdot\,,\,\cdot\,)$ stands for the usual Hausdorff distance, and $x \in D$ is said to be a *fixed point of* T if $x \in Tx$.

Theorem 5.3. *Let (M,d) be a complete $\mathbb{R}$-tree and $D \subseteq M$ nonempty. Let $T\colon D \to 2^M$ be a nonexpansive mapping with nonempty, bounded, closed and convex values such that the topological boundary of $T(x)$ is contained in D for each $x \in D$. Then T has a fixed point if and only if there is $x \in D$ such that the sequence (x_n) defined by $x_{n+1} := P_{T(x_n)}(x_n)$ is bounded and there is a unique $y \in D$ such that $d(A(x_n),y) = \operatorname{dist}(A(x_n),D)$.*

Proof. As in the previous proof, we make $p = A(x_n)$ the asymptotic center of the iterates of x. Let $z \in D$ and $z' = P_{T(z)}(z)$. It is enough to show that

$$d(p,z') \leq d(p,z) \quad \text{for any } z \in D. \tag{5.1}$$

We can assume that $z \notin T(z)$ since otherwise (5.1) easily follows. Consider now the sequence (x_n) given in the statement, we claim that

$$d(x_{n+1},z') \leq d(x_n,z) \tag{5.2}$$

holds for each n.

If $P_{T(x_n)}(z') = x_{n+1}$ then it must be the case that

$$d(z', x_{n+1}) = \operatorname{dist}(z', T(x_n)) \leq H(T(z), T(x_n)) \leq d(z, x_n).$$

Otherwise it must be the case that $x_{n+1} \in [x_n, z']$. Indeed, assume that this is not the case. Then $T(x_n) \cap [x_n, z'] = \emptyset$ and $x_{n+1} = P_{T(x_n)}(z')$, a contradiction.

Now, assume that $P_{T(z)}(x_{n+1}) = z'$, then

$$d(z', x_{n+1}) = \operatorname{dist}(x_{n+1}, T(z)) \leq H(T(z), T(x_n)) \leq d(z, x_n).$$

If this is not that case, $z' \in [x_{n+1}, z]$ and combining it with $x_{n+1} \in [x_n, z']$ we obtain

$$d(z, x_n) = d(z, z') + d(z', x_{n+1}) + d(x_{n+1}, x_n) > d(z', x_{n+1}),$$

which proves (5.2).

Now, from Lemma 3.5, there exists a subsequence $(x_{q(n)})$ such that

$$\lim_{n\to\infty} d(x_{q(n)}, p) = r(x_n) \quad \text{and} \quad \lim_{n\to\infty} P_{[z',p]}(x_{q(n)}) = p.$$

Clearly,

$$d(z', x_{q(n)}) \to d(z', p) + r(x_n) = d(z, p) + r(x_n) + \delta$$

where $\delta = d(p, z') - d(p, z)$.

Now, by (5.2), $d(x_{q(n)-1}, z) \geq d(x_{q(n)}, z')$ and so

$$\limsup d(x_{q(n)-1}, z) \geq d(z, p) + r(x_n) + \delta$$

which finally leads to

$$\limsup d(x_{q(n)-1}, p) \geq \limsup d(x_{q(n)-1}, z) - d(z, p) \geq r(x_n) + \delta.$$

Therefore $\delta \leq 0$ and so $d(p, z') \leq d(p, z)$ as we wanted to prove. $\square$

6. Appendix: Fixed points of semigroup of contractions

We close this work with an appendix where we study the existence of fixed points for strongly continuous semigroups of contractions defined on non-convex subsets of spaces with constant negative curvature and $\mathbb{R}$-trees. This problem is inspired by Rouhani's results for semigroups defined on non-convex subsets of Hilbert spaces [20] as well as results on extension of semigroups by Komura [15]. Let us begin with the definition of a strongly continuous semigroup of contractions. Suppose that X is a CAT(0) space and a set $D \subset X$.

Definition 6.1. A one parameter family $(S(t))_{t\geq 0}$ of mappings from D into D is called *a strongly continuous semigroup of contractions* on D if:

(a) $S(0)(x) = x$ for all $x \in D$;
(b) $S(t+s)(x) = S(t)(S(s)(x))$ for all $s, t \geq 0$ and $x \in D$;
(c) $\lim_{t\to t_0} S(t)(x) = S(t_0)(x)$ for all $t_0 \geq 0$ and $x \in D$;
(d) $d(S(t)(x), S(t)(y)) \leq d(x,y)$ for all $t > 0$ and $x, y \in D$.

A point $x \in D$ will be said to be a fixed point of the family $(S(t))_{t\geq 0}$ if $S(t)(x) = x$ for each t.

In the same way as in case of a bounded sequence of X we may consider $\{T(t) \colon t \geq 0\}$ a bounded (continuous) curve in X. For $x \in X$ set

$$r(x, (T(t))) = \limsup_{t\to\infty} d(x, T(t)).$$

The *asymptotic radius* $r(T(t))$ of $\{T(t) \colon t \geq 0\}$ is given by

$$r(T(t)) = \inf\{r(x, (T(t))) : x \in X\},$$

and the *asymptotic center* of $\{T(t) \colon t \geq 0\}$, which we will generically denote as $A(T(t))$, is given by the set

$$A(T(t)) = \{x \in X : r(x, (T(t))) = r(T(t))\}.$$

The proof of the following property follows the same patterns as in the case of bounded sequences (see, for instance, [12]).

Proposition 6.2. *Let $\{T(t) \colon t \geq 0\}$ be a bounded continuous curve in a space* CAT(0). *Then the asymptotic center of $\{T(t) \colon t \geq 0\}$ is a singleton. Moreover,*

$$A(T(t)) \in \overline{\mathrm{conv}}\{T(t) \colon t \geq 0\}.$$

Similar results as those from Sections 3 and 4 hold for nonlinear contraction semigroups.

Theorem 6.3. *Let D be a nonempty subset of a complete $CAT(0)$ space with constant curvature $\kappa \in (-\infty, 0)$ and $(S(t))_{t\geq 0}$ a strongly continuous semigroup of contractions from D into D. Then S has at least one fixed point if and only if there is $x \in D$ for which the curve $\{S(t)(x) \colon t \geq 0\}$ is bounded and there is a unique $y \in D$ such that*

$$d(A(S(t)(x)), y) = \mathrm{dist}(A(S(t)(x))), D),$$

where $A(S(T)(x))$ is the asymptotic center of the curve $\{S(t)(x) \colon t \geq 0\}$.

Proof. Let $A(S(t)(x)) =: p$. First we want to show that for each $s > 0$ any nonexpansive extension T of $S(s)$ to $D \cup \{p\}$ must verify that $Tp = p$. Notice such extensions exists from Theorem 3.4.

$$\limsup_{t\to\infty} d(T(p), S(t)(x))) = \lim_{n\to\infty} d(T(p), S(t_n)(x)).$$

Condition (d) in Definition 6.1 implies that

$$\lim_{n\to\infty} d(T(p), S(t_n)(x)) \le \liminf_{n\to\infty} d(p, S(t_n - s)(x)) \le \limsup_{t\to\infty} d(p, S(t)(x)),$$

what yields

$$\limsup_{t\to\infty} d(T(p), S(t)(x))) \le \limsup_{t\to\infty} d(p, S(t)(x)),$$

and, on account of Proposition 6.2, $T(p) = p$.

Now still for the same $s > 0$ and y chosen with respect to the assumptions

$$d(p, S(s)(y)) = d(T(p), T(y)) \le d(p, y),$$

so $S(s)(y) = y$. Since $s > 0$ was chosen without any additional assumption, we get $S(t)(y) = y$ for all $t \ge 0$. □

Remark 6.4. Notice that we have also shown that the point p in the above proof is an absolute fixed point for the family $(S(t))_{t\ge 0}$.

Corollary 6.5. *Let D be a nonempty subset of a complete $CAT(0)$ space with constant curvature $\kappa \in (-\infty, 0)$ and $(S(t))_{t\ge 0}$ a strongly continuous semigroup of contractions from D into D. Then S has at least one fixed point if and only if there is $x \in D$ for which the curve $\{S(t)(x): t \ge 0\}$ is bounded and for any $y \in \overline{\mathrm{conv}}\{S(t)(x): t \ge 0\}$ there is a unique $z \in D$ such that $d(y, z) = \mathrm{dist}(y, D)$.*

In the same way we can obtain equivalent results for $\mathbb{R}$-trees.

Theorem 6.6. *Let (M, d) be a complete $\mathbb{R}$-tree, D a nonempty subset of M and $(S(t))_{t\ge 0}$ a strongly continuous semigroup of contractions from D into D. Then S has at least one fixed point if and only if there is $x \in D$ such that the curve $\{S(t)(x): t \ge 0\}$ is bounded and there exists a unique $y \in D$ such that $d(A(S(T)(x)), y) = \mathrm{dist}(A(S(T)(x)), D)$, where, again, $A(S(T)(x))$ is the asymptotic center of the curve $\{S(t)(x): t \ge 0\}$.*

Remark 6.7. In the case when the curvature $\kappa \in (-\infty, 0)$ we have a second approach to the same problem. Instead of extending mappings we could work out of the notion of barycenter and follow an approach similar to that one leading to Corollary 4.14. This would lead to a result similar to Theorem 6.3 where the asymptotic center of the points in the curve would be replaced by certain barycenter obtained through integration in a similar way as it is done in [20]. Again, after Question 4.12, we do not know whether this would lead to a result different than Theorem 6.3.

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